

Stable Law Densities and Linear Relaxation Phenomena

Menachem Dishon

National Bureau of Standards, Gaithersburg, MD 20899

George H. Weiss

National Institutes of Health, Bethesda, MD 20205

and

John T. Bendler

General Electric Corporate Research & Development, Schenectady, NY 12301

Accepted: November 27, 1984

Stable law distributions occur in the description of the linear dielectric behavior of polymers, the motion of carriers in semi-conductors, the statistical behavior of neurons, and many other phenomena. No accurate tables of these distributions or algorithms for estimating the parameters in these relaxation models exist. In this paper we present tables of the functions

$$Q_\alpha(z) = \frac{1}{\pi} \int_0^\infty e^{-u^\alpha} \cos(zu) du$$

$$V_\alpha(z) = \frac{1}{\pi} \int_0^\infty e^{-u^\alpha} \sin(zu) du$$

together with related functional properties of $zQ_\alpha(z)$. These are useful in the estimation of the parameters in relaxation models for polymers and related materials. Values of the integral $Q_\alpha(z)$ are given for $\alpha=0.01, 0.02(0.02)0.1(0.1)1.0(0.2)2.0$ and those of $V_\alpha(z)$ are given for $\alpha=0.0(0.01)0.1(0.1)2.0$. A variety of methods was used to obtain six place accuracy. The tables can be used to sequentially estimate the three parameters appearing in the Williams-Watts model of relaxation. An illustration of this method applied to data in the literature is given.

1. Introduction

Stable law distributions and functionals of these distributions play an important role in a variety of scientific areas. They originated in a study of observation errors by Cauchy [1]¹, and many of their mathematical proper-

ties were elucidated by Lévy [2], and Khintchine and Lévy [3]. Applications of stable laws have appeared in the context of models for the broadening of spectral lines [4], fluctuations in stock market prices [5], statistical properties of neuronal activity [6], the motion of carriers through amorphous semiconductors [7], as well as in a variety of chemical physics problems [8]. Integrals of stable law densities have appeared in still another area of application: the theory of mechanical and electrical relaxation processes. The suggestion that relaxation processes in some electrical and mechanical systems could be described in terms of stable distributions can be traced back to experiments by Weber [9,10] and the Kohlraushes, father [11] and son [12]. Considerable recent interest in relaxation processes describable

About the Authors: Menachem Dishon, an applied mathematician, served as a guest worker in NBS' Scientific Computing Division. He has returned to his post as acting chief scientist, Ministry of Defense, Hakirya, Tel-Aviv, Israel. George H. Weiss is an applied mathematician and John T. Bendler is a physicist.

¹Bracketed figures indicate literature references.

in terms of stable laws has been inspired by experiments performed by Williams and Watts on polymers [13,14]. Many other examples of such applications to polymers and glasses occur in the literature, cf., for example, the work of Moynihan, Boesch, and Laberge [15] and Bendler [16]. The marriage between the mathematics of stable processes and the theory of dielectric relaxation processes has recently been proposed by Schlesinger and Montroll [17] who used the methodology of continuous time random walks [18] to try to model the underlying physical processes.

The relaxation of linear systems can be characterized by a frequency dependent dielectric constant or the appropriate mechanical analogue. This constant is generally expressible as

$$\frac{\epsilon(\omega) - \epsilon(\infty)}{\epsilon(0) - \epsilon(\infty)} = \epsilon'_n(\omega) - i\epsilon''_n(\omega) = - \int_0^\infty e^{-i\omega t} \frac{d\phi}{dt} dt \quad (1)$$

where $\phi(t)$ is the relaxation function which characterizes the particular material properties and $\epsilon'_n(\omega)$ and $\epsilon''_n(\omega)$ are the components of the normalized dielectric constant. It is found that for many polymers and glasses $\phi(t)$ can, to a good approximation, be chosen to have the form

$$\phi(t) = \exp[-(t/\tau)^\alpha] \quad (2)$$

where τ and α are constants that depend on the material. Data on polymers and glasses generally lead to α values in the range 0.2 to 0.8. Equations (1) and (2) together imply that, for $z = \omega\tau$,

$$\begin{aligned} \epsilon'_n(\omega) &= 1 - z \int_0^\infty e^{-u^\alpha} \sin(zu) du = 1 - \pi z V_\alpha(z) \\ \epsilon''_n(\omega) &= z \int_0^\infty e^{-u^\alpha} \cos(zu) du = \pi z Q_\alpha(z) \end{aligned} \quad (3)$$

where $Q_\alpha(z)$ and $V_\alpha(z)$ are the standard integrals

$$\begin{aligned} Q_\alpha(z) &= \frac{1}{\pi} \int_0^\infty e^{-u^\alpha} \cos(zu) du \\ V_\alpha(z) &= \frac{1}{\pi} \int_0^\infty e^{-u^\alpha} \sin(zu) du. \end{aligned} \quad (4)$$

The function $Q_\alpha(z)$ can be identified as a representation of a stable law density. In a recent paper Montroll and Bendler have presented a number of approximations to the function $Q_\alpha(z)$ for values of α useful for polymer applications [19]. Because $Q_\alpha(z)$ can be identified with a

stable density they were able to check their results against Holt and Crow's tables [20] for $\alpha = 1/4, 1/2$, and $3/4$. However, these tables are given to four places in fixed arithmetic. The variety of methods used in generating the tables has resulted in a lack of uniform accuracy.

Because of the wide range of applications of the functions $Q_\alpha(z)$ and $V_\alpha(z)$, and because there appear to be no accurate tabulations against which approximations can be checked, we present here tables of these functions for $\alpha = 0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1.0$ for $Q_\alpha(z)$, and for $\alpha = 0.0(0.01)0.1(0.1)2.0$ for $V_\alpha(z)$, accurate to six places in floating point.

2. Numerical Methods

Three methods were used to evaluate $Q_\alpha(z)$ and $V_\alpha(z)$ to 10 significant figures, which were truncated and rounded to six figures for the present tables. The first is the evaluation of a convergent series representation, the second is the evaluation of a divergent series, and the third is numerical integration in the region in which the divergent series does not yield the required accuracy at all, and the convergent series requires use of an unreasonably large number of terms. For $Q_\alpha(z)$ the following series converges for $0 < \alpha < 1$

$$Q_\alpha(z) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma(1+n\alpha)}{n! z^{1+n\alpha}} \sin\left(\frac{\pi n \alpha}{2}\right). \quad (5)$$

This series diverges when $1 < \alpha < 2$, but it is an asymptotic series, [21,22], and can be profitably used in this range for computation at sufficiently large values of z . The corresponding series that converges in $1 < \alpha < 2$, but diverges when $0 < \alpha < 1$, is

$$Q_\alpha(z) = \frac{1}{\pi \alpha} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma\left(\frac{2n-1}{\alpha}\right)}{(2n-2)!} z^{2(n-1)}. \quad (6)$$

Similar series can be derived for $V_\alpha(z)$. These are

$$V_\alpha(z) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(1+n\alpha)}{n! z^{1+n\alpha}} \cos\left(\frac{\pi n \alpha}{2}\right) \quad (7)$$

which converges for $0 < \alpha < 1$ and diverges when $1 < \alpha < 2$, and

$$V_\alpha(z) = \frac{1}{\pi \alpha} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma\left(\frac{2n}{\alpha}\right)}{(2n-1)!} z^{2n-1} \quad (8)$$

which diverges in $0 < \alpha < 1$ and converges in $1 < \alpha < 2$. When the appropriate series from eqs (5–8) requires a large number of terms to provide accurate results it is convenient to make the substitution $u = \tan \theta$ in eq (4) to find the following integral representations:

$$Q_\alpha(z) = \frac{1}{\pi} \int_0^{\pi/2} e^{-\tan^\alpha \theta} \cos(z \tan \theta) \frac{d\theta}{\cos^2 \theta} \quad (9)$$

$$V_\alpha(z) = \frac{1}{\pi} \int_0^{\pi/2} e^{-\tan^\alpha \theta} \sin(z \tan \theta) \frac{d\theta}{\cos^2 \theta}$$

which may be evaluated numerically.

Checks on the numerical calculations can be made for a few values of α for which both $Q_\alpha(z)$ and $V_\alpha(z)$ can be expressed in terms of known functions. For $\alpha = 1/2$ one finds

$$Q_{1/2}(z) = 2\pi\rho^3 \left\{ \left[\frac{1}{2} - C(\rho) \right] \cos\left(\frac{\pi\rho^2}{2}\right) + \left[\frac{1}{2} - S(\rho) \right] \sin\left(\frac{\pi\rho^2}{2}\right) \right\} \quad (10)$$

where

$$\rho = (2\pi z)^{-1/2}$$

$$C(\rho) = \int_0^\rho \cos\left(\frac{\pi t^2}{2}\right) dt, \quad S(\rho) = \int_0^\rho \sin\left(\frac{\pi t^2}{2}\right) dt \quad (11)$$

in terms of Fresnel integrals. The conjugate function, $V_{1/2}(z)$, can also be written in terms of ρ as

$$V_{1/2}(z) = 2\rho^2 \left[1 - \pi\rho \left\{ \left[\frac{1}{2} - S(\rho) \right] \cos\left(\frac{\pi\rho^2}{2}\right) - \left[\frac{1}{2} - C(\rho) \right] \sin\left(\frac{\pi\rho^2}{2}\right) \right\} \right]. \quad (12)$$

The functions in curly brackets in the expressions for $Q_{1/2}(z)$ and $V_{1/2}(z)$ are tabulated in Abramowitz and Stegun [23]. One can also verify the following special cases:

$$Q_1(z) = \frac{1}{\pi(1+z^2)}, \quad V_1(z) = zQ_1(z)$$

$$Q_2(z) = (4\pi)^{-1/2} \exp(-z^2/4),$$

$$V_2(z) = \pi^{-1} \exp(-z^2/4) \int_0^{z/2} \exp(t^2) dt \quad (13)$$

the expression for $V_2(z)$ being proportional to Dawson's

integral of argument $z/2$ [23]. In addition, Zolotarev [24] has found the value of $Q_{2/3}(z)$ in terms of Whittaker functions, but we have not evaluated his expression.

3. Computational Notes

All of the computations were performed in double precision in FORTRAN 77 on a Perkin-Elmer 3230 computer. Each complete set of tables, $Q_\alpha(z)$ and $V_\alpha(z)$, took between 1 and 1.5 minutes of CPU time. Tables 1 and 2 indicate when the appropriate series was used for computing $Q_\alpha(z)$ and $V_\alpha(z)$, respectively, as well as when numerical integration was required. Two points should be noted with respect to the two tables: 1) table entries for the critical values of z will change somewhat if other than six-digit accuracy is required, and 2) entries for critical z in tables 1 and 2 assume only restricted values, namely those values of z that appear in tables 3 and 4 (positioned at the end of this paper because of their length), where the $Q_\alpha(z)$ and $V_\alpha(z)$ are tabulated. Note that for some values of α the critical z values produce a range of overlap (for example, for $\alpha = 1.9$ in table 2); in these cases there is no need for numerical integration. But in other instances there is a gap between the critical values of z for use of the respective series, which may simply be due to the fact that no values of z are tabulated in the intermediate range. Hence, in these instances the calculation of $Q_\alpha(z)$ or $V_\alpha(z)$ for intermediate values of z may or may not require numerical integration. Since double precision was used in the calculations, some care

Table 1. Critical values of z for the use of eqs (5) and (6) or numerical integration to evaluate $Q_\alpha(z)$.

α	Use eq (6) for $z \leq$	Use eq (5) for $z \geq$	Use numerical integration for $z =$
≤ 0.3	—	0.001	—
0.4	0.002	0.004	0.003
0.5	0.01	0.02	—
0.6	0.05	0.10	—
0.7	0.1	0.15	—
0.8	0.25	0.35	0.30
0.9	0.50	0.65	0.55, 0.60
1.1	1.0	2.0	1.5
1.2	2.0	2.5	—
1.3	3.0	3.5	—
1.4	4.0	4.0	—
1.5	4.5	5.0	—
1.6	5.5	5.5	—
1.7	6.0	6.5	—
1.8	6.5	7.5	7.0
1.9	6.0	8.5	6.5–8.0

Table 2. Critical values of z for the use of eqs (7) and (8) or numerical integration to evaluate $V_a(z)$.

α	Use eq (8) for $z <$	Use eq (7) for $z >$	Use numerical integration for $z =$
≤ 0.3	—	0.001	—
0.4	0.001	0.01	0.002–0.005
0.5	0.01	0.025	0.015, 0.02
0.6	0.045	0.06	0.05, 0.055
0.7	0.1	0.15	—
0.8	0.25	0.35	0.3
0.9	0.5	0.65	0.55, 0.6
1.1	1.0	2.0	1.5
1.2	2.0	2.5	—
1.3	2.5	3.0	—
1.4	3.5	4.0	—
1.5	5.0	5.5	—
1.6	5.5	6.5	—
1.7	6.5	6.0	—
1.8	7.5	7.5	—
1.9	8.0	7.5	—
2.0	8.5	8.0	—

was required in the calculation of gamma function. We used the formula

$$\frac{1}{\Gamma(y+1)} = \sum_{n=0}^{2\alpha} a_n y^n \quad (14)$$

for $x = y + 1 \leq 3.21$, and

$$\Gamma(x) = (2\pi)^{1/2} x^{x-1/2} e^{-x} \sum_{n=0}^{\infty} c_n x^n \quad (15)$$

for $x \geq 3.21$. The coefficients $\{a_n\}$ and $\{c_n\}$ appearing in these formulae are to be found in the book by Luke [25]. The accuracy of formula [14] increases as $x \rightarrow 0$, while the accuracy of formula (15) increases as $x \rightarrow \infty$. For $x = 3.21$, both formulas yield the same accuracy.

When the series of eqs (5–8) were used to evaluate $Q_a(z)$ or $V_a(z)$, the number of terms used was a maximum of 150, which occurred near the switchover region for the two series. Most entries in the tables never required the evaluation of more than 20 terms. The numerical integrations of eq (9) were performed using Simpson's extended rule with a step size of $\pi/40,000$.

4. Application to Polymer Physics

The expressions in eq (3) allow us to estimate the three parameters that characterize polymers whose dielectric properties are described by eq (2) which is the Williams-

Watts relaxation function. Although it appears that all three parameters must be fit simultaneously, we will show that a small extension of tables 3 and 4 allows one to estimate the parameters separately. Since the most frequently measured property from which it is possible to determine the parameters α , τ , and $A = \epsilon(0) - 1$ is the dielectric loss function, i.e., $\epsilon''(\omega)$, we restrict our discussion to this function. As can be seen from eq (3), the dielectric loss function is proportional to $g_a(z) = zQ_a(z)$. This function is unimodal as a function of z as illustrated in figure 1. Several parameters can be defined that characterize the shape and properties of $g_a(z)$. These include $z_m(\alpha)$, the value of z at which $g_a(z)$ is maximized,

$$M(\alpha) = \max_z g_a(z) = g_a(z_m(\alpha)) \quad (16)$$

the height of the peak, and for $\theta < 1$ two sets of abscissas, $\{z_l(\theta, \alpha)\}$, $\{z_r(\theta, \alpha)\}$. These are, respectively, the values of z on the leading and trailing edges of $g_a(z)$ at which

$$g_a(z) = \theta M(\alpha). \quad (17)$$

These parameters are all illustrated in figure 1. For later

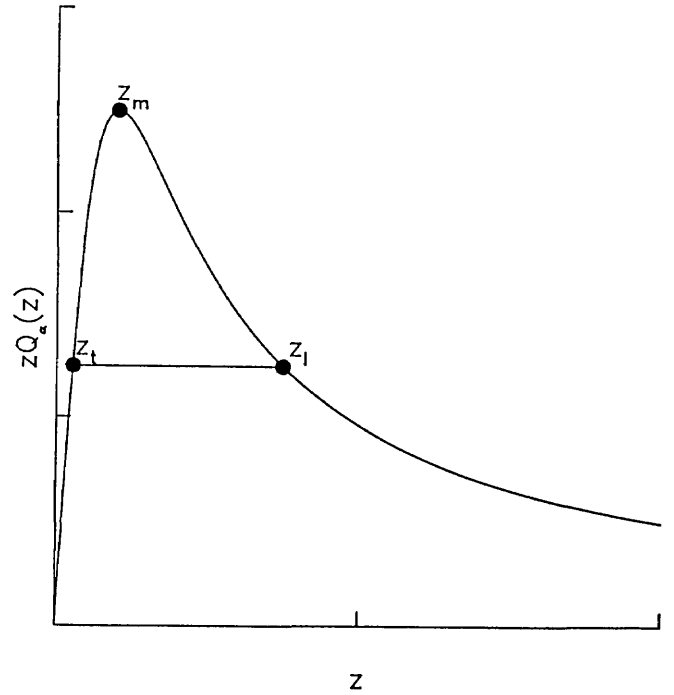


Figure 1—A typical curve of $g_a(z) = zQ_a(z)$ together with distinguished points useful for parameter estimation. The point z_m is the maximum, and $z_l(\theta, \alpha)$ and $z_r(\theta, \alpha)$ are the two solutions to $g_a(z) = \theta M(\alpha)$. In the figure θ was taken equal to $1/2$.

reference we give values of $z_m(\alpha)$ and $M(\alpha)$ in table 5 for $\alpha=0.05(0.05)1.00$. Table 6 contains values of

$$f(\theta, \alpha) = \ln[z_l(\theta, \alpha)/z_m(\alpha)] \quad (18)$$

for $\alpha=0.05(0.05)1.00$ and $\theta=0.1(0.1)0.9$ which will be used for data analysis, as will be explained in further

Table 5. Values of $z_m(\alpha)$ and the maximum $M(\alpha)$ for $\alpha=0.05(0.05)1.00$.

α	$z_m(\alpha)$	$M(\alpha)$
0.05	0.5741	0.9188(−2)
0.10	0.5885	0.1833(−1)
0.15	0.6041	0.2738(−1)
0.20	0.6207	0.3632(−1)
0.25	0.6380	0.4513(−1)
0.30	0.6561	0.5380(−1)
0.35	0.6751	0.6233(−1)
0.40	0.6948	0.7071(−1)
0.45	0.7154	0.7896(−1)
0.50	0.7370	0.8705(−1)
0.55	0.7595	0.9501(−1)
0.60	0.7830	0.1028(0)
0.65	0.8075	0.1105(0)
0.70	0.8329	0.1180(0)
0.75	0.8592	0.1253(0)
0.80	0.8863	0.1325(0)
0.85	0.9142	0.1394(0)
0.90	0.9425	0.1462(0)
0.95	0.9712	0.1528(0)
1.00	1.0000	0.1592(0)

detail below. The values of α cover the physically interesting range (0.2, 0.8) which has so far been found in polymers.

The procedure for parameter estimation involves three successive steps. Since τ , α , and A are initially unknown one can get an estimate of α independent of τ and A by first estimating the peak height, $z_m(\alpha)$, from the data and then solving for α from eq (18). Since $z = 2\pi f\tau$, the parameter τ drops out of eq (18) when the ratio of z 's is taken. Although it appears that there are many independent estimates of α , each of which corresponds to a different θ , the data may not be equally useful at all values of θ . At high frequencies the underlying physical assumption that the system has a single degree of freedom may be poor approximation. In particular, the data at θ values less than 0.5 may not fit the Williams-Watts model as well as data for $\theta > 0.5$. When a satisfactory value of α is determined, using eq (18), we may find τ from the relation

$$\hat{\tau} = z_m(\hat{\alpha})/(2\pi f_m) \quad (19)$$

where f_m is the frequency in Hz at which the peak maximum occurs. Finally, since an estimate of the peak height will generally be available, the estimate of the parameter A can be found from

$$\hat{A} = \max_f \epsilon''(2\pi f_m \hat{\tau})/M(\hat{\alpha}) \quad (20)$$

where relevant values of $M(\alpha)$ are given in table 5.

Table 6. Values of the function $f(\theta, \alpha) = \ln[z_l(\theta, \alpha)/z_m(\alpha)]$.

α	$\theta=0.1$	$\theta=0.2$	$\theta=0.3$	$\theta=0.4$	$\theta=0.5$	$\theta=0.6$	$\theta=0.7$	$\theta=0.8$	$\theta=0.9$
0.05	6.5303(1)	5.0611(1)	4.1593(1)	3.4833(1)	2.9237(1)	2.4287(1)	1.9654(1)	1.5039(1)	9.9471(0)
0.10	3.2670(1)	2.5323(1)	2.0813(1)	1.7432(1)	1.4633(1)	1.2157(1)	9.8394(0)	7.5297(0)	4.9815(0)
0.15	2.1793(1)	1.6895(1)	1.3888(1)	1.1633(1)	9.7667(0)	8.1152(0)	6.5691(0)	5.0280(0)	3.3273(0)
0.20	1.6353(1)	1.2679(1)	1.0424(1)	8.7325(0)	7.3322(0)	6.0932(0)	4.9332(0)	3.7766(0)	2.4999(0)
0.25	1.3085(1)	1.0146(1)	8.3420(0)	6.9892(0)	5.8690(0)	4.8778(0)	3.9497(0)	3.0244(0)	2.0025(0)
0.30	1.0902(1)	8.4535(0)	6.9504(0)	5.8235(0)	4.8905(0)	4.0649(0)	3.2919(0)	2.5210(0)	1.6697(0)
0.35	9.3382(0)	7.2399(0)	5.9523(0)	4.9871(0)	4.1881(0)	3.4812(0)	2.8195(0)	2.1596(0)	1.4308(0)
0.40	8.1600(0)	6.3250(0)	5.1993(0)	4.3558(0)	3.6578(0)	3.0405(0)	2.4627(0)	1.8866(0)	1.2502(0)
0.45	7.2385(0)	5.6087(0)	4.6095(0)	3.8611(0)	3.2421(0)	2.6948(0)	2.1827(0)	1.6724(0)	1.1086(0)
0.50	6.4962(0)	5.0310(0)	4.1333(0)	3.4615(0)	2.9061(0)	2.4154(0)	1.9565(0)	1.4993(0)	9.9427(−1)
0.55	5.8838(0)	4.5537(0)	3.7397(0)	3.1309(0)	2.6282(0)	2.1843(0)	1.7694(0)	1.3561(0)	8.9980(−1)
0.60	5.3686(0)	4.1517(0)	3.4078(0)	2.8521(0)	2.3937(0)	1.9893(0)	1.6116(0)	1.2355(0)	8.2024(−1)
0.65	4.9280(0)	3.8075(0)	3.1235(0)	2.6133(0)	2.1919(0)	1.8223(0)	1.4765(0)	1.1324(0)	7.5236(−1)
0.70	4.5459(0)	3.5084(0)	2.8764(0)	2.4057(0)	2.0183(0)	1.6774(0)	1.3594(0)	1.0431(0)	6.9377(−1)
0.75	4.2106(0)	3.2459(0)	2.6594(0)	2.2235(0)	1.8653(0)	1.5504(0)	1.2570(0)	9.6523(−1)	6.4276(−1)
0.80	3.9134(0)	3.0129(0)	2.4669(0)	2.0620(0)	1.7298(0)	1.4382(0)	1.1667(0)	8.9674(−1)	5.9811(−1)
0.85	3.6477(0)	2.8047(0)	2.2951(0)	1.9180(0)	1.6093(0)	1.3385(0)	1.0867(0)	8.3625(−1)	5.5887(−1)
0.90	3.4082(0)	2.6170(0)	2.1403(0)	1.7886(0)	1.5012(0)	1.2496(0)	1.0155(0)	7.8269(−1)	5.2431(−1)
0.95	3.1911(0)	2.4470(0)	2.0006(0)	1.6721(0)	1.4042(0)	1.1699(0)	9.5210(−1)	7.3521(−1)	4.9389(−1)
1.00	2.9932(0)	2.2924(0)	1.8738(0)	1.5668(0)	1.3170(0)	1.0986(0)	8.9559(−1)	6.9315(−1)	4.6715(−1)

This estimation procedure was applied to dielectric loss data of Sasabe and Moynihan [26] on polyvinyl acetate at 66.7 °C. The relevant data are the following:

	$\log_{10}f$	$\epsilon''(f)$	θ
1.	2.701	0.964	
2.	2.854	0.980	
3.	3.033	0.948	
4.	3.187	0.884	0.901
5.	3.560	0.665	0.678
6.	3.701	0.587	0.599
7.	3.857	0.510	0.520

The first three data points were used to estimate the peak location and height. These were found to be

$$\log_{10}\hat{f}_m = 2.839 \quad \hat{\epsilon}''(\max) = 0.981$$

following which the values of θ shown in data set were calculated. Equations (18–20) were then used to estimate α , τ , and A for the last four data points. The resulting estimates are the following:

	α	$\hat{\tau}(s)$	\hat{A}
4.	0.62	1.83(–4)	9.54
5.	0.60	1.81(–4)	9.54
6.	0.60	1.81(–4)	9.54
7.	0.60	1.81(–4)	9.54

These results should be contrasted with the estimates by Sasabe and Moynihan, $\hat{\alpha} = 0.59$ and $\hat{\tau} = 1.97 \times 10^{-4}$ s for the same set of data. The fitting procedure used by them required approximations suggested by Moynihan, Boesch, and Laberge [15], but these are of uncertain accuracy over the entire range of α because no accurate tables were available to check them.

References

[1] Cauchy, A. Oeuvres Complètes, 1 ser. 12, Sur les résultats moyens d'observations de même nature, et sur les résultats les probables, 94–104; Sur la probabilité des erreurs qui affectent des résultats moyens d'observations de même nature, 104–114. Gauthier-Villars, Paris (1900).

[2] Lévy, P. Theorie des erreurs. La loi de Gauss et les lois exceptionnelles. Bull. Societe Math. France **52**, 49–85 (1924).
[3] Khintchine, A. Ya, and P. Lévy. Sur les lois stables. Compt. Rend. Acad. Sci. Paris **202**, 374–376 (1936).
[4] Holstmark, J. Über die Verbreiterung von Spektrallinien. Ann. der Phys. **58**, 577–630 (1919).
[5] Mandelbrot, B. B. The variation of certain speculative prices. J. Bus. Univ. Chic. **36**, 394–419 (1963).
[6] Gerstein, G. L., and B. B. Mandelbrot. Random walk models for the spike activity of a single neuron. Biophys. J. **4**, 41–68 (1964).
[7] Scher, H. and E. W. Montroll. Anomalous transit-time dispersion in amorphous solids. Phys. Rev. **B12**, 2455–2477 (1975).
[8] Shlesinger, M. F. Electron scavenging in glasses. J. Chem. Phys. **70**, 4813–4818 (1979).
[9] Weber, W. Über die Elastizität der Seidenfäden. Pogg. Ann. der Phys. **4**, 247–261 (1835).
[10] Weber, W. Über die Elastizität fester Körper. Pogg. Ann. der Phys. **24**, 1–18 (1841).
[11] Kohlrausch, R. Theorie des Elektrischen Rückstandes in der Leidener Flasche. Pogg. Ann. der Phys. und Chemie. **91**, 179–214 (1854).
[12] Kohlrausch, F. Ueber die elastische Nachwirkung bei der Torsion, Pogg. Ann. der Phys. und Chemie. **119**, 337–338 (1863).
[13] Williams, G., and D. C. Watts. Non-symmetrical dielectric relaxation behaviour arising from a simple empirical decay function. Trans. Far. Soc. **66**, 80–85 (1970).
[14] Williams, G., and D. C. Watts. Further considerations of non symmetrical dielectric behaviour arising from a simple empirical decay function. Trans. Far. Soc. **67**, 1323–1335 (1971).
[15] Moynihan, C. T.; L. P. Boesch and N. L. Laberge. Decay function for the electric field relaxation in vitreous ionic conductors. Phys. and Chem. of Glasses **14**, 122–125 (1973).
[16] Bendler, J. T. Internal molecular motions and the elastic constants of polymer glasses. Macromol. **15**, 1325–1328 (1982).
[17] Shlesinger, M. F., and E. W. Montroll. On the Williams-Watts function of dielectric relaxation. Proc. Natl. Acad. Sci. USA **81**, 1280–1283 (1984).
[18] Montroll, E. W., and G. H. Weiss. Random walks on lattices. II. J. Math. Phys. **6**, 167–181 (1965).
[19] Montroll, E. W., and J. T. Bendler. On Levy (or stable) distributions and the Williams-Watts model of dielectric relaxation. J. Stat. Phys. **34**, 129–162 (1984).
[20] Holt, D. R., and E. L. Crow. Tables and Graphs of the Stable Probability Density Functions. J. Res. Natl. Bur. Stand. **77B**, 143–198 (1973).
[21] Bergström, H. On some expansions of stable distribution functions. Ark. Mat. **2**, 375–378 (1952).
[22] Feller, W. *An Introduction to Probability Theory and its Applications*. Vol. II 2nd ed. (John Wiley, New York) 1971.
[23] Abramowitz, M., and I. A. Stegun. *Handbook of Mathematical Functions*. AMS **55**, (Government Printing Office, Washington, DC) 1970.
[24] Zolotarev, V. M. (in Russian) Expression of the density of a stable distribution with exponent α greater than one by means of a frequency with exponent $1/\alpha$. Dokl. Akad. Nauk. USSR **98**, 735–738 (1954). English translation in Selected Trans. Math. Stat. and Prob., Inst. Math. Stat. and Am. Math. Soc. **1**, 163–167 (1961).
[25] Luke, T. L. *Mathematical Functions and Their Approximations*. (Academic Press, New York), 1975.
[26] Sasabe, H. and C. T. Moynihan. Structural relaxation in Poly (vinyl acetate). J. Poly. Sci. Phys. **16**, 1447–1457 (1978).

Z =	FUNCTION Q (ALPHA,Z) FOR					FUNCTION Q (ALPHA,Z) FOR					FUNCTION Q (ALPHA,Z) FOR				
	ALPHA =					ALPHA =					ALPHA =				
	0.100	0.200	0.300	0.400	0.500		0.600	0.700	0.800		0.600	0.700	0.800		
7.500	0.326110-02	0.349900-02	0.373490-02	0.396880-02	0.420070-02	0.443260-02	0.466450-02	0.489640-02	0.512830-02	0.536020-02	0.395710-00	0.418900-00	0.442090-00	0.465280-00	0.488470-00
7.600	0.327110-02	0.350900-02	0.374490-02	0.397880-02	0.421070-02	0.444260-02	0.467450-02	0.490640-02	0.513830-02	0.537020-02	0.396710-00	0.419900-00	0.443090-00	0.466280-00	0.489470-00
7.700	0.328110-02	0.351900-02	0.375490-02	0.398880-02	0.422070-02	0.445260-02	0.468450-02	0.491640-02	0.514830-02	0.538020-02	0.397710-00	0.420900-00	0.444090-00	0.467280-00	0.490470-00
7.800	0.329110-02	0.352900-02	0.376490-02	0.399880-02	0.423070-02	0.446260-02	0.469450-02	0.492640-02	0.515830-02	0.539020-02	0.398710-00	0.421900-00	0.445090-00	0.468280-00	0.491470-00
7.900	0.330110-02	0.353900-02	0.377490-02	0.400880-02	0.424070-02	0.447260-02	0.470450-02	0.493640-02	0.516830-02	0.540020-02	0.399710-00	0.422900-00	0.446090-00	0.469280-00	0.492470-00
8.000	0.331110-02	0.354900-02	0.378490-02	0.401880-02	0.425070-02	0.448260-02	0.471450-02	0.494640-02	0.517830-02	0.541020-02	0.400710-00	0.423900-00	0.447090-00	0.470280-00	0.493470-00
8.100	0.332110-02	0.355900-02	0.379490-02	0.402880-02	0.426070-02	0.449260-02	0.472450-02	0.495640-02	0.518830-02	0.542020-02	0.401710-00	0.424900-00	0.448090-00	0.471280-00	0.494470-00
8.200	0.333110-02	0.356900-02	0.380490-02	0.403880-02	0.427070-02	0.450260-02	0.473450-02	0.496640-02	0.519830-02	0.543020-02	0.402710-00	0.425900-00	0.449090-00	0.472280-00	0.495470-00
8.300	0.334110-02	0.357900-02	0.381490-02	0.404880-02	0.428070-02	0.451260-02	0.474450-02	0.497640-02	0.520830-02	0.544020-02	0.403710-00	0.426900-00	0.450090-00	0.473280-00	0.496470-00
8.400	0.335110-02	0.358900-02	0.382490-02	0.405880-02	0.429070-02	0.452260-02	0.475450-02	0.498640-02	0.521830-02	0.545020-02	0.404710-00	0.427900-00	0.451090-00	0.474280-00	0.497470-00
8.500	0.336110-02	0.359900-02	0.383490-02	0.406880-02	0.430070-02	0.453260-02	0.476450-02	0.499640-02	0.522830-02	0.546020-02	0.405710-00	0.428900-00	0.452090-00	0.475280-00	0.498470-00
8.600	0.337110-02	0.360900-02	0.384490-02	0.407880-02	0.431070-02	0.454260-02	0.477450-02	0.500640-02	0.523830-02	0.547020-02	0.406710-00	0.429900-00	0.453090-00	0.476280-00	0.499470-00
8.700	0.338110-02	0.361900-02	0.385490-02	0.408880-02	0.432070-02	0.455260-02	0.478450-02	0.501640-02	0.524830-02	0.548020-02	0.407710-00	0.430900-00	0.454090-00	0.477280-00	0.500470-00
8.800	0.339110-02	0.362900-02	0.386490-02	0.409880-02	0.433070-02	0.456260-02	0.479450-02	0.502640-02	0.525830-02	0.549020-02	0.408710-00	0.431900-00	0.455090-00	0.478280-00	0.501470-00
8.900	0.340110-02	0.363900-02	0.387490-02	0.410880-02	0.434070-02	0.457260-02	0.480450-02	0.503640-02	0.526830-02	0.550020-02	0.409710-00	0.432900-00	0.456090-00	0.479280-00	0.502470-00
9.000	0.341110-02	0.364900-02	0.388490-02	0.411880-02	0.435070-02	0.458260-02	0.481450-02	0.504640-02	0.527830-02	0.551020-02	0.410710-00	0.433900-00	0.457090-00	0.480280-00	0.503470-00
9.100	0.342110-02	0.365900-02	0.389490-02	0.412880-02	0.436070-02	0.459260-02	0.482450-02	0.505640-02	0.528830-02	0.552020-02	0.411710-00	0.434900-00	0.458090-00	0.481280-00	0.504470-00
9.200	0.343110-02	0.366900-02	0.390490-02	0.413880-02	0.437070-02	0.460260-02	0.483450-02	0.506640-02	0.529830-02	0.553020-02	0.412710-00	0.435900-00	0.459090-00	0.482280-00	0.505470-00
9.300	0.344110-02	0.367900-02	0.391490-02	0.414880-02	0.438070-02	0.461260-02	0.484450-02	0.507640-02	0.530830-02	0.554020-02	0.413710-00	0.436900-00	0.460090-00	0.483280-00	0.506470-00
9.400	0.345110-02	0.368900-02	0.392490-02	0.415880-02	0.439070-02	0.462260-02	0.485450-02	0.508640-02	0.531830-02	0.555020-02	0.414710-00	0.437900-00	0.461090-00	0.484280-00	0.507470-00
9.500	0.346110-02	0.369900-02	0.393490-02	0.416880-02	0.440070-02	0.463260-02	0.486450-02	0.509640-02	0.532830-02	0.556020-02	0.415710-00	0.438900-00	0.462090-00	0.485280-00	0.508470-00
9.600	0.347110-02	0.370900-02	0.394490-02	0.417880-02	0.441070-02	0.464260-02	0.487450-02	0.510640-02	0.533830-02	0.557020-02	0.416710-00	0.439900-00	0.463090-00	0.486280-00	0.509470-00
9.700	0.348110-02	0.371900-02	0.395490-02	0.418880-02	0.442070-02	0.465260-02	0.488450-02	0.511640-02	0.534830-02	0.558020-02	0.417710-00	0.440900-00	0.464090-00	0.487280-00	0.510470-00
9.800	0.349110-02	0.372900-02	0.396490-02	0.419880-02	0.443070-02	0.466260-02	0.489450-02	0.512640-02	0.535830-02	0.559020-02	0.418710-00	0.441900-00	0.465090-00	0.488280-00	0.511470-00
9.900	0.350110-02	0.373900-02	0.397490-02	0.420880-02	0.444070-02	0.467260-02	0.490450-02	0.513640-02	0.536830-02	0.560020-02	0.419710-00	0.442900-00	0.466090-00	0.489280-00	0.512470-00
10.000	0.351110-02	0.374900-02	0.398490-02	0.421880-02	0.445070-02	0.468260-02	0.491450-02	0.514640-02	0.537830-02	0.561020-02	0.420710-00	0.443900-00	0.467090-00	0.490280-00	0.513470-00

Z =

FUNCTION V (ALPHA,Z) FOR

Z =

ALPHA =

ALPHA =

ALPHA =

ALPHA =

ALPHA =

ALPHA =

0.000

0.010

0.020

0.030

0.040

0.050

0.060

0.070

0.080

0.090

0.100

0.110

0.120

0.130

0.140

0.150

0.160

0.170

0.180

0.190

0.200

0.210

0.220

0.230

0.240

0.250

0.260

0.270

0.280

0.290

0.300

0.310

0.320

0.330

0.340

0.350

0.360

0.370

0.380

0.390

0.400

0.410

0.420

0.430

0.440

0.450

0.460

0.470

0.480

0.490

0.500

0.510

0.520

0.530

0.540

0.550

0.560

0.570

0.580

0.590

0.600

0.610

0.620

0.630

0.640

0.650

0.660

0.670

0.680

0.690

0.700

0.710

0.720

0.730

0.740

0.750

0.760

0.770

0.780

0.790

0.800

0.810

0.820

0.830

0.840

0.850

0.860

0.870

0.880

0.890

0.900

0.910

0.920

0.930

0.940

0.950

0.960

0.970

0.980

0.990

1.000

1.010

1.020

1.030

1.040

1.050

1.060

1.070

1.080

1.090

1.100

1.110

1.120

1.130

1.140

1.150

1.160

1.170

1.180

1.190

1.200

1.210

1.220

1.230

1.240

1.250

1.260

1.270

1.280

1.290

1.300

1.310

1.320

1.330

1.340

1.350

1.360

1.370

1.380

1.390

1.400

1.410

1.420

1.430

1.440

1.450

1.460

1.470

1.480

1.490

1.500

1.510

1.520

1.530

1.540

1.550

1.560

1.570

1.580

1.590

1.600

1.610

1.620

1.630

1.640

1.650

1.660

1.670

1.680

1.690

1.700

1.710

1.720

1.730

1.740

1.750

1.760

1.770

1.780

1.790

1.800

1.810

1.820

1.830

1.840

1.850

1.860

1.870

1.880

1.890

1.900

1.910

1.920

1.930

1.940

1.950

1.960

1.970

1.980

1.990

2.000

2.010

2.020

2.030

2.040

2.050

2.060

2.070

2.080

2.090

2.100

2.110

2.120

2.130

2.140

2.150

2.160

2.170

2.180

2.190

2.200

2.210

2.220

2.230

2.240

2.250

2.260

2.270

2.280

2.290

2.300

2.310

2.320

2.330

2.340

2.350

2.360

2.370

2.380

2.390

2.400

2.410

2.420

2.430

2.440

2.450

2.460

2.470

2.480

2.490

2.500

2.510

2.520

2.530

2.540

2.550

2.560

2.570

2.580

2.590

2.600

2.610

2.620

2.630

2.640

2.650

2.660

2.670

2.680

2.690

2.700

2.710

2.720

2.730

2.740

2.750

2.760

2.770

2.780

2.790

2.800

2.810

2.820

2.830

2.840

2.850

2.860

2.870

2.880

2.890

2.900

2.910

2.920

2.930

2.940

2.950

2.960

2.970

2.980

2.990

3.000

3.010

3.020

3.030

3.040

3.050

3.060

3.070

3.080

3.090

3.100

3.110

3.120

3.130

3.140

3.150

3.160

3.170

3.180

3.190

3.200

3.210

3.220

3.230

3.240

3.250

3.260

3.270

3.280

3.290

3.300

3.310

3.320

3.330

3.340

3.350

3.360

3.370

3.380

3.390

3.400

3.410

3.420

3.430

FUNCTION V(ALPHA,Z) FOR

ALPHA =					ALPHA =				
0.050					0.100				
700.000	0.251140-03	0.236985D-03	0.247891D-03	0.289517D-03	0.269809D-03	0.190455D-01	0.182010D-03	0.219314D-01	0.234411D-01
750.000	0.111690-01	0.226173D-03	0.232040D-03	0.242066D-03	0.255108D-03	0.104330D-01	0.092800D-03	0.104330D-01	0.116970D-01
800.000	0.228217-01	0.026844D-03	0.026844D-03	0.274720D-03	0.282155D-03	0.044173D-01	0.039580D-03	0.044173D-01	0.051498D-01
850.000	0.071717-03	0.011817D-03	0.019443D-03	0.223338D-03	0.221155D-03	0.054283D-02	0.038180D-03	0.071999D-02	0.087551D-02
900.000	0.168195D-03	0.116718D-03	0.189013D-03	0.133978D-03	0.109313D-03	0.492711D-03	0.289232D-03	0.689336D-03	0.791280D-03
950.000	0.080170-01	0.119335D-03	0.116959D-03	0.124730D-03	0.124930D-03	0.371390D-02	0.425440D-02	0.550173D-02	0.641146D-02
1000.000	0.049174-02	0.081315D-04	0.105775D-04	0.164970D-04	0.098585D-04	0.336522D-02	0.425188D-02	0.545450D-02	0.680462D-02
1050.000	0.046019D-04	0.068614D-04	0.030909D-04	0.0744558D-04	0.079682D-04	0.307791D-02	0.389911D-02	0.442993D-02	0.498717D-02
1100.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.233945D-02	0.339322D-02	0.437789D-02	0.525773D-02
1150.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.205513D-02	0.031146D-02	0.037891D-02	0.042018D-02
1200.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.226011D-02	0.029299D-02	0.037288D-02	0.039399D-02
1250.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.216592D-02	0.027562D-02	0.031975D-02	0.035021D-02
1300.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.204604D-02	0.026957D-02	0.030648D-02	0.033064D-02
1350.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.193990D-02	0.026128D-02	0.029148D-02	0.031377D-02
1400.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.184299D-02	0.025603D-02	0.029148D-02	0.031377D-02
1450.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.175622D-02	0.024997D-02	0.028331D-02	0.030841D-02
1500.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.169889D-02	0.024357D-02	0.027559D-02	0.029168D-02
1550.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.163833D-03	0.023700D-03	0.026830D-03	0.028111D-03
1600.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.157408D-03	0.023030D-03	0.026159D-03	0.027477D-03
1650.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.150650D-03	0.022350D-03	0.025489D-03	0.026828D-03
1700.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.143572D-03	0.021660D-03	0.024819D-03	0.026179D-03
1750.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.136180D-03	0.020960D-03	0.024149D-03	0.025530D-03
1800.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.128480D-03	0.020260D-03	0.023479D-03	0.024881D-03
1850.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.120480D-03	0.019560D-03	0.022809D-03	0.024232D-03
1900.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.112180D-03	0.018860D-03	0.022139D-03	0.023583D-03
1950.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.103680D-03	0.018160D-03	0.021469D-03	0.022934D-03
2000.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.094980D-03	0.017460D-03	0.020799D-03	0.022285D-03
2050.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.086080D-03	0.016760D-03	0.020129D-03	0.021636D-03
2100.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.076980D-03	0.016060D-03	0.019459D-03	0.020987D-03
2150.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.067680D-03	0.015360D-03	0.018789D-03	0.020338D-03
2200.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.058180D-03	0.014660D-03	0.018119D-03	0.019689D-03
2250.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.048480D-03	0.013960D-03	0.017449D-03	0.019040D-03
2300.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.038580D-03	0.013260D-03	0.016779D-03	0.018391D-03
2350.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.028480D-03	0.012560D-03	0.016109D-03	0.017742D-03
2400.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.018180D-03	0.011860D-03	0.015439D-03	0.017093D-03
2450.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.007680D-03	0.011160D-03	0.014769D-03	0.016444D-03
2500.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.010460D-03	0.014099D-03	0.015794D-03
2550.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.009760D-03	0.013429D-03	0.015145D-03
2600.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.009060D-03	0.012759D-03	0.014496D-03
2650.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.008360D-03	0.012089D-03	0.013847D-03
2700.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.007660D-03	0.011419D-03	0.013198D-03
2750.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.006960D-03	0.010749D-03	0.012549D-03
2800.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.006260D-03	0.010079D-03	0.011900D-03
2850.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.005560D-03	0.009409D-03	0.011251D-03
2900.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.004860D-03	0.008739D-03	0.010602D-03
2950.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.004160D-03	0.008069D-03	0.010003D-03
3000.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.003460D-03	0.007399D-03	0.009404D-03
3050.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.002760D-03	0.006729D-03	0.008805D-03
3100.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.002060D-03	0.006059D-03	0.008206D-03
3150.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.001360D-03	0.005389D-03	0.007607D-03
3200.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000660D-03	0.004719D-03	0.007008D-03
3250.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.004049D-03	0.006409D-03
3300.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.003379D-03	0.005810D-03
3350.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.002709D-03	0.005211D-03
3400.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.002039D-03	0.004612D-03
3450.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.001369D-03	0.004013D-03
3500.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000699D-03	0.003414D-03
3550.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000029D-03	0.002815D-03
3600.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.002216D-03
3650.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.001617D-03
3700.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.001018D-03
3750.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000419D-03
3800.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00
3850.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00
3900.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00
3950.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00
4000.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00
4050.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00
4100.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00
4150.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00
4200.000	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00	0.000000D-00
4250.000	0.000000D-00	0.000000D-00	0.						

FUNCTION V(ALPHA,Z) FOR

Z =	ALPHA =	0.600	0.700	0.800	0.900	1.000	1.200	1.300	1.400	1.500
0.450	0.1957616+00	0.1182130+00	0.1571430+00	0.1261360+00	0.1261360+00	0.1261360+00	0.1261360+00	0.1261360+00	0.1261360+00	0.1261360+00
0.500	0.1757020+00	0.1150810+00	0.1533330+00	0.1249700+00	0.1249700+00	0.1249700+00	0.1249700+00	0.1249700+00	0.1249700+00	0.1249700+00
0.550	0.1627020+00	0.1128900+00	0.1516850+00	0.1232400+00	0.1232400+00	0.1232400+00	0.1232400+00	0.1232400+00	0.1232400+00	0.1232400+00
0.600	0.1485470+00	0.1106000+00	0.1499330+00	0.1214900+00	0.1214900+00	0.1214900+00	0.1214900+00	0.1214900+00	0.1214900+00	0.1214900+00
0.650	0.1390910+00	0.1146320+00	0.1489430+00	0.1259720+00	0.1259720+00	0.1259720+00	0.1259720+00	0.1259720+00	0.1259720+00	0.1259720+00
0.700	0.1272690+00	0.1103550+00	0.1452500+00	0.1606700+00	0.1606700+00	0.1606700+00	0.1606700+00	0.1606700+00	0.1606700+00	0.1606700+00
0.750	0.1647409+00	0.1177750+00	0.1635200+00	0.1610070+00	0.1610070+00	0.1610070+00	0.1610070+00	0.1610070+00	0.1610070+00	0.1610070+00
0.800	0.1909010+00	0.1146320+00	0.1489430+00	0.1529720+00	0.1529720+00	0.1529720+00	0.1529720+00	0.1529720+00	0.1529720+00	0.1529720+00
0.850	0.1722690+00	0.1103550+00	0.1452500+00	0.1606700+00	0.1606700+00	0.1606700+00	0.1606700+00	0.1606700+00	0.1606700+00	0.1606700+00
0.900	0.1490235+00	0.1126380+00	0.1456300+00	0.1603900+00	0.1603900+00	0.1603900+00	0.1603900+00	0.1603900+00	0.1603900+00	0.1603900+00
0.950	0.1201799+00	0.11081480+00	0.1456300+00	0.1603900+00	0.1603900+00	0.1603900+00	0.1603900+00	0.1603900+00	0.1603900+00	0.1603900+00
1.000	0.0955822+01	0.09695510+01	0.0955822+01	0.0955822+01	0.0955822+01	0.0955822+01	0.0955822+01	0.0955822+01	0.0955822+01	0.0955822+01
1.050	0.0702826+01	0.07173390+01	0.07173390+01	0.07173390+01	0.07173390+01	0.07173390+01	0.07173390+01	0.07173390+01	0.07173390+01	0.07173390+01
1.100	0.0477333+01	0.05474130+01	0.05474130+01	0.05474130+01	0.05474130+01	0.05474130+01	0.05474130+01	0.05474130+01	0.05474130+01	0.05474130+01
1.150	0.0143300+01	0.0143300+01	0.0143300+01	0.0143300+01	0.0143300+01	0.0143300+01	0.0143300+01	0.0143300+01	0.0143300+01	0.0143300+01
1.200	0.0022080+01	0.0022080+01	0.0022080+01	0.0022080+01	0.0022080+01	0.0022080+01	0.0022080+01	0.0022080+01	0.0022080+01	0.0022080+01
1.250	0.0011647+01	0.0011647+01	0.0011647+01	0.0011647+01	0.0011647+01	0.0011647+01	0.0011647+01	0.0011647+01	0.0011647+01	0.0011647+01
1.300	0.0004208+01	0.0004208+01	0.0004208+01	0.0004208+01	0.0004208+01	0.0004208+01	0.0004208+01	0.0004208+01	0.0004208+01	0.0004208+01
1.350	0.0002261+01	0.0002261+01	0.0002261+01	0.0002261+01	0.0002261+01	0.0002261+01	0.0002261+01	0.0002261+01	0.0002261+01	0.0002261+01
1.400	0.00011647+01	0.00011647+01	0.00011647+01	0.00011647+01	0.00011647+01	0.00011647+01	0.00011647+01	0.00011647+01	0.00011647+01	0.00011647+01
1.450	0.00004208+01	0.00004208+01	0.00004208+01	0.00004208+01	0.00004208+01	0.00004208+01	0.00004208+01	0.00004208+01	0.00004208+01	0.00004208+01
1.500	0.00002261+01	0.00002261+01	0.00002261+01	0.00002261+01	0.00002261+01	0.00002261+01	0.00002261+01	0.00002261+01	0.00002261+01	0.00002261+01
1.550	0.000011647+01	0.000011647+01	0.000011647+01	0.000011647+01	0.000011647+01	0.000011647+01	0.000011647+01	0.000011647+01	0.000011647+01	0.000011647+01
1.600	0.000004208+01	0.000004208+01	0.000004208+01	0.000004208+01	0.000004208+01	0.000004208+01	0.000004208+01	0.000004208+01	0.000004208+01	0.000004208+01
1.650	0.000002261+01	0.000002261+01	0.000002261+01	0.000002261+01	0.000002261+01	0.000002261+01	0.000002261+01	0.000002261+01	0.000002261+01	0.000002261+01
1.700	0.0000011647+01	0.0000011647+01	0.0000011647+01	0.0000011647+01	0.0000011647+01	0.0000011647+01	0.0000011647+01	0.0000011647+01	0.0000011647+01	0.0000011647+01
1.750	0.0000004208+01	0.0000004208+01	0.0000004208+01	0.0000004208+01	0.0000004208+01	0.0000004208+01	0.0000004208+01	0.0000004208+01	0.0000004208+01	0.0000004208+01
1.800	0.0000002261+01	0.0000002261+01	0.0000002261+01	0.0000002261+01	0.0000002261+01	0.0000002261+01	0.0000002261+01	0.0000002261+01	0.0000002261+01	0.0000002261+01
1.850	0.00000011647+01	0.00000011647+01	0.00000011647+01	0.00000011647+01	0.00000011647+01	0.00000011647+01	0.00000011647+01	0.00000011647+01	0.00000011647+01	0.00000011647+01
1.900	0.00000004208+01	0.00000004208+01	0.00000004208+01	0.00000004208+01	0.00000004208+01	0.00000004208+01	0.00000004208+01	0.00000004208+01	0.00000004208+01	0.00000004208+01
1.950	0.00000002261+01	0.00000002261+01	0.00000002261+01	0.00000002261+01	0.00000002261+01	0.00000002261+01	0.00000002261+01	0.00000002261+01	0.00000002261+01	0.00000002261+01
2.000	0.000000011647+01	0.000000011647+01	0.000000011647+01	0.000000011647+01	0.000000011647+01	0.000000011647+01	0.000000011647+01	0.000000011647+01	0.000000011647+01	0.000000011647+01

FUNCTION V(ALPHA,Z) FOR

[illegible][illegible]

FUNCTION V(ALPHA,Z) FOR

[illegible][illegible]