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Computation of Fresnel Integrals. II

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This paper describes an improved method for computing Fresnel integrals with an error of less than 1×10^{-9}. The method is based on a known approximate formula for a different integral which is due to Boersma and referenced by Abramowitz and Stegun.

Key words: computation; Fresnel integrals; rational approximations; series expansions.

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1. Improved Computation of Fresnel Integrals

In a previous paper [1], this author presented formulas for numerical computations of the Fresnel cosine and sine integrals,

$$C(x) = \int_0^x \mathrm{d}t \cos\left(\frac{\pi}{2} t^2\right) = \frac{1}{2} + f(x)\sin\left(\frac{\pi}{2} x^2\right) - g(x)\cos\left(\frac{\pi}{2} x^2\right), \tag{1a}$$

$$S(x) = \int_0^x \mathrm{d}t \sin\left(\frac{\pi}{2} t^2\right) = \frac{1}{2} - f(x)\cos\left(\frac{\pi}{2} x^2\right) - g(x)\sin\left(\frac{\pi}{2} x^2\right), \tag{1b}$$

to six significant figures by using the first three terms of the Taylor expansions of $C(x)$ and $S(x)$ for $|x| \leq 0.6$, the first three terms of the asymptotic expansions of the auxiliary functions $f(x)$ and $g(x)$ for $|x| \geq 3$, and modified rational approximations of $f(x)$ and $g(x)$ for the mid range. These formulas proved hard to use because they involve too many numerical constants.

It was found subsequently that a simpler and more accurate method of computation can be based on a formula derived by Boersma [2],

$$\int_0^u \frac{\mathrm{d}\tau \mathrm{e}^{-\mathrm{i}\tau}}{\sqrt{2\pi\tau}} = \frac{1-\mathrm{i}}{2} + \mathrm{e}^{-\mathrm{i}u}\sqrt{\frac{4}{u}} \sum_{n=0}^{11} (p_n + \mathrm{i}q_n)\left(\frac{4}{u}\right)^n, \quad u \geq 4, \tag{2a}$$

where p_n and q_n are numerical constants tabulated in Ref. [2] and the notation has been changed in order to avoid confusion with symbols used elsewhere in this paper. On substituting $t = \pi\tau^2/2$ and $x = \pi u^2/2$ this is transformed into

$$\int_0^x \mathrm{d}t \mathrm{e}^{-\mathrm{i}\pi t^2/2} = C(x) - \mathrm{i}S(x) = \frac{1-\mathrm{i}}{2} - \mathrm{e}^{-\mathrm{i}\pi x^2/2} \sum_{n=0}^{11} \frac{g_n - \mathrm{i}f_n}{x^{2n+1}}, \quad x \geq 1.6, \tag{2b}$$

$$f_n = (8/\pi)^{n+1/2} q_n, \quad g_n = -(8/\pi)^{n+1/2} p_n. \tag{2c}$$

Hence it follows by separation of real and imaginary parts and comparison with Eqs. (1a) and (1b) that

$$f(x) = \sum_{n=0}^{11} f_n x^{-2n-1}, \quad g(x) = \sum_{n=0}^{11} g_n x^{-2n-1}. \tag{2d}$$

The required coefficients, as computed from Boersma's data and Eq. (2c), are listed in Table 1. Equation (2d) was tested by computing sample values of $C(x)$ and $S(x)$ and comparing them to the values tabulated in Ref. [3] to eight digits. The agreement was perfect, which is consistent with Boersma's statement that the error of Eq. (2a) is less than 5×10^{-10}.

Table 1. Numerical values of f_n and g_n

n	f_n	g_n
0	0.318309844	0
1	9.34626E-08	0.101321519
2	−0.09676631	−4.07292E-05
3	0.000606222	−0.152068115
4	0.325539361	−0.046292605
5	0.325206461	1.622793598
6	−7.450551455	−5.199186089
7	32.20380908	7.477942354
8	−78.8035274	−0.695291507
9	118.5343352	−15.10996796
10	−102.4339798	22.28401942
11	39.06207702	−10.89968491

The above method is implicitly contained in Ref. [3], which mentioned Boersma's paper as well as the relationship between the integrals in Eqs. (1a), (1b), and (2a). Boersma also gave an approximation formula similar to Eq. (2a) for $|u| \leq 4$, or $|x| \leq 1.6$. This was not used in this work because in this range it is simpler to compute $C(x)$ and $S(x)$ by using their Taylor expansions [3],

$$C(x) = \sum_{n=0}^{\infty} c_n x^{4n+1}, \quad c_0 = 1,$$

$$c_{n+1} = \frac{-\pi^2(4n+1)c_n}{4(2n+1)(2n+2)(4n+5)}, \tag{3a}$$

$$S(x) = \sum_{n=0}^{\infty} s_n x^{4n+3}, \quad s_0 = \frac{\pi}{6},$$

$$s_{n+1} = \frac{-\pi^2(4n+3)s_n}{4(2n+2)(2n+3)(4n+7)}. \tag{3b}$$

These give results with errors less than 6×10^{-10} for $|x| \leq 1.6$ if the first 11 terms of the expansions are carried.

Software and algorithms for computing Fresnel integrals in Fortran and C (not based on this paper) are also available on the Internet [4,5].

2. References

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