A NEW METHOD OF DETERMINING THE FOCAL LENGTH OF A CONVERGING LENS.¹

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INTRODUCTION.

Incident to some work with a modified form of the Fabry-Perot interferometer, there has occurred to me a method for determining the focal length of a converging lens by measuring the linear diameter of a circular fringe in the real image formed by the lens in question. I have formulated the theory of the method and subjected it to experimental test. It is the purpose of this paper to present the method and discuss some of its features.

The method makes use of the system of concentric circular fringes obtained by the interference of the light reflected from a partial silver film with that reflected from a heavy polished silver film parallel to the first and behind it, the incident and reflected beams being perpendicular to these mirrors. The procedure for producing these fringes is precisely similar to that for producing the Fabry-Perot fringes except as to the degree of silvering and the fact that the reflected instead of the transmitted light is examined. Fringes by reflection from parallel silvered mirrors have been used by Hamy; and their theory and method of production have been given by him.² But they do not appear to have received as extensive an application in interference measurements as they merit. I have made some study of these fringes with apparatus somewhat different from that of Hamy; the silver films being supported on two plane plates instead of on a plane plate and a lens as was done by Hamy, and the light thrown into

¹ The substance of this paper, in a slightly modified form, was presented at a meeting of the Philosophical Society of Washington, February 13, 1909.
² J. de Physique, 4 série, 5, pp. 789-809; 1906.
the apparatus by a glass plate set at 45° with the normal to the films. The fringes were then observed by looking through the glass plate in a direction normal to the films. Under these conditions the fringes, as seen by the eye, are localized at infinity and a real image of them will be formed in the principal focal plane of a lens placed in front of the apparatus with its axis normal to the films. (See Fig. 1.) The important characteristics of these fringes are: (1) The outer edges of the bright rings are exceedingly sharp and well defined, so that measurements can be made upon them with great precision; (2) the appearance or disappearance of the bright point at the center, due to variation of the distance between the mirrors, is also very sharply defined, so that the difference of path at the center can be adjusted to this point of disappearance with great nicety. These properties of the fringes render them available for the present purpose. These peculiar properties of the fringes vary with the thickness of the partial silver film and with the intensity of the incident light. I have found that films transmitting less than 20 per cent of the incident light give the most satisfactory fringes. As the percentage transmission of the partial film is decreased, the fringes become sharper; but when reduced below 10 per cent the glare of light reflected from the partial film begins to interfere with the observation.

2. SUMMARY OF ADVANTAGES.

(a) The method gives the focal length properly so defined, i.e., the distance between the second principal point and the principal focus. (b) It is of general applicability for lenses of widely varying focal length. (c) It gives the focal length for a particular wave-length (spectrum line) very conveniently. Several determinations for selected wave-lengths thus give a definite and precise measure of the chromatism of the lens. The method therefore affords a convenient, definite, and simple test for achromatism. (d) Both the observation and calculation involved are exceedingly simple. The constant of the apparatus having been calculated once for all, subsequent determinations require the measurement of only one quantity, a length conveniently measured by a micrometer. The focal length is immediately given by the product of this measured length and the constant before men-
tioned. It is not necessary to make any measurement from any point on the lens or its mount. (e) The method does not require an optical bench, nor a long working distance, nor an "object."

3. THEORY OF THE METHOD.

The following argument assumes a previous knowledge of the theory of the Fabry-Perot fringes as given by Fabry and Perot, and Eversheim.

The method is justified by the following considerations:
(a) The real image of the fringe system due to the interference of the light from two accurately parallel mirrors is formed in the principal focal plane of the lens used to produce the image.
(b) The focal length, $F$, of the lens is given by

$$F = R_r \cot \frac{x_r}{2}$$

where $x_r$ is the angle subtended at the second principal point by the diameter, in the image, of the $r$-th ring from the center, counting the central bright point, zero;
$R_r$ is the linear radius of the same ring in the image, when the order of interference at the center is given by

$$P = r + N$$

where $N$ is the order of interference of the $r$-th ring from the center, $r$ being, by definition, an integer.

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5 The formula for wave lengths itself,

$$\lambda' = \frac{P \lambda}{P'} \left(1 + \frac{D^2 - D'^2}{8 F^2}\right)$$

might be transformed to give $F$, thus:

$$F = \sqrt{\frac{P \lambda (D^2 - D'^2)}{8 (P' \lambda' - P \lambda')}}$$

where $P$ is the order of interference for the ring of diameter, $D$, due to wave length $\lambda$, and $P'$, $D'$ $\lambda'$ are similarly related; but the advantages of the simpler formula presented in this paper are evident. To apply the above formula to determine $F$ to $\frac{1}{1000}$ would require all the labor and care necessary to determine $\lambda'$ to $\frac{1}{5000000}$. 
(c) The factor \( \cot \frac{x_r}{2} \) may be calculated from the formula

\[
\cot \frac{x_r}{2} = \frac{N}{\sqrt{P^2 - N^2}},
\]

or

\[
\cot \frac{x_r}{2} = \frac{P - r}{\sqrt{2 Pr - r^2}}.
\]

For, assuming \( P \cos \frac{x_r}{2} = N \),
we have

\[
\cos \frac{x_r}{2} = \frac{N}{P};
\]

and, therefore,

\[
\cot \frac{x_r}{2} = \frac{N}{\sqrt{1 - \frac{N^2}{P^2}}}
= \frac{N}{\sqrt{P^2 - N^2}}
\]

or

\[
\cot \frac{x_r}{2} = \frac{P - r}{\sqrt{2 Pr - r^2}} \quad \text{[from (3) and (2)].}
\]

(d) For large values of \( P \), the value of \( \delta \cot \frac{x_r}{2} \) (\( P \) and \( N \) both varying so that \( r \) remains constant) is very small. In other words, for different values of \( P \), the corresponding values of the diameter of the \( r \)-th ring from the center differ very little. This is true to such a degree that, except for lenses of very short focal length, it will be possible to choose \( P \) so that a value of it uncertain by several units will still be sufficiently accurate to serve in determining \( \cot \frac{x_r}{2} \) with the desired accuracy. Thus the labor of an exact determination of \( P \) is not, in general, involved in the method.

(e) The peculiar character of the fringes by reflection affords a means of making the adjustments and measurements with the accuracy requisite to make the method practicable.

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To summarize: The first two considerations justify the formula for focal length. The third consideration gives the formula for calculating the constant of the apparatus and shows that, for subsequent determinations, only one quantity need be measured. The fourth consideration justifies an approximation which enables the experimental part of the labor to be curtailed. The fifth consideration affords us the experimental means of making the measurements.

4. EXPERIMENTAL PROCEDURE.

The apparatus required is as follows:

(a) The interference apparatus, consisting of the two silver mirrors before mentioned, supported by glass plates separated by three invar buttons, and held in place in a frame by three small springs. It is such a contrivance as is used in wave-length measurements.\(^1\) (b) A Ramsden ocular, for examining the real image of the fringe system. (c) A micrometer screw and carriage to carry the ocular and move it through measured distances. (d) A suitable source of monochromatic light and means for illuminating the interference apparatus. (e) Suitable mountings for the interference apparatus and the micrometer permitting of their convenient displacement in both a horizontal and vertical direction, perpendicular to the line of sight.

The apparatus is arranged as shown in Fig. 1. The lens under test should be carried on the same support as the micrometer. The mounting should permit of bringing the axes of the ocular and lens into coincidence. There should be no chance for accidental displacement of the lens relative to the micrometer. The adjustments to be made preliminary to making a measurement are as follows: (a) The axes of the lens and ocular are made coincident for the central position of the ocular, by centering the circle of illumination on the cross-hairs with a low power ocular. (b) The micrometer ways and the mirrors are made roughly, parallel by sight. (c) The source and the illuminating devices are

\(^1\)For description of such an apparatus, see Fabry and Perot, Ann. de Chim. et de Phys. (7) 25, 107, or Baly, Spectroscopy, p. 302. We have placed the three buttons at the vertices of an isosceles right-triangle instead of the vertices of an equilateral triangle.
adjusted to throw the light normally on the mirrors and the lens and ocular are brought into position to receive the reflected light. 

(d) The ocular is focused on the cross-hairs and on the image of the fringes, the same colored light being used for both adjustments. (e) The common axis of the lens and ocular is tilted until the center of the ring system is in the center of the field (intersection of the cross-hairs). (f) The small springs holding the plates are adjusted until the two mirrors are accurately parallel and the bright point at the center has just disappeared. The test for parallelism is made by displacing the interference apparatus both horizontally and vertically in the plane of its mirrors. The rings should neither expand nor contract.

The foregoing adjustments having been made, the diameter of a suitable ring is determined from a series of readings of the micrometer screw for settings of the intersection of the cross-hairs on the sharp edge of the ring at opposite ends of a diameter.\(^8\)

We have yet to consider the determination of \(P\). It has been shown that, in general, \(P\) need be determined only to within a few units. Therefore, if \(t\) be the perpendicular distance between the mirrors and \(\lambda\) be the wave length, \(P\) may be determined with sufficient accuracy from

\[
\frac{2t}{\lambda} = P
\]

if \(t\) be known with an error not greater than a few microns. In some cases \(t\) could be determined by careful measurements of the invar buttons with a micrometer. I have determined \(t\) by means of the fringes of superposition,\(^9\) thus: The difference of path on the sliding interferometer was made equal to \(2t\), and then equal to \(4t\). The displacement of the carriage (obtained by reading a scale on the carriage by a microscope mounted on the bed) gave \(t\) with an error not greater than 2 microns. This is a very satisfactory method.

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\(^8\)The outside edge is used because it is the best defined line to set the cross-hairs upon, and corresponds, in phase, to the disappearance of the bright point at the center, which is likewise the most definite adjustment that can be made of the difference of path at the center. To apply the formula it is not necessary to assume that the difference of path at either of these points is truly integral, but only that they correspond in phase, i. e., that \(r\) is accurately integral.

A New Method for Focal Length.

5. DISCUSSION OF PRECISION ATTAINABLE.

(a) THEORETICAL CONSIDERATIONS.

We desire to determine to what degree the several errors of adjustment and measurement will affect the result; and to decide upon the most favorable circumstances for making the measurement.

The formula, \( F = R_r \cot \frac{x_r}{2} \) shows that the percentage accuracy of \( F \), will be fixed by the percentage accuracy of the factors \( \cot \frac{x_r}{2} \) and \( R_r \).

The factor \( 2R_r \) is a length, measured by the micrometer. Experiment shows that the average error of setting at each end is about 3 microns or less. To obtain an accuracy of about one part in a thousand \( 2R_r \) should be not less than a few millimeters, say 4 or 5.

The factor, \( \cot \frac{x_r}{2} \) is calculated from the formula,

\[
\cot \frac{x_r}{2} = \frac{N}{\sqrt{P^2 - N^2}}
\]

and is exact except for the error involved in adjusting \( P \) to the value \( r + N \). Using partial differentials to express the relation of a small error in \( P \) to the resultant error in \( F \), differentiate with respect to \( P \), regarding \( N \) as constant. From equations (1) and (3) it follows that

\[
\delta F = -F \frac{P}{P^2 - N^2} \delta P.
\]
Thus the resultant error in $F$ is a fraction of $F$ measured by
\[ \frac{P}{P^2 - N^2} \delta P. \]
When $P = N = r = 5$, then \[ \frac{P}{P^2 - N^2} = \frac{P}{10P - 25}. \]
For $P = 100$, we have
\[ \frac{P}{10P - 25} = 0.104; \tag{6} \]
and for greater values of $P$, the approximation to 0.100 becomes closer. Experiment shows that 0.01 is a safe maximum limit to set for $\delta P$. We have then,
\[ \delta F = -0.001 F. \tag{7} \]
In words, for $r = 5$ and $P > 100$, the error in $F$ due to error of adjustment in $P$ is not greater than about \[ \frac{1}{1000}. \] For greater values of $r$ this error will be less.

Having considered the error due to the accidental errors of the observations involved in the form of the formula, we next determine what error may be introduced by the approximation mentioned under 3(d). By differentiating with respect to $P$, regarding $r$ as constant, it follows from equations (i) and (4) that
\[ \delta F = F \left( \frac{1}{2P + \frac{r^2}{P} - 3r} \right) \delta P \tag{8} \]
which gives the error introduced in $F$ by calculating cot $\frac{x_r}{2}$ from a value of $P$ in error by $\delta P$, $N$ being treated as a function of $P$. If $r$ be taken about 5 or 10, then for all large values of $P$ (say $P > 1000$), it is evident that \[ \frac{1}{2P + \frac{r^2}{P} - 3r} = \frac{1}{2P} \]
very nearly. We have then,
\[ \frac{\delta F}{F} = \frac{\delta P}{2P} \tag{9} \]
from which, selecting the accuracy we desire for $F$, we can determine the permissible error in $P$, e. g., if the desired accuracy of $F$ is $\frac{1}{1000}$ and $P = 7000$, the permissible error in $P$ is 14. The permissible error in the distance between the mirrors would then be about 4 microns.

We have now considered all the errors due to the form of the formula and the approximation introduced; and have obtained the expressions which enable us to select favorable conditions for the measurements. If we select $\frac{1}{1000}$ as the desirable accuracy in $F$, these conditions may be summarized as follows:

$R$ should be greater than 2 mm,
$r$ should be greater than 5;
and if $P$ is determined only with mechanical precision
$P$ should be greater than 5000.

It is to be noted that the conditions for high accuracy require both $R$, and $r$ to be large and further, that $R$, is made large by making $r$ large, so that these conditions can be met simultaneously. It is also to be noted that the last condition requires $P$ to be large. For a given value or $r$, this condition would make $R$, small; but as $r$ can be chosen greater with advantage, so $R$ can be made greater. As small values of $P$ will need be used for small values of $F$, it may be necessary to determine $P$ exactly for very short focus lenses.

The method is subject to other possible sources of error as follows:

- The error in the adjustment of the parallelism of the mirrors.
- The error of focusing the ocular on the image.
- The error of adjustment of the axis of lens normal to the plane of the mirrors.
- The error in the planeness of the mirrors.
- The error due to variation in intensity of illumination.

The effects of these sources of error have been studied experimentally. The results so obtained will now be presented.
(b) EXPERIMENTAL RESULTS.

The focal length of a Goerz Dagor lens No. 0000 has been repeatedly determined by this method. Some of these determinations have been made in a way especially suited to obtain information relative to the effects of the sources of error above noted. The results of these determinations and other experiments relative to the same questions will first be given. Then the results not grouped under the above heads will be presented. Finally the results of a series of determinations of the same focal length by one of the old methods will be given as a check.

The following values have been obtained, the parallelism of the mirrors having been readjusted between each two determinations. \((P = 3328. \quad r = 10. \quad \lambda = 5876.\) Sliding interferometer used instead of the invar button standard.)

\[
\begin{align*}
42.95 & \text{ mm} \\
43.00 & \\
43.00 & \\
43.01 & \\
43.04 & \\
\hline
\text{Mean} & 43.00 \\
\text{Average deviation from mean} & \text{0.02}
\end{align*}
\]

The following values have been obtained, the ocular having been refocused on the image between each two determinations. \((P = 3327. \quad r = 10. \quad \lambda = 5876.\) Sliding interferometer.)

\[
\begin{align*}
43.14 & \\
43.09 & \\
43.04 & \\
43.09 & \\
43.04 & \\
\hline
\text{Mean} & 43.10 \\
\text{Average deviation from mean} & \text{0.04}
\end{align*}
\]
The following values have been obtained, the adjustment of the axis of the lens and ocular normal to the mirrors having been made between each two. \((P = 3328. \quad r = 10. \quad \lambda = 5876.\) Sliding interferometer.)

\[
\begin{align*}
43.03 \\
43.07 \\
43.08 \\
43.08 \\
\text{Mean} & \quad 43.06 \\
\text{Average deviation from mean} & \quad 0.02 \\
\end{align*}
\]

To detect any error due to errors in the plates, which would show in distortion of the rings, mutually perpendicular diameters were measured. A difference of only 0.1 per cent was found. Also, in another case the rear plate was turned about the line of sight through about 90° between successive determinations. The parallelism was readjusted and the ocular refocused. The difference between the two determinations was less than 0.1 per cent.

In order to test the effect of variation in illumination, measurements of the same diameter were made at two widely different intensities (a much greater range than would occur accidentally). A difference of about 0.3 per cent was obtained.

A number of determinations, not grouped under the above heads, have been made. The earliest determinations made without taking due care as to the conditions for accuracy and before practice in the method had been acquired, have been set aside.\(^{10}\) Results of all determinations made between December 19, 1908, and January 25, 1909, including those grouped above, are presented below. These determinations have been made under

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\(^{10}\) The mean of a series made about two or three months before the ones here discussed should perhaps be mentioned for the sake of a complete record, and as indicating that determinations made at different times may differ by about 0.5 per cent; but it should not be given great weight. The value was 42.81 mm.
varied conditions with different arrangements of apparatus. They are of different degrees of reliability but none have been withheld.

\[ F. \text{ in millimeters.} \]

\[
\begin{array}{cccc}
42.49 & 43.14 & 43.07 & 42.96 \\
42.95 & 43.09 & 43.08 & 43.05 \\
43.00 & 43.04 & 43.08 & 43.00 \\
43.00 & 43.09 & 43.09 & 42.86 \\
43.01 & 43.04 & 42.92 & 43.00 \\
43.04 & 43.03 & 42.98 & 43.00 \\
\end{array}
\]

\[
\begin{array}{c}
\text{Mean} \quad 43.01 \\
\text{Average deviation from mean} \quad 0.09 \\
\text{P. E. of mean} \quad \pm 0.013
\end{array}
\]

As a check on the determinations by the new method, the same focal length has been determined by method (6) p. 180, Kohlrausch, Physical Measurements. As object length, \( L \), the distance between the edges of two white paper riders on a steel meter bar was taken. The image length, \( l \), was measured by the same micrometer used in the new method. The object distance, \( A \), was determined by measuring, with a steel tape, the distance between two points on the table top, the two points having been fixed by dropping a plumb-line from the scale and from a point thought to be about the first principal point of the lens. All of these lengths were of such magnitude that the accidental errors would be less than 0.1 per cent. The focal length is given by

\[ F = A \frac{l}{L + l} \]

\(^{11}\) This value was obtained without due care as to parallelism of mirrors, and is subject to further error because of uncertainty in \( P \). It is not included in the mean given below.

\(^{12}\) These values were obtained from measurements made when the apparatus was illuminated by the diffused light from a ground glass, instead of bright light from a condensing lens.
The following values were obtained:

42.76. From mean of two measurements of \( l \) for \( L = 800 \) and \( A = 2913 \). Average deviation from mean = 0.04.

42.94. From mean of three measurements of \( l \) for \( L = 600 \) and \( A = 2913 \). Average deviation from mean = 0.03.

43.02. From mean of five measurements of \( l \) for \( L = 400 \) and \( A = 2913 \). Average deviation from mean = 0.11.

43.06. From mean of five measurements of \( l \) for \( L = 400 \) and \( A = 2857 \). Average deviation from mean = 0.01.

The different measurements of \( l \) were each made after refo-
cusing the ocular on the image. Attention is called to the fact
that the large values of \( L \) gave small values of \( F \). In these cases
the ends of the image were near the edge of the circle of illumina-
tion and settings were not made under the best conditions. For
the last two values, conditions were quite satisfactory. If we
assign a weight 1 to each of the first two values and 2 to each of
the last two, the most probable value from this data is 42.98 mm.

(c) CONCLUSIONS CONCERNING PRECISION OF THE METHOD.

From the discussion just given, the following conclusions are
evident:

1. The error due to departure of \( P \) from the exact value, \( r + N \);
and the error due to error in measurement of \( R \) can each be
made less than 0.1 per cent.

2. The error due to the approximation, \( 3 \ (d) \), can generally be
made less than 0.1 per cent. The importance of this error can
always be estimated, and, if necessary, it may be eliminated.

3. The average error due to readjustment of focus, is less than
0.1 per cent.

4. The average error due to readjustment of the parallelism of
mirrors, and the average error due to readjustment of axis of lens
normal to plane of mirrors, are each less than 0.05 per cent.

5. With the plates used, the error due to error in planeness of
the mirrors, was within the other errors above noted.

6. The difference in results due to a great change in illumina-
tion may be a few tenths of 1 per cent. Consequently the differ-

ences due to small departures from a normal working intensity will be very small and within the other errors.

7. The average error in a long series of determinations involving all the sources of error is less than 0.25 per cent.

8. The most probable value by the new method differs from the most probable value by the old method by less than 0.1 per cent.

9. The results show that a single determination, carefully made, would have given the true value, certainly to within 0.5 per cent; and that a comparatively short series is sufficient to establish the value with a probable error of about ±0.02 per cent.

6. SUGGESTION AS TO A PROPOSED ROUTINE METHOD.

If it should be desirable to have a rapid, convenient method for determination of focal lengths in a routine way, the manipulation of the method might be still further simplified. Instead of the standard with an air space, a single plane parallel plate of glass or fused quartz heavily silvered on one side and partially silvered on the other might be used. It would not be necessary to know \( P \), nor would \( P \) need be integral; for the diameter of any ring in the field might be measured with a lens of known focal length and the factor of the apparatus \( \cot \frac{x}{2} \) determined from this measurement. Other focal lengths could then be determined by a measurement of the same diameter. Standards of different thicknesses would be prepared to suit different focal lengths. A disadvantage of this modification is that the factor of the apparatus is a function of temperature. However, this factor could be determined at several points over the small range of room temperatures; and within certain limits the correction would be negligible. The modification obviates the readjustments for parallelism. Also, if \( F \) of the lens used to determine the factor of the apparatus has been previously determined as the mean of a large number of independent determinations, the error due to readjustment of the difference of path will be eliminated in the modified method, each single determination partaking of the accuracy gained by the multiplication of determinations. Further the accurate par-
allelism of the surfaces of the plate will be of little consequence, if diameters are always measured in the same direction across the plate.

The method is recommended as a simple, convenient, rapid and precise method for the determination of focal lengths in monochromatic light.

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