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Uniformly Best Constant Risk and Minimax Point Estimates[1]

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In this paper several types of point estimates are compared on the bases of their corresponding expected risk. It is shown that constant risk minimax estimates (which are always uniformly best constant risk estimates) exist, under certain conditions, for several frequently occurring types of parameters and general methods are obtained for constructing these minimax estimates.

1. Introduction

Let $x_1, \ldots, x_n$ denote n (not necessarily independent) observed values of a random variable ξ, which is distributed over a space S according to a distribution $P_\xi(x, \theta_1, \ldots, \theta_s)$. It is assumed that $P_\xi(x, \theta_1, \ldots, \theta_s)$ is completely specified except for the s unknown parameters $\theta_1, \ldots, \theta_s$. These parameters may be represented by a point $\theta=(\theta_1, \ldots, \theta_s)$ in the s-dimensional Euclidean parameter space Ω. Also $X=(x_1, \ldots, x_n)$ is a point in the n-dimensional Euclidean sample space, M. We shall assume that $P_\xi(x, \theta)$ is absolutely continuous, that is, ξ possesses an integrable probability density function $g_\xi(x, \theta)$. Let $p(X, \theta)=p(x_1, \ldots, x_n, \theta)$ denote the joint probability density function of the observations at $X\epsilon M$.

A statistical point estimate of a parameter θ_t, which ranges over a subset ω_t of one-dimensional Euclidean space, is a function $f_t(X)$ of the sample values that takes on values in ω_t. Let $W[f_t(X), \theta]$ be a nonnegative measurable function defined for all $\theta\epsilon\Omega$ and $X\epsilon M$. $W[f_t(X), \theta]$ is a weight function that represents the relative seriousness of taking $f_t(X)$ as the value of θ_t for any particular sample point X. The function

$$r_{f_t}(\theta)=\int_M W[f_t(X), \theta]\, p(X, \theta)\, dX$$

represents the risk or expected loss incurred by using $f_t(X)$ to estimate θ_t when θ is the true parameter point. Thus $r_{f_t}(\theta)$ is defined as the *risk function* of $f_t(X)$. The *expected risk* of $f_t(X)$, relative to an *a priori distribution* $\lambda(\theta)$ of θ is given by

$$R_{f_t}(\lambda)=\int_\Omega \int_M W[f_t(X), \theta]\, p(X, \theta)\, dX\, d\lambda(\theta).$$

We can now define the following classes of point estimates in terms of $r_{f_t}(\theta)$ and $R_{f_t}(\lambda)$.

(1) A *minimax* estimate of θ_t is one which minimizes $\sup r_{f_t}(\theta)$.

(2) A *constant risk* (CR) estimate f_t of θ_t is one such that $r_{f_t}(\theta)$ is constant.

(3) A *Bayes* estimate of θ_t, relative to an *a priori* distribution $\bar{\lambda}(\theta)$, is one that minimizes $R_{f_t}(\bar{\lambda})$.

(4) A *uniformly best* (UB) estimate of θ_t is one that minimizes $R_{f_t}(\lambda)$ for all possible *a priori* distributions $\lambda(\theta)$.

(5) A *uniformly best constant risk* (UBCR) estimate of θ_t is one that is a UB-estimate among all CR-estimates.

(6) A *constant risk minimax* (CRM) estimate is an estimate that is both a CR and a minimax estimate.

It is evident that a UB-estimate is preferable to any other, provided that one can be obtained. We will show that in several important cases it is reasonable to restrict our choice to CRM-estimates, since they possess certain desirable properties and, in most cases, are relatively easy to obtain. The concepts of a risk function, expected risk, and minimax estimates used here are due to Wald [5 to 8].

Let

$$\varphi_{f_t}(X)=\int_\Omega W[f_t(X), \theta]\, p(X, \theta)\, d\theta. \qquad (1)$$

Theorem 1.1 *Let $f_t(X)$ be a CR estimate of θ_t and suppose that for any other estimate $\bar{f}_t(X)$ there exists a probability measure $\bar{\lambda}(\theta)$ over Ω such that*

$$\int_\Omega r_{f_t}(\theta)\, d\bar{\lambda}(\theta) \leq \int_\Omega r_{\bar{f}_t}(\theta)\, d\bar{\lambda}(\theta). \qquad (2)$$

Then $f_t(X)$ is a minimax estimate.

Proof. Let $\bar{f}_t(X)$ be any other estimate and let $\bar{\lambda}(\theta)$ be a probability measure such that (2) is satisfied. Then

$$c-\sup_\theta r_{\bar{f}_t}(\theta) \leq \int_\Omega [c-r_{\bar{f}_t}(\theta)]\, d\bar{\lambda}(\theta)$$

$$\leq \int_\Omega [r_{f_t}(\theta)-r_{\bar{f}_t}(\theta)]\, d\bar{\lambda}(\theta) \leq 0$$

where $r_{f_t}(\theta)\equiv c$ (c a constant). Therefore

$$c=\sup_\theta r_{f_t}(\theta) \leq \sup_\theta r_{\bar{f}_t}(\theta),$$

and the theorem follows.

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Corollary 1.1 *Any CR-estimate which is a Bayes estimate relative to some probability measure* $\lambda'(\theta)$ *is a minimax estimate.*

Proof. This corollary follows immediately from theorem 1.1.

Theorem 1.2. *Any CR-estimate* $f_t(X)$ *that is a minimax estimate is a UBCR-estimate.*

Proof. Let $\bar{f}_t(X)$ be any other CR-estimate. Then

$$R_{\bar{f}_t}(\lambda)-R_{f_t}(\lambda)=\int_\Omega [r_{\bar{f}_t}(\theta)-r_{f_t}(\theta)]\,d\lambda(\theta)$$

$$=\bar{c}-c=\sup_\theta r_{\bar{f}_t}(\theta)-\sup_\theta r_{f_t}(\theta)\geq 0,$$

where $\lambda(\theta)$ is any probability measure over Ω, $r_{\bar{f}_t}(\theta)\equiv\bar{c}$, $r_{f_t}(\theta)\equiv c$, (c and $\bar{c}$ constants). Thus $f_t(X)$ is a UBCR-estimate.

Theorem 1.3. *Suppose that a CR-estimate* $f_t(X)$ *minimizes* $\phi_{f_t}(X)$ *for all* $X\epsilon M$ *and that at least one of the following conditions (A) and (B) is satisfied:*

(A) Ω *is compact;*

(B) $r_{f_t}(\theta)$ *and* $\phi_{f_t}(X)$ *are uniformly convergent over* M *and* Ω*, respectively. Then* $f_t(X)$ *is both a minimax and a UBCR-estimate of* θ_t.

Proof. Let $\bar{f}_t(X)$ be any other estimate of θ_t. Then

$$\phi_{\bar{f}_t}(X)-\phi_{f_t}(X)\geq 0 \qquad (3)$$

for all $X\epsilon M$. Let M_q and Ω_j be compact subsets of positive measure of M and Ω, respectively, such that

$$M_q<M_{q+1}, \qquad q=1,2,\ldots$$

$$\Omega_j<\Omega_{j+1}, \qquad j=1,2,\ldots$$

and

$$\lim_{q=\infty} M_q=M,$$

$$\lim_{j=\infty}\Omega_j=\Omega.$$

Since $W[f_t(X),\theta]$ is nonnegative and measurable, it follows from Fubini's Theorem (see, for example, [1] or [4]) that

$$\int_{M_q}\int_{\Omega_j} W[f_t(X),\theta]p(X,\theta)d\theta dX$$

$$=\int_{\Omega_j}\int_{M_q} W[f_t(X),\theta]p(X,\theta)dXd\theta \qquad (4)$$

for all j and q. By hypothesis,

$$r_{f_t}(\theta)\equiv c,$$

where c is a constant.

Suppose first that condition (A) is satisfied; that is, Ω is compact. Then from (4),

$$\int_M\int_\Omega W[f_t(X),\theta]p(X,\theta)d\theta dX$$

$$=\int_\Omega\int_M W[f_t(X),\theta]p(X,\theta)dXd\theta.$$

Since Ω is compact,

$$\left.\begin{aligned}&\int_M[\phi_{\bar{f}_t}(X)-\phi_{f_t}(X)]dX=\int_\Omega[r_{\bar{f}_t}(\theta)-r_{f_t}(\theta)]d\theta\\ &=\int_\Omega[r_{\bar{f}_t}(\theta)-c]d\theta\leq\Big[\sup_\theta r_{\bar{f}_t}(\theta)-c\Big]\int_\Omega d\theta.\end{aligned}\right\} \qquad (5)$$

But, it follows from (3) that the first integral in (5) is nonnegative and therefore

$$\sup r_{\bar{f}_t}(\theta)\geq c. \qquad (6)$$

Now suppose that Ω is not compact, but condition (B) is satisfied. From (3) it is seen that either

$$\phi_{\bar{f}_t}(X)-\phi_{f_t}(X)\equiv 0 \qquad (7)$$

for all $X\epsilon M$ or

$$\phi_{\bar{f}_t}(X)-\phi_{f_t}(X)>\epsilon>0 \qquad (7a)$$

over some set M' in M, where M' has finite positive measure m$'$.

First, consider the case where (7a) holds. Let M_q always be taken so that $M'<M_q$. Then, from (3), (4), and (7a), it follows that

$$\lim_{j=\infty}\int_{M_q}\int_{\Omega_j}\{W[\bar{f}_t(X),\theta]-W[f_t(X),\theta]\}p(X,\theta)\,d\theta dX$$

$$=\int_{M_q}\int_\Omega\{W[\bar{f}_t(X),\theta]-W[f_t(X),\theta]\}p(X,\theta)d\theta\,dX$$

$$=\int_{M_q}[\phi_{\bar{f}_t}(X)-\phi_{f_t}(X)]dX>\epsilon m'.$$

Let $\overline{W}_t-W_t=W[\bar{f}_t(X),\theta]-W[f_t(X),\theta]$. Since, by hypothesis,

$$\int_{M_q}[\overline{W}_t-W_t]p(X,\theta)\,dX$$

is uniformly convergent over M, there exists a q_0 such that

$$\left.\begin{aligned}&\bigg|\int_{M_q}[\overline{W}_t-W_t]p(X,\theta)\,dX\\ &\qquad-\int_M[\overline{W}_t-W_t]p(X,\theta)\,dX\bigg|<\frac{\epsilon}{2},\quad q\geq q_0\end{aligned}\right\}(8)$$

Also since, by hypothesis,

$$\int_{\Omega_j}[\overline{W}_t-W_t]p(X,\theta)\,d\theta$$

is uniformly convergent over Ω, there exists a j_0 such that

$$\left. \begin{aligned} &\left|\int_{\Omega_j}[\overline{W}_t-W_t]p(X,\theta)d\theta \right. \\ &\left. -\int_{\Omega}[\overline{W}_t-W_t]p(X,\theta)d\theta\right|<\frac{\epsilon m'}{2m_{\epsilon_0}},\ j\geq j_0, \end{aligned} \right\} \quad (9)$$

where m_{ϵ_0} is the measure of the set M_{ϵ_0}. Then, from (7a) and (9) it follows that

$$\int_{\Omega_{j_0}}\int_{M_{\epsilon_0}}[\overline{W}_t-W_t]p(X,\theta)dXd\theta$$

$$=\int_{M_{\epsilon_0}}\int_{\Omega_{j_0}}[\overline{W}_t-W_t]p(X,\theta)d\theta dX$$

$$=\int_{M_{\epsilon_0}}\left[\int_{\Omega}[\overline{W}_t-W_t]p(X,\theta)d\theta+\alpha\frac{\epsilon m'}{2m_{\epsilon_0}}\right]dX,\quad |\alpha|\leq 1,$$

$$=\int_{M_{\epsilon_0}}\int_{\Omega}[\overline{W}_t-W_t]p(X,\theta)d\theta dX+\alpha\frac{\epsilon m'}{2}$$

$$>\epsilon m'-\frac{\epsilon m'}{2}=\frac{\epsilon m'}{2}.$$

Therefore

$$\int_{\Omega_{j_0}}\int_{M_{\epsilon_0}}[\overline{W}_t-W_t]p(X,\theta)dXd\theta>\frac{\epsilon m'}{2}. \quad (10)$$

From (8) and (10) we have that

$$\int_{\Omega_{j_0}}\left[\int_{M}[\overline{W}_t-W_t]p(X,\theta)dX+\beta\frac{\epsilon}{2}\right]d\theta>\frac{\epsilon m'}{2},\ |\beta|<|1,$$

and thus

$$\left. \begin{aligned} &\left[\left(\sup_\theta r_{\overline{f}_t}(\theta)-c\right)+\beta\frac{\epsilon}{2}\right]\int_{\Omega_{j_0}}d\theta \\ &\geq\int_{\Omega_{j_0}}\left[(r_{\overline{f}_t}(\theta)-c)+\beta\frac{\epsilon}{2}\right]d\theta>\frac{\epsilon m'}{2}>0. \end{aligned} \right\} \quad (11)$$

If $\sup_\theta r_{\overline{f}_t}(\theta)-c<0$, (11) is impossible since ϵ is arbitrarily small, and therefore

$$\sup_\theta r_{\overline{f}_t}(\theta)-c\geq 0. \quad (12)$$

The proof that (12) is true for the case in which (7) holds is immediate. Hence, since $c\equiv\sup_\theta r_{\overline{f}_t}(\theta)$, $f_t(X)$ is a minimax estimate and by Theorem 1.2 is also a UBCR-estimate.

2. Classes of CR-Estimates

In this section we shall find classes of CR-estimates for several frequently occurring types of parameters.

2.1. *L*-Estimates of a Location Parameter

If $p(X,\theta)$ can be expressed in the form

$$p(x_1-\theta,\ldots,x_n-\theta) \quad (13)$$

then θ is called a *location* parameter. An estimate $f(x_1,\ldots,x_n)$ will be called an *L-estimate* of the location parameter θ provided

$$f(x_1+\tau,\ldots,x_n+\tau)=f(x_1,\ldots,x_n)+\tau \quad (14)$$

for any real τ.

2.2. *S*-Estimates of a Scale Parameter

If $p(X,\theta)$ can be expressed in the form

$$\theta^{-n}p\left(\frac{x_1}{\theta},\ldots,\frac{x_n}{\theta}\right),\qquad \theta\geq 0, \quad (15)$$

then θ is called a *scale* parameter. An estimate $f(x_1,\ldots,x_n)$ will be called an *S-estimate* of the scale parameter θ provided

$$f(\mu x_1,\ldots,\mu x_n)=\mu f(x_1,\ldots,x_n),\qquad \mu\geq 0 \quad (16)$$

and

$$f(x_1,\ldots,x_n)\geq 0. \quad (17)$$

2.3. *L*(*S*)-Estimates of a Location Parameter (Scale Parameter Unknown)

Suppose $p(X,\theta)$ is of the form

$$\theta_2^{-n}p\left(\frac{x_1-\theta_1}{\theta_2},\ldots,\frac{x_n-\theta_1}{\theta_2}\right),\qquad \theta_2\geq 0, \quad (18)$$

where θ_1 and θ_2 are unknown parameters. Then an estimate $f_1(x_1,\ldots,x_n)$ will be called an *L*(*S*)-*estimate* of the location parameter θ_1 (the scale parameter θ_2 unknown) provided that $f_1(X)$ is an *L-estimate*, that is, (14) is satisfied, and also

$$f_1(\mu x_1,\ldots,\mu x_n)=\mu f_1(x_1,\ldots,x_n) \quad (19)$$

for any real μ.

2.4. *S*(*L*)-Estimates of a Scale Parameter (Location Parameter Unknown)

Let $p(X,\theta)$ be of the form

$$\theta_2^{-n}p\left(\frac{x_1-\theta_1}{\theta_2},\ldots,\frac{x_n-\theta_1}{\theta_2}\right),\qquad \theta_2\geq 0 \quad (20)$$

where θ_1 and θ_2 are unknown parameters. Then an estimate $f_2(x_1,\ldots,x_n)$ is called an *S*(*L*)-*estimate* of the scale parameter θ_2 (the location parameter θ_1 unknown) provided that $f_2(x)$ is an *S-estimate*, that is, $f_2(X)$ satisfies (16) and (17), and also

$$f_2(x_1+\tau,\ldots,x_n+\tau)=f_2(x_1,\ldots,x_n) \quad (21)$$

for all real τ.

2.5. *D*-Estimates of the Difference Between Two Location Parameters

Let $p(X,Y,\theta,\delta)$ be the joint probability density function of $x_1,\ldots,x_m$ and $y_1,\ldots,y_m$ where

X and Y are samples from two populations with unknown location parameters θ and $\theta+\delta$, respectively. Then $p(X, Y, \theta, \delta)$ is of the form

$$p(x_1-\theta, \ldots, x_m-\theta, y_1-\theta-\delta, \ldots, y_n-\theta-\delta). \quad (22)$$

An estimate $f(X, Y)$ will be called a *D-estimate* of the difference δ provided

$$\begin{aligned} &f(x_1+\mu, \ldots, x_m+\mu, y_1+\lambda, \ldots, y_n+\lambda) \\ &=f(x_1, \ldots, x_m, y_1, \ldots, y_n)+(\lambda-\mu) \end{aligned} \quad (23)$$

for all real μ and λ.

2.6. *R*-Estimates of the Ratio of Two Scale Parameters

Let $p(X, Y, \theta, \rho)$ be of the form

$$\theta^{-m}(\rho\theta)^{-n} p\left(\frac{x_1}{\theta}, \ldots, \frac{x_m}{\theta}, \frac{y_1}{\rho\theta}, \ldots, \frac{y_n}{\rho\theta}\right), \qquad \theta, \rho>0 \quad (24)$$

where X and Y are samples from two populations with unknown scale parameters θ and $\rho\theta$, respectively. An estimate $f(X, Y)$ is called an *R-estimate* of the ratio ρ provided

$$\begin{aligned} &f(\mu x_1, \ldots, \mu x_m, \lambda y_1, \ldots, \lambda y_n) \\ &=\frac{\lambda}{\mu} f(x_1, \ldots, x_m, y_1, \ldots, y_n) \end{aligned} \quad (25)$$

for all $\mu, \lambda>0$

We now show that any estimate belonging to one of the classes 2.1 to 2.6 is a CR-estimate provided that the weight function W_t is of proper form. The following six theorems are stated as one.

Theorems 2.1 to 2.6. Let the density function p be of the form given in classes 2.1 to 2.6 and let the weight function W_t *be of the form* (1) $W[f(X)-\theta]$, (2) $W[\theta^{-1}f(X)]$, (3) $W[\theta_2^{-1}(f_1(X)-\theta_1)]$, (4) $W[\theta_2^{-1}f_2(X)]$, (5) $W[f(X,Y)-\delta]$, (6) $W[\rho^{-1}f(X,Y)]$. *Then, if* f_t *is an* (1) L, (2) S, (3) $L(S)$, (4) $S(L)$, (5) D, (6) *R-estimate, the risk function* $r_{f_t}(\theta)$ *is constant.*

Proof. We shall prove only theorem 2.3, as the others are proved in exactly the same manner. Consider the risk function

$$r_{f_1}(\theta_1, \theta_2)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} W\left[\frac{f_1(X)-\theta_1}{\theta_2}\right]\theta_2^{-n} \cdot p\left(\frac{x_1-\theta_1}{\theta_2}, \ldots, \frac{x_n-\theta_1}{\theta_2}\right)dx_1 \ldots dx_n. \quad (26)$$

Let

$$t_i=\frac{x_i-\theta_1}{\theta_2}, \quad (i=1,2, \ldots, n). \quad (27)$$

Since $f_1(X)$ is an $L(S)$-estimate

$$\begin{aligned} \frac{f_1(X)-\theta_1}{\theta_2}&=\frac{f_1(x_1, \ldots, x_n)-\theta_1}{\theta_2}=f_1\left(\frac{x_1-\theta_1}{\theta_2}, \ldots, \frac{x_n-\theta_1}{\theta_2}\right) \\ &=f_1(t_1, \ldots t_n). \end{aligned} \quad (28)$$

Thus, making the transformation (27) and using (28) in (26), we have

$$r_{f_1}(\theta)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}W[f_1(t_1, \ldots, t_n)]p(t_1, \ldots, t_n)dt_1 \ldots dt_n,$$

which is completely independent of θ and therefore $r_{f_1}(\theta)$ is constant.

3. CR-Minimax Estimates

As a direct consequence of Theorem 1.3 and Theorems 2.1 to 2.6 we have the following six theorems which are stated as one.

Theorems 3.1 to 3.6. If at least one of conditions (A) and (B) in Theorem 1.3 is satisfied and if the weight function W_t *is of the form* (1) $W[f(X)-\theta]$, (2) $W[\theta^{-1}f(X)]$, (3) $W[\theta_2^{-1}(f_1(X)-\theta_1)]$, (4) $W[\theta_2^{-1}f_2(X)]$, (5) $W[f(X,Y)-\delta]$, (6) $W[\rho^{-1}f(X,Y)]$, *then any* (1) L, (2) S, (3) $L(S)$, (4) $S(L)$, (5) D, (6) *R-estimate which minimizes* $(\phi_{f_t}X)$ *[as defined by formula (1)] for all X,* $(X, Y)\epsilon M$ *is a minimax (and also a UBCR) estimate of* θ_t.

Conversely, it has been shown by Kallianpur [3] that, "under mild restrictions", the minimax estimate in the above cases minimizes ϕ_{f_t} and also belongs to the corresponding class of CR-estimates. For example, in case 3.3, the minimax estimate of θ_1 minimizes $\phi_{f_1}(X)$ given in Theorem 3.3 and is an L(S) estimate.

4. Determination of General Classes of CR-Estimates

Suppose the joint probability density function $p(X, \theta)$ is of the form

$$p(X, \theta)=h(\eta(x_1, \theta), \ldots, \eta(x_n, \theta)) \prod_{i=1}^{n} \frac{\partial\eta(x_i, \theta)}{\partial x_i},$$

where $\eta(x_i, \theta)$ possesses the n first partial derivatives $\partial\eta(x_i, \theta)/\partial x_i$ continuous in x_i, $(i=1, 2, \ldots, n)$. Let M be an n-dimensional interval $(a_i \leq x_i \leq b_i)$ such that

$$\left.\begin{aligned} &\eta(a_i, \theta)\equiv c_i \\ &\eta(b_i, \theta)\equiv d_i, \qquad (i=1, 2, \ldots, n) \end{aligned}\right\} \quad (29)$$

where the a_i, b_i, c_i and d_i are constants (possibly infinite) which are independent of θ. Let $\zeta(x,y)$ and $\psi(\theta)$ be arbitrary functions such that the weight function $V[\zeta(f_t(X), \psi(\theta))]$ is non-negative and measurable over the product space $MX\Omega$. Then we define the risk function of an estimate $f_t(X)$ of θ_t to be

$$\left.\begin{aligned} r_{f_t}(\theta)&=\int_M V[\zeta(f_t(X), \psi(\theta))]p(X, \theta)dX \\ &=\int_M V[\zeta(f_t(X), \psi(\theta))]h(\eta(x_1, \theta), \ldots, \\ &\qquad \eta(x_n, \theta)) \prod_{i=1}^{n} \frac{\partial\eta(x_i, \theta)}{\partial x_i} dx_i. \end{aligned}\right\} \quad (30)$$

The following theorem yields a method for determining general classes of CR-estimates, that is, estimates that possess constant risk functions.

Theorem 4.1. *If* $f_i(X)=f_i(x_1, \ldots, x_n)$ *is such that*

$$f_i(\eta(x_1,\theta), \ldots, \eta(x_n,\theta))=\zeta(f_i(X),\psi(\theta)) \qquad (31)$$

then the risk function $r_{f_i}(\theta)$ *is constant.*

Proof. Let $f_i(X)$ be any estimate satisfying (31) and let

$$t_i=\eta(x_i,\theta), \qquad (i=1, 2, \ldots, n). \qquad (32)$$

Then, from (31) and (32) we have

$$\zeta(f_i(X), \psi(\theta))=f_i(t_1, \ldots, t_n). \qquad (33)$$

Let T denote the n-dimensional interval $(c_i \leq t_i \leq d_i)$. Applying the transformation (32) and using (29) and (33) in (30) we have

$$r_{f_i}(\theta)=\int_M V[\zeta(f_i(X),\psi(\theta)) \cdot h(\eta(x_1,\theta), \ldots, \eta(x_n,\theta)) \prod_{i=1}^{n} \frac{\partial \eta(x_i,\theta)}{\partial x_i} dx_i$$

$$=\int_T V[f_i(t_1, \ldots, t_n)]h(t_1, \ldots, t_n)\, dt_1 \ldots dt_n \equiv C$$

where C is a constant.

5. Examples

As an example to illustrate the usefulness of theorem 4.1, let

$$p(X,\theta)=\frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2}$$

and choose $V=(f-\theta)^2$. In this case we choose

$$\eta(x_i, \theta)\equiv x_i-\theta, \ \psi(\theta)\equiv\theta, \qquad \zeta(f(X), \psi(\theta))\equiv f(X)-\theta.$$

Then (31) becomes

$$f(x_1-\theta, \ldots, x_n-\theta)=f(x_1, \ldots, x_n)-\theta. \qquad (34)$$

Thus, for example, any weighted mean $\sum_{i=1}^{n} a_i x_i$, $\sum_{i=1}^{n} a_i=1$ is a CR-estimate of θ.

If we take

$$p(X,\theta)=\frac{1}{(2\pi\theta)^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{x_i^2}{\theta}}, \theta>0$$

and choose

$$V=\left(\frac{f-\theta}{\theta}\right)^2=\left(\frac{f}{\theta}-1\right)^2,$$

then we can take

$$\eta(x_i,\theta)\equiv\frac{x_i}{\theta^{1/2}}, \qquad \psi(\theta)\equiv\theta, \qquad \zeta(f(x), \psi(\theta))\equiv\frac{f}{\theta}$$

and (31) becomes

$$f\left(\frac{x_1}{\theta^{1/2}}, \ldots, \frac{x_n}{\theta^{1/2}}\right)=\frac{f(x_1, \ldots, x_n)}{\theta}. \qquad (35)$$

On the other hand, if we are estimating $\theta^{1/2}$ by $f(X)$, we take $\psi(\theta)\equiv\theta^{1/2}$ and (31) becomes

$$f\left(\frac{x_1}{\theta^{1/2}}, \ldots, \frac{x_n}{\theta^{1/2}}\right)=\frac{f(x_1, \ldots, x_n)}{\theta^{1/2}}. \qquad (36)$$

In these examples, conditions (34), (35), and (36) show the reasonableness and generality of the respective classes of CR-estimates.

Theorem 4.1 together with Theorem 1.3 can be used to obtain many results similar to Theorems 3.1 to 3.6, that is, to construct CR-minimax estimates and to throw light on their general desirability. The lower bound for $R_{f_i}(\lambda)$, where f_i is any CR-estimate, is readily seen to be $r_{\bar{f}_i}(\theta)$, where $\bar{f}_i$ is a CR-minimax estimate. This lower bound is

$$R_{\bar{f}_i}(\lambda)=\int_\Omega r_{\bar{f}_i}(\theta)\, d\lambda(\theta)\equiv\bar{c},$$

where $\bar{c}$ is a constant.

In a recent paper, [2], Hodges and Lehmann have illustrated some properties of minimax estimates. They mention, for example, that it has not been possible to obtain a general comparison between minimax estimates and unbiased estimates with uniformly smallest variance, if such exist. We can, imposing certain restrictions on the form of the probability density function $p(X,\theta)$, obtain the CR-minimax estimates with uniformly minimum variance and show that these estimates are unbiased. Also, a relationship exists between CR-minimax and maximum likelihood estimates.

6. References

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