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Some General Theorems on Iterants [1]

P. Stein [2]

If B is a square matrix, then it is known that a necessary and sufficient condition that $\lim_{n\to\infty} B^n=0$, is that the characteristic roots of B are all of modulus less than unity. An alternative condition is given in this paper, in terms of Hermitian matrices. Further, a generalization of the result is obtained that covers cases of matrices B whether B^n does or does not converge to 0, except for very special matrices.

Introduction. If B is a square matrix with real or complex elements, it is well known that a necessary and sufficient condition that $\lim_{n\to\infty} B^n=0$ is that the characteristic roots of B are all of modulus less than 1.

In this paper an alternative condition for the convergence of B^n to 0 will be given in terms of certain Hermitian and symmetric matrices. We also obtain a generalization of this result that covers matrices B when B^n does or does not converge to 0, except for a special class of such matrices B.

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We will consider square matrices B whose elements are either real or complex. The conjugate transpose of B will be denoted by B^*.

THEOREM 1. A necessary and sufficient condition that $\lim_{n\to\infty} B^n=0$ is that there exist a positive definite Hermitian matrix H for which $H-B^*HB$ is positive definite.

Corollary 1. If B is real, H may be taken real and symmetric.

Proof: Necessity: Let P be a nonsingular matrix such that

$$PBP^{-1}=K_1\dotplus K_2\dotplus \ldots \dotplus K_r,$$

where K_i is the Jordan normal form; i. e.

$$K_i=\lambda_i I^{n_i\times n_i}+U^{n_i\times n_i},$$

where $\sum_{i=1}^{r} n_i=n$, λ_i are the *not necessarily distinct* characteristic roots of B, and $U^{n_i\times n_i}$ is a matrix with units in the superdiagonal and zero elsewhere.

Let $\delta_i=\delta(\epsilon_i)$ be the diagonal matrix $(\epsilon_i^{n_i-1}, \epsilon_i^{n_i-2}, \ldots, 1)$ for i=1, 2, ... r.

If $Q=\delta_1\dotplus\delta_2\dotplus \ldots\dotplus\delta_r$, then

$$K=QPBP^{-1}Q^{-1}=N_1\dotplus N_2\dotplus \ldots \dotplus N_r, \text{ where}$$
$$N_i=\delta_i K_i\delta_i^{-1}=\lambda_i I+\epsilon_i U;$$

it being understood that I and U are of the correct order.

Note that

$$I-K^*K=(I-N_1^*N_1)\dotplus(I-N_2^*N_2)\dotplus \ldots \dotplus(I-N_r^*N_r) \tag{2}$$

is positive definite if and only if $(I-N_i^*N_i)$ is positive definite for all i. Clearly

$$I-N_i^*N_i=(1-\bar\lambda_i\lambda_i)I-\epsilon_i(\bar\lambda_iU+\lambda_iU^*+\epsilon_iU^*U) \tag{3}$$

will be positive definite if

$$1-\bar\lambda_i\lambda_i>\epsilon_i\left[\frac{y^*(\bar\lambda_iU+\lambda_iU^*)y}{y^*y}+\epsilon_i\frac{y^*U^*Uy}{y^*y}\right] \text{ for } y\neq 0. \tag{4}$$

If $M=\max|\lambda_i|$, we have

$$\frac{y^*(\bar\lambda_iU+\lambda_iU^*)y}{y^*y}<2M, \text{ also } \frac{y^*U^*Uy}{y^*y}<1 \text{ for all } y\neq 0;$$

hence

$$\epsilon_i\left[\frac{y^*(\bar\lambda_iU+\lambda_iU^*)y}{y^*y}+\epsilon_i\frac{y^*U^*Uy}{y^*y}\right]<\epsilon_iM+\epsilon_i^2 \text{ for all } y\neq 0. \tag{5}$$

Since $\lim_{n\to\infty} B^n=0$, $|\lambda_i|<1$; hence, from (3), (4), and (5), $I-N_i^*N_i$ is positive definite for sufficiently small values of ϵ_i; and so from (2), $I-K^*K$ is positive definite for such values of ϵ_i. A change of variable $y=QPx$ gives

$$y^*(I-K^*K)y=x^*(H-B^*HB)x, \text{ where } H=P^*Q^*QP.$$

Since H is clearly positive definite, the proof for the necessity part is complete.

Sufficiency: [3] Let H be *any* positive definite Hermitian matrix for which $H-B^*HB$ is positive definite. Since H is positive definite, $H=D^*D$, and by making the change of variables $Dx=y$,

$$x^*(H-B^*HB)x=y^*(I-K^*K)y>0, \qquad (K=DBD^{-1}). \tag{6}$$

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[2] University of Natal, Durban, South Africa, and University of California at Los Angeles.

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Now, if λ_i is any characteristic root of K (and hence of B), y_i an associated characteristic vector, we see that

$$y_i^*(I-K^*K)y_i = y_i^*y_i - \bar{\lambda}_i\lambda_i y_i^* y_i > 0.$$

Thus, $|\lambda_i| < 1$ for all characteristic roots of B, and hence B^n will converge to 0.

To prove the corollary, we suppose the elements of B real. Let H be the matrix of the theorem, then $H = A + iS$, where A is a real symmetric matrix, and S is a real skew-symmetric matrix. If H is positive definite, then it is known that A is positive definite. Again

$$H - B^*HB = H - B'HB = A - B'AB + i\,(S - B'SB).$$

$A - B'AB$ is symmetric and $S - B'SB$ is skew-symmetric. If $H - B'HB$ is positive definite, then $A - B'AB$ is positive definite. Hence we may use A in place of H in the theorem.

We give a sufficiency test for the nonconvergence of B^n to 0.

Theorem 2. If there exists a nonpositive-definite matrix H such that $H - B^*HB$ is positive definite, then $\lim_{n\to\infty} B^n \neq 0$.

For proof, we observe that if H is not positive definite, a vector x may be found such that $x^*Hx \leq 0$. Further, if $H - B^*HB$ is positive definite, the sequence $x^*B^{*n}HB^nx$ is decreasing. Hence $\lim_{n\to\infty} B^nx \neq 0$ and so $\lim_{n\to\infty} B^n \neq 0$.

It may be observed that the condition that $H - B^*HB$ should be positive definite may be weakened to $H - B^*HB$ at least positive-semi-definite, provided H is not positive-semi-definite.

Now we shall prove a generalization of the necessity part of theorem 1.

Theorem 3. Let B be a matrix whose characteristic roots of modulus 1 have multiplicity no greater than two. Then there exists a nonzero Hermitian matrix H_1 such that $H_1 - B^*H_1B \geq 0$.

Corollary 2. If B is real, H_1 may be taken real and symmetric. *Proof.* Using the expression (2) for $(I-K^*K)$, we see that

$$I - K^*K - K^*(I-K^*K)K = [I - N_1^*N_1 - N_1^*(I - N_1^*N_1)N_1] \dot{+} \ldots \dot{+} [I - N_r^*N_r - N_r^*(I - N_r^*N_r)N_r]$$

and again, this will be positive semidefinite if

$$(I - N_i^*N_i) - N_i^*(I - N_i^*N_i)N_i = (1 - \bar{\lambda}_i\lambda_i)^2 I - 2\epsilon_i(1 - \bar{\lambda}_i\lambda_i)E + \epsilon_i^2[\lambda_i U^*E + \bar{\lambda}_i EU + \epsilon_i U^*EU], \quad (8)$$

$\{E = (\bar{\lambda}_i U + \lambda_i U^* + \epsilon_i U^*U)\}$, is positive semidefinite.

Obviously, by a proper choice of ϵ_i, (8) can be made positive definite if B has no characteristic roots of modulus 1.

If B has a characteristic root such that $\bar{\lambda}_i\lambda_i = 1$, the right side of (8) vanishes for roots of multiplicity 1. For roots of multiplicity two, (8) becomes $\begin{pmatrix} 0 & 0 \\ 0 & 2\epsilon_i^2 \end{pmatrix}$ and hence can be made positive semidefinite.

Now a simple change of variables, $y = QPx$, as in Theorem 1, yields

$$y^*[(I-K^*K) - K^*(I-K^*K)K]y = x^*[(H - B^*HB) - B^*(H - B^*HB)B]x \geqq 0,$$

where $H = P^*Q^*QP$. Thus the H_1 of our theorem is chosen as $H - B^*HB$.

If the multiplicity of a root of modulus 1 is three or greater, the right side of (8) is not positive semidefinite since it will always contain the principal subminor $\epsilon_i^2\begin{pmatrix} 0 & \bar{\lambda}_i^2 \\ \lambda_i^2 & \epsilon_i^2 + 2\bar{\lambda}_i\lambda_i \end{pmatrix}$.

Hence the method used in the proof of this theorem does not yield an H_1 in these cases.

It may be observed that $\lim_{n\to\infty} B^n = 0$, if and only if H_1 is positive definite. For, if B has no roots of modulus equal to 1, then from the proof of the theorem it follows that $H_1 - B^*H_1B$ is positive definite, and the results follow from the sufficiency part of Theorem 1 and from Theorem 2. If B has a root of modulus 1, then since $H_1 = H - B^*HB$, we may show, as in the proof of the sufficiency part of Theorem 1, that H_1 is at best positive semidefinite, and hence also not positive definite. In this case also $\lim_{n\to\infty} B^n \neq 0$.

Corollary 2 may be proved in the same way as corollary 1 of theorem 1.

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