

- [18] Foster, F., and A. Hargreaves, The use of moments of x-ray intensity in space group determination I. Derivation of theoretical moments, *Acta Crystallogr.* **16**, 1124–1133 (1963); II. Practical application, *ibid* **16**, 1133–1138 (1963).
- [19] Shmueli, U., and U. Kaldor, Calculation of even moments of the trigonometric structure factor. Methods and results, *Acta Crystallogr.* **A37**, 76–80 (1981).
- [20] Shmueli, U., and U. Kaldor, Moments of the trigonometric structure factor, *Acta Crystallogr.* **A39**, 619–621 (1983).
- [21] Barakat, R., Isotropic random flights, *J. Phys.* **A6**, 796–804 (1973).
- [22] Barakat, R., First-order statistics of combined random sinusoidal waves with applications to laser speckle patterns, *Optica Acta* **21**, 921 (1974).
- [23] Hamming, R., *Numerical Methods for Scientists and Engineers*, (McGraw-Hill, New York, 2d ed. (1973).
- [24] *International Tables for x-ray Crystallography* ed. N.F.M. Henry, K. Lonsdale, Kynoch Press, Birmingham, 2d ed. (1965).
- [25] Rogers, D., and A. J. C. Wilson, The probability distribution of x-ray intensities. V. A note on some hypersymmetric distributions, *Acta Crystallogr.* **6**, 439–449 (1953).
- [26] Cochran, W., and M. M. Woolfson, The theory of sign relations between structure factors. *Acta Crystallogr.* **8**, 1–12 (1955).

DISCUSSION

of the Weiss-Shmueli paper,
Fourier Representations of Pdf's
Arising in Crystallography

E. Prince

Institute for Materials Science and Engineering
National Bureau of Standards

This interesting paper by Drs. Weiss and Shmueli represents a substantially exact solution of a problem that has concerned crystallographers for more than 35 years, the analysis in terms of atomic structure of x-ray diffraction data. (Similar information can be obtained from the diffraction of electrons and neutrons, but, for reasons that are both experimental and theoretical, this information is mainly used to supplement that obtained from x-ray diffraction, which remains the basic tool of the structural crystallographer.) The observed intensity in x-ray diffraction is given by

$$I = SL |F(\mathbf{h})|^2,$$

where S is a scale factor, L is a geometrical factor, and $F(\mathbf{h})$, commonly called the structure factor, is the Fourier transform of the electron density in a crystal. It may be written in the form

$$F(\mathbf{h}) = \int \rho(\mathbf{r}) \exp(2\pi i \mathbf{h} \cdot \mathbf{r}) d\mathbf{r}.$$

The density function, $\rho(\mathbf{r})$, in a crystal is periodic in three dimensions, so that it can be represented as a convolution of a function consisting of δ functions located at the nodes of a space lattice and a density defined in a small region known as a unit cell. Because of the periodicity the Fourier transform has appreciable values only at the nodes of a lattice in transform space, called by crystallographers the reciprocal lattice. Because it is a physical quantity, $\rho(\mathbf{r})$ is non-negative, and, furthermore, because a crystal is composed of atoms, it can be

represented as a sum of functions that are, at least to a good approximation, spherically symmetric about a finite set of nuclear positions. If we designate by f_j the Fourier transforms of these atomic functions, the Fourier transform of the crystal can be written

$$F(\mathbf{h}) = \sum_{j=1}^n f_j \exp(2\pi i \mathbf{h} \cdot \mathbf{r}_j),$$

which is the conventional structure factor formula.

If it were possible to measure the values of the structure factor throughout transform space, it would be possible to compute the inverse transform and determine the density function, $\rho(\mathbf{r})$, directly. However, $F(\mathbf{h})$ is, in general, a complex quantity, and, because it appears in the intensity formula only as $|F(\mathbf{h})|^2$, only its amplitude can be measured, and that only within a finite region of transform space. In the early days of structural crystallography this "phase problem" was treated by using chemical intuition to devise a trial model for which F could be calculated and then, using the calculated phases along with the observed amplitudes, to compute a density map. If the crystallographer was lucky, this map would show a sufficiently clear picture of the structure to suggest adjustments to the model, and several iterations of the process would converge to a structure that made chemical sense. In the days before high-speed, digital computers these computations were done on mechanical desk calculators using tabulated sines and cosines written on strips of cardboard known, after the two British crystallographers who introduced them, as Beevers-Lipson strips. The process was very laborious, and a single structure analysis could consume many months.

In studies, beginning in the 1940s, of the structures of boron hydrides chemical information to suggest a reasonable starting model was often not available. Even if the crystal possessed a center of symmetry, so that the imaginary parts of the contributions to F from pairs of atoms would cancel, thereby constraining F to have real values, the number of possible combinations of signs in the density summation could be enormous. It was realized, however, by Harker and Kasper [1]¹ that many of the sign combinations would result in violations of the non-negativity condition on the electron density, and they were able to derive a number of inequality conditions that must be satisfied by certain combinations of F values in order to keep the density positive. The use of the Harker-Kasper inequalities in the solution of the structure of decaborane, $B_{10}H_{14}$, in 1950 by Kasper, Lucht, and Harker [2] was the first successful application of direct methods to the determination of a crystal structure using diffraction data alone.

¹ Figures in brackets indicate literature references.

The Harker-Kasper inequalities were applicable only to centrosymmetric crystals, with their resultant real values of the structure factors. In view of the amount of labor involved in the computation of a density map it was certainly important to be able to determine whether a crystal did in fact have a center of symmetry. In the late 1940s this problem was attacked by Wilson and his coworkers [3]. In the limit of a large number of identical atoms distributed at random in the unit cell, the contributions of the individual atoms to F are random walk steps, and the central limit theorem can be invoked to show that the distribution of F is approximately normal with zero mean. $|F|^2$ is then distributed as χ^2 with one degree of freedom if the crystal has a center of symmetry and with two degrees of freedom otherwise. The presence of other symmetry operations, such as rotation axes or mirror planes, constrains certain subsets of the structure factors to be real, so that statistical tests on the observed intensities can be an aid to determining the proper symmetry group for a crystal.

The Harker-Kasper inequalities may be viewed as a limiting case of a more general problem, which may be stated as follows: Given the magnitudes of a set of structure factors and the phases of a subset of them, (It is always possible to assign the phases of three structure factors arbitrarily. This merely defines the origin.) what are the probability density functions for the phases of others? Harker and Kasper identified particular cases where a discrete phase could be assigned with unit probability. The more general problem was attacked by Hauptman and Karle² in a long series of papers, beginning in the early 1950s [4], in which they have developed increasingly powerful methods for defining narrow ranges within which phases are likely to lie with high probability.

Most of the statistical methods that have been developed for determining a structural model are based on assumptions similar to those used by Wilson (1949), namely, that the crystal was composed of a large number of nearly identical atoms located at random within the unit cell. There is, however, another limit in which the solution of the phase problem is well known. This is the case (such as a simple metal) where the unit cell contains only one atom. In this limit all structure factors are identical in both magnitude and phase. Structure studies of very large molecules, such as proteins, have depended heavily on the preparation of crystals in which the unit cell contains one heavy atom, or a few at most, along with the very large number of atoms of carbon, nitrogen, and oxygen. The distribution of in-

² While this discussion was in press, the importance of the work of Hauptman and Karle was recognized with the announcement that the 1985 Nobel Prize in Chemistry was awarded to them.

tensities in this case cannot be similar to that in either limiting case, but must rather represent some sort of intermediate situation. Crystallographers have used various approximate methods to treat these real situations, and the results are strongly dependent on the validity of the approximations. The results given here by Weiss and Shmueli provide an accurate solution to which the approximate methods may be compared, and for this reason they are of tremendous interest to the crystallographic community.

References

- [1] Harker, D., and J. S. Kasper, *Acta Cryst.* **1**, 70–75 (1948).
- [2] Kasper, J. S.; C. M. Lucht and D. Harker, *Acta Cryst.* **3**, 436–455 (1950).
- [3] Wilson, A. J. C., *Acta Cryst.* **2**, 318–321 (1949).
- [4] Hauptman, H., and J. Karle, *Solution of the Phase Problem I. The Centrosymmetric Crystal*, ACA Monograph No. 3, New York, Polycrystal Book Service (1953).