A Note on Bounds of Multiple Characteristic Roots of a Matrix

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If \( A = (a_{ij}) \) is an \( n \times n \) matrix and if \( C_i \) are the circles, center \( a_{ii} \) and radii \( \sum_{s \neq i}^{n} |a_{is}| \), and if

\[ \lambda \] is a characteristic root with \( m \) independent characteristic vectors, Olga Taussky proved the following two results:

1. If \( \lambda \) lies outside all but one circle \( C_i \), then \( m \) cannot be greater than 1.
2. If \( m = n - 1 \), then \( \lambda \) is an inner or boundary point of at least \( m \) circles \( C_i \).

In this note the gap between these two results is closed, and it is shown that \( \lambda \) lies in at least \( m \) circles \( C_i \), for all finite values of \( m \) and \( n \), \( m \leq n \).

We require this lemma:

**Lemma.** If \( X_i, i = 1, 2, \ldots, m \), are \( m \) independent vectors with components \( x_{is}, s = 1, 2, \ldots, n, n \geq m \), we may construct a set of \( m \) independent vectors \( Y_i \), with components \( y_{is} \), which are linear combinations of the vectors \( X_i \) and which have the property that we may select components of maxima moduli corresponding to each \( Y_i \), so that no two such selected components have the same subscripts.

We may suppose \( m \geq 2 \).

We choose \( Y_1 = X_1 \). Let \( y_{i1} = x_{i1} \) be a component of maximum modulus of \( Y_1 \). Choose \( \alpha_1 \) and \( \alpha_2 \) so that

\[ \alpha_1 y_{i1} + \alpha_2 x_{i2} = 0. \]

and

\[ Y_2 = \alpha_1 Y_1 + \alpha_2 X_2. \]

Since \( y_{i1} \neq 0 \), \( \alpha_2 \neq 0 \), and so since \( X_1 \) and \( X_2 \) are linearly independent, \( Y_2 \neq 0 \). Let \( y_{i2} \) be a component of maximum modulus of \( Y_2 \). By (1) and (2) \( y_{i2} \neq 0 \), hence \( s_2 \neq s_1 \). Further, \( Y_1 \) and \( Y_2 \) are linear combinations of the vectors \( X_1 \) and \( X_2 \) and are independent. The construction is thus complete for two independent vectors \( Y_i \). If \( m \geq 3 \), we choose three numbers \( \beta_1, \beta_2, \beta_3 \) so that

\[ \beta_1 y_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} = 0 \]

\[ \beta_1 y_{i2} + \beta_2 y_{i2} + \beta_3 x_{i3} = 0 \]

and

\[ Y_3 = \beta_1 Y_1 + \beta_2 Y_2 + \beta_3 X_3. \]

Since

\[ y_{i1} y_{i2} y_{i3} y_{i4} = y_{i1} y_{i2} \neq 0, \]

\[ y_{i2} y_{i3} y_{i4} y_{i3} \neq 0 \]

\[ \beta_3 \neq 0 \), and \( \gamma \neq 0 \). The argument used above may now be repeated to show that if \( y_{i3} \) is a component of maximum modulus of \( Y_3 \), then \( s_3 \neq s_1, s_3 \neq s_2 \). Further, \( Y_3 \) has the other properties required of \( Y_i \).
This would complete the construction for three vectors \( X_i \).

If \( m \geq 4 \), the other properties may be continued step by step till all the independent vectors \( Y_i \) are exhausted. This completes the proof of the Lemma. To complete the proof of Theorem C, given the set of \( m \) independent characteristic vectors \( X_i \), corresponding to the characteristic root \( \lambda \), we construct the set \( Y_i \) of the Lemma. Since \( Y_i \) are linear combinations of \( X_i \), they are also characteristic vectors corresponding to the characteristic root \( \lambda \). Hence we have the system of equations

\[
\sum_{s=1}^{n} a_{s,t} y_{is} = \lambda y_{ts}, \quad i = 1, 2, \ldots, m; t = 1, 2, \ldots, n.
\]

In particular we have

\[
\sum_{s=1}^{n} a_{s,t} y_{is} = \lambda y_{ts},
\]

and so

\[
(\lambda - a_{s,t}) y_{is} = \sum_{s=1}^{n} a_{s,t} y_{is}.
\]

Dividing through by \( y_{is} \), and taking the moduli of the two sides, since \( |y_{is}| \geq |y_{st}|, s = 1, 2, \ldots, n \) we get that \( \lambda \) lies in the circle \( C_{s,t} \).

Since \( s_t \neq s_i, i \neq j \), we conclude that \( \lambda \) lies in \( m \) different circles \( C_i \). This concludes the proof of the theorem.

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