Some General Theorems on Iterants

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If $B$ is a square matrix, then it is known that a necessary and sufficient condition that 
\[ \lim_{n \to \infty} B^n = 0, \]
is that the characteristic roots of $B$ are all of modulus less than unity. An alternative condition is given in this paper, in terms of Hermitian matrices. Further, a generalization of the result is obtained that covers cases of matrices $B$ whether $B^n$ does or does not converge to 0, except for very special matrices.

Introduction. If $B$ is a square matrix with real or complex elements, it is well known that a necessary and sufficient condition that 
\[ \lim_{n \to \infty} B^n = 0, \]
is that the characteristic roots of $B$ are all of modulus less than 1.

In this paper an alternative condition for the convergence of $B^n$ to 0 will be given in terms of certain Hermitian and symmetric matrices. We also obtain a generalization of this result that covers matrices $B$ when $B^n$ does or does not converge to 0, except for a special class of such matrices $B$.

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We will consider square matrices $B$ whose elements are either real or complex. The conjugate transpose of $B$ will be denoted by $B^*$. 

**THEOREM 1.** A necessary and sufficient condition that $\lim_{n \to \infty} B^n = 0$ is that there exist a positive definite Hermitian matrix $H$ for which $H - B^*H B$ is positive definite.

**Corollary 1.** If $B$ is real, $H$ may be taken real and symmetric.

**Proof:** Necessity: Let $P$ be a nonsingular matrix such that 
\[ P B P^{-1} = K_1 + K_2 + \ldots + K_r, \]
where $K_i$ is the Jordan normal form; i.e., 
\[ K_i = \lambda_i I + U^{\infty}i, \]
where $\sum_{i=1}^{r} n_i = n$, $\lambda_i$ are the not necessarily distinct characteristic roots of $B$, and $U^{\infty}i$ is a matrix with units in the superdiagonal and zero elsewhere.

Let $\delta = \delta(e_i)$ be the diagonal matrix $(e_i, e_i, \ldots, e_i, 1)$ for $i = 1, 2, \ldots, r$.

If $Q = \delta + \delta + \ldots + \delta$, then 
\[ K = Q P B P^{-1} Q^{-1} = N_1 + N_2 + \ldots + N_r, \]
where 
\[ N_i = \delta K_i \delta^{-1} = \lambda_i I + e_i U; \]
it being understood that $I$ and $U$ are of the correct order.

Note that 
\[ I - K^*K = (I - N_1^*N_1) + (I - N_2^*N_2) + \ldots \]
\[ + (I - N_r^*N_r) \] (2)
is positive definite if and only if $(I - N_i^*N_i)$ is positive definite for all $i$. Clearly 
\[ I - N_i^*N_i = (1 - \lambda_i \lambda_i) I - e_i (\lambda_i U + \lambda_i U^* + e_i U U) \] (3)
will be positive definite if 
\[ 1 - \lambda_i \lambda_i > e_i \left[ \frac{y^*(X_i U + \lambda_i U^*)y}{y^*y} + e_i \frac{y^*U^*Uy}{y^*y} \right] \] for $y \neq 0$. (4)

If $M = \max |\lambda_i|$, we have 
\[ y^*(X_i U + \lambda_i U^*)y < 2M, \text{ also } \frac{y^*U^*Uy}{y^*y} < 1 \text{ for all } y \neq 0; \]
hence 
\[ e_i \left[ \frac{y^*(X_i U + \lambda_i U^*)y}{y^*y} + e_i \frac{y^*U^*Uy}{y^*y} \right] < e_i M + e_i \] for all $y \neq 0$. (5)

Since $\lim_{n \to \infty} B^n = 0$, $|\lambda_i| < 1$; hence, from (3), (4), and (5), $I - N_i^*N_i$ is positive definite for sufficiently small values of $e_i$; and so from (2), $I - K^*K$ is positive definite for such values of $e_i$. A change of variable $y = Q P x$ gives 
\[ y^*(I - K^*K)y = x^*(H - B^*H B)x, \text{ where } H = P^*Q^*Q P. \]

Since $H$ is clearly positive definite, the proof for the necessary part is complete.

Sufficiency: Let $H$ be any positive definite Hermitian matrix for which $H - B^*H B$ is positive definite. Since $H$ is positive definite, $H = D^*D$, and by making the change of variables $Dx = y$,
\[ x^*(H - B^*H B)x = y^*(I - K^*K)y > 0, \quad (K = DRD^{-1}). \] (6)

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Now, if $\lambda_i$ is any characteristic root of $K$ (and hence of $B$), $y_i$ an associated characteristic vector, we see that

$$y_i^*(I-K*K)y_i=y_i^*(1-\lambda_i^2)y_i>0.$$ 

Thus, $|\lambda_i|<1$ for all characteristic roots of $B$, and hence $B^n$ will converge to 0.

To prove the corollary, we suppose the elements of $B$ real. Let $H$ be the matrix of the theorem, then $H=A+iS$, where $A$ is a real symmetric matrix, and $S$ is a real skew-symmetric matrix. If $H$ is positive definite, then it is known that $A$ is positive definite. Again

$$H-B*HB=H-B'BHB=A-B'AB+i(S-B'SB).$$

$A-B'AB$ is symmetric and $S-B'SB$ is skew-symmetric. If $H-B*HB$ is positive definite, then $A-B'AB$ is positive definite. Hence we may use $A$ in place of $H$ in the theorem.

We give a sufficiency test for the nonconvergence of $B^n$ to 0.

Theorem 2. If there exists a nonpositive-definite matrix $H$ such that $H-B*HB$ is positive definite, then

$$\lim_{n \to \infty} B^n=0.$$ 

For proof, we observe that if $H$ is not positive definite, a vector $x$ may be found such that $x^*Hx<0$. Further, if $H-B*HB$ is positive definite, the sequence $x^*(H-B*HB)x$ is decreasing. Hence $\lim_{n \to \infty} B^n=0$ and so $\lim_{n \to \infty} B^n=0$.

It may be observed that the condition that $H-B*HB$ should be positive definite may be weakened to $H-B*HB$ at least positive-semidefinite, provided $H$ is not positive-semidefinite.

Now we shall prove a generalization of the necessity part of theorem 1.

Theorem 3. Let $B$ be a matrix whose characteristic roots of modulus 1 have multiplicity no greater than two. Then there exists a nonzero Hermitian matrix $H_i$ such that $H_i-B*H_iB \geq 0$.

Corollary 2. If $B$ is real, $H_i$ may be taken real and symmetric. Proof. Using the expression (2) for $(I-K*K)$, we see that

$$I-K*K-K^*(I-K*K)K=[I-N_i*N_i-N_i*(I-N_i*N_i)N_i]+\ldots+[I-N_i*N_i-N_i*(I-N_i*N_i)N_i]$$

and again, this will be positive semidefinite if

$$I-N_i*N_i-N_i*(I-N_i*N_i)N_i=(1-\lambda_i^2)I-2\varepsilon_i(1-\lambda_i^2)E+\varepsilon_i[I*U*E+\lambda_iEU+\varepsilon_i U*EU],$$

$$\{E=(\lambda_iU+\lambda_iU*+\varepsilon_i U*U), \text{ is positive semidefinite.}\}$$

Obviously, by a proper choice of $\varepsilon_i$, (8) can be made positive definite if $B$ has no characteristic roots of modulus 1.

If $B$ has a characteristic root such that $\lambda_i=1$, the right side of (8) vanishes for roots of multiplicity 1.

For roots of multiplicity two, (8) becomes

$$\begin{pmatrix} 0 & 0 \\ 0 & 2\varepsilon_i^2 \end{pmatrix}$$

and hence can be made positive semidefinite.

Now a simple change of variables, $y=QPz$, as in Theorem 1, yields

$$y^*(I-K*K)-K^*(I-K*K)Ky^*-(H-B*HB)-B*(H-B*HB)B|x \geq 0,$$

where $H=\Re Q*QP$. Thus the $H_i$ of our theorem is chosen as $H-B*HB$.

The multiplicity of a root of modulus 1 is three or greater, the right side of (8) is not positive semidefinite since it will always contain the principal subminor $\varepsilon_i^2(\lambda^2_i+2\lambda_i+\lambda_i^2)$.

Hence the method used in the proof of this theorem does not yield an $H_i$ in these cases.

It may be observed that $\lim_{n \to \infty} B^n=0$, if and only if $H_i$ is positive definite. For, if $B$ has no roots of modulus equal to 1, then from the proof of the theorem it follows that $H_i-B*H_iB$ is positive definite, and the results follow from the sufficiency part of Theorem 1 and from Theorem 2. If $B$ has a root of modulus 1, then since $H_i=H-B*HB$, we may show, as in the proof of the sufficiency part of Theorem 1, that $H_i$ is at best positive semidefinite, and hence also not positive definite. In this case also $\lim_{n \to \infty} B^n=0$.

Corollary 2 may be proved in the same way as corollary 1 of theorem 1.

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