Table of the Zeros and Weight Factors of the First Twenty Hermite Polynomials[1]

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The chief use of this table of the zeros and weight factors of the Hermite polynomials is in the calculation of integrals over the interval $[-\infty, \infty]$, when the integrand is either the product of e^{-x^2} and a polynomial, or may be closely approximated by e^{-x^2} times a polynomial. The zeros and weight factors, $x_i^{(n)}$ and $\alpha_i^{(n)}$, respectively, together with the auxiliary quantities $\beta_i^{(n)}=\alpha_i^{(n)} \exp [(x_i^{(n)})^2]$, which are useful in computation, are all tabulated here for the first twenty Hermite polynomials. The zeros $x_i^{(n)}$ and $\beta_i^{(n)}$ are given to 13 or more decimals, and the weight factors $\alpha_i^{(n)}$ to 13 significant figures. Although other shorter tables have appeared, this present table will enable one to cope with problems requiring much higher degree and accuracy, both in problems involving direct quadratures and those arising in the numerical solution of linear integral equations for the range $[-\infty, \infty]$. Thus the use of this table in any direct quadrature can give exact accuracy as far as the 39th degree (the only inexactitude arising from the use of rounded values of $x_i^{(n)}$ and $\alpha_i^{(n)}$).

1. Introduction

The main purpose of the table of the zeros $x_i^{(n)}$ and weight factors ("Christoffel numbers") $\alpha_i^{(n)}$ of the Hermite polynomials is to provide the basic quantities that occur in the quadrature formula

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx=\sum_{i=1}^{n} \alpha_i^{(n)} f(x_i^{(n)})+R_n, \tag{1}$$

where

$$R_n=\frac{\pi^{\frac{1}{2}} f^{(2n)}(\xi)}{2^n(2n)(2n-1)\ldots(n+2)(n+1)}$$

for some ξ, $-\infty<\xi<\infty$ [2, p. 101–102, 369].[2]

When the integrand does not contain e^{-x^2} explicitly, but behaves like e^{-x^2} times a polynomial, instead of (1), there is

$$\int_{-\infty}^{\infty} F(x)\,dx=\int_{-\infty}^{\infty} e^{-x^2}[e^{x^2}F(x)]\,dx\cong\sum_{i=1}^{n}\alpha_i^{(n)} e^{x_i^{(n)2}}F(x_i^{(n)}). \tag{2}$$

For dealing with quadratures of the type in (2), it is convenient to follow the example of A. Reiz [1] and to tabulate also the quantities $\beta_i^{(n)}=\alpha_i^{(n)} \exp [(x_i^{(n)})^2]$. From the form of the remainder term R_n, (1) is seen to be exact (save for the use of rounded values of $x_i^{(n)}$ and $\alpha_i^{(n)}$) when $f(x)$ is a polynomial of at most the $(2n-1)$th degree.

2. Numerical Solution of Integral Equations

Besides problems involving direct quadratures, there are those arising in the numerical solution of linear integral equations of the second kind for integrals over the range $[-\infty, \infty]$, namely,

$$\phi(y)=f(y)+\lambda\int_{-\infty}^{\infty} f(x)K(y,x)\,dx. \tag{3}$$

By considering $e^{x^2} f(x)K(y,x)$ as a polynomial in x, and making use of (2) for $y=x_i^{(n)}$, $i=1, 2, \ldots, n$, the approximate solution of (3) is reduced to the solution of a set of only n linear equations for the (approximate values of) $f(x_i^{(n)})$. These values of $f(x_i^{(n)})$ can then be used in (2) to find $f(y)$ for any value of y. The advantage in the use of (2) instead of an equally spaced quadrature formula for solving (3) is that here only n points are needed to give accuracy obtainable by approximating $e^{x^2} f(x)K(y,x)$ as a polynomial of the $(2n-1)$th degree in x. For a full description of the method, including examples, see A. Reiz [1], especially p. 4 to 10, 16 to 21.

3. Important Properties of Hermite Polynomials

The Hermite polynomials may be defined by the formula

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}), \tag{4}$$

or by the generating function

$$e^{-z^2+2zx}=\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!}. \tag{5}$$

They are given explicitly by the formula

$$H_n(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k}. \tag{6}$$

The numerical values of the coefficients of $H_n(x)$ for the first 30 values of n are contained in H. E. Salzer [3]. The polynomials $H_n(x)$ satisfy the differential equation

$$H_n''(x)-2xH_n'(x)+2nH_n(x)=0, \tag{7}$$

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[2] Figures in brackets indicate the literature references at the end of this paper.

the recurrence formula

$$H_{n+1}(x)-2xH_n(x)+2nH_{n-1}(x)=0, \qquad n\geqq 2, \tag{8}$$

and the differential-recurrence formula

$$H_n'(x)=2nH_{n-1}(x). \tag{9}$$

The Hermite polynomials have the orthogonality property

$$\int_{-\infty}^{\infty} e^{-x^2}H_n(x)\,H_m(x)\,dx=\pi^{\frac{1}{2}}2^n n!\delta_n^m. \tag{10}$$

A most important consequence of (10) is (1), where the $\alpha_i^{(n)}$ are defined by

$$\alpha_i^{(n)}=\frac{1}{H_n'(x_i^{(n)})}\int_{-\infty}^{\infty}\frac{e^{-x^2}H_n(x)}{x-x_i^{(n)}}\,dx. \tag{11}$$

The $\alpha_i^{(n)}$ are also expressible in the following form [2, p. 344], [4, p. 311]:

$$\alpha_i^{(n)}=\frac{\pi^{\frac{1}{2}}2^{n+1}n!}{\{H_n'(x_i^{(n)})\}^2}, \tag{12}$$

which, from (9), can be written as

$$\alpha_i^{(n)}=\frac{\pi^{\frac{1}{2}}2^{n-1}(n-1)!}{n\{H_{n-1}(x_i^{(n)})\}^2} \tag{13}$$

the latter form being more convenient for computation.

4. Description of Tables

This table gives the zeros $x_i^{(n)}$, the weight factors $\alpha_i^{(n)}$ and $\beta_i^{(n)}$, for the first 20 Hermite polynomials. The absolute values of the zeros, $|x_i^{(n)}|$, are tabulated to 15 decimals for $n=1(1)13$, to 14 decimals for $n=14(1)16$, and to 13 decimals for $n=17(1)20$. To each nonvanishing $|x_i^{(n)}|$ there corresponds the two zeros $x_i^{(n)}=+|x_i^{(n)}|$ and $x_i^{(n)}=-|x_i^{(n)}|$. The weight factors $\alpha_i^{(n)}=\alpha_{n-i+1}^{(n)}$ are tabulated to 13 significant figures. For a given n, zeros having the same absolute value are associated with the same weight factor. The auxiliary quantities $\beta_i^{(n)}$ are tabulated to 13 decimals.

5. Method of Computation and Checking

Since $H_n(x)$ is an odd or even polynomial for n odd or even, respectively, it was convenient to change the variable to $X=4x^2$ and to calculate $X_i^{(n)}=4x_i^{(n)2}$, where the nonvanishing $X_i^{(n)}$ are the zeros of polynomials $A_{[n/2]}(X)$, n odd, and $B_{[n/2]}(X)$, n even, of degree $[n/2]$ in X. Thus the degree of the polynomials $A_{[n/2]}(X)$ and $B_{[n/2]}(X)$ did not exceed 10, and the coefficients were considerably smaller than those of $H_n(x)$. A first approximation to $X_i^{(n)}$ was obtained from E. R. Smith [5], which gives the zeros of

$$h_n(x)=(-1)^n e^{x^2/2}\frac{d^n(e^{-x^2/2})}{dx^n}$$

by taking twice the square of the latter. The polynomials $A_{[n/2]}(X)$ and $B_{[n/2]}(X)$ were tabulated for five or six values of X in the neighborhood of each $X_i^{(n)}$, at intervals of 0.0001 in X. The zeros $X_i^{(n)}$ were calculated by a formula for inverse interpolation given in H. E. Salzer [6], and then were checked by direct interpolation in the polynomials $A_{[n/2]}(X)$ and $B_{[n/2]}(X)$, using the Gregory-Newton formula with advancing differences. The accuracy of the zero $X_i^{(n)}$ was determined from the approximate relation:

$$\text{error in } X_i^{(n)}\cong\frac{0.0001}{\Delta}F(X_i^{(n)}),$$

where $F(X_i^{(n)})$ denotes either $A_{[n/2]}(X_i^{(n)})$ or $B_{[n/2]}(X_i^{(n)})$, and Δ is the difference between two consecutive entries of $F(X)$ on either side of $X_i^{(n)}$.

The values of $x_i^{(n)}$ were found from $x_i^{(n)}=\pm\frac{1}{2}\sqrt{X_i^{(n)}}$, and checked back by squaring to get $X_i^{(n)}=4x_i^{(n)2}$. Additional checks upon the $x_i^{(n)}$ were performed by calculating

$$\sum_{i=1}^{n} x_i^{(n)2}=\frac{n(n-1)}{2}, \tag{14}$$

$$\prod_{i=1}^{n}{}' x_i^{(n)}=(-1)^{(n-1)/2}\frac{(n+1)!}{((n+1)/2)!}\text{ for } n \text{ odd}, \tag{15a}$$

where π' denotes the omission of the factor $x_i^{(n)}=0$, and

$$\prod_{i=1}^{n} x_i^{(n)}=(-1)^{n/2}\frac{n!}{(n/2)!}\text{ for } n \text{ even}. \tag{15b}$$

The weight factors $\alpha_i^{(n)}$ were calculated from (13), using

$$H_{n-1}(x_i^{(n)})=B_{[(n-1)/2]}(X_i^{(n)})\text{ for } n \text{ odd},$$

and

$$H_{n-1}(x_i^{(n)})=\sqrt{X_i^{(n)}}\,A_{[(n-1)/2]}(X_i^{(n)})\text{ for } n \text{ even}.$$

Formula (12) was used to check the $\alpha_i^{(n)}$, in the form

$$\alpha_i^{(n)}=\frac{\pi^{\frac{1}{2}}2^{n-3}n!}{\{X_i^{(n)}A'_{[n/2]}(X_i^{(n)})\}^2}\text{ for } n \text{ odd}, i\neq\frac{1}{2}(n+1) \tag{12a}$$

and

$$\alpha_i^{(n)}=\frac{\pi^{\frac{1}{2}}2^{n-3}n!}{X_i^{(n)}\{B'_{[n/2]}(X_i^{(n)})\}^2}\text{ for } n \text{ even}. \tag{12b}$$

Additional over-all checks upon $\alpha_i^{(n)}$ were performed, using (1) for $P(x)=1, x^2, x^4, \ldots, x^{2m}$, $2^m<2n-1<2^{m+1}$, and in a few cases for $P(x)=x^{2^m+2^p}$, $2^m+2^p<$

$2n-1<2^{m+1}$, the left member of (1) equal to $\Gamma\left(k+\frac{1}{2}\right)$ for $P(x)=x^{2k}$. The choice of these different powers of x for $P(x)$ was to ensure that every significant figure of the $\alpha_i^{(n)}$ was covered in the checks.

The auxiliary quanitites $\beta_i^{(n)}$ were computed by multiplying the weight factors $\alpha_i^{(n)}$ by $e^{x_i^{(n)2}/4}$, where the $e^{x_i^{(n)2}/4}$ were obtained from the NBS tables of the exponential function [7] and were checked by the relation $\log \beta_i^{(n)}=\log \alpha_i^{(n)}+X_i^{(n)}/4$, where the logarithms were obtained from the NBS tables [8].

The functions $x_i^{(n)}$, $\alpha_i^{(n)}$, and $\beta_i^{(n)}$ were calculated to about two more places than are tabulated, and are available in manuscript form. All entries given here, which were obtained by rounding the manuscript table, are correct to within about a unit in the last place.

The functions in this present table were compared with those in three much shorter similar tables given by R. E. Greenwood and J. J. Miller [9], A. Reiz [1], and Z. Kopal [10], and the few errors of more than a unit in the last place in those three tables were noted and reported to the journal "Mathematical Tables and Other Aids to Computation." Also, the values of $X_i^{(n)}$ for $n=16$ upon the worksheets were compared with those in J. Barkley Rosser [11], with perfect agreement. Finally, the zeros in the table of the Harvard University Laboratory [28] were checked by calculating $\sqrt{2}|x_i^{(n)}|$, to give complete agreement.

6. Illustration

As an example of the use of (1) for quadratures, suppose that one wished to evaluate numerically, say to 13 places, the integral $\int_0^\infty e^{-t^2}J_0(t)\,dt$, where $J_0(t)$ denotes the Bessel function of order zero. It happens that that integral is known to be $\frac{1}{2}\sqrt{\pi}\,e^{-1/8}I_0(1/8)$, where $I_0(t)$ denotes the modified Bessel function of order zero (see [26, p. 394]). As the integrand $e^{-t^2}J_0(t)$ is even, and as $\alpha_i^{(n)}J_0(x_i^{(n)})=\alpha_{n-i+1}^{(n)}J_0(-x_i^{(n)})$, from (1), whenever n is even,

$$\int_0^\infty e^{-t^2}J_0(t)\,dt\cong\Sigma\alpha_i^{(n)}J_0(x_i^{(n)}), \tag{16}$$

where the summation covers only the positive values of $x_i^{(n)}$. In the present example the error in (16) (from the term R_n in (1) and the inequality $|J_0^{(m)}(x)|\leq 1$) does not exceed 10^{-18} for n as small as 10. For the right member of (16), $J_0(x_i^{(10)})$ was computed to as many places as were sufficient to obtain $\alpha_i^{(n)}J_0(x_i^{(n)})$ to 13 places. The $J_0(x_i^{(10)})$ was calculated by interpolation in the Bessel function table of the Harvard Computation Laboratory [27]. The following are the results of these computations (the extra fourteenth place in $\alpha_i^{(10)}J_0(x_i^{(10)})$ was carried to avoid the cumulation of rounding errors):

i	$J_0(x_i^{(10)})$	$\alpha_i^{(10)}J_0(x_i^{(10)})$
6	0. 97081 99870 413	0. 59302 76641 6692
7	0. 74887 18398 349	0. 17983 30434 9660
8	0. 36515 41522 17	0. 01236 93767 8926
9	−0. 06452 02154 9	−0. 00008 66923 1312
10	−0. 37050 3331	−0. 00000 28308 0582

$$\sum_{i=6}^{10}\alpha_i^{(10)}J_0(x_i^{(10)})=0.78515\ 05503\ 338.$$

This answer differs by only a unit in the thirteenth place from the true value of $\frac{1}{2}\sqrt{\pi}\,e^{-1/8}I_0(1/8)$ which was found (employing the power series for $I_0(t)$) to be, to 13 decimals, 0.78515 05503 339.

7. Table of the Zeros and Weight Factors of the First Twenty Hermite Polynomials

$\lvert x_i^{(n)}\rvert$	$\alpha_i^{(n)}$	$\beta_i^{(n)}$
$n=1$	$n=1$	$n=1$
0. 00000 00000 00000	1. 7724 53850 906	1. 77245 38509 055
$n=2$	$n=2$	$n=2$
0. 70710 67811 86548	0. 88622 69254 528	1. 46114 11826 611
$n=3$	$n=3$	$n=3$
0. 00000 00000 00000	1. 1816 35900 604	1. 18163 59006 037
1. 22474 48713 91589	0. 29540 89751 509	1. 32393 11752 136
$n=4$	$n=4$	$n=4$
0. 52464 76232 75290	0. 80491 40900 055	1. 05996 44828 950
1. 65068 01238 85785	0. (1)81312 83544 725	1. 24022 58176 958
$n=5$	$n=5$	$n=5$
0. 00000 00000 00000	0. 94530 87204 829	0. 94530 87204 829
0. 95857 24646 13819	0. 39361 93231 522	0. 98658 09967 514
2. 02018 28704 56086	0. (1)19953 24205 905	1. 18148 86255 360

$\lvert x_i^{(n)} \rvert$	$a_i^{(n)}$	$\beta_i^{(n)}$
$n=6$	$n=6$	$n=6$
0. 43607 74119 27617	0. 72462 96952 244	0. 87640 13344 362
1. 33584 90740 13697	0. 15706 73203 229	0. 93558 05576 312
2. 35060 49736 74492	0. (2)45300 09906 509	1. 13690 83326 745
$n=7$	$n=7$	$n=7$
0. 00000 00000 00000	0. 81026 46175 568	0. 81026 46175 568
0. 81628 78828 58965	0. 42560 72526 101	0. 82868 73032 836
1. 67355 16287 67471	0. (1)54515 58281 913	0. 80718 46002 252
2. 65196 13568 35233	0. (3)97178 12450 995	1. 10133 07296 103
$n=8$	$n=8$	$n=8$
0. 38118 69902 07322	0. 66114 70125 582	0. 76454 41286 517
1. 15719 37124 46780	0. 20780 23258 149	0. 79289 00483 864
1. 98165 67566 95843	0. (1)17077 98300 741	0. 86675 26065 634
2. 93063 74202 57244	0. (3)19960 40722 114	1. 07193 01442 480
$n=9$	$n=9$	$n=9$
0. 00000 00000 00000	0. 72023 52156 061	0. 72023 52156 061
0. 72355 10187 52838	0. 43265 15590 026	0. 73030 24527 451
1. 46855 32892 16668	0. (1)88474 52739 438	0. 76460 81250 946
2. 26658 05845 31843	0. (2)49436 24275 537	0. 84175 27014 787
3. 19099 32017 81528	0. (4)39606 97726 326	1. 04700 35809 767
$n=10$	$n=10$	$n=10$
0. 34290 13272 23705	0. 61086 26337 353	0. 68708 18539 513
1. 03661 08297 89514	0. 24013 86110 823	0. 70329 63231 049
1. 75668 36492 99882	0. (1)33874 39445 548	0. 74144 19319 436
2. 53273 16742 82790	0. (2)13436 45746 781	0. 82066 61264 048
3. 43615 91188 37738	0. (5)76404 32856 233	1. 02545 16913 657
$n=11$	$n=11$	$n=11$
0. 00000 00000 00000	0. 65475 92869 146	0. 65475 92869 146
0. 65680 95668 82100	0. 42935 97523 561	0. 66096 04194 410
1. 32655 70844 94933	0. 11722 78751 677	0. 68121 18810 667
2. 02594 80158 25755	0. (1)11911 39544 491	0. 72195 36247 284
2. 78329 00997 81652	0. (3)34681 94663 233	0. 80251 68688 510
3. 66847 08465 59683	0. (5)14396 60393 714	1. 00652 67881 724
$n=12$	$n=12$	$n=12$
0. 31424 03762 54359	0. 57013 52362 625	0. 62930 78743 695
0. 94778 83912 40164	0. 26049 23103 642	0. 63962 12320 203
1. 59768 26351 52605	0. (1)51607 98561 588	0. 66266 27732 669
2. 27950 70805 01060	0. (2)39053 90584 629	0. 70522 03661 122
3. 02063 70251 20890	0. (4)85736 87043 588	0. 78664 39394 633
3. 88972 48978 69782	0. (6)26585 51684 356	0. 98969 90470 923
$n=13$	$n=13$	$n=13$
0. 00000 00000 00000	0. 60439 31879 211	0. 60439 31879 211
0. 60576 38791 71060	0. 42161 62968 985	0. 60852 95837 033
1. 22005 50365 90748	0. 14032 33206 870	0. 62171 60552 868
1. 85310 76516 01512	0. (1)20862 77529 617	0. 64675 94633 158
2. 51973 56856 78238	0. (2)12074 59992 719	0. 69061 80348 378
3. 24660 89783 72410	0. (4)20430 36040 271	0. 77258 08233 517
4. 10133 75961 78640	0. (7)48257 31850 073	0. 97458 03956 399
$n=14$	$n=14$	$n=14$
0. 29174 55106 7256	0. 53640 59097 121	0. 58406 16005 220
0. 87871 37873 2940	0. 27310 56090 642	0. 59110 66670 432
1. 47668 27311 4114	0. (1)68505 53422 347	0. 60637 97391 261
2. 09518 32585 0772	0. (2)78500 54726 458	0. 63290 06064 723
2. 74847 07249 8540	0. (3)35509 26135 519	0. 67770 67591 924
3. 46265 69336 0227	0. (5)47164 84355 019	0. 75998 70873 976
4. 30444 85704 7363	0. (8)86285 91168 125	0. 96087 87030 257

$\lvert x_i^{(n)}\rvert$	$\alpha_i^{(n)}$	$\beta_i^{(n)}$
$n=15$	$n=15$	$n=15$
0. 00000 00000 0000	0. 56410 03087 264	0. 56410 03087 264
0. 56506 95832 5558	0. 41202 86874 989	0. 56702 11534 466
1. 13611 55852 1092	0. 15848 89157 959	0. 57619 33502 835
1. 71999 25751 8649	0. (1)30780 03387 255	0. 59302 74497 642
2. 32573 24861 7386	0. (2)27780 68842 913	0. 62066 26085 270
2. 96716 69279 0560	0. (3)10000 44412 325	0. 66616 60051 091
3. 66995 03734 0445	0. (5)10591 15547 711	0. 74860 73660 169
4. 49999 07073 0939	0. (8)15224 75804 254	0. 94886 89708 276
$n=16$	$n=16$	$n=16$
0. 27348 10461 3815	0. 50792 94790 166	0. 54737 52050 378
0. 82295 14491 4466	0. 28064 74585 285	0. 55244 19573 675
1. 38025 85391 9888	0. (1)83810 04139 899	0. 56321 78290 882
1. 95178 79909 1625	0. (1)12880 31153 551	0. 58124 72754 009
2. 54620 21578 4748	0. (3)93228 40086 242	0. 60973 69582 560
3. 17699 91619 7996	0. (4)27118 60092 538	0. 65575 56728 761
3. 86944 79048 6012	0. (6)23209 80844 865	0. 73824 56222 777
4. 68873 89393 0582	0. (9)26548 07474 011	0. 93687 44928 841
$n=17$	$n=17$	$n=17$
0. 00000 00000 000	0. 53091 79376 249	0. 53091 79376 249
0. 53163 30013 427	0. 40182 64694 704	0. 53307 06545 736
1. 06764 87257 435	0. 17264 82976 701	0. 53976 31139 085
1. 61292 43142 212	0. (1)40920 03414 976	0. 55177 73530 782
2. 17350 28266 666	0. (2)50673 49957 628	0. 57073 92941 245
2. 75776 29157 039	0. (3)29864 32866 978	0. 59989 27826 678
3. 37893 20911 415	0. (5)71122 89140 021	0. 64629 17002 129
4. 06194 66758 755	0. (7)49770 78981 631	0. 72874 83705 871
4. 87134 51936 744	0. (10)45805 78930 799	0. 92625 41399 895
$n=18$	$n=18$	$n=18$
0. 25826 77505 191	0. 48349 56947 255	0. 51684 68364 816
0. 77668 29192 674	0. 28480 72856 700	0. 52063 49466 761
1. 30092 08583 896	0. (1)97201 74764 132	0. 52858 94429 188
1. 83553 16042 616	0. (1)18640 04238 754	0. 54157 86786 621
2. 38629 90891 667	0. (2)18885 22630 268	0. 56127 90455 498
2. 96137 75055 316	0. (4)91811 26867 929	0. 59095 30034 631
3. 57376 90684 863	0. (5)18106 54481 093	0. 63763 01720 062
4. 24811 78735 681	0. (7)10467 20579 579	0. 71909 33831 053
5. 04836 40088 745	0. (11)78281 99772 116	0. 91639 35375 519
$n=19$	$n=19$	$n=19$
0. 00000 00000 000	0. 50297 48882 762	0. 50297 48882 762
0. 50352 01634 239	0. 39160 89886 130	0. 50461 53313 522
1. 01036 83871 343	0. 18363 27013 070	0. 50967 93750 983
1. 52417 06193 935	0. (1)50810 38690 905	0. 51863 31937 003
2. 04923 17098 506	0. (2)79888 66777 723	0. 53240 23605 516
2. 59113 37897 945	0. (3)67087 75214 072	0. 55269 46209 671
3. 15784 88183 476	0. (4)27209 19776 316	0. 58277 95297 628
3. 76218 73519 640	0. (6)44882 43147 223	0. 62965 66326 368
4. 42853 28066 038	0. (8)21630 51009 864	0. 71188 18743 371
5. 22027 16905 375	0. (11)13262 97094 499	0. 90719 87960 928
$n=20$	$n=20$	$n=20$
0. 24534 07083 009	0. 46224 36696 006	0. 49092 15006 667
0. 73747 37285 454	0. 28667 55053 628	0. 49384 33852 721
1. 23407 62153 953	0. 10901 72060 200	0. 49992 08713 363
1. 73853 77121 166	0. (1)24810 52088 746	0. 50967 90271 175
2. 25497 40020 893	0. (2)32437 73342 238	0. 52408 03509 486
2. 78880 60584 281	0. (3)22833 86360 163	0. 54485 17423 644
3. 34785 45673 832	0. (5)78025 56478 532	0. 57526 24428 525
3. 94476 40401 156	0. (6)10860 69370 769	0. 62227 86961 914
4. 60368 24495 507	0. (9)43993 40992 273	0. 70433 29611 769
5. 38748 08900 112	0. (12)22293 93645 534	0. 89859 19614 532

8. References

In addition to the references cited in the preceding text, there are mentioned here a few articles and books that cover many essential properties of Hermite polynomials. A detailed bibliography on Hermite polynomials is given in J. A. Shohat [12]. The standard work on the theory is G. Szegö [2].

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