Generalization of S. Bernstein's Polynomials to the Infinite Interval

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Let $P(u,x)=e^{-xu}\sum_{v=1}^{\infty}\frac{(ux)^v}{v!}f\left(\frac{v}{u}\right), u>0.$

The paper studies the convergence of $P(u,x)$ to $f(x)$ as $u\to\infty$. The results obtained are generalized analogs, for the interval $0\leq x\leq\infty$, of known properties of S. Bernstein's approximation polynomials in a finite interval.

1. With a function $f(t)$ in the closed interval [0,1], S. Bernstein in 1912 associated the polynomials

$$B_n(t)=\sum_{v=0}^{n}\binom{n}{v}t^v(1-t)^{n-v}f(v/n),\ n=1,2,3,\cdots \tag{1}$$

He proved that if $f(t)$ is continuous in the closed interval [0,1], then $B_n(t)\to f(t)$ uniformly, as $n\to\infty$. This yields a simple constructive proof of Weierstrass's approximation theorem.

More generally the following theorems hold:

Theorem A. If $f(t)$ is bounded in [0,1] and continuous at every point of $[a,b]$, where $0\leqq a<b\leqq 1$, then $B_n(t)\to f(t)$ uniformly in $[a,b]$. (See [6], p. 66)[1].

Theorem B. If $f(t)$ is bounded in [0,1] and continuous at a point τ, then $B_n(\tau)\to F(\tau)$. (See [3], p. 112).

Theorem C. If $f(t)$ satisfies a Lipschitz-Hölder condition

$$|f(t)-f(t')|<c|t-t'|^{\lambda}, 0<\lambda\leqq 1,$$

then $|f(t)-B_n(t)|<c_2n^{-\lambda/2}, c_1, c_2$ constants (see [7], p. 53;4). $B_n(t)$ is a linear transform of the function $f(t)$; for the infinite interval $(0,\infty)$ we define an analogous transform:

$$P(u;f)=e^{-ux}\sum_{v=0}^{\infty}\frac{1}{v!}(ux)^vf(v/u),\ u>0.^2. \tag{2}$$

We shall prove corresponding theorems of approximation for this transform; $u\to\infty$ corresponds to $n\to\infty$ in eq 1. We also sharpen theorem B to *uniform* convergence *at* the point τ.

Definition: A set of continuous functions $P(u,x)$ is said to converge uniformly to the value S at a point $x=\zeta$, as $u\to\infty$ if $P(u_n,x_n)\to S$, whenever $x_n\to\zeta$ and $u_n\to\infty$, as $n\to\infty$. An equivalent formulation is: to any $\epsilon>0$ there exists a $\delta(\epsilon)$ and an $\eta(\delta,\epsilon)$ so that $|P(u,x)-S|<\epsilon$ for $|x-\zeta|<\delta$ and $u>\eta$.

2. In this section we introduce some lemmas for later application.

Lemma 1. For $\lambda>0$, $u>0$,

$$\sum_{|v-u|\geqq\lambda}\frac{u^v}{v!}<\lambda^{-2}ue^u. \tag{3}$$

The following identity is easily verified:

$$\sum_{v=0}^{\infty}(v-u)^2\frac{u^v}{v!}=ue^u; \tag{4}$$

it follows that

$$\lambda^2\sum_{|v-u|\geqq\lambda}\frac{u^v}{v!}>\sum_{v=0}^{\infty}(v-u)^2\frac{u^v}{v!}=ue^u.$$

This proves lemma 1.

Lemma 2. For $u\geqq 0$

$$\sum_{0}^{\infty}|v-u|\frac{u^v}{v!}\leqq\sqrt{ue^u}. \tag{5}$$

[1] Figures in brackets indicate the literature references at the end of this paper.

[2] M. Kac also considered the transform (2) independently, from a simillar point of view.

By Schwarz's inequality and by eq 4

$$\left(\sum_0^\infty |v-u| \frac{u^v}{v!}\right)^2 \leqq \left\{\sum_0^\infty (v-u)^2 \frac{u^v}{v!}\right\} \left(\sum_0^\infty \frac{u^v}{v!}\right) = ue^{2u};$$

this proves lemma 2.

Observe that

$$\sum_{v=0}^\infty (v-u) \frac{u^v}{v!} = 0, \tag{6}$$

thus, if u is a positive integer

$$\sum_0^\infty |v-u| \frac{u^v}{v!} = \sum_{v\leqq u} (u-v) \frac{u^v}{v!} + \sum_{v\geq u} (v-u) \frac{u^v}{v!} =$$

$$2\sum_{v\leqq u} (u-v) \frac{u^v}{v!} = 2u \sum_{v\leqq u} \frac{u^v}{v!} - 2u \sum_{v\leqq u-1} \frac{u^v}{v!} =$$

$$2u \frac{u^u}{u!} \sim \frac{2\sqrt{u}}{\sqrt{2\pi}} e^u,$$

by Stirling's formula. Thus the estimate (5) is the sharpest possible, except for a constant factor.

3. *Theorem 1.* Suppose that $f(x)$ is bounded in every finite interval; if $f(x) = 0(x^k)$ for some $k>0$ as $x\rightarrow\infty$ and if $f(x)$ is continuous at a point ζ, then $P(u;f)$ converges uniformly to $f(x)$ at $x=\zeta$.

Consider

$$e^{ux}\{P(u;f)-f(x)\} = \sum_{v=0}^\infty \{f(v/u)-f(x)\}(ux)^v \frac{1}{v!} =$$

$$\sum_{|v/u-x|\leqq\delta} + \sum_{|v/u-x|>\delta} = S_1 + S_2, \text{ say}$$

and assume that $|x-\zeta|<\delta$.

Let

$$\max |f(x)-f(\xi)| = m(\delta,\xi) = m(\delta),$$

for

$$|x-\xi| \leq \delta,$$

then $m(\delta)\rightarrow 0$ as $\delta\rightarrow 0$. Now

$$f(v/u)-f(x) = f(v/u)-f(\zeta)+f(\zeta)-f(x),$$

and

$$\frac{v}{u}-\zeta = \frac{v}{u}-x+x-\zeta. \tag{7}$$

In S_1 $\left|\frac{v}{u}-x\right| \leqq \delta$,

hence, from eq 7

$$|v/u-\zeta| \leqq 2\delta$$

and

$$|f(v/u)-f(x)| \leqq m(2\delta)+m(\delta) \leqq 2m(2\delta).$$

Hence

$$|S_1| < 2m(2\delta) \sum_0^\infty \frac{(ux)^v}{v!} = 2m(2\delta)e^{ux}.$$

Next write

$$S_2 = \sum_{v<u(x-\delta)} + \sum_{v>u(x+\delta)} = S_3 + S_4, \text{ say.}$$

Then

$$|S_3| < \sum_{v<u(x-\delta)} \frac{(ux)^v}{v!} |f(v/u)-f(x)|.$$

Let

$$\sup |f(x)| = U(\delta), \text{ for } x \leqq \delta.$$

Then

$$|S_3| < 2U(\zeta+\delta) \sum_{ux-v>u\delta} \frac{(ux)^v}{v!}.$$

Applying lemma 1 with $\lambda = u\delta$, we get

$$|S_3| < 2U(\zeta+\delta) \frac{uxe^{ux}}{u^2\delta^2} = 2U(\zeta+\delta) \frac{xe^{ux}}{u\delta^2}.$$

Finally, assuming $u(x+\delta)>k$,

$$S_4 = 0\left(\sum_{v>u(x+\delta)} \frac{(ux)^v}{v!}\left(\frac{v}{u}\right)^k\right) = 0\left(\sum_{v>u(x+\delta)} x^k \frac{(ux)^{v-k}}{(v-k)!}\right)$$

$$= 0\left(\sum_{\mu>u(x+\delta)-k} x^k \frac{(ux)^\mu \rho}{\mu!}\right).$$

We apply again lemma 1, with $\lambda = u\delta - k > 0$,

then

$$S_4 = 0\left(x^k \frac{uxe^{ux}}{(u\delta-k)^2}\right) = 0\left(\frac{ue^{ux}}{(u\delta-k)^2}\right), \; u\rightarrow\infty.$$

Summarizing, we find

$$P(u;f(x))-f(x) = 0\left\{m(2\delta)+\frac{1}{u\delta^2}+\frac{u}{(u\delta-k)^2}\right\}.$$

Letting $u\rightarrow\infty$ for a fixed δ,

$$\limsup |P(u;f(x))-f(x)| \leqq 0(m(2\delta)), u\rightarrow\infty, |x-\zeta| \leqq \delta,$$

from which our theorem follows.

It can be shown easily that uniform convergence at each point of a closed set D implies uniform convergence over the set D. A similar argument applies to the transform (1), thus sharpening the theorems A and B.

4. *Theorem 2.* If $f(x)$ satisfies the Lipschitz-type condition

$$|f(x_1)-f(x_2)|<\gamma\frac{|x_2-x_1|^\rho}{(x_1+x_2)^{\rho/2}},\ 0<x_1<x_2<\infty,\quad (8)$$

γ, ρ constants, $0<\rho\leqq 1$,

then

$$|P(u;f(x))-f(x)|\leqq\gamma u^{-\rho/2},$$

uniformly for $0<x<\infty$, as $u\to\infty$.

We have, for $\rho=1$

$$|P(u;f)-f(x)|\leqq e^{-ux}\sum_0^\infty (ux)^v\frac{1}{v!}|f(v/u)-f(x)|$$

$$<e^{-ux}\gamma\sum_0^\infty\frac{(ux)^v}{v!}\frac{|v/u-x|}{(v/u+x)^{\frac{1}{2}}}=\gamma\frac{e^{-ux}}{\sqrt{u}}\sum_0^\infty\frac{(ux)^v}{v!}\frac{|v-ux|}{(v+ux)^{\frac{1}{2}}}$$

$$<\frac{\gamma e^{-ux}}{u\sqrt{x}}\sum_0^\infty\frac{(ux)^v}{v!}|v-ux|\leqq\frac{\gamma}{\sqrt{u}},$$

by lemma 2. This proves theorem 2 for $\rho=1$. Now from Hölder's inequality, for $0<\rho<1$

$$e^{ux}|P(u;f)-f(x)|=\sum_0^\infty\frac{(ux)^{v(1-\rho)}}{(v!)^{1-\rho}}\frac{(ux)^\rho}{(v!)^\rho}|f(v/u)-f(x)|$$

$$\leqq\left(\sum_0^\infty\frac{(ux)^v}{v!}\right)^{1-\rho}\left\{\sum\frac{(ux)^v}{v!}|f(v/u)-f(x)|^{1/\rho}\right\}^\rho.$$

Assume that

$$|f(x_1)-f(x_2)|<\gamma\frac{|x_2-x_1|^\rho}{(x_1+x_2)^{\rho/2}},$$

then

$$e^{ux}|P(u;f)-f(x)|<\exp((1-\rho)ux)\gamma\left(\sum\frac{(ux)^v}{v!}\cdot\frac{|v/u-x|}{(v/u+x)^{1/2}}\right)^\rho$$

$$<\gamma\exp((1-\rho)ux)\left(\frac{1}{u\sqrt{x}}\sqrt{uxe^{ux}}\right)^\rho=\gamma u^{-\rho/2}e^{ux}.$$

This completes the proof of theorem 2.

Let $\rho=1$, and $f(x)=c-x$, for $0\leqq x\leqq c$, c a positive constant; $f(x)=0$ for $x\geqq c$. Now the condition (8) is satisfied. Furthermore

$$P(u;f)-f(c)=P(u;f)=e^{-uc}\sum_{v\leqq uc}\frac{(uc)^v}{v!}(c-v/u)=\frac{1}{u}e^{-uc}\sum_{v\leqq uc}(uc-v)\frac{(uc)^v}{v!}.$$

Let $[uc]=k$, then

$$u^{1/2}P(u,c)>u^{-1/2}e^{-k-1}\sum_{v=0}^k(k-v)\frac{k^v}{v!},$$

and, from section 2

$$\sum_{v=0}^k(k-v)\frac{k^v}{v!}=\frac{k^{k+1}}{k!}\sim\left(\frac{k}{2\pi}\right)^{1/2}e^k.$$

Thus

$$\liminf_{u\to\infty}u^{\frac{1}{2}}P(u,c)>0.$$

This proves that for $\rho=1$ the order of the estimate in theorem 2 is the sharpest possible. We do not know of a similar example for $\rho<1$. For Bernstein's polynomials an exact result has been given by M. Kac [4].

5. Suppose that $f(x)$ is continuous in the infinite interval $(0,\infty)$.

Let

$$x=\log\frac{1}{t},\ 0\leqq t\leqq 1,$$

$f(x)=f(\log 1/t)=\phi(t)$ is continuous in $0\leqq t\leqq 1$. Given $\epsilon>0$, we can find a polynomial $\sum_0^n a_kt^k=p_n(t)$ so that $|\phi(t)-p_n(t)|<\epsilon$. It follows that

$$|f(x)-p_n(e^{-x})|<\epsilon,\ 0<x<\infty.$$

Now

$$P(u;f)=e^{-xu}\sum_0^\infty\frac{(ux)^v}{v!}f(v/u),$$

and for $p_n(e^{-x})$

$$P(u;p_n)=e^{-xu}\sum_0^\infty\frac{(ux)^v}{v!}\sum_0^n a_ke^{-kv/u}$$

$$=e^{-xu}\sum_0^n a_k\sum_0^\infty\frac{(ux\exp(-k/u))^v}{v!}$$

$$=e^{-xu}\sum_0^n a_k\exp(ux\exp(-k/u))$$

$$=\sum_0^n a_ke^{-ux(1-\exp-k/u)}.$$

Clearly for $u\to\infty$, $P(u;p_n)\to p_n(e^{-x})$ uniformly in $(0,\infty)$; furthermore

$$f(x)=p_n(e^{-x})+\epsilon_n(x),\ |\epsilon_n(x)|<\epsilon,$$

and

$$P(u;f)=P(u,f-p_n)+P(u,p_n).$$

Here

$$|P(u;f-p_n)|<\epsilon,$$

hence

$$|P(u;f)-f(x)|<\epsilon+|P(u;p_n)-f(x)|\leqq\epsilon+$$
$$|P(u;p_n)-p_n(e^{-x})|+|p_n(e^{-x})-f(x)|.$$

Thus, the theorem:

Theorem 3. If $f(x)$ is continuous in $(0,\infty)$ then $P(u;f)\rightarrow f(x)$ uniformly in $(0,\infty)$.

6. *Theorem* 4. If $f(x)$ is r-times differentiable, $f^{(r)}(x)=0(x^k)$ as $x\rightarrow\infty$, for some $K>0$, and if $f^{(r)}(x)$ is continuous at a point ζ, then $P^{(r)}(u;f)$ converges uniformly to $f^{(r)}(x)$ at $x=\zeta$.

We write $1/u=h$, and introduce the notation

$$\Delta f(vh)=f(\overline{v+1}h)-f(vh).$$

$$\Delta^2 f(vh)=\Delta\Delta f(vh)=f(\overline{v+2}h)-2f(\overline{v+1}h)+f(vh).$$

$$\Delta^r f(vh)=\Delta\Delta^{r-1}f(vh)$$
$$=\sum_{k=0}^{r}(-1)^k\binom{r}{k}f(\overline{v+k}h),\ r\geqq 0.$$

$$P(1/h;f)=Q(h;f)=e^{-x/h}\sum_{v=0}^{\infty}1/v!\left(\frac{x}{h}\right)^v f(vh).$$

Lemma 3. We have

$$e^{x/h}\frac{d^r}{dx^r}Q(h;f)=\sum_0^\infty\Delta^r f(vh)\,1/v!\left(\frac{x}{h}\right)^v h^{-r}.$$

Differentiation gives

$$\frac{d}{dx}Q(h;f)=e^{-x/h}\sum_0^\infty\frac{1}{(v-1)!}\left(\frac{x}{h}\right)^{v-1}\frac{1}{h}f(vh)-$$
$$\frac{1}{h}e^{-x/h}\sum_0^\infty\frac{1}{v!}\left(\frac{x}{h}\right)^v f(vh)$$
$$=\frac{1}{h}e^{-x/h}\sum_0^\infty\frac{1}{v!}\left(\frac{x}{h}\right)^v\Delta f(vh).$$

The lemma now follows by induction.
It is known that

$$\frac{\Delta^r f(vh)}{h^r}=f^{(r)}(\eta),$$

where

$$vh<\eta<(v+r)h.$$

Now

$$D_rQ(h;f)-f^{(r)}(x)$$
$$=e^{-x/h}\sum_0^\infty\left\{\frac{h^{-r}\Delta^r f(vh)-f^{(r)}(x)}{v!}\right\}\left(\frac{x}{h}\right)^v$$
$$=e^{-x/h}\left\{\sum_{|vh-x|\leqq\delta}+\sum_{|vh-x|>\delta}\right\},$$

where we assume that $|x-\zeta|<\delta$. Using the same device as in the proof of theorem 1, we get theorem 4. For Bernstein's polynomials see G. Lorentz [5], and his reference to Wigert's work.

For $\rho=1$ theorem 2 and formula (6) suggest the following proposition:

Theorem 5. If $f(x)$ is bounded in every finite interval, if it is differentiable at a point $\zeta>0$, and if $f(x)=0(x^k)$ for some $k>0$, $x\rightarrow\infty$, then

$$u^{1/2}\{P(u;f(\zeta))-f(\zeta)\}\rightarrow 0,\ u\rightarrow\infty.$$

Let

$$\max\left|\frac{f(\zeta+h)-f(\zeta)}{h}-f'(\zeta)\right|=\mu(\delta,\zeta)=\mu(\delta),$$

then $\mu(\delta)\rightarrow 0$ as $\delta\rightarrow 0$. We may write

$$f(\zeta+h)-f(\zeta)=hf'(\zeta)+h\epsilon(\zeta,h),$$

where

$$|\epsilon(\zeta,h)|\leqq\mu(\delta)\text{ for }|h|\leqq\delta.$$

Now

$$P(u;f(\zeta))-f(\zeta)=e^{-u\zeta}\sum_0^\infty\frac{(u\zeta)^v}{v!}\left\{\left(\frac{v}{u}-\zeta\right)f'(\zeta)+\right.$$
$$\left.\left(\frac{v}{u}-\zeta\right)\epsilon_v(u)\right\},$$

where

$$|\epsilon_v(u)|\leqq\mu(\delta)\text{ for }\left|\frac{v}{u}-\zeta\right|\leqq\delta.$$

Utilizing formula (6) we get

$$P(u;f)-f(\zeta)=\frac{1}{u}e^{-u\zeta}\sum_0^\infty\frac{(u\zeta)^v}{v!}(v-u\zeta)\epsilon_v(u)$$
$$=\frac{1}{u}e^{-u\zeta}\left\{\sum_{|v-u\zeta|\leqq\delta u}+\sum_{|v-u\zeta|>\delta u}\right\}.$$

Using the same device as in the proof of theorem 1, and employing lemma 2, we can complete the proof of theorem 3.

The result can be generalized to higher derivatives. We restrict ourselves here to the case that $f''(\zeta)$ exists. Thus,

$$f(\zeta+h)-f(\zeta)=hf'(\zeta)+\frac{1}{2}h^2\{f''(\zeta)+\epsilon(\zeta,h)\},$$

where

$$|\epsilon(\zeta,h)|\leqq\eta(\delta) \text{ for } |h|\leqq\delta, \text{ and } \eta(\delta)\to 0,\ \delta\to 0.$$

Now

$$P\{u;f(\zeta)\}-f(\zeta)=e^{-u\zeta}\sum_0^\infty \frac{(u\zeta)^v}{v!}\left\{\left(\frac{v}{u}-\zeta\right)f'(\zeta)+\right.$$

$$\left.\frac{1}{2}\left(\frac{v}{u}-\zeta\right)^2 f''(\zeta)\right\}+e^{-u\zeta}\sum_0^\infty \frac{(u\zeta)^v}{v!}\frac{1}{2}(v/u-\zeta)^2\epsilon_v(u),$$

where

$$|\epsilon_v(u)|\leqq\eta(\delta) \text{ for } \left|\frac{v}{u}-\zeta\right|\leqq\delta. \tag{9}$$

It follows from formulas 4 and 6 that

$$P(u;f(\zeta))-f(\zeta)=\frac{\zeta}{2u}f''(\zeta)+$$

$$\frac{e^{-u\zeta}}{2u^2}\sum_0^\infty\frac{(u\zeta)^v}{v!}(v-u\zeta)^2\epsilon_v(u),$$

or

$$u\{P(u;f)-f(\zeta)\}=\frac{1}{2}\zeta f''(\zeta)+$$

$$\frac{e^{-u\zeta}}{2u}\sum_0^\infty\frac{(u\zeta)^v}{v!}(v-u\zeta)^2\epsilon_v(u).$$

We write

$$\sum_0^\infty\frac{(u\zeta)^v}{v!}(v-u\zeta)^2\epsilon_v(u)=\sum_{|v-u\zeta|\leqq u\delta}+\sum_{|v-u\zeta|>u\delta}$$

$$=T_1+T_2, \text{ say.}$$

From (9) and (4),

$$|T_1|<\eta(\delta)u\zeta e^{u\zeta}. \tag{10}$$

Hence,

$$\frac{e^{-u\zeta}}{2u}|T_1|<\frac{1}{2}\eta(\delta)\zeta.$$

Next write

$$T_2=\sum_{v<u(\zeta-\delta)}+\sum_{v>u(\zeta+\delta)}=T_3+T_4, \text{ say,}$$

and note that

$$\frac{1}{2}\left(\frac{v}{u}-\zeta\right)^2\epsilon_v(u)=f\left(\frac{v}{u}\right)-f(\zeta)-(v/u-\zeta)f'(\zeta)-$$

$$\frac{1}{2}\left(\frac{v}{u}-\zeta\right)^2 f''(\zeta).$$

Let

$$\sup|f(x)|=M(\zeta),\ x\leqq\zeta$$

then

$$|T_3|<\left\{2M(\zeta)+\zeta|f'(\zeta)|+\frac{1}{2}\zeta^2|f''(\zeta)|\right\}$$

$$\sum_{u\zeta-v>u\delta}u^2\xi^2\frac{1}{v!}(u\zeta)^v,$$

We now employ the formula (see e.g. [2], p. 200)

$$\sum_{|v-u|>\delta u}e^{-u}\frac{u^v}{v!}=0\left(\exp\left(-\frac{1}{3}\delta^2u\right)\right), u\to\infty$$

It follows that

$$e^{-u\zeta}\sum_{u\zeta-v>u\delta}\frac{(u\zeta)^v}{v!}=0\left(\exp\left(-\frac{1}{3}\frac{\delta^2}{\zeta}u\right)\right),$$

so that

$$\frac{e^{-u\zeta}}{u}T_3=0\left\{u\exp\left(-\frac{1}{3}\frac{\delta^2}{\zeta}u\right)\right\}.$$

Finally, in view of $f(x)=0(x^k)$, we have for $v>u\zeta$ and $k\geq 2$

$$(v-u\zeta)^2\epsilon_v(u)=0\left(u^2\frac{v^k}{u^k}\right),$$

hence

$$T_4=0\left(\sum_{v-u\zeta>u\delta}\frac{(u\zeta)^v}{v!}\frac{v^k}{u^k}\right)=0\left(\sum_{v-u\zeta>u\delta}u^2\frac{(u\zeta)^{v-k}}{(v-k)!}\right)$$

$$=0\left(\sum_{v-u\zeta>u\delta-k}u^2\frac{(u\zeta)^v}{v!}\right)=0\left\{u^2\exp\left(u\zeta-\frac{\delta^2}{3\zeta}u\right)\right\}.$$

Thus,

$$\frac{e^{-u\zeta}}{u}T_4=0\left\{u\exp\left(-\frac{1}{3\zeta}\delta^2u\right)\right\}. \tag{11}$$

Summarizing, from (10) and (11),

$$\limsup|u\{P(u;f(\zeta)-f(\zeta)\}-\frac{1}{2}\zeta f''(\zeta)|\leqq\delta.$$

But δ is arbitrarily small, hence the theorem:

Theorem 6. If $f(x)$ is bounded in every finite interval, if it is twice differentiable at a point $\zeta>0$, and if for some $k>0$ $f(x)=0(x^k)$, $x\to\infty$, then

$$u\{P(u;f(\zeta))-f(\zeta)\}\to\frac{1}{2}\zeta f''(\zeta),\ u\to\infty.$$

Analogous theorems for Bernstein's polynomials were given in [1] and [8].

8. In the terminology of probability distribution the Bernstein polynomial corresponds to the binomial distribution. The distribution function is

$$\sum_{v\leq r}\binom{n}{v}t^v(1-t)^{n-v}=F_n(r);$$

the linear functional $B_n(t;f)$ is

$$\int_0^1 f(r/n)dF_n(r)=\sum_{v=0}^{n} f(v/n) \binom{n}{v} t^v(1-t)^{n-v}.$$

Similarly, $P(u;f)$ corresponds to the Poisson distribution; the distribution function is

$$\sum_{v\leq r} \frac{x^v}{v!} e^{-x}=G(r,x),$$

and

$$P(u;f)=\int_0^\infty f(r/u)dG(r,ux)=\sum_{v=0}^{\infty} e^{-ux} \frac{(ux)^v}{v!} f(v/u);$$

here the term with the largest weight has the index $v\sim ux$. In the Bernoulli polynomial the term with the largest weight has the index $v\sim tn$.

If instead of a function f we consider a sequence S_0, S_1, S_2, . . . , then to the transform (1) corresponds

$$\sum_{v=0}^{n} \binom{n}{v} t^v(1-t)^{n-v}S_v, n\to\infty, \tag{9}$$

which defines the generalized Euler summability, and to the transform (2) corresponds

$$e^{-x}\sum_{v=0}^{\infty} \frac{x^v}{v!} S_v, x\to\infty,$$

which gives Borel's summability method.

9. To approximate a function $f(x)$ over the whole real axis, we write

$$f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}=f_1(x)+f_2(x), \text{ say;}$$

obviously

$$f_1(-x)=f_1(x), f_2(-x)=-f_2(x).$$

Now

$$P(u;f)=P(u;f_1)+P(u;f_2);$$

here

$$P(u;f_1)=e^{-ux}\sum_0^\infty \frac{(ux)^v}{v!} f_1(v/u),$$

and if we change x into $-x$, u into $-u$, we get

$$P(-u;f_1(-x))=P(u;f_1(x)),$$

so that our previous results are directly applicable. Similarly,

$$P(-u;f_2(-x))=-P(u;f_2(x));$$

thus for negative values of x we need only change u into $-u$, and revert to our previous results.

10. It follows from a well-known property of the Beta function that

$$\binom{n}{v}\int_0^1 t^v(1-t)^{n-v}dt=\frac{1}{n+1}, v=0, 1, \cdots, n;$$

hence

$$\int_0^1 B_n(t)dt=\frac{1}{n+1}\sum_{v=0}^{n} f(v/n),$$

so that for any Riemann integrable function

$$\int_0^1 B_n(t)dt\to\int_0^1 f(t)dt.$$

Similarly, at first formally

$$\int_0^\infty P(u;f)dx=\sum_0^\infty \frac{u^v}{v!}f(v/u) \int_0^\infty e^{-xu}x^v dx=$$
$$\frac{1}{u}\sum_0^\infty f(v/u);$$

the interchange of integration and summation is legitimate if the series $\sum f(v/u)$ is convergent. Thus, the formula

$$\int_0^\infty P(u;f)dx=\frac{1}{u}\sum_0^\infty f(v/u)$$

is valid if both sides exist. However, it is a delicate question under what conditions

$$\lim_{u\to\infty} \frac{1}{u}\sum_0^\infty f(v/u)\to\int_0^\infty f(x)dx.$$

An extensive literature deals with this question.

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