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Cross-Sectional Correction for Computing Young's Modulus From Longitudinal Resonance Vibrations of Square and Cylindrical Rods

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The cross-sectional correction involved in the calculation of Young's modulus from the longitudinal resonance vibrations of both square and cylindrical bars has been determined by an empirical method.

On an order of accuracy of 1 part in 1,000, Bancroft's correction, developed for longitudinal waves in cylinders of infinite length was found to be satisfactory. For this purpose the thickness of square bars is related to the diameter of cylindrical bars of the same length by $4t^2=3d^2$.

For accuracies of 1 part in 10,000, modifications in Bancroft's correction must be applied. These modifications take a different form for the square rods than for the cylindrical rods. The relation, $2l=n\lambda$ held for both shapes and on the higher order of accuracy, i.e., the cross-sectional correction was the same for the fundamental and overtones on specimens of the same effective length.

1. Introduction

This is the fourth in a series of papers [1,2,3] [1] dealing with the empirical establishment of accurate [2] relations for computing elastic moduli from the various types of resonance vibrations of isotropic solids. The experimental technique and general approach have been fully described in these previous papers and will not be elaborated here. In one of these papers [2] the problem of the cross-sectional correction factors involved in the computation of Young's modulus from both the fundamental and overtones of the longitudinal resonance vibrations of cylindrical bars has already been treated. The present investigation extends this treatment by considering longitudinal vibrations of square as well as cylindrical bars; and on a higher order of accuracy for both shapes (by a factor of 10) than in the previous work.

Bancroft [4] has obtained an accurate numerical solution of the Pochhammer-Chree equations for the case of longitudinal waves in cylinders of infinite length. However, it is known [5] that these equations cannot be applied rigorously for longitudinal resonance vibrations in bars of finite length (where the ends of the bar are at zero stress). The problem of the present investigation is to determine experimentally the degree to which Bancroft's solution, developed for longitudinal waves in cylinders of infinite length, fits the case for longitudinal resonance vibrations in cylindrical and square bars of finite length; and to what degree modification is required.

[1] Figures in brackets indicate the literature references at the end of this paper.
[2] As accurate as the determination of the relevant parameters permit.

2. Experiment

Of the eight steel specimens used in this investigation, (table 1) the first six were the same ones used in a previous study [3], and have the same designation. A description of their method of preparation and characterization is given in that reference. This description applies also to the remaining two specimens. The longitudinal modes of vibration were obtained concurrently with the torsional modes discussed in [3]. Consequently, the advantage which was gained in the previous study by cutting down certain larger specimens after a series of resonance frequencies had been determined, and obtaining the resonances on the shortened specimens thus formed (enabling one to obtain a fairly large number of experimental points from a fairly small piece of original stock), is also retained here.

The number of experimental points obtained here, however, was more limited than in the study on torsion because the fundamental and overtones of the longitudinal resonance frequencies of bars are higher than the corresponding torsional ones. It is recalled from the previous paper that the upper frequency limit of the resonance determinations is about 50 kc/s. Consequently, fewer overtones of longitudinal resonances could be obtained before this upper frequency limit was attained; and for the shortest specimen (A_D, actually a cube) for which only the fundamental torsional resonance could be determined, not even the fundamental longitudinal resonance frequency could be detected.

The same precautions to insure accuracy, including waiting for the specimens to attain thermal equilibrium with a controlled ambient temperature, that were taken for the torsional resonance determinations, were also used here. The accuracy of these resonance determinations, was conservatively estimated to be about 1 part in 10,000, as was the case for torsional resonance.

3. Results and Analysis

The frequency of the fundamental and overtones of the longitudinal resonance vibrations of all the specimens is given in table 1.

Table 1. *Dimensions and longitudinal resonance frequencies of specimens of this investigation*

Specimen*	Length	Thickness	Resonance frequencies (c/s)			
			f_1**	f_2	f_3	f_4
	cm	*cm*				
A	28.0902	3.4960	9043.6	18090.8	26890	35520
A 1	19.9753	3.4960	12981.2	25988	39010	----------
A 12	16.1737	3.4960	15953.1	31555	46234	----------
A 2	8.2450	3.4960	30932	60437	----------	----------
B	5.7142	2.5737	44807	----------	----------	----------
C	9.2341	2.5737	27927	55075	----------	----------
D	9.2340	2.2286	27924	55057	----------	----------
F	5.7744	2.2385	44785	----------	----------	----------

*Specimens B and C are cylindrical; all others are square in cross section.
**f_1 = fundamental resonance frequency, f_2 = first overtone, etc.

We proceed with an analysis of the data by recalling certain well-known relations, some of which have been used previously [2] in studying this mode of vibration. First, the relation between the velocity, v_0, of a longitudinal wave in an infinitely thin rod, of infinite length, and the Young's modulus, E, and density, ρ, of the medium is given by

$$v_0=\sqrt{E/\rho}. \tag{1}$$

In a cylinder of finite diameter, d, and infinite length, the velocity of a longitudinal wave, v, is reduced from v_0. Bancroft's [4] numerical solution of the Pochhammer-Chree equations for this particular case has already been mentioned. His results can be conveniently expressed in the form of a table, such as table 2, which gives numerical values of the reduction factor,

$$K_n=(v_B/v_0)^2, \tag{2}$$

as a function of d/λ (λ being the wavelength of the wave) and Poisson's ratio, μ, of the medium; v_B is the velocity from Bancroft's correction.

For square bars, no theory comparable in accuracy to that developed for cylinders (for this cross-sectional correction) appears to be available. However, it seems that the correction factors from table 2 could be applied to square bars if an appropriate assumption were made as to what cross-sectional dimension of a square bar should be taken to correspond to d for a cylinder of the same length. There appears to be some theoretical justification [5] for assuming that the correction would be the same if the polar moments of inertia of the square and circular cross-sectional areas were the same. This assumption was adopted here. Such a condition would require the following relation between the thickness, t, of a square bar, and the diameter, d, of a cylinder of the same length.

$$d^2=\frac{4}{3}\,t^2. \tag{3}$$

In longitudinal resonance vibrations in cylindrical or square rods (associated with standing waves) the following set of relations are usually adopted,

$$\lambda=2l/n,\ v_s=f_n\lambda=2lf_n/n \tag{4}$$

where l is the length of the specimen, v_s the "velocity" of the wave, f the longitudinal resonance frequency, and the letter n, either as a subscript, or independently, indicates the order of the resonance vibration; for the fundamental, $n=1$; for the first overtone, $n=2$, etc.

Assuming that $v_B=v_s$, and combining eqs (1), (2), and (4), one obtains the following relation,

$$E/\rho=(1/K_n)(2lf_n/n)^2. \tag{5}$$

Here the subscript in K_n takes on the added significance of indicating the order of vibration to which this correction factor applies.

All the parameters in the parentheses in eq (5) may be determined experimentally and E/ρ for a carefully selected group of specimens from the same source should be the same.

The remainder of this paper, then, reduces itself to the problem of determining the degree to which the factor, K_n, developed for longitudinal waves in cylinders of infinite length also applies to longitudinal resonance vibrations (in rods of finite length). This is equivalent to finding the degree to which the assumption that $v_B=v_s$ holds. Also to be tested is the assumption that square and cylindrical rods of the same length, having the same polar moments of inertia of cross sectional area, undergo the same reduction in velocity.

In figure 1, the data of table 1 are plotted on a scale comparable in precision (about 1 part in 1,000) to that of the previous study (fig. 1 of [2]). The square of the parameters comprising the ordinate and abscissa are selected, rather than the first power, as in the previous study, to show the approximately linear relationship that then exists between these variables. Such a presentation is in conformity with table 2 which facilitates accurate interpolation. For the highest accuracy of interpolation from Bancroft's values Aitken's [6] method of interpolation must be used. In order to include square as well as cylindrical specimens, β, in the figure, is chosen so that $\beta^2=(d/2)^2$, or $t^2/3$.

The line in the figure is obtained by plotting Bancroft's values from table 2 for an appropriate value of μ. Such a value may be selected in two independent ways, and the degree to which the values so obtained agree provides a check of the consistency of the data. On the one hand, one may select that value of μ from table 2 for which the associated values of ordinate and abscissa give the best fit to the experimental data. The value of μ so obtained was 0.2906. On the other hand, one may extrapolate the experimental values to obtain the best estimate of v_0; then, knowing ρ for the specimens[3] from the previous study [3], E may be computed from eq (1). Since G for these specimens is also known from [3], μ may be computed from the well-known relation, $\mu=(E/2G)-1$. The value of μ so obtained was 0.2880. This was the value used in the figure. The two values are seen to be in excellent agreement with each other and also with the ones given in the literature for steel.

Examination of figure 1 leads to the following conclusions:

(1) The data of the previous study for cylinders is confirmed in that Bancroft's cross-sectional correction for longitudinal waves in cylinders of infinite length may also be applied to longitudinal resonance vibrations (in cylinders of finite length).

(2) For square specimens, this correction also applies, if the assumption previously made is adopted, namely, that $d^2=(4/3)t^2=4\beta^2$.

The above two conclusions hold only on the order of accuracy of figure 1, i.e., 1 part in 1,000. Small deviations are noted when the data are plotted on a more expanded scale as will be shown in figures 2 and 3.

(3) For both square as well as cylindrical specimens, the observation previously made for cylindrical specimens with respect to overtones still holds. This is that the points for overtones fall on the same line as for fundamental resonance vibrations. This means that the relation, $\lambda=2l/n$, holds and that the correction factor for the overtones of a longer specimen is the same as for that of a shorter specimen having the same effective length (i.e., having the same value of abscissa).

In figure 2, the experimental points are plotted on a more extended scale, comparable to the full accuracy of the data itself, namely, about 1 part in 10,000. A convenient way to present such a plot is in terms of possible departures of the experimental points from the velocities resulting from Bancroft's correction, v_B. This would correspond geometrically to making a horizontal line of the one given in figure 1 at $(v/v_B)^2=1$, and plotting the ratio $(v/v_B)^2$ as a function of $(\beta n/l)^2$ for all the experimental points in table 1.

When this procedure is followed, certain significant departures from Bancroft's correction become evident. The upper line shows that the cross-sectional correction factor for cylinders deviates linearly from Bancroft's correction as $(\beta n/l)^2$ increases.

[3] Actually, since the same set of specimens is involved, ρ should be the same for all specimens and need not be known. Poisson's ratio may then also be computed using v_0 (longitudinal) and v_t (torsional) in place of E and G, respectively; v_t (torsional) is also known from [3].

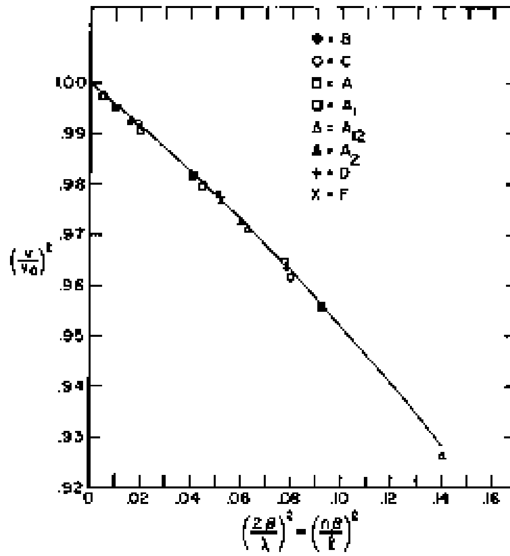

FIGURE 1. *Plot showing the reduction in longitudinal velocity, v, in a square or cylindrical rod of finite cross section from v_0, the corresponding velocity in an infinitely thin rod of the same length l.*

β is related to the diameter, d, of cylinders, and the thickness, t, of square rods by $4\beta^2=d^2$ and $3\beta^2=t^2$. Also the wavelength, $\lambda=2l/n$ (n being the overtone). The line is drawn through Bancroft's values for Poisson's ratio=0.2880, showing the agreement between Bancroft's correction, developed for longitudinal waves in cylinders of infinite length, and resonance vibrations in cylinders or square bars of finite length on this scale of accuracy (1 part in 1,000).

TABLE 2. *Bancroft's cross-sectional correction factor, $K_n=(v_B/v_0)^2$ [a], as a function of Poisson's ratio, μ, and diameter to wavelength ratio, d/λ*

d/λ	μ / $(d/\lambda)^2$	.10	.15	.20	.25	.30	.35	.40
0.00	0.0000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
.05	.0025	0.99965	0.99972	0.99960	0.99922	0.99898	0.99845	0.99802
.10	.0100	.99860	.99886	.99798	.99688	.99549	.99379	.99208
.15	.0225	.99682	.99736	.99533	.99277	.98967	.98509	.98202
.20	.0400	.99780	.99508	.99188	.98670	.98117	.97450	.96772
.25	.0625	.99632	.99164	.98679	.97930	.96958	.95975	.94903
.30	.0900	.99421	.98728	.97810	.96702	.95435	.94049	.92571
.35	.1225	.98114	.98065	.96758	.95208	.93486	.91660	.89770
.40	.1600	.98651	.97146	.96310	.93238	.91031	.88770	.86503
.45	.2025	.97913	.96776	.93300	.90068	.88003	.85372	.82812
.50	.2500	.96621	.95846	.90502	.87283	.84372	.81499	.78780

[a] v_0 is the velocity of a longitudinal wave in an infinitely thin cylinder, and v_B is the velocity in a cylinder of finite diameter and infinite length.

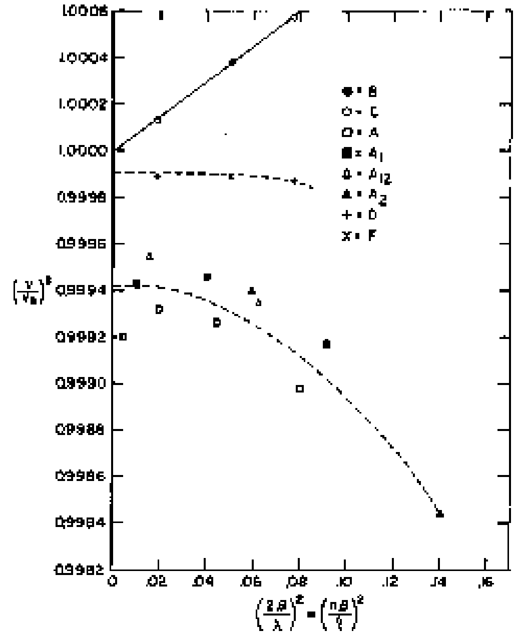

FIGURE 2. *Expanded plot showing departure of longitudinal velocities, associated with resonance vibrations, from values obtained using Bancroft's correction for longitudinal waves, v_B, as a function of cross section to length.*

Upper line is a least square solution of points for cylindrical specimens while lower two curves are simply drawn through experimental points for smaller and larger square specimens.

This line is obtained by a least squares fit of the experimental points yielding the following relation:

$$(v/v_B)^2 = 1 + 0.0075(nd/2l)^2. \qquad (6)$$

The two lower curves are for the square specimens. The fact that two such curves, one for the larger specimens and one for the smaller ones, appear is disconcerting but presents no serious problem. If the reasonable assumption is made that the separation of these curves is caused by real differences in intrinsic elastic moduli (or E/ρ) of the large and small specimens due to differences in work hardening or some other cause [4] and the two curves are brought into coincidence by changing the base value for the lower curve, say, then figure 3 results. Following the procedure of the other figures, the abscissa is again squared in order to obtain an approximately linear relationship. The line in figure 3 is also a least square solution of the experimental points, represented by the equation,

$$(v/v_B)^2 = 1 - 0.055[1/3(nt/l)^2]^2. \qquad (7)$$

It is emphasized that these departures from Bancroft's correction do not mean that this factor is in

[4] It is recalled from [3] that the large and small square specimens were cut from opposite ends of the original stock. It is also important to remember that these differences only show themselves on the highly expanded scale of figure 2.

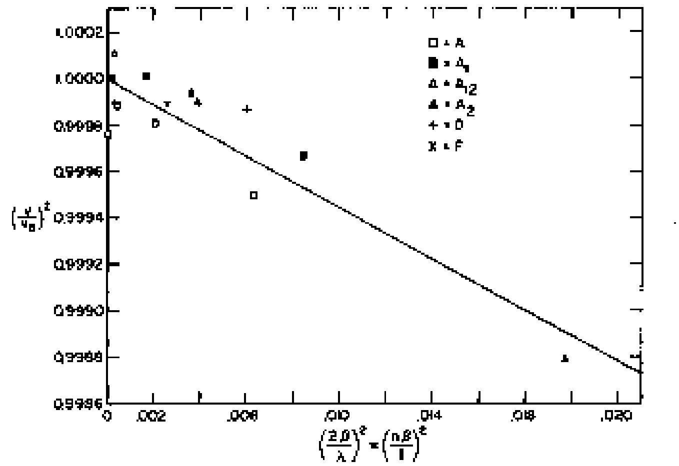

FIGURE 3. *Expanded plot showing reduction in longitudinal velocities of square bars, associated with longitudinal resonance vibrations, from that obtained using Bancroft's correction, after adjustment in the base value of the larger specimens.*

The line is a least square solution of the experimental points.

error. Rather they are a measure of the degree to which the experimental points depart from Bancroft's values when the conditions of his analysis are not fulfilled.

It is also noted that the third observation on figure 1 concerning the identity of the correction factor for overtones and fundamental resonance vibrations still holds on the expanded scale of figures 2 and 3. It is recalled from the previous study [3] that this was not the case for torsional resonance vibrations in square rods in which the cross-sectional correction for overtones followed a different pattern from that for the fundamental of specimens of the same effective length. For cylindrical specimens in torsional resonance, on the other hand, no cross-sectional correction at all is required theoretically either for the fundamental or for overtones and this is borne out by experiment.

4. Summary

The foregoing analysis may now be summarized in the following way.

In making the cross-sectional correction for longitudinal resonance vibrations of square or cylindrical bars, Bancroft's theoretical correction (developed for longitudinal waves in cylinders of infinite length) may be safely used if accuracy not higher than 1 part in 1,000 is sought. Bancroft's values are presented in such a manner that linear interpolations can also be made to this order of accuracy.

In making a similar correction to an accuracy of 1 part in 10,000, an adjustment in Bancroft's correction factor is required. In computing E, from this mode of vibration, then, the following two equations fit the data.