Improvements in the Application Of the Numerical Method of Characteristics To Predict Attenuation in Unsteady Partially Filled Pipe Flow

Volume 91 Number 3 May–June 1986

John A. Swaffield and Katrina Maxwell-Standing

Brunel University, Uxbridge, Middlesex, United Kingdom

The use of linear interpolation and simplified iteration procedures are shown to introduce inaccuracies to the rectangular grid method of characteristics, particularly when applied to subcritical flows. Comparisons of experimental and computational results are presented illustrating the use of Everett and Newton-Gregory interpolation, in addition to a more complex iteration procedure, to substantially improve the method's ability to maintain both steady uniform flows under subcritical conditions, and retain wave steepness during propagation along the drainage pipe. The results presented will be directly applicable to the building drainage network model previously developed at Brunel University with the support of NBS CBT grant aid.

Key words: integration; interpolation; method of characteristics; rectangular grid (R.G.).

Accepted: February 25, 1986

1. Introduction

A computer model that will simulate the propagation of steep fronted waves along a drainage pipe is being developed from recent research by Bridge [1][1] and Swaffield et al. [2]. The rectangular grid (R.G.) method of characteristics, employed to solve numerically the appropriate unsteady flow equations, was used to model the attenuation of the flow profiles in the pipe. Several papers describe the derivation and applications of the numerical solutions of the governing equations for transient partially filled pipe flow [1,2,3]. This paper presents results of an investigation into the computational problems identified by Swaffield and Bridge. One example is the collapse of the water surface profile over time during the passage of steady flows under subcritical conditions. It was felt that these inaccuracies were probably caused by the linear interpolation and simplified iteration procedures being used within the method of characteristics.

Numerous computer runs were made to determine if the accuracy of the method could be improved by employing a nonlinear interpolation procedure and by using a more complex iteration procedure. The computed results were compared to data from experiments carried out on the Brunel test rig.

About the Authors: John A. Swaffield was with Brunel when the work described was performed. He is now a professor in the Department of Building, Heriot-Watt University, Edinburgh, Scotland. Katrina Maxwell-Standing is with Brunel's Mechanical Engineering Department. The research was conducted under NBS Grant No. 60NANB5D0510.

[1] Figures in brackets indicate literature references.

NOTATION

A = water depth cross-sectional area (m^2)

c = local wavespeed (m/s)

D = pipe diameter (m)

f_n = $f(x_n)$

g = acceleration due to gravity (m/s^2)

h = constant step length (m)

S_0 = pipe slope

S_f = slope of energy grade line

T = water surface width (m)

t = time (s)

$TFAC$ = time factor

V = local flow average velocity (m/s)

x = distance from upstream boundary (m)

y = local flow depth (m)

$\delta^i f_n$ = i^{th} central difference at $x = x_n$.

2. The Rectangular Grid Method of Characteristics

The St. Venant equations are a pair of quasi-linear hyperbolic partial differential equations. The equation of continuity is expressed

$$V\frac{\partial A}{\partial x} + A\frac{\partial V}{\partial x} + T\frac{\partial y}{\partial t} = 0 \tag{1}$$

and the dynamic equation,

$$g\,(S_f - S_0) + g\frac{\partial y}{\partial x} + V\frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} = 0. \tag{2}$$

The St. Venant equations can be transformed into their characteristics in a number of ways; Fox [3] gives one method. The equations describing the positive and negative characteristics are:

$$\frac{dV}{dt} + \frac{g}{c}\frac{dy}{ct} + g\,(S_f - S_0) = 0 \tag{3}$$

$$\frac{dx}{dt} = V + c \tag{4}$$

$$\frac{dV}{dt} - \frac{g}{c}\frac{dy}{dt} + g\,(S_f - S_0) = 0 \tag{5}$$

$$\frac{dx}{dt} = V - c \tag{6}$$

where $c = g\sqrt{A/T}$. The positive characteristic is described by eqs (3 and 4) and the negative characteristic by (5 and 6).

These equations are then numerically integrated. An approximation is introduced in the integration of some terms as the variation of V and y is not known as a function of time. Referring to figure 1, if the velocity and depth are known at points R and S at time level $t - \Delta t$ then a first-order approximation leads to the following set of equations which may be written in terms of the unknown depth and velocity at point P at time level t,

$$V_P - V_R + \frac{g}{c_R}(y_P - y_R) + g\,(S_R - S_0)\,\Delta t = 0 \tag{7}$$

$$x_P - x_R = (V_R + c_R)\,\Delta t \tag{8}$$

$$V_P - V_S + \frac{g}{c_s}(y_P - y_S) + g\,(S_S - S_0)\,\Delta t = 0 \tag{9}$$

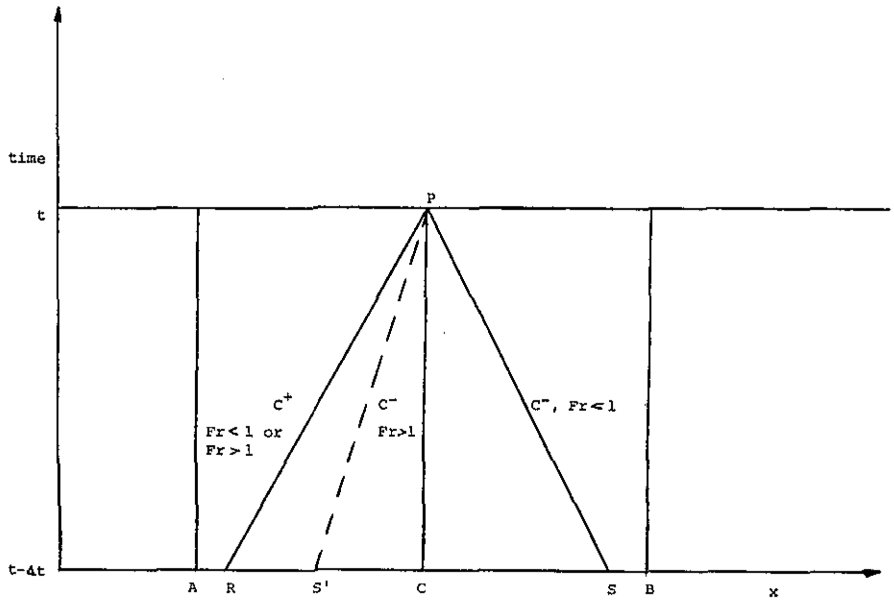
$$x_P - x_S = (V_S - c_S)\,\Delta t\ . \tag{10}$$

These equations are paired such that eq (7) is true only if eq (8) is satisfied and similarly for eqs (9 and 10). In most cases a first order integration is satisfactory, however attention must be given to the choice of the time step to ensure a stable solution, as mentioned in [3]. Generally, applying the Courant condition,

$$\Delta t = \frac{\Delta x}{(V + c)_{max}} \tag{11}$$

ensures that the points R and S fall within $\pm\ \Delta x$ of point P. Since the velocity and celerity both vary throughout the duration of the analysis the maximum value of each is found at every time step.

The algebraic solution of eqs (7–10) reduces to a problem of determining the values of x_R, y_R, V_R, x_S, y_S and V_S since x_p, t_p, t_R and t_S are known values. Referring to figure 1, x_R and x_S are determined by using eqs (4 and 6) respectively to estimate the t-Δt line intersections of the positive and negative characteristtics passing through point P. The quantities y_R, y_S, V_R and V_S are then determined by interpolation between grid points A and C and C and B respectively for this example of subcritical flow.



Fr = Froude number

If Fr< 1 then flow is subcritical and S lies between C and B.

If Fr >1 then flow is supercritical and S' lies between A and C.

Figure 1–Specified time interval technique.

Finally the solution to point P is obtained through the simultaneous solution of eqs (7 through 10).

3. Inaccuracies in the Method of Characteristics

A summary of the initial and boundary conditions required to apply the R.G. method of characteristics to drainage flow is presented in [1]. The method described above was used in a model simulating the movement of a hydraulic jump in a drainage pipe. Like Streeter and Wylie [4], the positions of x_R and x_s were determined by assuming that the slope of the characteristic lines at P were equal to those at C. The depth and velocity at points R and S were determined by linearly interpolating between grid points A and C and C and B (subcritical flow). However, it was found that two problems arose. The first being that the water surface profile downstream of the jump collapsed over time, and the second that the hydraulic jump acquired a positive velocity during the passage of steady flows. Harris [5] mentions that the Streeter-Wylie approximations are reasonable provided that the values of y and V at A, B and C do not differ greatly. The values of depth and velocity across a hydraulic jump or a steep-fronted wave do differ greatly; therefore, these approximations are not valid for the case being considered and a more exact method must be employed.

4. Improvement of the R.G. Method of Characteristics

4.1 Integration

There are several ways by which the R.G. method of characteristics can be made more accurate. One method [6] is to apply the trapezoidal rule when integrating the characteristic equations. However, the resulting equations must be solved by an implicit method. The implicit scheme is fairly complicated for open channel flow problems and is seldom used as it requires an iteration procedure. In addition, the benefits derived from the more accurate integration of the characteristic equations are often lost in the interpolation methods required to evaluate conditions at R and S.

4.2 Iteration

Accuracy can also be increased by using an iterative method to determine the positions of x_R and x_S from the slopes of the characteristics. A simplified iterative method similar to one developed by Lister [7] has been used to improve the computer model. The position of x_R is initially determined by assuming that the slope of the characteristic line at P is equal to that at C; y_R and V_R are then found from a nonlinear interpolation of values at surrounding gridpoints. (The exact interpolation method will be discussed below.) x_R is then recalculated assuming that the slope of the characteristic line from P to R is equal to that of the previous value of x_R:

$$x_P - x_R{}^{(k+1)} = (V_R^k + c_R^k)\, dt \;. \tag{12}$$

This process is repeated until the difference between successive iterations of x_R is less than 0.001 meters. The same process is used to find x_S.

4.3 Interpolation

The inaccuracies caused by the effect of linearly interpolating between grid points to find the values of the depth and velocity at points R and S at each time step are illustrated in figure 2. The figure shows a section of backwater profile at the end of a pipe in which the flow is subcritical. Alternatively, this can be considered as the backwater profile between a hydraulic jump and a junction. For steady uniform flow the backwater profile should remain constant. However, linear interpolation between points A and C and points B and C produce values for depth at points R and S which are lower than the actual values. This leads to an underestimation of the depth at P at the next time step, which in turn leads to an overestimation of the velocity at P. These interpolation errors build up as the calculations continue in time causing the backwater profile to collapse and the discharge to increase. These effects have also been noted by Fox (see [1]).

Quadratic interpolation [7–9] and Langrangian [10–12] interpolation procedures have been used in the past to increase the accuracy of the R.G. method of characteristics. Jolly and Vevjevich [10] found that the Langrangian interpolation yielded more accurate results than linear interpolation. It was found that accuracy was not improved by using a more sophisticated scheme than third-order Langrangian interpolation equations combined with an iterative procedure to find the positions of R and S.

Although the Langrange method minimizes the amount of calculations it has the disadvantage that it gives no indication of the accuracy available. As a result, difference methods tend to be used in exploratory work and the Langrange methods in well-understood routine work [13]. In general, the error in using a difference method is of the order of the first term to be omitted. Formulae which involve central differences give the best accuracy for a given amount of computation. They also have the philosophical merit of giving equal weight to the tabulated data on each side of the relevant data point. The practical interpolation formulae are those of Everett and Bessel. If third differences are negligible for the accuracy required, then Bessel's formula is simpler and hence better. If third differences are greater than 60 units in the last decimal they cannot be neglected. Thus Everett's formula is best for our purposes [14]. Everett's formula involves only even-order central differences:

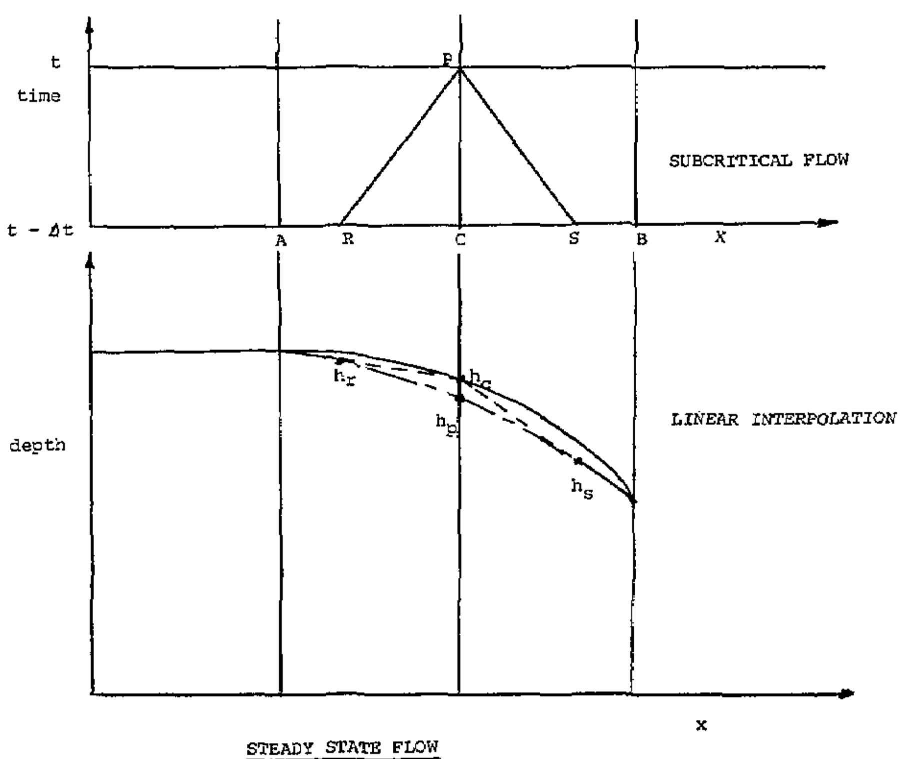


Figure 2–Interpolation error on the backwater profile.

$$P(x)=E_0f_0+E_2\delta^2f_0+E_4\delta^4f_0+\ldots$$
$$+F_0f_1+F_2\delta^2f_1+F_4\delta^4f_1+\ldots \tag{13}$$

where

$$E_0=1-u \tag{14}$$

$$E_2=\frac{-u(1-u)(2-u)}{3!} \tag{15}$$

$$E_4=\frac{-(-1-u)(u)(1-u)(2-u)(3-u)}{5!} \tag{16}$$

$$F_0=u \tag{17}$$

$$F_2=\frac{u(u^2-1)}{3!} \tag{18}$$

$$F_4=\frac{u(u^2-1)(u^2-4)}{5!} \tag{19}$$

and

$$u=\frac{(x-x_0)}{h},\quad 0<u<1. \tag{20}$$

Near the ends of the pipe where central differences cannot be obtained the Newton-Gregory forward backward difference formulae are used. The Newton-Gregory forward difference formula is expressed by:

$$P(x)=P(x_0+hu)=f_0+u\,\Delta f_0+\frac{1}{2}\left[u(u-1)\Delta^2 f_0\right]$$
$$+\frac{1}{3!}\left[u(u-1)(u-2)\Delta^3 f_0\right]+\ldots$$
$$\ldots+\frac{1}{n!}\left[u(u-1)\ldots(u-n+1)\Delta^n f_0.\right] \tag{21}$$

and the backward difference formula by:

$$P(x)=P(x_0+hu)=f_0+u\,\nabla f_0+\frac{1}{2}\left[u(u+1)\nabla^2 f_0\right]$$
$$+\frac{1}{3!}\left[u(u+1)(u+2)\nabla^3 f_0\right]+\ldots$$
$$\ldots+\frac{1}{n!}\left[u(u+1)\ldots(u+n-1)\nabla^n f_0\right]. \tag{22}$$

5. Comparison of the Results

An example of the effects of linear interpolation can be seen in figure 3 which shows subcritical partially filled flow with a wave entering the pipe at a time of 0.23 seconds. As time progressed the backwater profile collapsed. A time factor, *TFAC*, of 5.0 was used in this example. The use of a time factor causes the time step eq (11) to become smaller:

$$\Delta t=\frac{\Delta x}{(V_{\max}+c_{\max})\,TFAC}\,. \tag{23}$$

It is important to determine a value of *TFAC* that results in a converged solution, whilst not requiring excessive computation time. If *TFAC* is chosen sufficiently large then a converged solution will always be achieved, provided that the results are not dominated by rounding and truncation errors. Generally, the smallest *TFAC* that will allow stability is

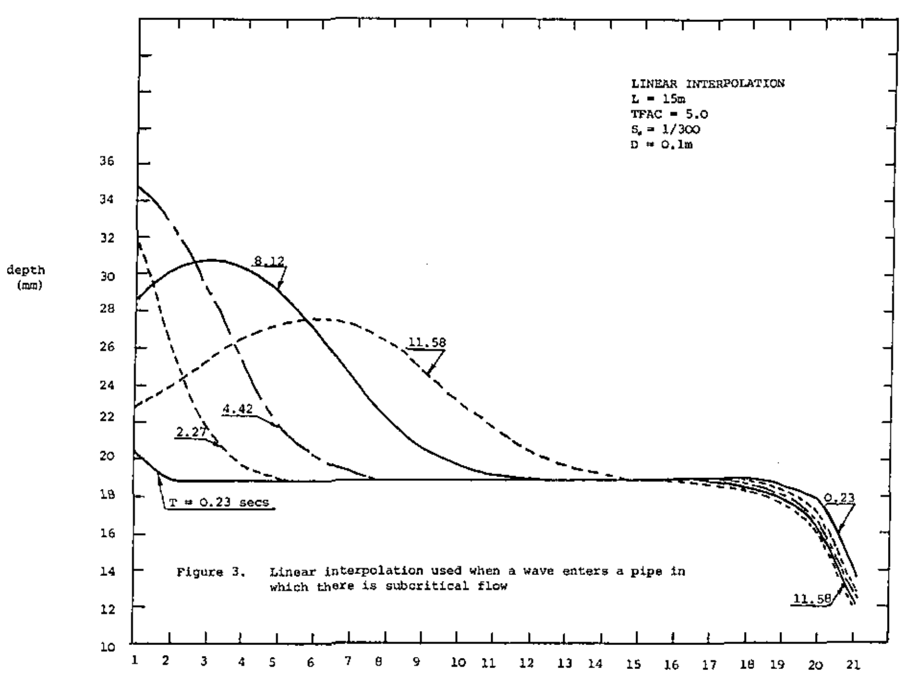


Figure 3–Linear interpolation used when a wave enters a pipe in which there is subcritical flow.

used. The time factors used in these results were determined by trial and error.

Figure 4 shows the effects of using Everett and Newton-Gregory interpolation on the same flow conditions of the previous example. A *TFAC* of 1.0 is the smallest *TFAC* that would allow stability, from eq (11). It can be seen that the backwater profile remained unchanged with increasing time. On comparing the two figures it can be seen that the linear method tended to flatten out the wave whereas Everett's method retained the wave's steepness. Everett's method also remained stable with a *TFAC* of 1.0. When linear interpolation was used with a *TFAC* of 1.0 a discontinuity developed on the wave front at a time of 11.31 seconds as shown in figure 5.

Figure 6 shows the experimentally measured depth profile at a distance of 8.0 meters from the entrance of the pipe against time. It can be seen that in this case both Everett's method and the linear interpolation method used with a *TFAC* of 1.0 retained the wave's steepness and compare most

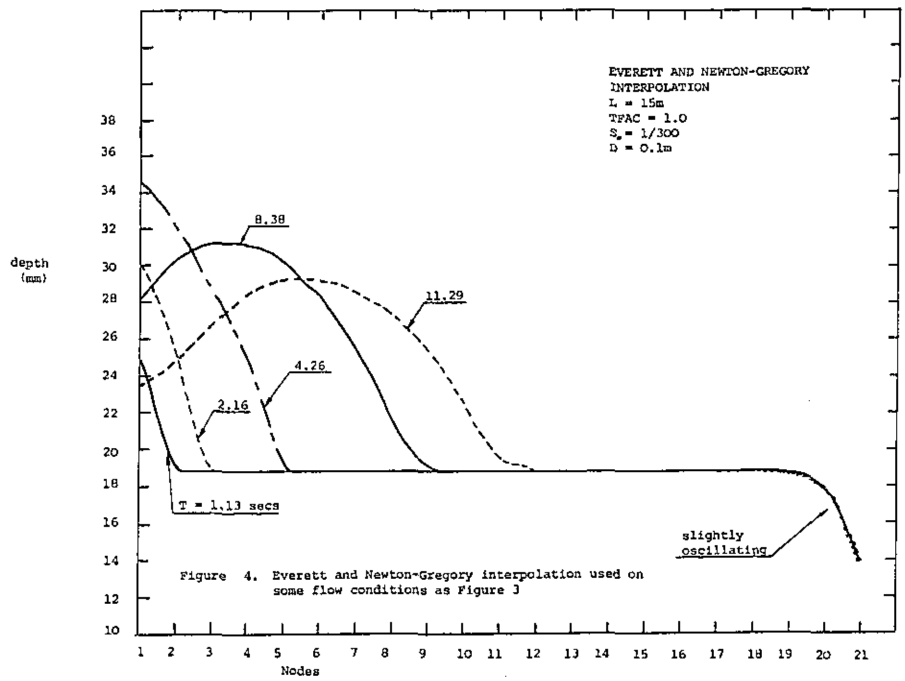


Figure 4–Everett and Newton-Gregory interpolation used on some flow conditions as figure 3.

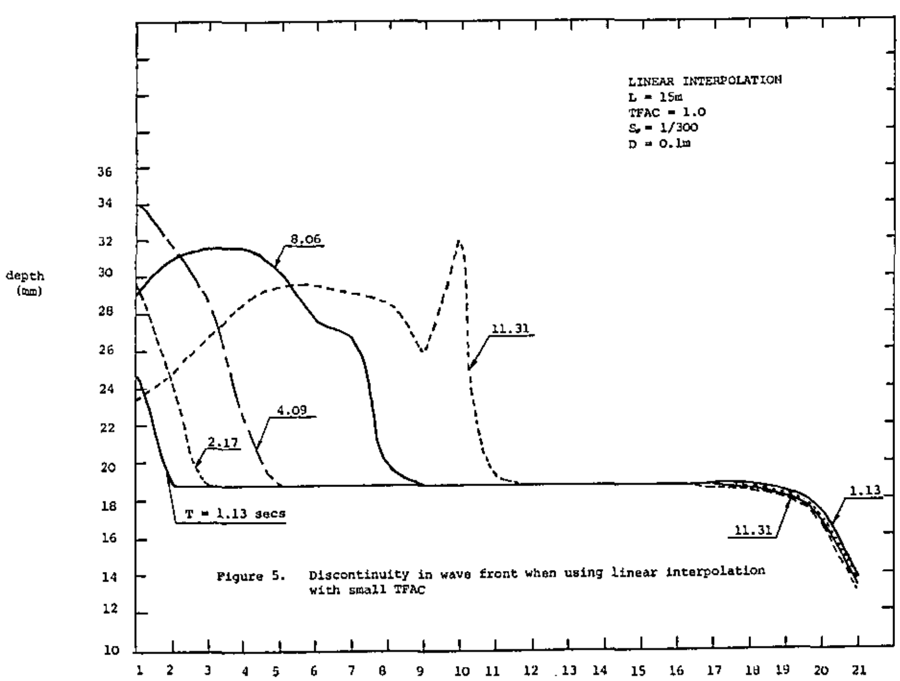


Figure 5–Discontinuity in wave front when using linear interpolation with small *TFAC*.

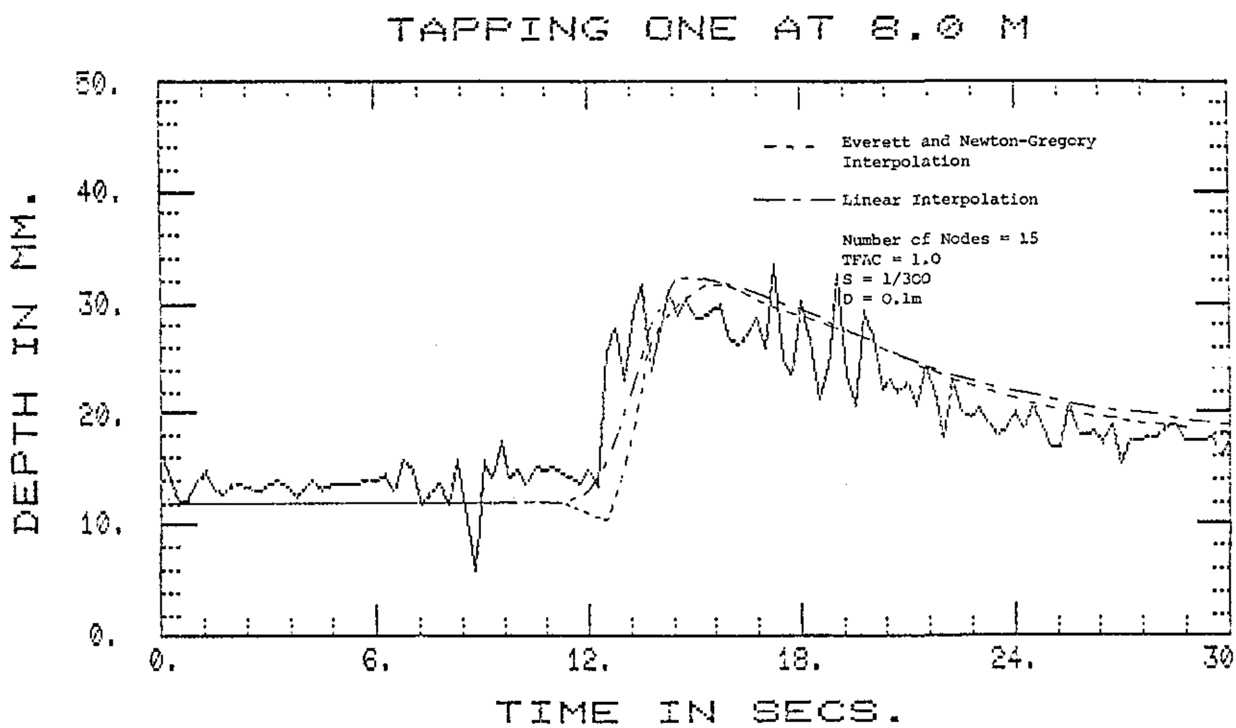


Figure 6—Comparison of experimental and computed results used to predict wave attenuation.

favorably with the experimental data from the Brunel test rig.

An increasing backwater profile is created where a drainage pipe joins a junction. Figure 7 shows the propagation of a wave entering such a pipe in a case where Everett and Newton-Gregory interpolation with a *TFAC* of 3.0 was used. It can be seen that the backwater profile remained steady and the steepness of the wave was retained. When linear interpolation was used with the same *TFAC* the backwater profile increased as time progressed. Thus Everett's method is clearly an improvement

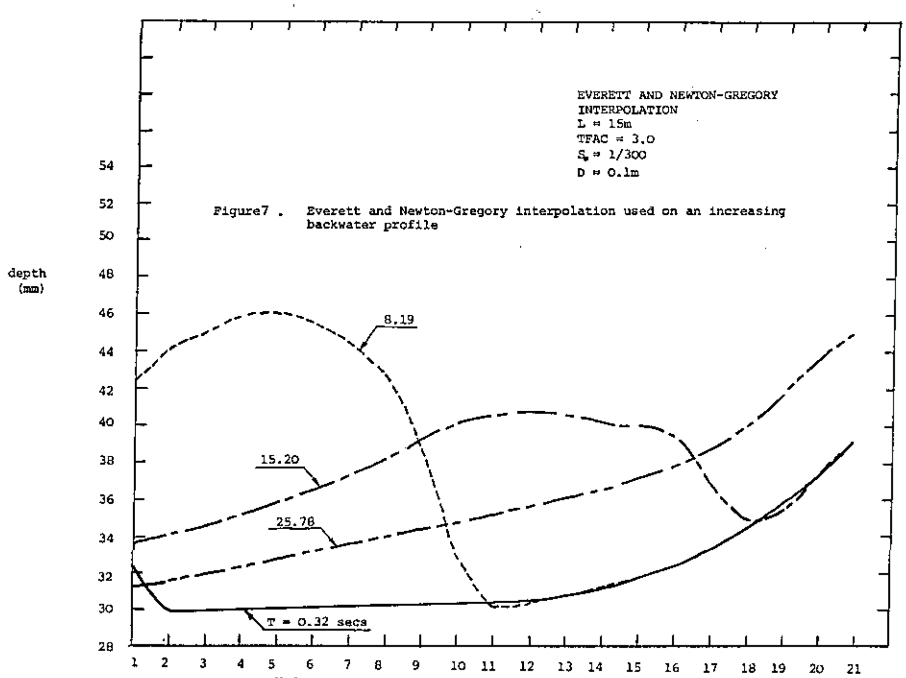


Figure 7—Everett and Newton-Gregory interpolation used on an increasing backwater profile.

to linear interpolation as it retains the backwater profile and wave steepness when run with a small *TFAC*.

6. Conclusions

A computer model, based on the rectangular grid method of characteristics that will simulate the propagation of steep fronted waves along a drainage pipe is being developed from recent research by Bridge and Swaffield [1,2]. Experimental tests carried out on the Brunel test rig as well as numerous computer runs have shown that the accuracy of the model has been considerably improved by including an interation procedure to find the positions of the space time coordinates R and S and also by using Everett's interpolation to determine the height and velocity at points R and S. The improvements made have solved the problem of the decreasing backwater profile during steady uniform flow noted by Bridge. The improvements also retain the steepness of waves shown in test data as they travel along the branch pipe.

References

[1] Bridge, S., A study of unsteady flow wave attenuation in partially filled pipe networks. Unpublished PhD thesis, Brunel University, May 1984.

[2] Swaffield, J. A.; S. Bridge and L. S. Galowin, Mathematical modelling of time dependent wave attenuation and discrete solid body transport in gravity driven partially filled pipe flows. Finite Elements in Water Resources. Proc. 4th Int. Conf. Hanover, Germany, 1982.

[3] Fox, J. A., Hydraulic Analysis of Unsteady Flow in Pipe Networks. MacMillan, London, 1977.

[4] Streeter, V. L., and E. B. Wylie, Hydraulic Transients. McGraw-Hill, New York, 1967.

[5] Harris, G. S., Real time routing of flood hydrographs in storm savers. J. Hyd. Div., A.S.C.E., 96, HY6, 1970.

[6] Wylie, E. B., Unsteady free surface flow computations. J. Hyd. Div., A.S.C.E., 96, HY11, 1970.

[7] Lister, M., The numerical solutions of hyperbolic partial differential equations by the method of characteristics. Mathematical Methods for Digital Computers, Ralston and Wilf, eds., John Wiley & Sons, 1960.

[8] Mozayeng, B., and C. S. Song, Propagation of flood waves in open channels. J. Hyd. Div., A.S.C.E., 95, HY3, 1969.

[9] Zovne, J. J., and C. S. Martin, Simulation of transient supercritical channel flow. J. Hyd. Div., A.S.C.E., 105, HY7, 1979.

[10] Jolly, J. P., and V. Yevjevich, Simulation accuracies of gradually varied flow. J. Hyd. Div., A.S.C.E., 100, HY7, 1974.

[11] Sivaloganathan, K., Comparative study of characteristic methods for free surface flow computations. Proc. Int. Conf. on Urban Storm Drainage, Southampton, 1978.

[12] Sivaloganathan, K., Channel flow computations using characteristics. J. Hyd. Div., A.S.C.E., 105, HY7, 1979.

[13] Hamming, R. W., Numerical Methods for Scientists and Engineers, McGraw-Hill, 1973.

[14] Spencer, A. J. M. et al. Engineering Mathematics. Van Nostrand Reinhold, England, 1977.