FORMULÆ AND TABLES FOR THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE.

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TABLES OF CONSTANTS AND FUNCTIONS USEFUL IN THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE.

INTRODUCTION.

A great many formulæ have been given for calculating the mutual and self-inductance of the various cases of electrical circuits occurring in practice. Some of these formulæ have subsequently been shown to be wrong, and of those which are correct and applicable to any given case there is usually a choice, because of the greater accuracy or greater convenience of one as compared with the others. For the convenience of those having such calculations to make we have brought together in this paper all the formulæ with which we are acquainted which are of value in the calculation of mutual and self-inductance, particularly in nonmagnetic circuits where the frequency of the current is low enough to assure sensibly uniform distribution of current. A considerable number of formulæ which have been shown to be unreliable or which have been replaced by others that are less complicated or more accurate have been omitted, although in most cases we have given references to such omitted formulæ. Where several formulæ are applicable to the same case we have pointed out the especial advantage of each and indicated which one is best adapted to precision work.

In the second part of the paper we give a large number of examples to illustrate and test the formulæ. Some of these examples are taken from previous papers by the present authors, but many are new. We have given the work in many cases in full to serve as a guide in such calculations in order to make the formulæ as useful as possible to students and others not familiar with such calculations, and also to facilitate the work of checking up the results by anyone going over the subject. We have been impressed with the advantage of this in reading the work of others.

In the appendix to the paper are a number of tables that will be found useful in numerical calculations of inductance.

In most cases we have given the name of the author of a formula in connection with the formula. This is not merely for the sake of historical interest, or to give proper credit to the authors, but also because we have found it helpful to distinguish in this way the various formulæ instead of denoting each merely by a number. The formulæ of sections 8 and 9, which are taken largely from a paper by one of the present authors,[1] are, however, not so designated, although the authorship of those that are not new is indicated where known.

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[1] Rosa, this Bulletin, **4**, p. 301; 1907.

I. FORMULÆ.

1. MUTUAL INDUCTANCE OF TWO COAXIAL CIRCLES.

MAXWELL'S FORMULÆ IN ELLIPTIC INTEGRALS.

The first and most important of the formulæ for the mutual inductance of coaxial circles is the formula in elliptic integrals given by Maxwell:[2]

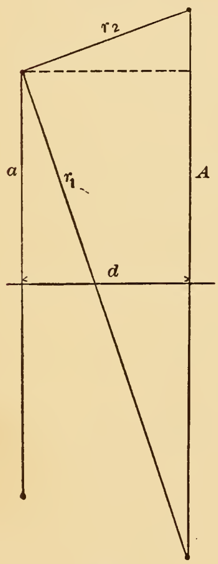

Fig. 1.

$$M=4\pi\sqrt{Aa}\left\{\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right\} \qquad [1]$$

in which A and a are the radii of the two circles, d is the distance between their centers, and

$$k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+d^2}}=\sin\gamma$$

F and E are the complete elliptic integrals of the first and second kind, respectively, to modulus k. Their values may be obtained from the tables of Legendre, or the values of $M\div 4\pi\sqrt{Aa}$ may be obtained from Table I in the appendix of this paper, the values of γ being the argument.

The notation of Maxwell is slightly altered in the above expressions in order to bring it into conformity with the formulæ to follow.

Formula (1) is an absolute one, giving the mutual inductance of two coaxial circles of any size at any distance apart. If the two circles have equal or nearly equal radii, and are very near each other, the quantity k will be very nearly equal to unity and γ will be near to 90°. Under these circumstances it may be difficult to obtain a sufficiently exact value of F and E from the tables, as the quantities are varying rapidly and the tabular differences are very large. Under such circumstances the following formula, also given by Maxwell[2] (derived by means of Landen's transformation), is more suitable:

$$M=8\pi\frac{\sqrt{Aa}}{\sqrt{k_1}}\left\{F_1-E_1\right\} \qquad [2]$$

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[2] Electricity and Magnetism, Vol. II, § 701.

in which F_1 and E_1 are complete elliptic integrals to modulus k_1, and

$$k_1 = \frac{r_1 - r_2}{r_1 + r_2} = \sin \gamma_1$$

r_1 and r_2 are the greatest and least distances of one circle from the other (Fig. 1); that is,

$$r_1 = \sqrt{(A+a)^2 + d^2}$$
$$r_2 = \sqrt{(A-a)^2 + d^2}$$

The new modulus k_1 differs from unity more than k, hence γ_1 is not so near to 90° as γ and the values of the elliptic integrals can be taken more easily from the tables than when using formula (1) and the modulus k.

Another way of avoiding the difficulty when k is nearly unity is to calculate the integrals F and E directly, and thus not use the tables of elliptic integrals, expanding F and E in terms of the complementary modulus k', where $k' = \sqrt{1-k^2}$. The expressions for F and E are very convergent when k' is small.

$$\left.\begin{aligned}
F = {} & \log\frac{4}{k'} + \frac{1^2}{2^2}k'^2\left(\log\frac{4}{k'} - \frac{2}{1.2}\right) \\
& + \frac{1^2}{2^2}\frac{3^2}{4^2}k'^4\left(\log\frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4}\right) \\
& + \frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5^2}{6^2}k'^6\left(\log\frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{2}{5.6}\right) \\
& + \frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5^2}{6^2}\frac{7^2}{8^2}k'^8\left(\log\frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{2}{5.6} - \frac{2}{7.8}\right) \\
& + \cdots\cdots\cdots \\
E = {} & 1 + \frac{1}{2}k'^2\left(\log\frac{4}{k'} - \frac{1}{1.2}\right) \\
& + \frac{1^2}{2^2}\frac{3}{4}k'^4\left(\log\frac{4}{k'} - \frac{2}{1.2} - \frac{1}{3.4}\right) \\
& + \frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5}{6}k'^6\left(\log\frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{1}{5.6}\right) \\
& + \frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5^2}{6^2}\frac{7}{8}k'^8\left(\log\frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{2}{5.6} - \frac{1}{7.8}\right) \\
& + \cdots\cdots\cdots
\end{aligned}\right\} \quad [3]$$

WEINSTEIN'S FORMULA.

Weinstein[3] gives an expression for the mutual inductance of two coaxial circles, in terms of the complementary modulus k' used in the preceding series (3). Substituting in equation (1) the values of F and E given above we have Weinstein's equation, which is as follows:

$$M=4\pi\sqrt{Aa}\left\{\left(1+\frac{3}{4}k'^2+\frac{33}{64}k'^4+\frac{107}{256}k'^6+\frac{5913}{16384}k'^8+\ .\ \right)\left(\log\frac{4}{k'}-1\right)\right.$$
$$\left.-\left(1+\frac{15}{128}k'^4+\frac{185}{1536}k'^6+\frac{7465}{65536}k'^8+\ .\ .\ .\right)\right\} \qquad [4]$$

This expression is rapidly convergent when k' is small, and hence will give an accurate value of M when the circles are near each other. Otherwise formula (1) may be more suitable.

NAGAOKA'S FORMULÆ.

Nagaoka[4] has given formulæ for the calculation of the mutual inductance of coaxial circles, without the use of tables of elliptic integrals. These formulæ make use of Jacobi's q-series, which is very rapidly convergent. The first is to be used when the circles are not near each other, the second when they are near each other. Either may be employed for a considerable range of distances between the extremes, although the first is more convenient. The *first* formula is as follows:

$$\begin{aligned} M&=16\pi^2\sqrt{Aa}.q^{\frac{3}{2}}(1+\epsilon)\\ &=4\pi\sqrt{Aa}\{4\pi q^{\frac{3}{2}}(1+\epsilon)\} \end{aligned} \qquad [5]$$

where A and a are the radii of the two circles. The correction term ϵ can be neglected when the circles are quite far apart.

$$\epsilon=3q^4-4q^6+9q^8-12q^{10}+\ .\ .\ .\ .$$
$$q=\frac{l}{2}+\left(\frac{l}{2}\right)^5+15\left(\frac{l}{2}\right)^9+\ .\ .\ .\ .$$
$$l=\frac{1-\sqrt{k'}}{1+\sqrt{k'}} \qquad k'=\frac{r_2}{r_1}=\frac{\sqrt{(A-a)^2+d^2}}{\sqrt{(A+a)^2+d^2}}$$

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[3] Wied. Ann. **21**, p. 344; 1884.

[4] Phil. Mag., **6**, p. 19; 1903.

d being the distance between the centers of the circles, and k' the complementary modulus occurring in equations (3) and (4).

Nagaoka's *second* formula is as follows:

$$M=4\pi\sqrt{Aa}\cdot\frac{1}{2(1-2q_1)^2}\left\{\log\frac{1}{q_1}\left[1+8q_1(1-q_1+4q_1^2)\right]-4\right\} \qquad [6]$$

$$q_1=\frac{l_1}{2}+2\left(\frac{l_1}{2}\right)^5+15\left(\frac{l_1}{2}\right)^9+\ \cdot\ \cdot\ \cdot$$

$$l_1=\frac{1-\sqrt{k}}{1+\sqrt{k}} \qquad k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+d^2}}$$

k is the modulus of equation (1), but is employed here to obtain the value of the q-series instead of the values of the elliptic integrals employed in (1). This formula is ordinarily simpler in use than it appears, because some of the terms in the expressions above are usually negligible.

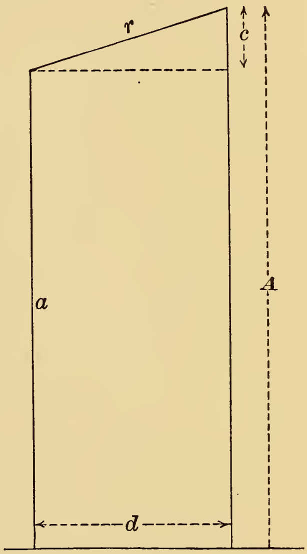

Fig. 2.

MAXWELL'S SERIES FORMULA.

Maxwell[5] obtained an expression for the mutual inductance between two coaxial circles in the form of a converging series which is often more convenient to use than the elliptical integral formula, and when the circles are nearly of the same radii and relatively near each other the value given is generally sufficiently exact. In the following formula a is the smaller of the two radii, c is their difference, $A-a$, d is the distance apart of the circles as before, and $r=\sqrt{c^2+d^2}$. The mutual inductance is then

$$M=4\pi a\left\{\log\frac{8a}{r}\left(1+\frac{c}{2a}+\frac{c^2+3d^2}{16a^2}-\frac{c^3+3cd^2}{32a^3}+\ \cdot\ \cdot\ \cdot\right)\right.$$
$$\left.-\left(2+\frac{c}{2a}-\frac{3c^2-d^2}{16a^2}+\frac{c^3-6cd^2}{48a^3}-\ \cdot\ \cdot\ \cdot\right)\right\} \qquad [7]$$

[5] Electricity and Magnetism, Vol. II, § 705.

When c and d are small compared with a, we have for an approximate value of the mutual inductance the following simple expression:[6]

$$M=4\pi a\left\{\log\frac{8a}{r}-2\right\} \qquad [8]$$

When the two radii are equal, as is often the case in practice, the equation (7) is somewhat simplified, as follows:

$$M=4\pi a\left\{\log\frac{8a}{d}\left(1+\frac{3d^2}{16a^2}\right)-\left(2+\frac{d^2}{16a^2}\right)\right\} \qquad [9]$$

The above formulæ (7) and (9) are sufficiently exact for very many cases, the terms omitted in the series being unimportant when $\frac{c}{a}$ and $\frac{d}{a}$ are small. For example, if $\frac{d}{a}$ is 0.1, the largest term neglected in (9) is less than two parts in a million. If, however $d=a$, this term will be more than one per cent, and the formula will be quite inexact.

Coffin[7] has extended Maxwell's formula (9) for two equal circles by computing three additional terms for each part of the expression. This enables the mutual inductance to be computed with considerable exactness up to $d=a$. Formula (1) is exact, as stated above, for all distances, and either it or (5) should be used in preference to (10) when d is large. Coffin's formula is as follows:

$$\begin{aligned}M=4\pi a\Big\{\log\frac{8a}{d}&\left(1+\frac{3d^2}{16a^2}-\frac{15d^4}{8\times128a^4}+\frac{35d^6}{128^2a^6}-\frac{1575d}{2\times128^3a^8}+\ \ldots\right)\\ &-\left(2+\frac{d^2}{16a^2}-\frac{31d^4}{16\times128a^4}+\frac{247d^6}{6\times128^2a^6}-\frac{7795d^8}{8\times128^3d^8}+\ \ldots\right)\Big\}\end{aligned} \qquad [10]$$

We have extended Maxwell's formula (7) for unequal circles as follows:[8]

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[6] This is equivalent to the approximate formula given by Wiedemann, $M=4\pi a\left\{\log\frac{2l}{c}-2.45\right\}$, where l is the circumference of the smaller circle and c is the same as r above.

[7] J. G. Coffin, this Bulletin, 2, p. 113; 1906.

[8] This Bulletin, 2, p. 364; 1907.

$$M=4\pi a\left\{\log\frac{8a}{r}\left(1+\frac{c}{2a}+\frac{c^2+3d^2}{16a^2}-\frac{c^3+3cd^2}{32a^2}+\frac{17c^4+42c^2d^2-15d^4}{1024a^4}\right.\right.$$
$$\left.-\frac{19c^5+30c^3d^2-45cd^4}{2048a^5}+\ldots\right)-\left(2+\frac{c}{2a}-\frac{3c^2-d^2}{16a^2}+\frac{c^3-6cd^2}{48a^3}\right.$$
$$\left.\left.+\frac{19c^4+534c^2d^2-93d^4}{6144a^4}-\frac{379c^5+3030c^3d^2-1845cd^4}{61440a^5}\right)\right\} \quad [11]$$

When $c=0$, this gives the first part of series (10). When $d=0$, the case of two circles in the same plane, with radii a and $a+c$, we have

$$M=4\pi a\left\{\log\frac{8a}{c}\left(1+\frac{c}{2a}+\frac{c^2}{16a^2}-\frac{c^3}{32a^3}+\frac{17c^4}{1024a^4}-\frac{19c^5}{2048a^5}+\ldots\right)\right.$$
$$\left.-\left(2+\frac{c}{2a}-\frac{3c^2}{16a^2}+\frac{c^3}{48a^3}+\frac{19c^4}{6144a^4}-\frac{379c^5}{61440a^5}+\ldots\right)\right\} \quad [12]$$

These formulæ (11) and (12) give the mutual inductance with great precision when the coils are not too far apart. The degree of convergence, of course, indicates approximately in any case the accuracy of the result.

The necessity for accurate formulæ for the mutual inductance of coaxial circles, which arises in connection with the development and testing of other formulæ as well as in the determination of the mutual inductance of coils by the methods of the next section, is fully met by the preceding formulæ. It is only necessary to use a sufficient number of decimal places to get any required accuracy when using absolute formulæ like (1) and (2), and some of the series formulæ give very high accuracy in many cases. The considerable number of formulæ available in most cases makes it possible to check important calculations by independent formulæ, and in general to choose for any particular case the formula that is on the whole best adapted.

For illustrations and tests of the above formulæ see examples 1–11, page 65.

2. MUTUAL INDUCTANCE OF TWO COAXIAL COILS.

ROWLAND'S FORMULA.

Let there be two coaxial coils of mean radii A and a, axial breadth of coils b_1 and b_2, radial depth c_1 and c_2, and distance apart of their mean planes d. Suppose them uniformly wound with n_1 and n_2 turns of wire. The mutual inductance M_0 of the two central turns of the coils (Fig. 3), will be given by formula (1) or (4), and the mutual inductance M of the two coils of n_1 and n_2 turns will then be, to a *first approximation*,

$$M = n_1 n_2 M_0$$

The following second approximation was obtained by Rowland by means of Taylor's theorem, following Maxwell, § 700:

$$M = M_0 + \frac{1}{24}\left\{(b_1^2 + b_2^2)\frac{d\,M_0}{dx^2} + c_1^2\frac{d^2 M_0}{da^2} + c_2^2\frac{d^2 M_0}{dA^2}\right\}$$

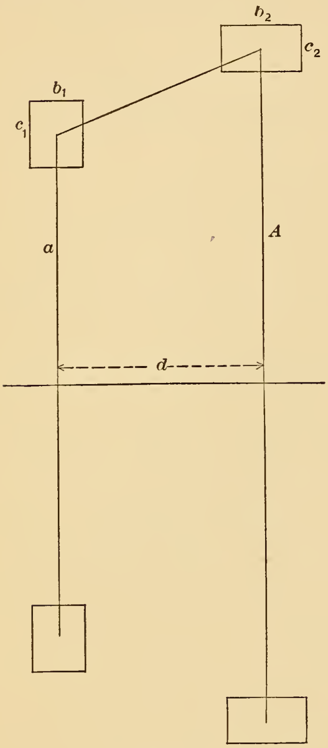

Fig. 3.

If the two coils are of equal radii but unequal section,

$$M = M_0 + \frac{1}{24}\left\{(b_1^2 + b_2^2)\frac{d^2 M_0}{dx^2} + (c_1^2 + c_2^2)\frac{d^2 M_0}{da^2}\right\} \qquad [13]$$

If the two coils are of equal radii and equal section, this becomes

$$M = M_0 + \frac{1}{12}\left\{b^2\frac{d^2 M}{dx^2} + c^2\frac{d^2 M}{da^2}\right\} \qquad [14]$$

The value of M_0 is preferably calculated by formula (1), but any one of the foregoing formulæ for the mutual inductance of coaxial circles adapted to the particular case may be used. The correction terms will be calculated by means of the following:

$$\left.\begin{aligned}\frac{d^2M_0}{da^2}&=\pi\frac{k}{a}\left\{(2-k^2)\,F-\left(2-k^2\frac{1-2k^2}{1-k^2}\right)E\right\}\\ \frac{d^2M_0}{dx^2}&=\pi\frac{k^3}{a}\left\{F-\frac{1-2k^2}{1-k^2}E\right\}\end{aligned}\right\}. \quad [15]$$

The equation (14) is equivalent to Rowland's equation, where 2ξ and 2η are the breadth and depth of the section of the coil, instead of b and c, except that there is an error in the formula as printed in Rowland's[9] paper, ξ and η being interchanged. The equations (15) are equivalent to those given by Rowland, being somewhat simpler.[10]

Formula (14) gives a very exact value for the mutual inductance of two coils, provided the cross sections are relatively small and the distance apart d is not too small. But when b or c is large or d is small the fourth differential coefficients which have been neglected become appreciable and the expression may not be sufficiently exact.

RAYLEIGH'S FORMULA.

Maxwell[11] gives a formula, suggested by Rayleigh, for the mutual inductance of two coils, which has a very different form from Rowland's, but is nearly equivalent to it when the coils are not near each other. It has been used by Rayleigh in calculating the mutual inductance of a Lorenz apparatus and by Glazebrook (Phil. Trans., 1883) in calculating the mutual inductance of parallel coils of rectangular section employed in a determination of the ohm. It may also be employed in calculating the attraction between two coils.[12] It is sometimes called the formula of quadratures, and is as follows:[13]

$$M=\frac{1}{6}\left(M_1+M_2+M_3+M_4+M_5+M_6+M_7+M_8-2M_0\right) \quad [16]$$

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[9] Collected Papers, p. 162. Am. Jour. Sci. [3], XV, 1878.

[10] Gray, Absolute Measurements, Vol. II, Part II, p. 322.

[11] Electricity and Magnetism, Vol. II, Appendix II, Chapter XIV.

[12] Gray, Absolute Measurements, Vol. II, Part II, p. 403.

[13] This Bulletin, 2, p. 370–372; 1906.

where M_1 is the mutual inductance of the circle O_2 and a circle through the point 1 of radius $A-\frac{c_1}{2}$, and similarly for the others, Fig. 4.

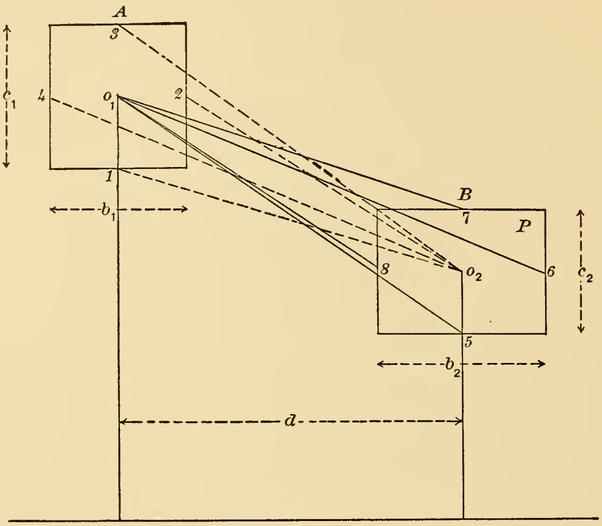

Fig. 4.

For two coils of equal radii and equal section this becomes

$$M=\frac{1}{3}\left(M_1+M_2+M_3+M_4-M_0\right) \qquad [17]$$

Equation (16) is Rayleigh's formula, or the formula of quadratures. Instead of computing the correction to M_0 by means of the differential coefficients (13), eight additional values are computed, corresponding to the mutual inductances of the single turns at the eight numbered points indicated in Fig. 4, each with reference to the central turn of the other coil. These M's may be computed by formulæ (7) and (9) or (10) and (11), and the values of the constants for the case of two coils of *equal radii* are given in the following table, the radius being a in every case.

	Axial distance.	Radial distance.	r
Using (7)	d	$-\frac{c_1}{2}$	$\sqrt{d^2+\frac{c_1^2}{4}}$
"	d	$+\frac{c_1}{2}$	$\sqrt{d^2+\frac{c_1^2}{4}}$
"	d	$-\frac{c_2}{2}$	$\sqrt{d^2+\frac{c_2^2}{4}}$
"	d	$+\frac{c_2}{2}$	$\sqrt{d^2+\frac{c_2^2}{4}}$
Using (9)	$d-b_1/2$	0	
"	$d+b_1/2$	0	
"	$d+b_2/2$	0	
"	$d-b_2/2$	0	

MAGNITUDE OF THE ERRORS IN ROWLAND'S AND RAYLEIGH'S FORMULÆ.

The error ϵ_1 in equation (17), for two coils of equal radii a, distance between centers being d, and section $b \times c$, depends on the dimensions of the coil in a manner shown by the following expression:[14]

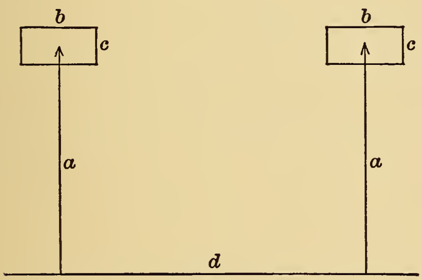

Fig. 5.

$$\epsilon_1 \propto 4\pi a\left\{\frac{3b^4+3c^4-20b^2c^2}{480d^4}\right\} \quad [18]$$

For a square coil the correction is a negative quantity, showing that M by equation (17) is too large, and the error is proportional to the fourth power of $\frac{1}{d}$, the reciprocal of the distance between the mean planes of the coils. For a rectangular coil in which b is greater than c the correction is negative so long as b is not more than 2.5 times c. When b is still larger with respect to c the correction becomes plus, the value of M by (17) being too small.

Thus, for a coil of cross section 4 sq. cm, we get the following values of the numerator of (18) as we vary the shape of cross section, keeping $bc=4$.

[14] This Bulletin, 2, p. 373; 1906.

Dimensions of coil.		Error proportional to
$b=2$	$c=2$	$-\ 224$
$b=2.5$	$c=1.6$	$-\ 183$
$b=3$	$c=1.33$	$-\ 67.5$
$b=4$	$c=1$	$+\ 451$
$b=8$	$c=0.5$	$+11,988$

Thus we see that the value of M as given by the formula of quadratures may be too large or too small according to the shape of the section, and that the error is proportional directly to the fourth power of the dimensions of the section and inversely to the fourth power of the distance between the mean planes of the coils. When the section is small and d large the error will become negligible.

The error by Rowland's formula is[14]

$$\epsilon_2 \propto 4\pi a \frac{6}{d^4}\left\{\frac{b^4+c^4}{360}-\frac{b^2c^2}{144}\right\} \propto 4\pi a\left\{\frac{8b^4+8c^4-20b^2c^2}{480d^4}\right\} \qquad [19]$$

This is negative for a square coil, but smaller than ϵ_1. For a coil of section such that $b=c\sqrt{2}$, the error is zero, and for sections such that $\frac{b}{c}>\sqrt{2}$, the error is positive. Thus, for a coil of cross section 4 sq. cm, we get the following values of the numerator of (19) which is proportional to the error by Rowland's formula.

Dimensions of coil.		Error proportional to—
$b=2$	$c=2$	$-\ 64$
$b=2.5$	$c=1.6$	$+\ 45$
$b=3$	$c=1.33$	$+\ 353$
$b=4$	$c=1$	$+\ 1,736$
$b=8$	$c=0.5$	$+32,448$

Thus the error is smaller by Rowland's formula for coils having square or nearly square section, but larger for coils having rectangular sections not nearly square.

LYLE'S FORMULA.

Professor Lyle[15] has recently proposed a very convenient method for calculating the mutual inductance of coaxial coils, which gives very accurate results for coils at some distance from each other.

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[14] This Bulletin, **2**, p. 373; 1906.

[15] Phil. Mag., **3**, p. 310; 1902. Also this Bulletin, **2**, p. 374–378; 1906.

The mutual inductance is calculated from formula (1) or any other formula for two coaxial circles, using, however, a modified radius r instead of the mean radius a, r being given by the following equation when the section is square, b being the side of the square section:

$$r = a\left(1 + \frac{b^2}{24a^2}\right) \qquad [20]$$

If the coil has a rectangular section not square, it can be replaced by two filaments, the distance apart of the filaments being called the *equivalent breadth* or the *equivalent depth* of the coil.

$$\left.\begin{aligned} \beta^2 &= \frac{b^2 - c^2}{12}, \ 2\,\beta \text{ is the equivalent breadth of A} \\ \delta^2 &= \frac{c^2 - b^2}{12}, \ 2\,\delta \text{ is the equivalent depth of B} \end{aligned}\right\} \qquad [21]$$

The equivalent radius of A is given by the same expression which holds for a square coil, viz:

$$r = a\left(1 + \frac{c^2}{24a^2}\right)$$

In the coil B the equivalent filaments have radii $r+\delta$ and $r-\delta$, respectively, where

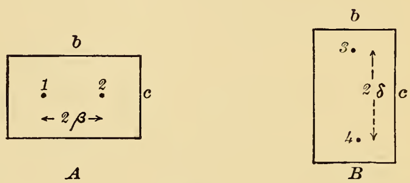

Fig. 6.

$$r = a\left(1 + \frac{b^2}{24a^2}\right)$$

The mutual inductance of two coils may now be readily calculated. If each has a square section, it is necessary only to calculate the mutual inductance of the two equivalent filaments. For coils of rectangular sections, as A, B, the mutual inductance will be the sum of the mutual inductances of the two filaments of A on the two filaments of B, counting $n/2$ turns in each. Or, it is n_1n_2 times the mean of the four inductances M_{13}, M_{14}, M_{23}, M_{24}, where M_{13} is the mutual inductance of filament 1 on filament 3, etc.

Lyle's method is of special value in computing mutual inductances because it applies to coils of unequal as well as of equal radii.

ROSA'S FORMULÆ.[16]

Writing the mutual inductance of two coaxial coils of equal radii and equal section as $M=M_0+\varDelta M$, where M_0 is the mutual inductance of the central circles of the two equal coils of sections $b\times c$, Fig. 5, and $\varDelta M$ is the correction for the section of the coil, the value of $\varDelta M$ is as follows:

$$\varDelta M=4\pi a n^2\left\{\frac{3b^2+c^2}{96a^2}\cdot\log\frac{8a}{d}-\frac{11b^2-3c^2}{192a^2}+\frac{b^2-c^2}{12d^2}+\frac{2b^4+2c^4-5b^2c^2}{120d^4}\right.$$
$$+\frac{6b^4+6c^4+5b^2c^2}{5760a^2d^2}+\frac{3b^6-3c^6+14b^2c^4-14b^4c^2}{504d^6}+\frac{7c^2d^2}{1024a^4}\left(\log\frac{8a}{d}-\frac{163}{84}\right)$$
$$\left.-\frac{15b^2d^2}{1024a^4}\left(\log\frac{8a}{d}-\frac{97}{60}\right)\right\} \qquad [22]$$

For a square section, when $b=c$, this becomes

$$\varDelta M=\frac{\pi b^2n^2}{6a}\left\{\log\frac{8a}{d}-1-\frac{a^2b^2}{5d^4}-\frac{3d^2}{16a^2}\left(\log\frac{8a}{d}-\frac{4}{3}\right)+\frac{17b^2}{240d^2}\right\} \qquad [23]$$

The last two terms of equation (23) are relatively small, so that we may write, *approximately:*

$$\varDelta M=\frac{\pi b^2n^2}{6a}\left\{\log\frac{8a}{d}-1-\frac{a^2b^2}{5d^4}\right\} \qquad [24]$$

These expressions for $\varDelta M$ are very exact where the coils are near together or even where they are separated by a considerable distance, but become less exact as d is greater. They are therefore most reliable where formulæ (14), (17), and (20) are least reliable. As formula (24) is exact enough for most purposes, it affords a very easy method of getting the correction for equal coils of square section.

Stefan's formula for the mutual inductance of two equal coaxial coils (originally published[17] without demonstration) is incorrect and is not given here. It resembles equation (22), but is seriously in error for coils at considerable distances.

[16] This Bulletin, **4**, p. 348, (38) and (39).

[17] Wied. Annalen, **22**, p. 107; 1884.

THE ROSA-WEINSTEIN FORMULA.

Weinstein's formula[18] for the mutual inductance of equal coaxial coils has been revised and corrected by Rosa, and the value of ΔM, the correction for section, expressed separately. The expression for ΔM is as follows:[19]

$$\Delta M = 4\pi a n_1 n_2 \sin\gamma \left\{ (F - E)\left(A + \frac{c^2}{24a^2}\right) + EB \right\} \qquad [25]$$

where F and E are the complete elliptic integrals to modulus $\sin\gamma$, Fig. 7 (as in equation 1) and

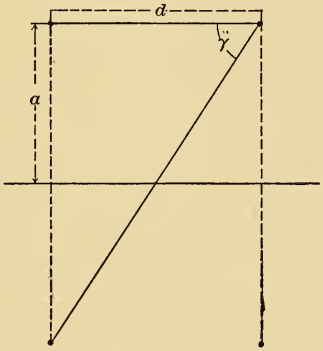

Fig. 7.

$$A = \frac{\cos^2\gamma}{12d^2}\left(a_1 - a_2 - a_3 + (2a_2 - 3a_3)\cos^2\gamma + 8a_3\cos^4\gamma\right)$$

$$B = \frac{\sin^2\gamma}{12d^2}\left(a_1 + \frac{a_2}{2} + 2a_3 + (2a_2 + 3a_3)\cos^2\gamma + 8a_3\cos^4\gamma\right)$$

The values of a_1, a_2, and a_3 are as follows:

$a_1 = b^2 - c^2 + \frac{c^4}{30a^2}$	For square section:	$a_1 = \frac{b^4}{30a^2}$
$a_2 = \frac{5b^2c^2 - 4c^4}{60a^2}$	" " "	$a_2 = \frac{b^4}{60a^2}$
$a_3 = \frac{2b^4 + 2c^4 - 5b^2c^2}{20d^2}$	" " "	$a_3 = -\frac{b^4}{20d^2}$

Formula (25) is a very exact formula for all positions of the two coils, except when they are very close together.

Weinstein's original formula,[18] which is much less accurate than (25) for coils relatively near together, is not here given.

[18] Wied. Annalen, **21**, p. 350; 1884.

[19] This Bulletin, **4**, p. 342, equation (20).

USE OF FORMULÆ FOR SELF-INDUCTANCE IN CALCULATING MUTUAL INDUCTANCE.

One can sometimes obtain the mutual inductance of adjacent coils, or of coils at a distance from one another, by means of a formula for the self-inductance of coils. Thus, suppose we have a coil of rectangular section, which we subdivide into three equal parts, 1, 2, 3, Fig. 8. Let L be the self-inductance of the whole coil, L_1 be the self-inductance of any one of the three equal smaller coils, and L_2 be the self-inductance of two adjacent coils taken together. Also let M_{12} be the mutual inductance of coil 1 on coil 2, or of coil 2 on coil 3, and M_{13} be the mutual inductance of coil 1 on coil 3. Then,

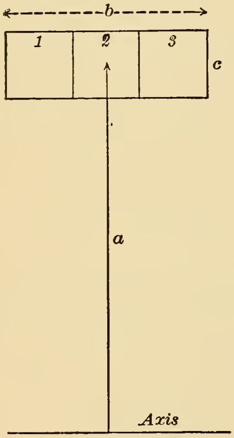

Fig. 8.

$$L = 3L_1 + 4M_{12} + 2M_{13}$$

$$\text{Also, } L_2 = 2L_1 + 2M_{12}$$

$$\left.\begin{aligned} \therefore\ M_{12} &= \frac{L_2 - 2L_1}{2} \\ \text{and } M_{13} &= \frac{L + L_1 - 2L_2}{2} \end{aligned}\right\} \qquad [26]$$

Formula (26) will thus enable us to find the mutual inductance of two coils of equal radii adjacent or near each other by the calculation of self-inductances from such formulæ as those of Weinstein (67) and Stefan (69). These latter formulæ are not, however, exact enough when the section is large to permit us to apply them to coils at any considerable distance from one another.

GEOMETRIC MEAN DISTANCE FORMULA.

The mutual inductance of two coaxial coils adjacent or very near can sometimes be obtained by means of the geometric mean distances. This method is accurate only when the sections are very small relatively to the radius. It can often be used to advantage in testing other formulæ, but not often in determining the mutual inductance of actual coils.

Formula (7) gives the mutual inductance of two very near coaxial coils in terms of the geometric mean distance, if r be replaced by R, the geometric mean distance of the two sections. Formula (7) gives M_0 if r be used, where r is the distance between centers. Thus,

$$\Delta M = 4\pi a n^2\left(1+\frac{c}{2a}\right)\log\frac{r}{R} \qquad [27]$$

For coils A and C, $R<r$ and ΔM is positive; $R=0.99770\ r$
" " A " B, $R>r$ and ΔM is negative; $R=1.00655\ r$

The same formula may also be used for squares not adjacent, but only when quite near.[20]

For illustrations and tests of the above formulæ see examples 12–21, pages 71–77.

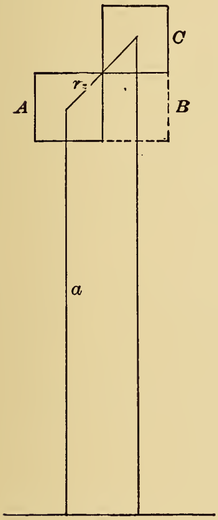

Fig. 9.

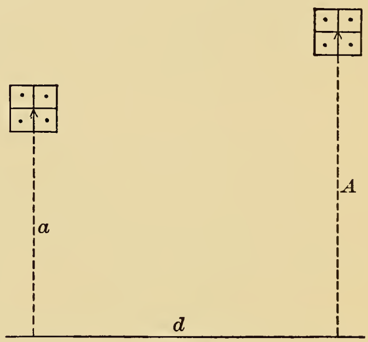

Fig. 9a.

The preceding formulæ can be used with entire satisfaction to calculate the mutual inductance of coaxial coils, especially those of coils of equal radii. Formulæ (16), (17), (20), and (21) apply also to coils of unequal radii, but unfortunately they are not as accurate as some of the others, except when the coils are relatively distant or have very small cross sections. The difficulty can be overcome by subdividing each of the two coils into two, four, or more equal parts, and taking the sum of the mutual inductances of all of the parts of one on all the parts of the other. This is a laborious operation, but in important cases it should be done. As the subdivision is carried further the results will approach a final value, and hence the results themselves show when the subdivision has been carried far enough.

[20] For other values of the geometric mean distances of squares in a plane see this Bulletin, **3**, p. 1; 1907.

Thus, suppose two coils A, B of square section are subdivided into four equal parts and by the method of Lyle, formula (20), the mutual inductance of the whole of B is computed on each of the four parts of A. If the sum differs appreciably from the result obtained by taking A and B as wholes in one calculation, then the four parts of B may be taken separately with respect to the separate parts of A. If one is doubtful whether this is sufficiently accurate, one of the sections of A may be subdivided further and calculated with respect to one section of B, to see whether there is any appreciable difference due to this further subdivision. For coils of equal radii very accurate results for near coils can be obtained much more easily by using some of the other formulæ.

3. MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS.

There are several formulæ for the calculation of the mutual inductance of coaxial solenoids. Although few of these formulæ are exact, the approximate formulæ often permit inductances to be calculated with very great accuracy by using a sufficient number of terms of the series by which they are expressed.

MAXWELL'S FORMULA.[21]

CONCENTRIC, COAXIAL SOLENOIDS OF EQUAL LENGTH.

The mutual inductance M of two coaxial solenoids of equal length is given by the following expression, due to Maxwell, where A and a are the radii of the outer and inner solenoids, respectively, l is the common length, and n_1 and n_2 the number of turns of wire per cm on the single layer winding of the outer and inner solenoids, respectively:

$$M = 4\pi^2 a^2 n_1 n_2 [l - 2A\alpha]$$

where

$$\left.\begin{aligned} \alpha = \frac{l-r+A}{2A} - \frac{a^2}{16A^2}\left(1 - \frac{A^3}{r^3}\right) - \frac{a^4}{64A^4}\left(\frac{1}{2} + 2\frac{A^5}{r^5} - \frac{5}{2}\frac{A^7}{r^7}\right) \\ - \frac{35a^6}{2048A^6}\left(\frac{1}{7} - \frac{8A^7}{7r^7} + \frac{4A^9}{r^9} - \frac{3A^{11}}{r^{11}}\right) + \ldots \end{aligned}\right\} \quad [28]$$

Putting

$$M = M_0 - \varDelta M$$

[21] Electricity and Magnetism, Vol. II, § 678.

$M_0 = 4\pi^2 a^2 n_1 n_2 l$ is the mutual inductance of an infinite outer solenoid and the finite inner solenoid, while ΔM is the correction due to the ends.

Equation (28) is Maxwell's expression, except that we have carried it out further than Maxwell did. This expression for M is rapidly convergent when a is considerably smaller than A, Fig. 10. Equation (28) shows that the mutual inductance is proportional to $l - 2A\alpha$; or the length l must be reduced by $A\alpha$ on each end. When a is small and l is large α is 1/2, approximately. That is, the length l is reduced by A, the radius of the outer solenoid.

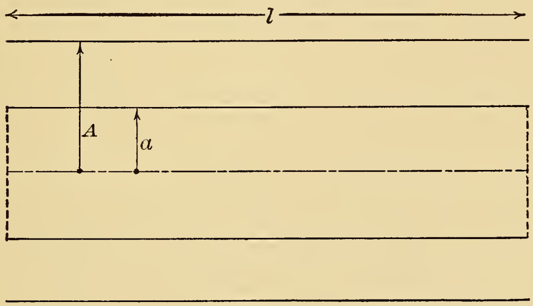

Fig. 10.

For the case of two coils each of more than one layer the above formula may be used, A and a being the mean radii, and n_1 and n_2 the total number of turns per cm in all the layers. The result will be only approximate, but usually less in error than if one uses the formula of Maxwell § 679 quoted by Mascart and Joubert.[22]

When the solenoids are very long in comparison with the radii, formula (28) may be simplified by omitting the terms in A/l, A^3/r^3, A^5/r^5, etc. The expression for α then becomes

$$\alpha = \frac{1}{2} - \frac{a^2}{16A^2} - \frac{a^4}{128A^4} - \frac{5a^6}{2048A^6} - \cdots \qquad [29]$$

Heaviside[23] gives an extension of formula (29), but as it neglects $\frac{A}{l}$, $\frac{A^3}{r^3}$, etc., the additional terms are of no importance, being smaller than the terms already neglected in (29).

[22] Electricity and Magnetism, Vol. I, p. 533.

[23] There are some misprints in Heaviside, 2, p. 277. The radius of the *inner* solenoid should be c_2, of the *outer* c_1, and ρ is c_2^2/c_1^2.

RÒITI'S FORMULA.

For a pair of concentric, coaxial solenoids of which the inner solenoid is considerably shorter than the outer, we have the following formula:[24]

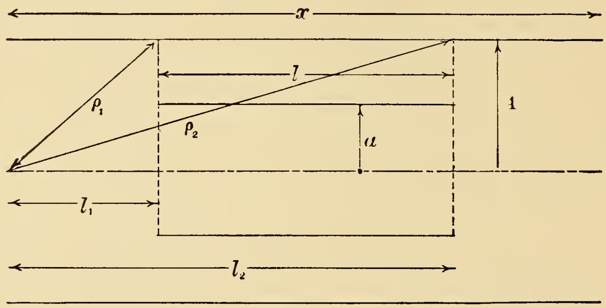

Fig. 11.

$$M=4\pi^2a^2n_1n_2\Big[\rho_2-\rho_1+\frac{a^2A^2}{8}\left(\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right)-\frac{a^4A^2}{16}\left(\frac{1}{\rho_1^5}-\frac{1}{\rho_2^5}\right)$$
$$+\frac{5a^4A^4}{64}\left(\frac{1}{\rho_1^7}-\frac{1}{\rho_2^7}\right)+\frac{5a^6A^2}{128}\left(\frac{1}{\rho_1^7}-\frac{1}{\rho_2^7}\right)$$
$$-\frac{35a^6A^4}{256}\left(\frac{1}{\rho_1^9}-\frac{1}{\rho_2^9}\right)+\frac{105a^6A^6}{1024}\left(\frac{1}{\rho_1^{11}}-\frac{1}{\rho_2^{11}}\right)+\ldots\Big] \quad [30]$$

in which

$$\rho_1=\sqrt{l_1^2+A^2} \text{ where } l_1=\frac{x-l}{2}$$

$$\rho_2=\sqrt{l_2^2+A^2} \quad \text{"} \quad l_2=\frac{x+l}{2}$$

$l=l_2-l_1=$ length of inner solenoid.

$x=$ length of outer solenoid and A and a the radii.

This is for many cases a very convenient and very accurate formula.

[24] For the derivation and extension of this formula see this Bulletin, 3, pp. 309–310; 1907. This formula was originally given (without proof and without the last three terms) in this Bulletin, 2, p. 130; 1906.

GRAY'S FORMULA.

Gray[25] gives a general expression for the mutual kinetic energy of two solenoidal coils which may or may not be concentric, and their axes may be at any angle ϕ. The most important case in practice is when the two coils are concentric and coaxial. In that case the zonal harmonic factors in each term reduce to unity, and half the terms become zero. Putting the current in each equal to unity, the mutual kinetic energy becomes the mutual inductance M.

Let $2x$ = the length of outer solenoid
$2l$ = " " " inner "
A = radius of outer "
a = " " inner "
n_1 = number of turns per cm on outer solenoid
n_2 = " " " " " " inner "

Gray's expression with these changes becomes

$$M=\pi^2a^2A^2n_1n_2[K_1k_1+K_3k_3+K_5k_5+\ .\ .\ .\ ] \qquad [31]$$

where K_1, K_3, etc., are functions of x and A, and k_1, k_3, etc., are functions of l and a.[26] When the ratio of the length of the winding of the outer coil to the radius is $\sqrt{3}$ to 1, $K_5=0$, and if the same condition holds for the inner coil, $k_3=0$. If in addition a is considerably smaller than A, the terms of higher order become negligible and (31) reduces to

$$M=\frac{2\pi^2a^2N_1N_2}{d} \qquad [32]$$

where d is half the diagonal of the outer coil, $=\sqrt{x^2+A^2}$. When the dimensions depart slightly from these theoretical ratios the small correction terms to (32) can be calculated.[26]

SEARLE AND AIREY'S FORMULA.

The following expression for the mutual inductance of two concentric, coaxial solenoidal coils has been given by Searle and Airy.[27]

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[25] Absolute Measurements, **2**, Part I, p. 274, equation 53.
[26] Rosa, this Bulletin, **3**, p. 221.
[27] The Electrician (London), **56**, p. 318; 1905.

$$M = g_1G_1 + g_3G_3 + g_5G_5 + g_7G_7 + \dots$$

$$= \frac{2\pi^2a^2N_1N_2}{d}\left[1 - \frac{A^2}{2d^4}\cdot\frac{4l^2-3a^2}{4} - \frac{A^2(4x^2-3A^2)}{8d^8}\cdot\frac{8l^4-20l^2a^2+5a^4}{8}\right.$$

$$\left. - \frac{A^2(8x^4-20x^2A^2+5A^4)}{16d^{12}}\cdot\frac{(64l^6-336l^4a^2+280l^2a^4-35a^6)}{64} - \dots\right] \quad [33]$$

The notation of (33) differs slightly from that used by Searle and Airey.

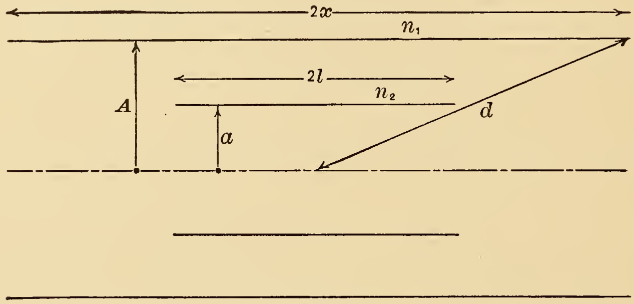

Fig. 12.

Equation (33) has been extended and put for greater convenience in calculation into the following form:[28]

$$M = \frac{2\pi^2a^2N_1N_2}{d}\left[1 + \frac{A^2a^2}{8d^4}L_2 + \frac{A^4a^4}{32d^8}X_2L_4\right.$$
$$\left. + \frac{A^6a^6}{32d^{12}}X_4L_6 + \frac{A^8a^8}{32d^{16}}X_6L_8 + \dots\right] \quad [34]$$

where

$$L_2 = 3 - 4\frac{l^2}{a^2}$$

$$X_2 = 3 - 4\frac{x^2}{A^2}$$

$$L_4 = \frac{5}{2} - 10\frac{l^2}{a^2} + 4\frac{l^4}{a^4}$$

$$X_4 = \frac{5}{2} - 10\frac{x^2}{A^2} + 4\frac{x^4}{A^4}$$

$$L_6 = \frac{35}{16} - \frac{35}{2}\frac{l^2}{a^2} + 21\frac{l^4}{a^4} - 4\frac{l^6}{a^6}$$

$$X_6 = \frac{35}{16} - \frac{35}{2}\frac{x^2}{A^2} + 21\frac{x^4}{A^4} - 4\frac{x^6}{A^6}$$

$$L_8 = \frac{63}{32} - \frac{105}{4}\frac{l^2}{a^2} + 63\frac{l^4}{a^4} - 36\frac{l^6}{a^6} + 4\frac{l^8}{a^8}$$

[28] Rosa, this Bulletin, **3**, p. 224.

This reduces to (32) when the terms after the first are negligible, as they are when the conditions assumed for (32) are fulfilled. The above expressions for L_2, X_2 show what these conditions are in order to make the second and third terms zero. If l^2/a^2 is slightly more or less than ¾, (34) gives the value of the second term which is neglected in (32), etc.

COHEN'S FORMULA.[29]

This is an absolute formula for two coaxial, concentric solenoids of lengths $2l_1$ and $2l_2$, Fig. 13.

$$\left.\begin{aligned} M &= 4\pi n_1 n_2 (V - V_1) \\ V &= -(A^2-a^2)c\left[F\{F(k',\theta)-E(k',\theta)\}-E\,F(k',\theta)\right] \\ &\quad + \frac{c^4-(A^2-6Aa+a^2)c^2-2(A^2-a^2)^2}{3\sqrt{(A+a)^2+c^2}}.F \\ &\quad + \frac{2(A^2+a^2)-c^2}{3}\sqrt{(A+a)^2+c^2}.\,E - c(A^2-a^2)\frac{\pi}{2} \end{aligned}\right\} \quad [35]$$

V_1 is obtained from V by replacing c by c_1,

$$c = l_1 + l_2, \qquad c_1 = l_1 - l_2,$$

F and E are the complete elliptic integrals of the first and second kind to modulus k, where $k^2 = \dfrac{4Aa}{(A+a)^2+c^2}$

$F(k',\theta)$ and $E(k',\theta)$ are the incomplete elliptic integrals of modulus k' and amplitude θ,

$$k'^2 = 1 - k^2 = 1 - \frac{4Aa}{(A+a)^2+c^2}$$

$$= \frac{(A-a)^2+c^2}{(A+a)^2+c^2}$$

$$\sin^2\theta = \frac{(A^2-a^2)^2+c^2(A-a)^2}{(A^2-a^2)^2+c^2(A+a)^2}$$

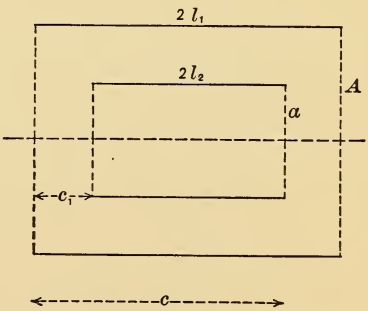

Fig. 13.

[29] This Bulletin, **3**, p. 301; 1907.

RUSSELL'S FORMULÆ.[30]

Russell's formula for coaxial solenoids in the notation of this paper is

$$M=4\pi^2a^2n_1n_2\Bigg[R_1\Bigg\{1-\frac{1}{2}q_2k_1^{\,2}-\frac{1}{2}\cdot\frac{1}{4}q_3k_1^{\,4}-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}q_4k_1^{\,6}$$

$$-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}q_5k_1^{\,8}-\frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}q_6k_1^{\,10}-\ \ldots\ \Bigg\}$$

$$-R_2\Bigg\{1-\frac{1}{2}q_2k_2^{\,2}-\frac{1}{2}\cdot\frac{1}{4}q_3k_2^{\,4}-\text{terms with above coefs.}\Bigg\}\Bigg] \qquad [36]$$

where

$$R_1^{\,2}=(A+a)^2+(l_1+l_2)^2 \qquad k_1^{\,2}=\frac{4\,Aa}{R_1^{\,2}}$$

$$R_2^{\,2}=(A+a)^2+(l_1-l_2)^2 \qquad k_2^{\,2}=\frac{4\,Aa}{R_2^{\,2}}$$

$$q_n=\frac{(A+a)^2}{4\,Aa}q_{n-1}-\frac{1}{n}\cdot\frac{1}{2}\cdot\frac{3\cdot 5\ldots\ldots 2n-3}{4\cdot 6\ldots\ldots 2n-2}\cdot\frac{A}{a}$$

$$q_2=\frac{(A+a)^2}{4\,Aa}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{A}{a}$$

$$q_3=\frac{(A+a)^2}{4\,Aa}q_2-\frac{1}{3}\cdot\frac{1\cdot 3}{2\cdot 4}\cdot\frac{A}{a}$$

etc.

A and a are the radii of the outer and inner cylinders respectively, $2l_1$ and $2l_2$ their lengths, Fig. 13, and n_1, n_2 the number of turns of wire per cm in the two windings. This formula applies only when the inner coil is shorter than the outer. For two coils of equal length the second part of the above formula is not convergent, and hence it must be replaced by an expression in elliptic integrals. The formula thus becomes (equation 42 in Russell's paper)

$$M=4\pi^2a^2n_1n_2\Bigg[R_1\Bigg\{1-\frac{1}{2}\,q_2k_1^{\,2}-\frac{1}{8}q_3k_1^{\,4}=\ldots\text{ as above}\Bigg\}\Bigg]$$

$$+\frac{8\pi Aa}{3(A+a)}n_1n_2[(A^2+a^2)(F-E\,)-2\,AaF] \qquad [37]$$

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[30] Alexander Russell, Phil. Mag., Apr. 1907, p. 420.

This formula gives an accurate result for equal solenoids of considerable length, but Maxwell's formula (28) is just as accurate and much more convenient.

For short coils neither (36) nor (37) will apply, but for that case as well as other cases Russell's general formula may be used. As the latter is equivalent to (35) it is not here given.

ROSA'S FORMULA FOR SINGLE LAYER COILS OF EQUAL RADII AND EQUAL BREADTH.

The mutual inductance of two coaxial single layer coils of equal radii and equal breadth is given by the following expression:

$$M = M_0 + \varDelta M$$

where M_0 is the mutual inductance of the two parallel circles at the centers of the coils and $\varDelta M$ is given by the following expression:[31]

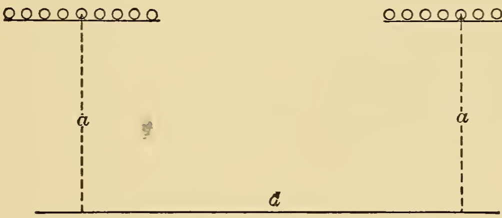

Fig. 14.

$$\varDelta M = 4\pi n^2 a\left[\frac{b^2}{12d^2} + \frac{b^2}{32a^2}\left(\log\frac{8a}{d} - \frac{11}{6}\right) - \frac{15b^2d^2}{1024a^4}\left(\log\frac{8a}{d} - \frac{97}{60}\right)\right.$$
$$+ \frac{175b^2d^4}{2(128)^2a^6}\left(\log\frac{8a}{d} - \frac{54}{35}\right) - \frac{3675b^2d^6}{(128)^3a^8}\left(\log\frac{8a}{d} - \frac{3793}{2520}\right) \qquad [38]$$
$$\left. + \frac{b^4}{960a^2d^2} + \frac{b^4}{60d^4} - \frac{b^4}{1024a^4}\left(\log\frac{8a}{d} - \frac{187}{60}\right) + \frac{b^6}{168d^6} + \frac{b^8}{360d^8}\right]$$

This expression will give a very accurate value of $\varDelta M$ for two coils not nearer together than their breadth if a is considerably greater than b, the breadth of the coil.

OTHER FORMULÆ.

Himstedt has given several formulæ for different cases of coaxial solenoids. The first[32] is for the case of a short secondary on the

[31] Rosa, this Bulletin, 2, p. 351.

[32] Wied. Annalen, 26, p. 551; 1885.

outside of a long primary. The formula is very complicated and the calculation tedious. By putting the shorter coil inside, the formula of Ròiti or of Searle and Airey may be used to much better advantage.

Himstedt's second expression is for the case of two coaxial solenoids not concentric, the distance between their mean planes having any value; the radius of one is supposed to be considerably smaller than the other. This also is a very complicated formula, involving second and fourth derivatives of expressions containing the elliptic integrals F and E. Gray's general equation is much simpler to calculate. This is not, however, an important case in practice, and we do not therefore give Himstedt's equation. Himstedt's third equation is general and applies to two coaxial solenoids of nearly equal or very different radii, which may or may not be concentric. This expression of Himstedt's consists of four terms, each of which is a somewhat complicated expression involving both complete and incomplete elliptic integrals. A less inconvenient general expression for M in elliptic integrals is given above (35).

For illustrations and tests of the above formulæ see examples 22–24, page 77.

4. THE MUTUAL INDUCTANCE OF A CIRCLE AND A COAXIAL SINGLE LAYER COIL.

LORENZ'S FORMULA.

The problem of finding the mutual inductance of a circle and a coaxial single layer winding was first solved by Lorenz.[33] Assuming that the current was uniformly distributed over the surface of the cylinder, forming a current sheet, he integrated over the length of the cylinder the expression for the mutual inductance of a circular element and the given circle. This expression is an elliptic integral. Lorenz reduced the integrated form to a series and gave the following formula for the mutual inductance of the disk and solenoid of what is now called the Lorenz apparatus. He called it, however, the constant of the apparatus instead of mutual inductance, and

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[33] Wied. Annalen, **25**, p. 1; 1885.
Ouvres Scientifiques, **2**, p. 162.

denoted it by C. It is of course the whole number of lines of magnetic force passing through the disk due to unit current in the surrounding solenoid.

$$\left.\begin{aligned} M&=\frac{\pi q r^2}{d}\left[Q(a_1)+Q(a_2)\right] \\ Q(a)&=2\pi q\sqrt{\frac{a-1}{a}}\left[1+\frac{3}{8}\frac{q^2}{a^2}+\frac{5}{16}\frac{q^4}{a^4}\left(\frac{7}{4}-a\right)\right. \\ &\qquad\left.+\frac{35}{128}\frac{q^6}{a^6}\left(\frac{33}{8}-\frac{9}{2}a+a^2\right)+\ldots\right] \end{aligned}\right\}\quad[39]$$

ρ = radius of the disk, Fig. 1.
r = radius of the solenoid.
$2x$ = length of winding of solenoid.
$q=\rho/r$ = ratio of the two radii.
$d=\frac{2x}{n}$ = distance between centers of successive turns of wire.
$a=\frac{x^2+r^2}{r^2}$

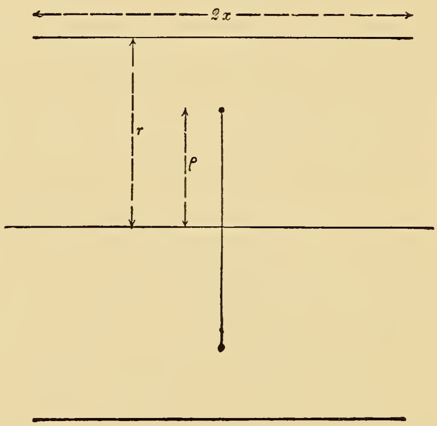

Fig. 15.

If the disk be not exactly in the mean plane of the solenoid, and x_1 be the distance from the plane of the disk to one end of the solenoid and x_2 to the other,

$$a_1=\frac{x_1^2+r^2}{r^2} \qquad a_2=\frac{x_2^2+r^2}{r^2}$$

Then $Q(a_1)$ is found by substituting the values of a_1 in equation (39) above, and $Q(a_2)$ by using the value of a_2 for a in the same equation. The sum of these two quantities multiplied by $\frac{\pi q r^2}{d}$ gives the constant of the instrument; that is, the mutual inductance sought.

As Lorenz gave the expression for the general term of (39), his equation can be extended. The following is the general term:

$$Q(a)=2\pi\sum_{m=0}^{m=\infty} q^{2m+1}\frac{1.3\ldots.2m-1}{2.4\ldots.2m}\cdot\frac{1}{1.2\ldots.(m+1)}\cdot\frac{d^m}{da^m}\left(\frac{a-1}{a}\right)^{m+\frac{1}{2}}$$

JONES'S FORMULÆ.

Two solutions of the above problem were given by Jones,[34] both in terms of elliptic integrals. The current was considered to flow not in a current sheet, but along a spiral winding or helix. The first solution was in the form of a series, convergent only when O_1A, Fig. 16, is less than the difference in the radii of inner and outer coils; that is, when O_1A is less than $A-a$. As this is a serious limitation, and the formula is a laborious one to use, it is not here given. The second solution is exact and general, and is in terms of elliptic integrals of all three kinds. The second formula is as follows:

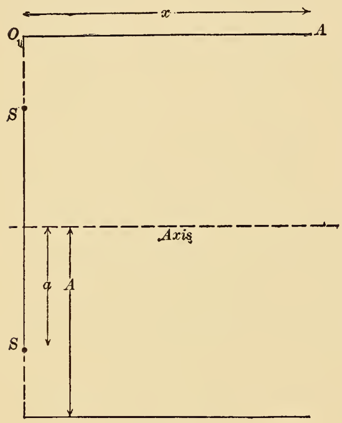

Fig. 16.

$$M_\theta = \Theta\,(A+a)\,ck\left\{\frac{F-E}{k^2}+\frac{c'^2}{c^2}(F-\Pi)\right\} \qquad [40]$$

M_θ = mutual inductance of helix O_1A, Fig. 16, with respect to the disk S in the plane of one end.

$\Theta = 2\pi n$, $1/n$ = pitch of winding, Θ = whole angle of winding.

F, E, and Π are the complete elliptic integrals to modulus k, where

$$k^2=\frac{4Aa}{(A+a)^2+x^2}=\sin^2\gamma,\quad c^2=\frac{4Aa}{(A+a)^2}\quad c'^2=1-c^2.$$

Π, the complete elliptic integral of the third kind, can be expressed in terms of incomplete integrals of the first and second kinds, and the value of M_θ can then be calculated by the help of Legendre's tables; see example 27. The calculation is, however, extremely tedious, especially when the value is to be determined with high precision.

Campbell has given Jones's formula (40) a slightly different form,[35] somewhat more convenient in calculation, as follows:

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[34] J. V. Jones, Proc. Roy. Soc., **63**, p. 198; 1898. Also, Trans. Roy. Soc., **182**, A; 1891. Jones's first formula was given in Phil. Mag., **27**, p. 61; 1889.

[35] A. Campbell, Proc. Roy. Soc., A, **79**, p. 428; 1907. There is a misprint in the formula as given in Campbell's paper. It was, however, used correctly in the numerical calculations given in the paper.

$$M=2\pi n_1 n_2(A+a)\left\{\frac{c}{k}(F-E)+\frac{A-a}{b}\psi\right\} \qquad [41]$$

where n_1 is the same as n above, the number of turns per cm on the solenoid, n_2 is the number of turns in the secondary coil (in the above case it was taken as one), A is the greater and a the less of the two radii (in the above case A was the radius of the solenoid and a of the circle within), and

$$\psi=F(k)E(k',\beta)-$$

$$[F(k)-E(k)]F(k',\beta)-\frac{\pi}{2}$$

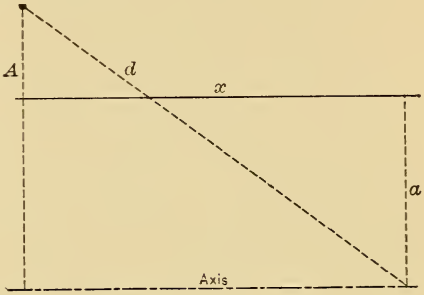


Fig. 17.

where $F(k)$ and $E(k)$ are the complete elliptic integrals to modulus k, and $F(k',\beta)$ and $E(k',\beta)$ are the incomplete elliptic integrals to modulus k' and amplitude β; $k'=\cos\gamma$, $\beta=c'/k'$; k, c, and c' are given above. If a secondary circle or coil has a radius greater than that of the solenoid, the same formula can be used if A is taken for the radius of the larger secondary and a is the radius of the solenoid.

ROSA'S FORMULA.[36]

The following formula gives the mutual inductance of a single layer coil of length x and a coaxial circle of radius a in the plane of one end of the coil, as shown in Fig. 16. It is the same quantity represented by M of equations (39) and (41) and M_θ of (40).

$$M_{OA}=\frac{2\pi^2a^2N}{d}\left[1+\frac{3}{8}\frac{a^2A^2}{d^4}+\frac{5}{64}\frac{a^4A^4}{d^8}X_2+\frac{35}{512}\frac{a^6A^6}{d^{12}}X_4+\frac{63}{1024}\frac{a^8A^8}{d^{16}}X_6\right.$$

$$\left.+\frac{231}{4096}\frac{a^{10}A^{10}}{d^{20}}X_8+\frac{429}{16384}\frac{a^{12}A^{12}}{d^{24}}X_{10}+\ \ldots\right] \qquad [42]$$

$$X_2=3-4\frac{x^2}{A^2}$$

$$X_4=\frac{5}{2}-10\frac{x^2}{A^2}+4\frac{x^4}{A^4}$$

[36] This Bulletin, 3, p. 209; 1907.

$$X_6=\frac{35}{16}-\frac{35}{2}\frac{x^2}{A^2}+21\frac{x^4}{A^4}-4\frac{x^6}{A^6}$$

$$X_8=\frac{63}{32}-\frac{105}{4}\frac{x^2}{A^2}+63\frac{x^4}{A^4}-36\frac{x^6}{A^6}+4\frac{x^8}{A^8}$$

$$X_{10}=\frac{231}{128}-\frac{1155}{32}\frac{x^2}{A^2}+\frac{1155}{8}\frac{x^4}{A^4}-165\frac{x^6}{A^6}+55\frac{x^8}{A^8}-4\frac{x^{10}}{A^{10}}$$

a = radius of disk or circle S, Fig. 2.
A = radius of the solenoid.
x = length O_1A of one end of the solenoid.
$d=\sqrt{x^2+A^2}$ = half the diagonal of the solenoid.
N is the whole number of turns of wire in the length x.

This formula is very easy to use in numerical calculation, notwithstanding it looks somewhat elaborate. The logarithm of $\frac{a^2A^2}{d^4}$, multiplied by 2, 3, 4, etc., gives the logarithm of the corresponding factor in each of the other terms. Similarly, the various terms X_2, X_4 etc., contain only powers of $\frac{x^2}{A^2}$ besides the numerical coefficients. It is hence a far simpler matter to compute M with high precision by this formula than by Jones's formula, the latter containing as it does elliptic integrals of all three kinds and involving the tedius interpolations for incomplete elliptic integrals.

If the secondary circle has a larger radius than the solenoid, A will be the radius of the circle and a the radius of solenoid. In every case A is the greater and a the less of the two radii, and d is $\sqrt{A^2+x^2}$.

Equation (42) may be written

$$M=\frac{2\pi^2a^2n_1x}{d}S$$

where n_1 is the number of turns of wire per cm, x is the length of the coil, Fig. 16, and S is the value of the quantity in brackets in (42), which is always somewhat greater than unity. This may also be put as follows:

$$M=a^2n_1\left(\frac{2\pi^2x}{d}\right)S=a^2n_1RS$$

or, [43]

$$M=a^2n_1K$$

The quantity R depends on x/d; that is, only upon the shape of the solenoid. S depends upon x/A, a/A, and A/d; that is, upon the relative sizes of the inner circle and the solenoid and the shape of the solenoid. If we have the value of RS, or K of equation (43) for a given solenoid and circle, we can get M by multiplying by a^2n_1, and for any other system of similar shape but different size by multiplying the same value of K by a^2n_1. The values of the constant K for various values of a/A and x/A are given in Table III, page 113.

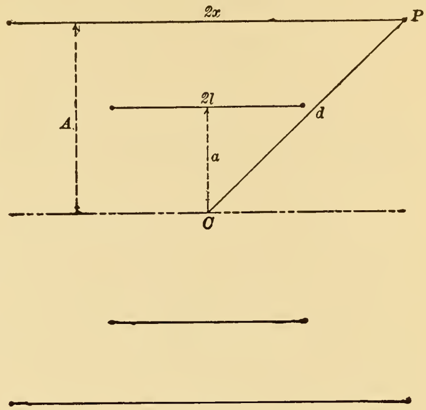


Fig. 18.

If the disk or circle be in the center of a solenoid of length $2x$ (Fig. 18), the value of M is of course double that given by using x. If it be not quite in the center, the value of M must be calculated for each end separately.

For illustrations and tests of the above formulæ see examples 25, 26, and 27, page 83.

5. THE SELF-INDUCTANCE OF A CIRCULAR RING OF CIRCULAR SECTION.

KIRCHHOFF'S FORMULA.

The formula for the self-inductance of a circle was first given by Kirchhoff[37] in the following form:

$$L=2l\left\{\log\frac{l}{\rho}-1.508\right\} \qquad [44]$$

where l is the circumference of the circular conductor and ρ is the radius of its cross section. This is equivalent to the following:

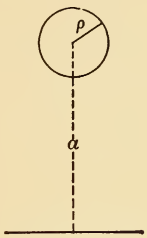
$$L=4\pi a\left\{\log\frac{8a}{\rho}-1.75\right\} \qquad [45]$$

[37] Pogg. Annalen, **121**, p. 551; 1864.

a being the radius of the circle, Fig. 19. These formulæ are approximate, being more nearly correct as the ratio ρ/a is smaller.

MAXWELL'S FORMULA.

A more accurate expression, obtained by means of Maxwell's principle of the geometrical mean distance, is the following:



$$L=4\pi a\left\{\left(1+\frac{3}{16}\frac{R^2}{a^2}\right)\log\frac{8a}{R}-\left(2+\frac{R^2}{16a^2}\right)\right\} \qquad [46]$$

Substituting in this equation the value of the geometrical mean distance for a circular area, $R=\rho e^{-\frac{1}{4}}=.7788\rho$, we obtain[38]

$$L=4\pi a\left\{\left(1+0.1137\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-.0095\frac{\rho^2}{a^2}-1.75\right\} \qquad [47]$$


Fig. 19.

This is a very accurate formula for circles in which the radius of section ρ is very small in comparison with the radius a of the circle. The geometrical mean distance R has, however, been computed on the supposition of a linear conductor, and can only be applied to circles of relatively small value of ρ/a, and the square of the geometrical mean distance is used for the arithmetical mean square distance in the second order terms. We must therefore expect an appreciable error in formula (47) when the ratio ρ/a is not very small. Formulæ (44), (45), and (47) have been deduced on the supposition of a uniform distribution of the current over the cross section of the ring.

If the ring is a hollow circular thin tube, or if the current in the ring is alternating and of extremely high frequency, so that it can be regarded as flowing on the surface of the ring, the geometrical mean distance for the section would be the radius ρ, and we should have instead of (47) the following by substituting $R=\rho$,

$$L=4\pi a\left\{\left(1+\frac{3}{16}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-\frac{\rho^2}{16a^2}-2\right\} \qquad [48]$$

[38] Wied. Annalen, **53**, p. 928; 1894.

In the case of solid rings carrying alternating currents of moderate frequency the value of L would be somewhere between the values given by (47) and (48).

WIEN'S FORMULÆ.

Max Wien[38] has given the most accurate formula for the self-inductance of a circle, as follows:

$$L=4\pi a\left\{\left(1+\frac{1}{8}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-.0083\frac{\rho^2}{a^2}-1.75\right\} \qquad [49]$$

It will be noticed that the formula differs very slightly from (47). Neglecting the terms in ρ^2/a^2 we obtain from either (47) or (49) Kirchhoff's approximate formula.

If the current be not distributed uniformly over the section of the wire, but the current density at any point is proportional to the distance from the axis of the ring, Wien's formula for the self-inductance is

$$L=4\pi a\left\{\left(1+\frac{3}{8}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-.092\frac{\rho^2}{a^2}-1.75\right\} \qquad [50]$$

which differs very slightly from (49).

This would apply to the case of a ring revolving about a diameter in a uniform magnetic field.

As would be expected, (50) gives a greater value than (49).

RAYLEIGH AND NIVEN'S FORMULA.

Rayleigh and Niven gave[39] the following formula for a circular coil of n turns and of circular section:[40]

$$L=4\pi n^2 a\left\{\left(1+\frac{\rho^2}{8a^2}\right)\log\frac{8a}{\rho}+\frac{\rho^2}{24a^2}-1.75\right\} \qquad [51]$$

When $n=1$, this will be the self-inductance of a single circular ring. It agrees with Wien's, except as to one term, which is

$$+\frac{\rho^2}{24a^2} \text{ instead of } -0.0083\frac{\rho^2}{a^2}.$$

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[38] Wied. Annalen, **53**, p. 928; 1894.

[39] Rayleigh's Collected Papers, Vol. II, p. 15.

[40] Neglecting the correction for effect of insulation and shape of section of the separate wires.

If used for a coil of more than one turn, the expression for L (whether obtained from (51) or from one of the preceding more accurate expressions) must be corrected for the space occupied by the insulation between the wires and for the shape of the section.[41]

J. J. THOMSON'S FORMULA FOR RING OF ELLIPTICAL SECTION.

If the circular ring has an elliptical section the approximate formula for its self-inductance (corresponding to (45) for a circular section) is[42]

$$L=4\pi a\left\{\log\frac{16a}{\alpha+\beta}-1.75\right\} \qquad [52]$$

where α and β are the semi-axes of the ellipse, and a is the mean radius of the circular ring.

The formulæ of Minchin,[43] Hicks,[44] and Blathy[45] we have elsewhere[46] shown to be incorrect, and hence they are not here given.

6. THE SELF-INDUCTANCE OF A SINGLE LAYER COIL OR SOLENOID.

The following approximate formula for the self-inductance of a long solenoid is often given:

$$L=4\pi^2a^2n_1^2l \qquad [53]$$

where a is the mean radius, n_1 is the number of turns of wire per cm, and l is the length, supposed great in comparison with a. There is a considerable error in this formula, due to the end effect, but the variations in L due to changes in l are almost exactly proportional to the changes in l, and hence this formula may be used for calculating the corresponding variations in L.

RAYLEIGH AND NIVEN'S FORMULÆ.

The following formula[47] for the self-inductance of a single layer winding on a solenoid is very accurate when the length b is small compared with the radius a, Fig. 20:

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[41] See Rosa, this Bulletin, **3**, p. 1; 1907.
[42] J. J. Thomson, Phil. Mag., **23**, p. 384; 1886.
[43] Phil. Mag., **37**, p. 300; 1894.
[44] Phil. Mag., **38**, p. 456; 1894.
[45] London Electrician, **24**, p. 630; April 25, 1890.
[46] This Bulletin, **4**, p. 149; 1907.
[47] Proc. Roy. Soc., **32**, pp. 104–141; 1881. Rayleigh's Collected Papers, **2**, p. 15.

$$L_s = 4\pi a n^2 \left\{ \log \frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2}\left(\log \frac{8a}{b} + \frac{1}{4}\right)\right\} \qquad [54]$$

n is the whole number of turns of wire on the coil, and the radius is measured to the center of the wire. The length b is the *mean over-all length including the insulation on the first and last wires* if the coil is wound closely with insulated wire. See also page 41.

The self-inductance L_s is, however, not the actual self-inductance of the coil, but the current sheet value; that is, it is the value of the self-inductance if the winding were of infinitely thin tape, so that the current would cover the entire length b. To get the actual self-inductance L for any given case one must correct L_s by formula (59) below. The same remark applies to all the formulæ in this section for L_s. The approximate formula (53) is too rough to make it worth while to apply such a correction.

For a coil in which the axial dimension b is zero and the radial depth is c, the following current sheet formula of Rayleigh and Niven gives the self-inductance:

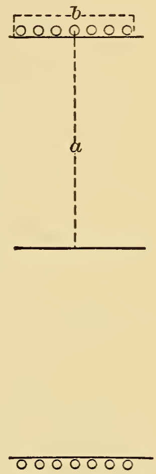

Fig. 20.

$$L_s = 4\pi a n^2 \left\{ \log \frac{8a}{c} - \frac{1}{2} + \frac{c^2}{96a^2}\left(\log \frac{8a}{c} + \frac{43}{12}\right)\right\} \qquad [55]$$

This is not an important case in practice.

Formulæ (54) and (55) may be obtained from (67) by making first $c=0$ and then $b=0$.

COFFIN'S FORMULA.

Coffin[48] has extended formula (54) so that it is very accurate for coils of length as great as the radius, and sufficiently accurate for most purposes for coils considerably longer than this.

$$L_s = 4\pi a n^2 \left\{ \log \frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2}\left(\log \frac{8a}{b} + \frac{1}{4}\right) - \frac{1}{1024}\frac{b^4}{a^4}\left(\log \frac{8a}{b} - \frac{2}{3}\right) \right.$$

$$\left. + \frac{10}{131072}\frac{b^6}{a^6}\left(\log \frac{8a}{b} - \frac{109}{120}\right) - \frac{35}{4194304}\frac{b^8}{a^8}\left(\log \frac{8a}{b} - \frac{431}{420}\right)\right\} \qquad [56]$$

[48] This Bulletin, 2, p. 113; 1906.

LORENZ'S FORMULA.

Lorenz first gave[49] an exact formula for the self-inductance of a single layer solenoid. It is, like the others, a current sheet formula, and requires correction by (59) for a winding of wire, but applies to a solenoid of any length. Changing the notation slightly Lorenz's formula as originally given is as follows:

$$L_s = \frac{32}{3}\frac{\pi n^2 a^3}{b^2}\left\{\frac{2k^2-1}{k^3}E + \frac{1-k^2}{k^3}F - 1\right\} \qquad [57]$$

where $k = \frac{4a^2}{4a^2+b^2}$ and F and E are complete elliptic integrals of the first and second kind to modulus k, and a, b, and n are the radius,

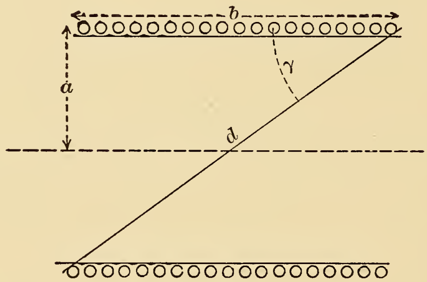

Fig. 21.

length, and whole number of turns of wire, respectively. By simple substitutions the formula may be put into the following form, where d is the diagonal of the solenoid $= \sqrt{4a^2+b^2}$;

$$L_s = \frac{4\pi n^2}{3b^2}\left\{d\,(4a^2-b^2)E + db^2F - 8a^3\right\} \qquad [58]$$

Coffin derived[50] an expression for L in elliptic integrals which is equivalent to (58), and also obtained (58) from an expression[51] attributed to Kirchhoff.

Formula (58) may be written

[49] Wied. Annal., 7, p. 161; 1879. Oeuvres Scientifiques de L. Lorenz, Tome 2, 1, p. 196.
[50] This Bulletin, 2, p. 123, equation (31).
[51] This Bulletin, 2, p. 127, equation (36). The notation is slightly different.

$$L_s=an^2\left[\frac{8\pi}{3}\left\{\sqrt{1+\frac{b^2}{4a^2}}\left(\frac{4a^2}{b^2}-1\right)E+\sqrt{1+\frac{b^2}{4a^2}}F-\frac{4a^2}{b^2}\right\}\right]$$

or $L_s=an^2Q$ [58]*a*

where a is the radius of the solenoid, n is the whole number of turns on the coil, and Q is the function of $\frac{2a}{b}$ $(=\tan\gamma)$ contained in the square brackets. We have calculated Q for various values of $\tan\gamma$ from 0.2 to 4.0 and given them in Table IV, p. 114. This table will be found useful in calculating L_s for solenoids when $\tan\gamma$ has one of the values given in the table, as all calculation of elliptic integrals is avoided. In problems where the length and diameter can be chosen at will, as in the designing of apparatus, this method of calculating L will be most frequently useful. The values of the constant Q given in the table have been computed with great care, so that they give very accurate values of L_s, for long as well as short solenoids.

In calculating the value of L_s by means of formula (54), (56), (58), or (58*a*) one should use for the length b the *over-all length including the insulation* ($A\ B$, Fig. 22, and not $a\ b$) for a close winding of insulated wire, or *n times the pitch* for a uniform winding of bare or covered wire, which is, of course, the same as the length from center to center of $n+1$ turns. The radius a is the mean radius to the center of the wire. The same method of taking the breadth and depth b and c applies in the formulæ of section 7. See also remarks under example 24.

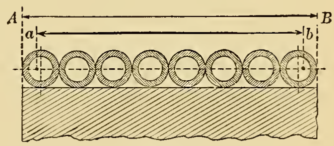

Fig. 22.

ROSA'S CORRECTION FORMULA.

Rosa has shown[52] that the above formulæ (54 to 58) apply accurately only to a winding of infinitely thin strip which completely covers the solenoid (the successive turns being supposed to meet at the edges without making electrical contact) and so realizing the uniform distribution of current over the cylindrical surface which has been assumed in the derivation of all the formulæ. A winding of insulated wire or of bare wire in a screw thread may have a greater or less self-inductance than that given by the current sheet formulæ above according to the ratio of the diameter of the wire to

[52] This Bulletin, 2, pp. 161–187; 1906.

the pitch of the winding. Putting L for the actual self-inductance of a winding and L_s for the current sheet value given by one of the above formulæ,

$$L = L_s - \Delta L$$

The correction ΔL is given by the following expression:

$$\Delta L = 4\pi a n\ [A + B] \qquad [59]$$

where as above a is the radius, n the whole number of turns of wire and A and B are constants given in Tables VII and VIII, pp. 116 and 117.

The correction term A depends on the size of the (bare) wire (of diameter d) as compared with the pitch D of the winding; that is, on the value of the ratio d/D. For values of d/D less than 0.58, A is negative, and in such cases when the numerical values of A are greater than the value of B, which is always positive, the correction ΔL will be negative, and hence L will be *greater* than L_s. See examples 32 and 33.

THE SUMMATION FORMULA FOR L.[53]

If we have a single layer winding on a cylinder the self-inductance is equal to the sum of the self-inductances of the separate turns plus the sum of the mutual inductances of each wire on all the others. Thus if there are n turns

$$L = nL_1 + 2(n-1)M_{12} + 2(n-2)M_{13} + 2(n-3)M_{14} + \ \ldots\ 2M_{1n} \qquad [60]$$

where L_1 is the self-inductance of a single turn, M_{12} is the mutual inductance of the first and second turns or any two adjacent turns, M_{13} is the mutual inductance of the first and third or of any two turns separated by one, etc., and M_{1n} is the mutual inductance of the first and last turns. For a coil of four turns this becomes

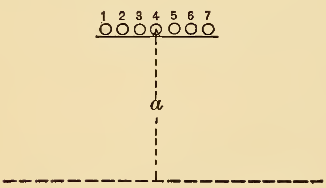

Fig. 23.

$$L = 4L_1 + 6M_{12} + 4M_{13} + 2M_{14}$$

L_1 should be calculated by formula (49) or any formula for circles, and M_{12}, etc., by (9) or (10). When the number of turns on the coil is small formula (60) is very convenient, and gives very accurate results.

[53] Kirchhoff, Gesammelte Abhandlungen, p. 177.

STRASSER'S FORMULA.

Strasser[54] has derived a formula for the self-inductance of a single layer coil of few turns from (60) by substituting for L_1 its value as given by formula (45) and for the various M's their values as given by (9). Strasser's formula with slight correction and some changes in notation is as follows:[55]

$$L=4\pi a\left[n\left(\log\frac{8a}{\rho}-1.75\right)+n(n-1)\left(\log\frac{8a}{d}-2\right)-A\right.$$

$$\left.+\frac{d^2}{8a^2}\left\{\left(3\log\frac{8a}{d}-1\right)\left(\frac{n^2(n^2-1)}{12}\right)-B\right\}\right] \qquad [61]$$

where n is the whole number of turns, d is the pitch, or distance between the centers of two adjacent turns, a is the mean radius of the coil, ρ is the radius of the section of the wire, and A and B are constants given by Table V, page 115, for values of n up to 30. For coils of a larger number of turns (or indeed any number of turns) the value of L can be accurately calculated by (69) and (72) or by (58) and (59).

SELF-INDUCTANCE OF TOROIDAL COIL OF RECTANGULAR SECTION.

The first approximation to the self-inductance of a toroidal coil (that is, a circular solenoid) of rectangular section, wound with a single layer of n turns of wire is

$$L_s=2n^2h\log\frac{r_2}{r_1} \qquad [62]$$

where h is the axial depth of the coil, and r_1 and r_2 are the inner and outer radii of the ring, Fig. 24. Formula (62) is exact for a toroidal core enveloped by a current sheet, or for a winding of n turns of infinitely thin tape covering the core completely, the core within the current sheet being h cm in axial height and (r_2-r_1) cm in radial breadth.

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[54] Wied. Annal., **17**, p. 763; 1905.

[55] Strasser uses the formula for L as: $L=4\pi a\left(\log\frac{a}{\rho}+0.333\right)$. This is not quite correct. It should be

$$L_1=4\pi a\left(\log\frac{8a}{\rho}-1.75\right)=4\pi a\log\left(\frac{a}{\rho}-1.75+\log_e 8\right)=4\pi a\left(\log\frac{a}{\rho}+0.32944\right).$$

When the core is wound with round insulated wire, the self-inductance is affected by those lines of force within the cross section of the wire itself, and by those linked with each separate turn of wire in addition to those running through the core. Rosa has shown[56] that the total self-inductance may be more or less than the current sheet value given by (62) according to the size of the wire and the pitch of the winding. In every case, however, the correct value of the self-inductance is derived from the current sheet value

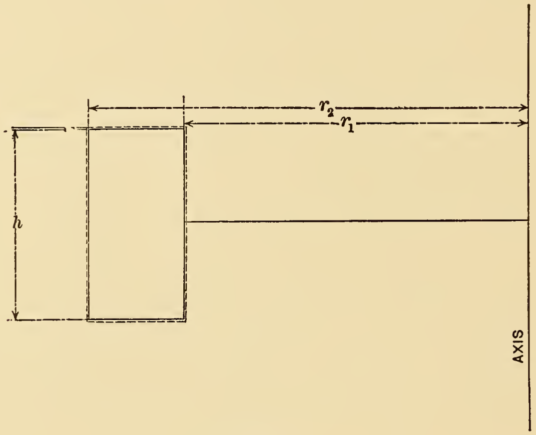

Fig. 24.

L_s by subtracting a correction term ΔL, which is equal to twice the length of the wire multiplied by the sum of two quantities A and B. Thus

$$L = L_s - 2nl(A+B) \qquad [63]$$

where n is the whole number of turns in the winding, l is the length of one turn, A is a quantity, depending on the diameter of the wire and the pitch of the winding, given in Table VII, and B is 0.332. When A is negative and greater than B, L is greater than L_s. This occurs when the pitch of the winding is more than 2.5 times the diameter of the (uncovered) wire.

Fröhlich's formula[57] based on the assumption that a winding of round wires is equivalent to a thick current sheet has been shown to be incorrect.[58]

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[56] This Bulletin, **4**, p. 141; 1907. [57] Wied. Annal., **63**, p. 142; 1897.
[58] This Bulletin, **4**, p. 141; 1907.

7. THE SELF-INDUCTANCE OF A CIRCULAR COIL OF RECTANGULAR SECTION.

MAXWELL'S APPROXIMATE FORMULA.

Maxwell first gave[59] an approximate formula for the important case of a circular coil or conductor of rectangular section, Fig. 25, as follows:

$$L=4\pi an^2\left(\log\frac{8a}{R}-2\right) \qquad [64]$$

where R is the geometrical mean distance of the cross section of the coil or conductor. The current is supposed uniformly distributed over this section.

The value of R for any given shape of rectangular section is given by (103). Its value for several particular cases is given in the table of page 60. It is very nearly proportional to the perimeter of the rectangle and approximately equal to 0.2235 $(\alpha+\beta)$ where α and β are the length and breadth of the rectangle.

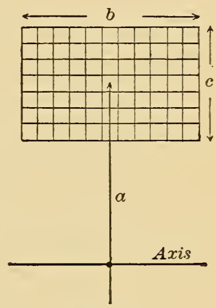

Fig. 25.

Formula (64) is derived from (8) by putting R, the geometrical mean distance of the area of the section of the coil from itself, in place of r, the distance between two circles. If we use (9) instead of (8) for this purpose, we shall have a closer approximation to the value of L. Thus,

$$L=4\pi an^2\left\{\log\frac{8a}{R}\cdot\left(1+\frac{3R^2}{16a^2}\right)-\left(2+\frac{R^2}{16a^2}\right)\right\} \qquad [65]$$

We have placed R^2 in place of d^2 in the second order terms, which is of course not strictly correct, as we should use an arithmetical mean square distance instead of a geometrical mean square distance. (See p. 63.) Nevertheless, (65) is a much closer approximation than (64).

PERRY'S APPROXIMATE FORMULA.

Professor Perry has given[60] the following empirical expression for the self-inductance of a short circular coil of rectangular section:

[59] Elect. and Mag., §706.

[60] John Perry, Phil. Mag., **30**, p. 223; 1890.

$$L=\frac{4\pi n^2a^2}{0.2317a+0.44b+0.39c} \qquad [66]$$

in which n is the whole number of turns of wire, a the mean radius, b the axial breadth, c the radial depth. As in all the formulæ of this paper, the dimensions are in centimeters and the value of L is in centimeters. This formula gives a good approximation to L as long as b and c are small compared with a.

WEINSTEIN'S FORMULA.

Maxwell's more accurate expression for the self-inductance of a circular coil of rectangular section[61] was not quite correct. The investigation was repeated by Weinstein,[62] who gave the following formula:

$$L_u=4\pi an^2\,(\lambda+\mu)$$

where

$$\lambda=\log\frac{8a}{c}+\frac{1}{12}-\frac{\pi x}{3}-\frac{1}{2}\log\,(1+x^2)+\frac{1}{12x^2}\log\,(1+x^2)$$
$$+\frac{1}{12}x^2\log\left(1+\frac{1}{x^2}\right)+\frac{2}{3}\left(x-\frac{1}{x}\right)\tan^{-1}x,$$

$$\mu=\frac{c^2}{96a^2}\left[\left(\log\frac{8a}{c}-\frac{1}{2}\log\,(1+x^2)\right)(1+3x^2)+3.45x^2+\frac{221}{60}\right.$$
$$\left.-1.6\pi x^3+3.2x^3\tan^{-1}x-\frac{1}{10}\frac{1}{x^2}\log(1+x^2)+\frac{1}{2}x^4\log\left(1+\frac{1}{x^2}\right)\right] \qquad [67]$$

b and c are the breadth and depth of the coil and $x=\frac{b}{c}$.

Weinstein's formula for the case of a square section, where $b=c$, reduces to the following simpler expression:

$$L_u=4\pi an^2\left\{\left(1+\frac{c^2}{24a^2}\right)\log\frac{8a}{c}+.03657\frac{c^2}{a^2}-1.194914\right\} \qquad [68]$$

This is a very accurate formula as long as c/a is a small quantity. The current is supposed distributed uniformly over the section of the coil, and hence for a winding of round insulated wire correction must be made by formula (72).

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[61] Phil. Trans., 1865, and collected works.
[62] Wied. Annal., **21**, p. 329; 1884.

STEFAN'S FORMULA.

Stefan[63] simplified Weinstein's expression (67) by collecting together terms depending on the ratio of b to c and computing two short tables of constants y_1 and y_2. His formula is as follows:

$$L=4\pi a n^2\left\{\left(1+\frac{3b^2+c^2}{96a^2}\right)\log\frac{8a}{\sqrt{b^2+c^2}}-y_1+\frac{b^2}{16a^2}y_2\right\} \qquad [69]$$

The values of y_1 and y_2 are given in Table VI, page 115, as functions of $x=b/c$ or c/b; that is, x is the ratio of the breadth to the depth of the section, or vice versa, being always less than unity.

For the method of taking the dimensions b and c of the cross section see p. 99, section 6. Also example 40, p. 99.

LONG COIL OF RECTANGULAR SECTION; I. E., SOLENOID OF MORE THAN ONE LAYER.

ROSA'S METHOD.

When the coil is so long that the formula of Stefan is no longer accurate, the self-inductance may be accurately calculated by a method given by Rosa.[64]

In Figs. 26, 27, and 28 are shown three coils, having the same length and mean radius. The first is a single winding of thin tape and the self-inductance, calculated by a current sheet formula, is L_s. The second is a single layer of wire of square section (length b, depth c, and b/c turns) and its self-inductance is L_u, the current being supposed uniformly distributed over the area of the square conductors. The third is a winding of round insulated wire of length b, depth c, and any number of layers, and its self-inductance is L. These different self-inductances are related as follows:

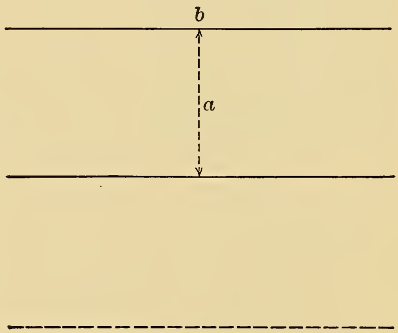

Fig. 26.

$$L_s-\Delta_1 L=L_u$$
$$L_u+\Delta_2 L=L$$
$$\therefore\ L=L_s-\Delta_1 L+\Delta_2 L$$

L_s is calculated by any current sheet formula as (54), (56), (57), or

[63] Wied. Annal., **22**, p. 113; 1884.

[64] This Bulletin, **4**, p. 369; 1907.

(58). The correction $\Delta_1 L$ for the depth of the coil is given by the following formula:

$$\Delta_1 L = 4\pi a n' \ [A_s + B_s] \qquad [70]$$

This formula has the same form as (59), but some of the quantities have a different meaning; a is the mean radius as before, n' is b/c, the number of square conductors in the length b, Fig. 26, and A_s and B_s are given in Tables IX and X.

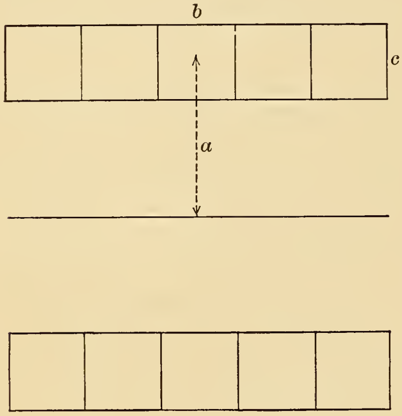

Fig. 27.

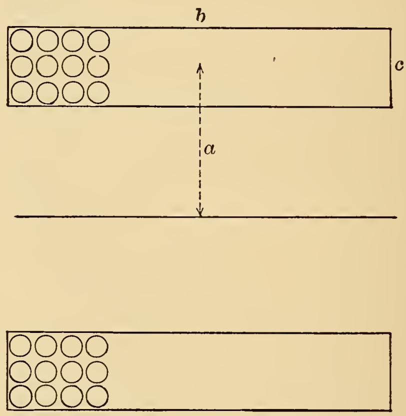

Fig. 28.

The correction $\Delta_2 L$ is calculated in precisely the same way as for a short coil, as described below, formula (72). The above formula for $\Delta_1 L$ gives a very accurate value of the correction to be applied to L_s to obtain L_u, and permits a test to be made for the error of Stefan's formula when applied to longer coils than the latter is intended for. Such a calculation shows that for a coil as long as its diameter Stefan's formula (and Weinstein's also, of course) is 1 per cent in error, giving too large a value.

COHEN'S APPROXIMATE FORMULA.

Cohen has given the following approximate formula[65] for the self-inductance of a long coil or solenoid of several layers:

$$L = 4\pi^2 n^2 m \left\{ \frac{2a_0^4 + a_0^2 l^2}{\sqrt{4a_0^2 + l^2}} - \frac{8a_0^3}{3\pi} \right\} + 8\pi^2 n^2 \Bigg\{ [(m-1)a_1^2 + (m-2)a_2^2 + \dots] \left(\sqrt{a_1^2 + l^2} - \frac{7}{8} a_1 \right) + \frac{1}{2} [m(m-1)a_1^2 + \dots] \left(\frac{a_1 \delta a}{\sqrt{a_1^2 + l^2}} - \delta a \right) \Bigg\} \qquad [71]$$

[65] This Bulletin, **4**, p. 389; 1907.

where a_0 is the mean radius of the solenoid, $a_1, a_2, \ldots a_m$ are the mean radii of the various layers, m is the number of layers and δa is the distance between centers for any two consecutive layers.

For long solenoids, where the length is, say, four times the diameter, we can neglect the last term in equation (71).

This formula is sufficiently accurate for most purposes; it will give results accurate to within one half of one per cent even for short solenoids, where the length is only twice the diameter.

MAXWELL'S CORRECTION FORMULA.[66]

GIVING THE VALUE OF $\Delta_2 L$.

Maxwell has shown that when a coil of rectangular section (Fig. 28) is wound with round insulated wire and the self-inductance is calculated by a formula in which the current is assumed to be distributed uniformly over the section, as in Weinstein's and Stefan's, the calculated value L_u is subject to three corrections, each of which tends to increase the calculated value of the self-inductance. Thus:

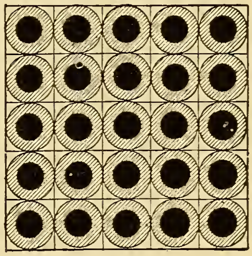

Fig. 29.

$$L = L_u + \Delta_2 L$$

and $$\Delta_2 L = 4\pi a n \left\{ \log_e \frac{D}{d} + 0.13806 + E \right\} \quad [72]$$

Maxwell showed that the first term takes account of the effect of the insulation, d and D being the diameters of the bare and covered wire respectively. The second correction term (0.13806) reduces from a square section to a circular section for the conductor. The third correction term E takes account of the differences in the mutual inductances of the separate turns of wire on one another when the wire has a round section from what the mutual inductances would be if the wire were of square section and no space was occupied by insulation. This term was stated by Maxwell to be equal to $-.01971$; it was subsequently stated by Stefan to be equal to $+.01688$. Rosa has shown[67] that its value is variable, depending on the number of turns of wire in the coil and the shape of the cross section of the latter, and has given the values of E for a number of particular cases.

[66] Elect. and Mag., vol. 2, § 693.

[67] This Bulletin, 3, p. 37.

From the following table one can interpolate for E for any particular case not included in the table.

Summary of the values of E found for the various cases considered:

2 turns		$E=$	.006528
3 "	(one layer)	$E=$	.009045
4 "	(two layers)	$E=$	.01691
4 "	(one layer)	$E=$	.01035
8 "	(two layers)	$E=$	.01335
10 "	(one layer)	$E=$	.01276
20 "	(one layer)	$E=$	.01357
16 "	(four layers)	$E=$	.01512
100 "	(ten layers)	$E=$	.01713
400 "	(20×20)	$E=$	.01764
1,000 "	(50×20)	$E=$	.01778
Infinite number of turns		$E=$	.01806

8. SELF AND MUTUAL INDUCTANCE OF LINEAR CONDUCTORS.[68]

SELF-INDUCTANCE OF A STRAIGHT CYLINDRICAL WIRE.

The self-inductance of a length l of straight cylindrical wire of radius ρ is

$$L=2\left[l\log\frac{l+\sqrt{l^2+\rho^2}}{\rho}-\sqrt{l^2+\rho^2}+\frac{l}{4}+\rho\right] \qquad [73]$$

$$=2l\left[\log\frac{2l}{\rho}-\frac{3}{4}\right]\text{ approximately.} \qquad [74]$$

Where the permeability of the wire is μ, and that of the medium outside is unity,

$$L=2l\left[\log\frac{2l}{\rho}-1+\frac{\mu}{4}\right] \qquad [75]$$

This formula was originally given by Neumann.

For a straight cylindrical tube of infinitesimal thickness, or for alternating currents of great frequency, when there is no magnetic field within the wire, the self-inductance is

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[68] See paper by E. B. Rosa, this Bulletin, **4**, p. 301; 1907.

$$L=2l\left[\log\frac{2l}{\rho}-1\right] \qquad [76]$$

This is obtained by subtracting from (74) $l/2$ or from (75) $\mu\, l/2$, the magnetic flux within the conductor due to unit current.

THE MUTUAL INDUCTANCE OF TWO PARALLEL WIRES.

The mutual inductance of two parallel wires of length l, radius ρ, and distance apart d is the number of lines of force due to unit current in one which cut the other when the current disappears. This is

$$M=2\left[l\log\frac{l+\sqrt{l^2+d^2}}{d}-\sqrt{l^2+d^2}+d\right] \qquad [77]$$

$$\therefore M=2l\left[\log\frac{2l}{d}-1+\frac{d}{l}\right] \text{ approximately} \qquad [78]$$

when the length l is great in comparison with d.

Equation (77), which is an exact expression when the wires have no appreciable cross section, is not an exact expression for the mutual inductance of two parallel cylindrical wires, but is not appreciably in error even when the section is large and d is small if l is great compared with d.

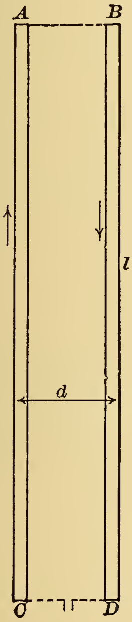

Fig. 30.

THE SELF-INDUCTANCE OF A RETURN CIRCUIT.

If we have a return circuit of two parallel wires each of length l (the current then flowing in opposite direction in the two wires) the self-inductance of the circuit, neglecting the effect of the end connections shown by dotted lines, Fig. 30, will be very approximately

$$L=4l\left[\log\frac{d}{\rho}+\frac{\mu}{4}-\frac{d}{l}\right] \qquad [79]$$

In the usual case of $\mu=1$ this will be, when d/l is small

$$L=4l\left[\log\frac{d}{\rho}+\frac{1}{4}\right] \qquad [80]$$

If the end effect is large, as when the wires are relatively far apart, use the expression for the self-inductance of a rectangle below (86); or, better, add to the value of (79) the self-inductance of AB+CD, using equation (71) in which $l=2$AB.

[This is equivalent to the following formula in which the logarithms are common:

$$L = 0.7411 \log_{10} \frac{d}{\rho} + .0805 \text{ in millihenrys per mile of conductor,}$$

$$= 0.4605 \log_{10} \frac{d}{\rho} + .050 \text{ in millihenrys per kilometer of conductor,}$$

d and ρ being expressed in centimeters, inches, or any other unit.]

MUTUAL INDUCTANCE OF TWO LINEAR CONDUCTORS IN THE SAME STRAIGHT LINE.

The mutual inductance of two adjacent linear conductors of lengths l and m in the same straight line is

$$M_{lm} = l \log \frac{l+m}{l} + m \log \frac{l+m}{m}, \text{ approximately.} \qquad [81]$$

This approximation is very close indeed if the radius of the conductor (which has been assumed zero) is very small.

THE SELF-INDUCTANCE OF A STRAIGHT RECTANGULAR BAR.

The self-inductance of a straight bar of rectangular section is, to within the accuracy of the approximate formula (75), the same as the mutual inductance of two parallel straight filaments of the same length separated by a distance equal to the geometrical mean distance of the cross section of the bar. Thus,

$$L = 2l\left[\log \frac{2l}{R} - 1 + \frac{R}{l}\right] \qquad [82]$$

where R is the geometrical mean distance of the cross section of the rod or bar. If the section is a square, $R = .447\ a$, a being the side of the square. If the section is a rectangle, the value of R is given by Maxwell's formula (103).

This is equivalent to the following:

$$L = 2l\left[\log \frac{2l}{\alpha+\beta} + \frac{1}{2} + \frac{0.2235(\alpha+\beta)}{l}\right] \qquad [83]$$

In the above formula L is the self-inductance of a straight bar or wire of length l and having a rectangular section of length α and breadth β.

TWO PARALLEL BARS. SELF AND MUTUAL INDUCTANCE.

The mutual inductance of two parallel straight, square, or rectangular bars is equal to the mutual inductance of two parallel wires or filaments of the same length and at a distance apart equal to the geometrical mean distance of the two areas from one another. This is very nearly equal in the case of square sections to the distance between their centers for all distances, the g. m. d. being a very little

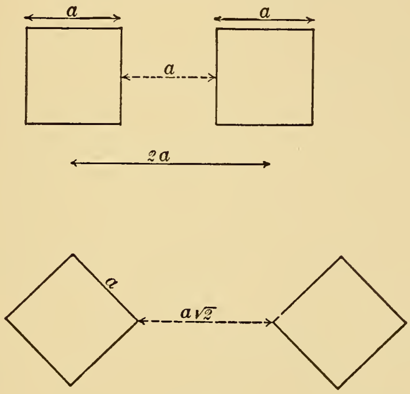

Fig. 31.

greater for parallel squares, and a very little less for diagonal squares[69] (Fig. 31). We should, therefore, use equation (78) with d equal to the g. m. d. of the sections from one another; that is, substantially, to the distances between the centers.

The self-inductance of a return circuit of two such parallel bars is equal to twice the self-inductance of one minus twice their mutual inductance. That is,

$$L = 2[L_1 - M]$$

in which L_1 is calculated by (83) and M by (78).

SELF-INDUCTANCE OF A SQUARE.

The self-inductance of a square may be derived from the expressions for the self and mutual inductance of finite straight wires from the consideration that the self-inductance of the square is the sum of the self-inductances of the four sides minus the mutual inductances. That is,

$$L = 4L_1 - 4M$$

[69] Rosa, this Bulletin, **3**, p. 1.

the mutual inductance of two mutually perpendicular sides being zero. Substituting a for l and d in formulæ (73) and (77) we have, neglecting ρ^2/a^2, $L = 8a\left(\log\frac{a}{\rho}+\frac{\rho}{a}-.524\right)$ [84]

where a is the length of one side of the square and ρ is the radius of the wire. If we put $l=4a=$ whole length of wire in the square,

$$L=2l\left(\log\frac{l}{\rho}+\frac{4\rho}{l}-1.910\right)$$

$$\text{or, } L=2l\left(\log\frac{l}{\rho}-1.910\right), \text{ approximately.} \qquad [85]$$

Formulæ (84) and (85) were first given by Kirchhoff[70] in 1864.

SELF-INDUCTANCE OF A RECTANGLE.

(*a*) *The conductor having a circular section.*

The self-inductance of the rectangle of length a and breadth b is

$$L=2(L_a+L_b-M_a-M_b)$$

where L_a and L_b are the self-inductances of the two sides of length a and b taken alone, M_a and M_b are the mutual inductances of the two opposite pairs of length a and b, respectively.

From (73) and (77) we therefore have, neglecting ρ^2/a^2, and putting d for the diagonal of the square $=\sqrt{a^2+b^2}$

$$L=4\left[(a+b)\log\frac{2ab}{\rho}-a\log(a+d)-b\log(b+d)\right.$$
$$\left.-\frac{7}{4}(a+b)+2(d+\rho)\right] \qquad [86]$$

(*b*) *The conductor having a rectangular section.*

For a rectangle made up of a conductor of rectangular section $\alpha\times\beta$,

$$L=4\left[(a+b)\log\frac{2ab}{\alpha+\beta}-a\log(a+d)-b\log(b+d)\right.$$
$$\left.-\frac{a+b}{2}+2d+0.447(\alpha+\beta)\right] \qquad [87]$$

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[70] Gesammelte Abhandlungen, p. 176. Pogg. Annal., **121**, 1864.

where as before d is the diagonal of the square. This is equivalent to Sumec's exact formula [71] ($6a$).

For $a=b$, a square,

$$L=8a\left[\log\frac{a}{a+\beta}+0.2235\frac{a+\beta}{a}+0.726\right] \qquad [88]$$

If $a=\beta$, that is, the section of the conductor is a square,

$$L=8a\left[\log\frac{a}{a}+.447\frac{a}{a}+.033\right] \qquad [89]$$

MUTUAL INDUCTANCE OF TWO EQUAL PARALLEL RECTANGLES.

For two equal parallel rectangles of sides a and b and distance apart d the mutual inductance, which is the sum of the several mutual inductances of parallel sides, is,

$$\begin{aligned}M=4&\left[a\log\left(\frac{a+\sqrt{a^2+d^2}}{a+\sqrt{a^2+b^2+d^2}}\cdot\frac{\sqrt{b^2+d^2}}{d}\right)\right.\\&\left.+b\log\left(\frac{b+\sqrt{b^2+d^2}}{b+\sqrt{a^2+b^2+d^2}}\cdot\frac{\sqrt{a^2+d^2}}{d}\right)\right]\\&+8\left[\sqrt{a^2+b^2+d^2}-\sqrt{a^2+d^2}-\sqrt{b^2+d^2}+d\right]\end{aligned} \qquad [90]$$

For a square, where $a=b$, we have

$$\begin{aligned}M=8&\left[a\log\left(\frac{a+\sqrt{a^2+d^2}}{a+\sqrt{2a^2+d^2}}\cdot\frac{\sqrt{a^2+d^2}}{d}\right)\right]\\&+8\left[\sqrt{2a^2+d^2}-2\sqrt{a^2+d^2}+d\right]\end{aligned} \qquad [91]$$

Formula (90) was first given by F. E. Neumann[72] in 1845.

SELF AND MUTUAL INDUCTANCE OF THIN TAPES.

The self-inductance of a straight thin tape of length l and breadth b (and of negligible thickness) is equal to the mutual inductance of two parallel lines of distance apart R_1 equal to the geometrical mean distance of the section, which is $0.22313b$, or $\log R=\log b-\frac{3}{2}$.

[71] Elektrotech. Zs., p. 1175; 1906.

[72] Allgemeine Gesetze der Inducirten Ströme, Abh. Berlin Akad.

Thus we have approximately,

$$L=2l\left[\log\frac{2l}{R_1}-1\right]$$

$$=2l\left[\log\frac{2l}{b}+\frac{1}{2}\right] \qquad [92]$$

If the thickness of the tape is not negligible this formula becomes, when a is the thickness of the tape,

$$L=2l\left[\log\frac{2l}{b}-\frac{a}{b}+\frac{1}{2}\right] \qquad [93]$$

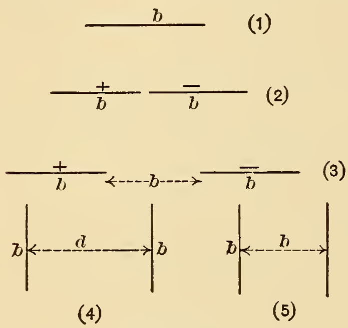

Fig. 32.

A closer approximation to L is given by (83) in which a is the thickness and β is the breadth of the tape. For two such tapes in the same plane, coming together at their edges without making electrical contact, the mutual inductance is

$$\left.\begin{aligned} M&=2l\left[\log\frac{2l}{R_2}-1\right]\\ &=2l\left[\log\frac{2l}{b}-0.8863\right]\end{aligned}\right\} \qquad [94]$$

where R_2 is the geometrical mean distance of one tape from the other, which in this case is $0.89252b$. For a return circuit made up of these two tapes the self-inductance is

$$L=2L_1-2M$$

$$=4l\left(\log\frac{R_2}{R_1}\right)=4l\log_e 4 \qquad [95]$$

$$=5.545\times \textit{length of one tape.}$$

Thus the self-inductance of such a circuit is independent of the width of the tapes. If the tapes are separated by the distance b equal to the width of the tapes, $R_2=1.95653$ and $L=8.685\ l$.

If the two tapes are not in the same plane but parallel,

$$L=2L_1-2M=4l\log\frac{R_2}{R_1} \qquad [96]$$

and when the distance apart is equal to the breadth of the tapes we have

$$\log \frac{R_2}{R_1} = \frac{\pi}{2}$$

and

$$L = 4l\frac{\pi}{2} = 2\pi\, l \qquad [97]$$

In this case also the self-inductance [2π cm per unit of length] of the pair of thin strips is independent of their width so long as the distance apart is equal to their width. Formula (96) may be employed to calculate the self-inductance of a non-inductive shunt made up of a sheet of thin metal doubled on itself.

CONCENTRIC CONDUCTORS.

The self-inductance of a thin, straight tube of length l and radius a_2, when a_2/l is very small, is given by (76),

$$L_2 = 2l\left[\log \frac{2l}{a_2} - 1\right]$$

The mutual inductance of such a tube on a conductor within it is equal to its self-inductance, since all the lines of force due to the outer tube cut through the inner when they collapse on the cessation of current. The self-inductance of the inner conductor, suppose a solid cylinder, is

$$L_1 = 2l\left[\log \frac{2l}{a_1} - \frac{3}{4}\right]$$

If the current goes through the latter and returns through the outer tube, the self-inductance of the circuit is

$$L = L_1 + L_2 - 2M = L_1 - L_2$$

since M equals L_2

$$\therefore\ L = 2l\left[\log \frac{a_2}{a_1} + \frac{1}{4}\right] \qquad [98]$$

This result can also be obtained by integrating the expression for the force outside a_1 between the limits a_1 and a_2, and adding the term for the field within a_1, there being no magnetic field outside a_2.

If the outer tube has a thickness $a_3 - a_2$ and the current is distributed uniformly over its cross section the self-inductance will be a little greater, the geometrical mean distance from a_1 to the tube, which is more than a_2 and less than a_3, being given by the expression

$$\log a_g = \frac{a_3^2 \log a_3 - a_2^2 \log a_2}{a_3^2 - a_2^2} - \frac{1}{2}$$

Putting this value of log a in (56) in place of log a_2, we should have the self-inductance of the return circuit.

If the current is alternating and of very high frequency, the current would flow on the outer surface of a_1 and on the inner surface of the tube, and L for the circuit would be

$$L = 2l \log \frac{a_2}{a_1} \qquad [99]$$

MULTIPLE CONDUCTORS.

If a current be divided equally between two wires of length l, radius ρ and distance d apart, the self-inductance of the divided conductor is the sum of their separate self-inductances plus twice their mutual inductance.

Thus, when d/l is small,

$$L = 2l\left[\log \frac{2l}{(\rho d)^{\frac{1}{2}}} - \frac{7}{8}\right] = 2l\left[\log \frac{2l}{(r_g d)^{\frac{1}{2}}} - 1\right] \qquad [100]$$

where r_g, the g. m. d. of the section of the wire is 0.7788ρ for a round section.

If there are three straight conductors in parallel and distance d apart, the self-inductance is similarly

$$L = 2l\left[\log \frac{2l}{(r_g d^2)^{\frac{1}{3}}} - 1\right] \qquad [101]$$

The expression $(r_g d^2)^{\frac{1}{3}}$ is the g. m. d. of the multiple conductor.

9. FORMULÆ FOR GEOMETRICAL AND ARITHMETICAL MEAN DISTANCES.

GEOMETRICAL MEAN DISTANCES.

Maxwell showed how to calculate mutual and self-inductances in several important cases by means of what he called the geometrical mean distances, either of one conductor from another or of a conductor from itself. On account of the importance of this method we give below some of the most useful of these formulæ. The geometrical mean distance of a point from a line is the *n^{th} root of the product of the n distances from the point P to the various points in the line, n being increased to infinity in determining the value of R.* Or, the logarithm of R is the *mean value of log d for all the infinite values of the distance d.* Similarly, the geometrical mean distance of a line from itself is the *n^{th} root of the product of the n distances between all the various pairs of points in the line, n being infinity.*[73]

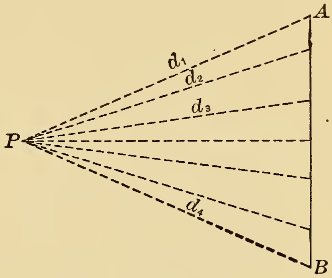

Fig. 33.

Similar definitions apply to the g. m. d. of one area from another, or of an area from itself.

The geometrical mean distance R of a *line* of length a from itself is given by

$$\left.\begin{aligned} \log R &= \log a - \frac{3}{2} \\ R &= ae^{-\frac{3}{2}} \\ \text{or } R &= 0.22313a \end{aligned}\right\} \quad [102]$$

The g. m. d. of a *rectangular area* of sides a and b from itself is given by

$$\log R = \log \sqrt{a^2+b^2} - \frac{1}{6}\frac{a^2}{b^2}\log\sqrt{1+\frac{b^2}{a^2}} - \frac{1}{6}\frac{b^2}{a^2}\log\sqrt{1+\frac{a^2}{b^2}}$$
$$+\frac{2}{3}\frac{a}{b}\tan^{-1}\frac{b}{a}+\frac{2}{3}\frac{b}{a}\tan^{-1}\frac{a}{b}-\frac{25}{12} \quad [103]$$

When the area is a *square*, and hence $a=b$,

$$\left.\begin{aligned} \log R &= \log a + \frac{1}{3}\log 2 + \frac{\pi}{3} - \frac{25}{12} \\ \therefore R &= 0.44705\, a \end{aligned}\right\} \quad [104]$$

[73] Rosa, this Bulletin, 4, p. 326.

For a *circular area* of radius a,

$$\left.\begin{aligned} \log R &= \log a - \frac{1}{4} \\ R &= ae^{-\frac{1}{4}} \\ R &= 0.7788\, a \end{aligned}\right\} \quad [105]$$

For an *ellipse* of semi-axes a and b,

$$\log R = \log \frac{a+b}{2} - \frac{1}{4} \quad [106]$$

An approximate expression for the g. m. d. of a *rectangular area* of length a and breadth b is

$$R = 0.2235(a+b) \quad [107]$$

which is nearly true for all values of a and b; that is, the geometrical mean distance of the rectangular area from itself is approximately proportional to the perimeter of the rectangle. The following table gives the ratio $R/(a+b)$ for a series of rectangles of different proportions, from a square to a ratio of 20 to 1 between length and breadth, and finally when the breadth is infinitesimal in comparison with the length. By interpolating for any other case between the values given in the table one can obtain a quite accurate value without the trouble of calculating it by formula (103).

Geometrical Mean Distances of Rectangles of Different Proportions.

[a and b are the Length and Breadth of the Rectangles. R is the Geometrical Mean Distance of its Area.]

Ratio	R	$\frac{R}{a+b}$
1 :1	0.44705a	0.22353
1.25:1	0.40235a	0.22353
1.5 :1	0.37258a	0.22355
2 :1	0.33540a	0.22360
4 :1	0.27961a	0.22369
10 :1	0.24596a	0.22360
20 :1	0.23463a	0.22346
1 :0	0.22315a	0.22315

The g. m. d. of an *annular ring* of radii a_1 and a_2 from itself is given by

$$\log R = \log a_1 - \frac{a_2^4}{(a_1^2 - a_2^2)^2} \log \frac{a_1}{a_2} + \frac{1}{4} \frac{3a_2^2 - a_1^2}{a_1^2 - a_2^2} \qquad [108]$$

The g. m. d. of a *line* of length a from a second line of the same length, distant in the same straight line na, center to center, is given by the following formula:

Fig. 34.

$$\log R_n = \frac{(n+1)^2}{2} \log (n+1)a - n^2 \log na + \frac{(n-1)^2}{2} \log (n-1)a - \frac{3}{2} \qquad [109]$$

This formula is equivalent to the following, which is more convenient for calculation for all values of n greater than one.[74]

$$\log R_n = \log n - \left[\frac{1}{12n^2} + \frac{1}{60n^4} + \frac{1}{168n^6} + \frac{1}{360n^8} + \frac{1}{660n^{10}} + \cdots\cdots \right] \qquad [110]$$

This formula is very convergent, and only two or three terms are generally required.

The following values of the geometrical mean distances (calling a unity) were calculated from the above formulæ, all after the second being obtained by (110):

$R_0 = 0.22313$	$R_5 = 4.98323$
$R_1 = 0.89252$	$R_6 = 5.98610$
$R_2 = 1.95653$	$R_7 = 6.98806$
$R_3 = 2.97171$	$R_8 = 7.98957$
$R_4 = 3.97890$	$R_9 = 8.99076$

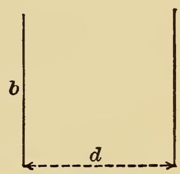

Fig. 34*a*.

If the strips are parallel and at distance d, the g. m. d. is given by

$$\log R = \frac{d^2}{b^2} \log d + \frac{1}{2}\left(1 - \frac{d^2}{b^2}\right) \log (b^2 + d^2) + 2\frac{d}{b} \tan^{-1}\frac{b}{d} - \frac{3}{2} \qquad [111]$$

If $d = b$,

$$\log R = \log b + \frac{\pi}{2} - \frac{3}{2} \qquad [112]$$

The g. m. d. from a point O_2 outside a circle to the circumference of the circle, or to the entire area of the circle *is the distance d from* O_2 *to the center of the circle.*

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[74] Rosa, this Bulletin, **2**, p. 168; 1906.

(1) The g. m. d. from the center O_1 to the circumference is of course the radius a. (2) The g. m. d. of any point (as O_3) within the circle from the circumference is also a. (3) The g. m. d. of any point on the circumference (as O_4) from all other points of the circumference is also a.

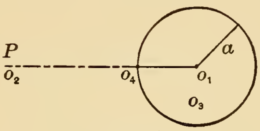

Fig. 35.

(4) Therefore the g. m. d. of a circular line of radius a from itself is a; that is,

$$R = a \qquad [113]$$

for each of the four cases named above.

The g. m. d. of a point outside a circular ring from the ring *is the distance d to the center of the ring*. The g. m. d. of any point O_1, O_3, etc., within the ring is given by

$$\log R = \frac{a_1^2 \log a_1 - a_2^2 \log a_2}{a_1^2 - a_2^2} - \frac{1}{2} \qquad [114]$$

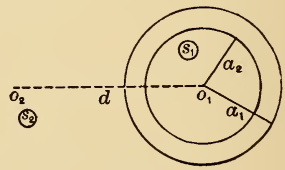

Fig. 36.

The same expression gives the g. m. d. of any figure, as S_1, within the ring from the ring. The g. m. d. of an external figure, as S_2, from the annular ring is equal to the g. m. d. of the center O_1 from the figure S_2.

The g. m. d. from one circular area to another is the distance between their centers; that is,

$$R = d \qquad [115]$$

Fig. 37.

for the area S_1 with respect to S_2 as it is for the point O_1 with respect to S_2.

The g. m. d. of a line of length a from a second parallel line of length a' located symmetrically (Fig. 38) is given by Gray[75], equation (114). The g. m. d. of a line from a parallel and symmetrically

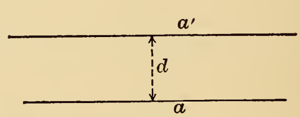

Fig. 38.

[75] Absolute Measurements, Vol. II, Part I.

There are a number of misprints in equations 104, 109, 111, and 113 of Gray. The sign of the first term of equation 104 should be +. The signs before p^2 in the coefficients of the log in the first four terms of equation 109 should be all minus; thus $\frac{1}{4}(\beta^2-p^2), -\frac{1}{4}(\alpha^2-p^2), -\frac{1}{4}[(a-\beta)^2-p^2], +\frac{1}{4}[(a-\alpha)^2-p^2]$. Similarly in equation 111 the coefficients of the first two terms should be $\frac{1}{2}(\beta^2-p^2)$ and $-\frac{1}{2}(\alpha^2-p^2)$. In equation 113 the coefficient of β^4 in each of the first four terms should be $\frac{1}{6}$ instead of $\frac{1}{2}$ and the first term should have $\log[(p+b+b')^2+\beta^2]$ instead of $\log[(p+b+b')^2-\beta^2]$.

situated rectangle is given by Gray's equation (112). The g. m. d. of two unequal rectangles from one another is given by Gray's equation (113).[76]

The g. m. d. of two adjacent rectangles and of two obliquely situated rectangles are given by Rosa,[77] equations (8*a*) and (17). As these expressions are somewhat lengthy and not often required they are not repeated here. The values of the g. m. d. for two equal squares in various relative positions to one another have been accurately calculated[78] by these formulæ, and the results used in the determination[79] of the correction term E of formula (72).

ARITHMETICAL MEAN DISTANCES.

In the determination of self and mutual inductances by the method of geometrical mean distances it has been shown[80] that more accurate formulæ can be obtained by the use of certain arithmetical mean distances and arithmetical mean square distances taken in connection with geometrical mean distances.

The arithmetical mean distance of a point from a line is the arithmetical mean of the n distances of the point from the various points of the line, n being infinite. Similarly, the arithmetical mean distance of a line from itself is the *arithmetical mean of the distances of the n pairs of points in the line from one another, n being infinite.*

The a. m. d. of a line of length b from itself is[81]

$$S_2=\frac{b}{3} \qquad [116]$$

that is, while the g. m. d. of a line from itself is 0.22313 times its length, the a. m. d. is one-third the length.

The arithmetical mean square distance of a line from itself is of course larger than the square of the a. m. d. Putting S_2^2 for the arithmetical mean square distance (a. m. s. d.)

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[76] Also by Rosa, equation (8). This Bulletin, 3, p. 6.
[77] This Bulletin, 3, pp. 7 and 12.
[78] This Bulletin, 3, pp. 9–19.
[79] This Bulletin, 3, p. 37.
[80] Rosa, this Bulletin, 4, p. 326–32.
[81] Rosa, this Bulletin 4, p. 326.

$$S_2^2=\frac{b^2}{6}, \text{ or } \sqrt{S_2^2}=\frac{b}{\sqrt{6}} \qquad [117]$$

The arithmetical mean distance of a point in the circumference of a circle from the circle is the same as the a. m. d. of the circle from itself; that is, for a circle of radius a,

$$S_1=S_2=\frac{4}{\pi}a \qquad [118]$$

The arithmetical mean square distance is

$$S_2^2=2a \text{ and } \sqrt{S_2^2}=a\sqrt{2} \qquad [119]$$

(The g. m. d. for this case is $R=a$, equation (113).)

The arithmetical mean distance of an external point P from the circumference of a circle is

$$S_1=\sqrt{d^2+a^2} \qquad [120]$$

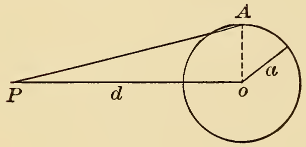

Fig. 39.

which is the distance PA.

The arithmetical mean distance from P to the entire area of the circle is

$$S_1=\sqrt{d^2+\frac{a^2}{2}} \qquad [121]$$

(The g. m. d. for each of these cases is $R=d$, equation (115).)

For the proof of these and other expressions for the arithmetical mean distances and applications of their use see the article referred to above.

II. EXAMPLES TO ILLUSTRATE AND TEST THE FORMULÆ.

1. COAXIAL CIRCLES.

EXAMPLE 1. MAXWELL'S FORMULA (1). FOR ANY TWO COAXIAL CIRCLES.

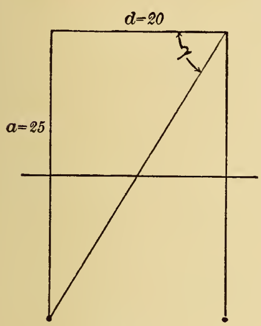

Fig. 40.

Let $a=A=25$ cm, Fig. 40,
$d=20$ cm.

$$k=\frac{50}{\sqrt{2500+400}}=0.9284766=\sin\gamma$$

$$\gamma=68^\circ\ 11'\ 54.''88=68^\circ.198578.$$

From Legendre's tables, we obtain

$$\log F=0.3852191 \qquad \left(\frac{2}{k}-k\right)F-\frac{2}{k}E=0.5318500$$
$$\log E=0.0547850$$

$$4a=100 \qquad \therefore\ M=167.0856 \text{ cm}.$$

To facilitate calculations in such problems as this, we have prepared Table II, which gives F and log F, E and log E, as functions of tan γ. In the above case $\tan\gamma=\frac{50}{20}=2.5$, and from Table II we can take the values of log F and log E directly, avoiding the calculation of γ and the interpolation for log F and log E in Legendre's tables (or Table XIII). This is only applicable for circles of equal radii, and is especially advantageous when tan γ is one of the values given in the table, when interpolation is unnecessary.

The above problem may also be calculated by means of Table I, taken from Maxwell, as follows:

$$\log_{10}\frac{M}{4\pi a} \text{ for } 68^\circ.1 \qquad =\bar{1}.7230634$$

$$\text{for } 68^\circ.2 \qquad =\underline{\bar{1}.7258281}$$

$$\text{for } 68^\circ.198578=\bar{1}.7257888=\log\frac{M}{4\pi a}$$

$\therefore\ M=167.0855$ cm, agreeing almost exactly with the above value.

The calculation of mutual inductance by the above methods is simplest for circles not near each other, as then the values of log F, log E, and log $\frac{M}{4\pi\sqrt{Aa}}$ are very exact when taken by simple interpolation. When γ is nearly 90°, however, second and third differences have to be used in interpolation.

EXAMPLE 2. MAXWELL'S SECOND EXPRESSION (2). FOR CIRCLES NEAR EACH OTHER.

$$\text{Let } a = A = 25 \text{ cm}, \ d = 1 \text{ cm}$$

$$\text{In this case } k = \sin\gamma = \frac{50}{\sqrt{2501}} = 0.9998002 \qquad \gamma = 88^\circ\ 51'\ 14''$$

This value of γ is so nearly 90° that it is difficult to obtain accurate values of F and E from tables of elliptic integrals, or of $\frac{M}{4\pi a}$ from Maxwell's table.

We may therefore use formula (2) instead of (1).

$$r_1 = \sqrt{2501} = 50.01 \text{ nearly}, \ r_2 = 1.0$$

$$\therefore\ k_1 = \sin\gamma_1 = \frac{r_1 - r_2}{r_1 + r_2} = \frac{49.01}{51.01} = 0.960792$$

$$\gamma_1 = 73^\circ\ 54'\ 9.''7 = 73^\circ\!.9027$$

From Legendre's tables or Table XIII, for $\gamma_1 = 73^\circ\!.9027$,

$$F_1 = 2.7024553$$
$$E_1 = 1.0852167$$
$$F_1 - E_1 = 1.6172386$$

$$\frac{8\pi\sqrt{Aa}}{\sqrt{k_1}} = \frac{200\pi}{\sqrt{.960792}} \qquad \therefore \frac{8\pi\sqrt{Aa}}{\sqrt{k_1}}\left(F_1 - E_1\right) = M = 1036.667 \text{ cm.}$$

EXAMPLE 3. FORMULA (3). SERIES FOR F AND E, CIRCLES NEAR EACH OTHER.

Suppose that, in the last example, we calculate F and E by means of formula (3), instead of taking them from Table XIII.

$$A = a = 25, \ d = 1.$$

$$k'^2 = 1 - k^2 = 1 - \frac{2500}{2501} = \frac{1}{2501}$$

$$\therefore\ F = 5.298947 \qquad E = 1.000960$$

If these values of F and E be substituted in formula (1), k being 0.9998002, we obtain the same value of M as by formula (2).

EXAMPLE 4. FORMULA (3). SECOND CASE, CIRCLES NOT NEAR.

$$A=25,\ a=20,\ d=10 \text{ cm.} \quad \text{(See Fig. 1.)}$$

$$k^2=\frac{4\times 20\times 25}{(45)^2+(10)^2}=\frac{16}{17} \qquad \therefore k'^2=\frac{1}{17}$$

$$\log\frac{4}{k'}=\frac{1}{2}\log(16\times 17)=\frac{1}{2}\log_e 272 \qquad =2.8029010$$

$$\frac{k'^2}{4}\left(\log\frac{4}{k'}-1\right) \qquad = .0265132$$

$$\frac{9k'^4}{64}\left(\log\frac{4}{k'}-\frac{7}{6}\right) \qquad = .0007962$$

$$\frac{25k'^6}{256}\left(\log\frac{4}{k'}-\frac{111}{90}\right) \qquad = .0000312$$

$$\frac{1225k'^8}{16384}\left(\log\frac{4}{k'}-1.27\right) \qquad = \underline{.0000014}$$

$$\therefore F \ =2.8302430$$

$$1+\frac{k'^2}{2}\left(\log\frac{4}{k'}-\frac{1}{2}\right) \qquad =1.0677324$$

$$\frac{3k'^4}{16}\left(\log\frac{4}{k'}-\frac{13}{12}\right) \qquad = .0011156$$

$$\frac{15k'^6}{128}\left(\log\frac{4}{k'}-1.20\right) \qquad = .0000381$$

$$\frac{175k'^8}{2048}\left(\log\frac{4}{k'}-1.25\right) \qquad = \underline{.0000017}$$

$$\therefore E \ =1.0688878$$

To find the value of M we now use equation (1).

$$\left\{\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right\}=0.885388$$

Multiplying by $4\pi\sqrt{Aa}=4\pi\sqrt{500}$ gives

$$M=248.7875 \text{ cm.}$$

EXAMPLE 5. WEINSTEIN'S FORMULA (4). FOR ANY COAXIAL CIRCLES NOT TOO FAR APART.

Take the same circles as in example 4.

$$A=25,\ a=20,\ c=5,\ d=10.$$

$$k'^2=\frac{1}{17},\ \log\frac{4}{k}-1=1.802901$$

$$1+\frac{3}{4}k'^2=1.0441176 \qquad 1+\frac{15}{128}k'^4= \quad 1.0004053$$

$$\frac{33}{64}k'^4= \quad .0017842 \qquad \frac{185}{1536}k'^6= \quad .0000245$$

$$\frac{107}{256}k'^6= \quad .0000851 \qquad \frac{7465}{65536}k'^8= \quad \underline{.0000012}$$

$$1.0004310=C$$

$$\frac{5913}{16384}k'^8= \quad \underline{.0000042}$$

$$\text{Sum} \quad =1.0459911=B$$

$$B\log\left(\frac{4}{k'}-1\right)=1.8858184;\ \left\{B\log\left(\frac{4}{k'}-1\right)-C\right\}=0.8853874$$

Multiplying by $4\pi\sqrt{500}$ gives $M=248.7873$ cm, agreeing almost exactly with the value previously found, example 4.

EXAMPLE 6. NAGAOKA'S Q-SERIES FORMULA (5). FOR CIRCLES NOT NEAR EACH OTHER.

$$A=a=25, \quad d=20 \quad \text{See Fig. 40.}$$

$$\sqrt{k'}=\sqrt{\cos\gamma}=\left(\frac{20}{\sqrt{2900}}\right)^{\frac{1}{2}}= \quad 0.6094183$$

$$\frac{l}{2}=\frac{1-\sqrt{k'}}{2(1+\sqrt{k'})}=\frac{1}{2}\,\frac{0.3905817}{1.6094183}= \quad 0.1213425$$

$$2\left(\frac{l}{2}\right)^5= \quad .0000526$$

$$15\left(\frac{l}{2}\right)^9= \quad \underline{.0000000}$$

$$\therefore q= \quad 0.1213951$$

$$3q^4=+ \quad .0006516$$

$$-4q^6=- \quad .0000128$$

$$+9q^8=+ \quad \underline{.0000004}$$

$$\epsilon= \quad .0006392$$

$$1+\epsilon= \quad \underline{1.0006392}$$

$$\log(1+\epsilon)= \quad 0.0002773$$

$$\log q^{\frac{3}{2}}= \quad \bar{2}.6263018$$

$$\log 16\pi\sqrt{Aa}= \quad \underline{3.0992099}$$

$$\log\frac{M}{\pi}= \quad 1.7257890$$

$\therefore M=167.0855$ cm, as previously found by formula (1), example 1.

EXAMPLE 7. NAGAOKA'S SECOND FORMULA (6). FOR CIRCLES NEAR EACH OTHER.

$$A = a = 25,\ d = 4$$

$$k = \sin\gamma = \frac{50}{\sqrt{2516}};\quad \sqrt{k} = 0.99840637$$

$$l_1 = \frac{1-\sqrt{k}}{1+\sqrt{k}} = \frac{0.00159363}{1.9984064};\quad \frac{l_1}{2} = 0.00039872 = q_1$$

as $\left(\frac{l_1}{2}\right)$ and higher powers can be neglected.

$$\frac{1}{q_1} = 2508.04,\ \log_e\left(\frac{1}{q_1}\right) = 7.827238$$

$$(1 - q_1 + 4q_1^2) = 0.9996019$$

$$8q_1 = 0.00318976$$

$$\left\{\log_e\left(\frac{1}{q_1}\right)[1 + 8q_1(1 - q_1 + 4q_1^2)] - 4\right\} = 3.852195 = A$$

$$\frac{1}{2(1-2q_1)^2} = 0.5007985 = B$$

$$4\sqrt{Aa} = 100 = C$$

$$\text{Product } A \times B \times C = M = 606.0679 \text{ cm.}$$

There is a difficulty in using the above formula, owing to the fact that when k is nearly unity the numerator of the expression for l_1 is small, and unless the value of k is carried out to about eight decimal places the value of M may be appreciably in error. For approximate calculations a seven-place table of logarithms is sufficient, and it is not very troublesome to carry out this one number to the necessary number of places for precision calculations. Or, k can easily be computed to any degree of accuracy without logarithms. The same thing applies to formula (3), where k' must be computed with great precision when it is quite small.

Using Table II for the above problem, where $\tan\gamma = 12.5$, we have $\log F = 0.5932708$ and $\log E = 0.0047004$. Using these values in formula (1) we obtain for the mutual inductance

$$M = 606.0666 \text{ cm}$$

which differs from the value by Nagaoka's formula by 2 parts in a million.

EXAMPLE 8. MAXWELL'S SERIES FORMULA (7). FOR ANY TWO COAXIAL CIRCLES NEAR EACH OTHER.

$$A=26,\ a=25,\ d=1,\ c=1, \text{ and } r=\sqrt{2}$$

$$\text{Since } r=\sqrt{2},\ \log_e\frac{8a}{r}=\log_e\frac{200}{\sqrt{2}}=4.951744$$

$$1+\frac{c}{2a} \quad = \quad 1.020000 \qquad 2+\frac{c}{2a} \quad = \quad 2.020000$$

$$\frac{c^2+3d}{16a^2} \quad = \quad .000400 \qquad -\frac{3c^2-d^2}{16a^2}=-\ .000200$$

$$-\frac{c^3+3cd^2}{32a^3}=-\ \underline{.000008} \qquad -\frac{c^3-6cd^2}{48a^3}=-\ \underline{.000010}$$

$$1.020392=B \qquad 2.019790=C$$

$$B\log\frac{8a}{r} \quad = \quad 5.052310$$

$$C \quad = \quad \underline{2.019790}$$

$$\left\{B\log\frac{8a}{r}-C\right\}= \quad 3.032520 \quad \text{Multiply by } 4\pi a=100\pi \text{ and}$$

$$M=952.6943 \text{ cm.}$$

This formula would be less accurate for the circles of problem 4, but is accurate for circles close together, as this problem shows.

EXAMPLE 9. MAXWELL'S FORMULA (9). FOR CIRCLES OF EQUAL RADII NEAR EACH OTHER.

$$A=a=25, \quad d=1$$

$$\frac{8a}{d}=200, \ \log_e 200=5.298317$$

$$\log_e\frac{8a}{d}\left(1+\frac{3d^2}{16a^2}\right)=1.000300\times 5.298317=5.29990$$

$$\left(2+\frac{d^2}{16a^2}\right)=\frac{2.00010}{3.29980}$$

Multiply by $4\pi a=100\pi$

$$M=1036.663 \text{ cm.}$$

nearly agreeing with the more exact value found under problem 2.

This is a very simple and convenient formula for equal circles, and gives approximate results for circles still farther apart than in this problem.

EXAMPLE 10. COFFIN'S FORMULA (10). EXTENSION OF FORMULA (9) FOR CIRCLES OF EQUAL RADII.

$$A=a=25, \quad d=16$$

$$\frac{8a}{d}=12.5, \quad \log_e 12.5 \quad =2.5257286$$

$$\text{First series of terms}=B=1.074478$$

$$\text{Second series of terms}=C=2.023220$$

$$\therefore \left\{B \log \frac{8a}{d}-C\right\}=0.690620$$

$$4\pi a=100\pi \quad \therefore M=216.9647 \text{ cm.}$$

This agrees with the value given by formula (1) within 1 part in 200,000. As the distance apart of the circles increases the accuracy by this formula of course gradually decreases.

EXAMPLE 11. FORMULA (11). EXTENSION OF MAXWELL'S FORMULA (7) FOR CIRCLES OF UNEQUAL RADII.

$$A=25, \quad a=20, \quad c=5, \quad d=10.$$

$$r=\sqrt{c^2+d^2}=5\sqrt{5}, \quad \log_e\frac{8a}{r}=\log_e\frac{32}{\sqrt{5}}= \; 2.6610169$$

$$\text{First series of terms}=B \log_e \frac{8a}{r}=3.112060$$

$$\text{Second series of terms}=C \qquad =2.122114$$

$$0.989946$$

multiplying by $4\pi a=80\pi$, $M=248.8006$ cm.

This result is correct to 1 part in 19,000 (see examples 4 and 5). Using only the first three terms for B and C (that is, formula 8), the result would be too large by 1 part in 1750.

2. MUTUAL INDUCTANCE OF COILS OF RECTANGULAR SECTIONS.

EXAMPLE 12. ROWLAND'S FORMULA (14). FOR COAXIAL COILS OF EQUAL RADII.

$$A=a=25, \qquad b=c=2 \text{ cm}, \qquad d=10.$$

The mutual inductance of the two coils is $M=M_0+\Delta M$.

We find M_0 by formula 1, 5, or 10, and ΔM by 14 and 15.

$$M_0 = 107.4885\pi$$

$$k = \sin\gamma = \frac{50}{\sqrt{2600}} = 0.9805807$$

$$k^2 = 0.9615383$$

$$\log_{10}F = 0.4821754$$

$$\log_{10}E = 0.0207625$$

By Table II, since $\tan\gamma = 5$, $\log F = 0.4821752$ and $\log E = 0.0207626$. These slight differences in the logarithms obtained in the two different ways amount to scarcely one part in two million of F and E, respectively, and may usually be neglected. If more accurate values are required they may be obtained by carrying the interpolations further in Legendre's table, provided the angle γ is obtained with sufficient accuracy.

Substituting these values in formula (15) we obtain

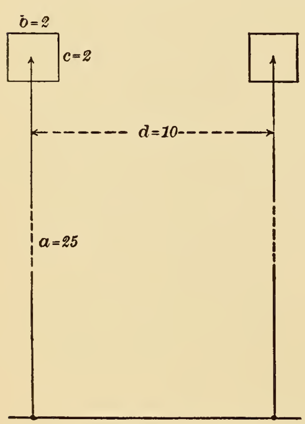

Fig. 41.

$$\frac{d^2M}{da^2} = -0.9081\,\pi$$

$$\frac{d^2M}{dx^2} = +1.0639\,\pi \qquad b^2 = c^2 = 4$$

Substituting these values in formula (14) we obtain

$$\Delta M = .05193\,\pi$$

$$\therefore M = M_0 + \Delta M = (107.4885 + 0.519)\pi$$
$$= 337.8481 \text{ cm.}$$

The correction ΔM thus amounts to about 1 part in 2,000 of M. At a distance $d = 20$ cm, the correction is over 1 part in 1,000. For a coil of section 4×4 cm at $d = 10$, ΔM would be four times as large as the value above, or about 1 part in 500, and at 20 cm 1 part in 250.

EXAMPLE 13. RAYLEIGH'S FORMULA (17). FOR COAXIAL COILS OF EQUAL RADII.

$$A = a = 25, \quad b = 4, \quad c = 1, \quad d = 10$$

We now find by formula (1) in accordance with formula (17) the mutual inductance of the following pairs of circles:

O, 1 when $a=25$, $A=25.5$, $d=10$; O, 4 when $a=25$, $A=24.5$, $d=10$; O, 2 when $a=A=25$ and $d=8$; O, 3 when $A=a=25$, $d=12$ and finally O, O′ when $A=a=25$, $d=10$. Thus:

$$
\begin{aligned}
M_1 &= 109.3217\pi \\
M_4 &= 105.4287\pi \\
M_2 &= 127.3949\pi \\
M_3 &= 91.9206\pi \\
\hline
 & \, 434.0659\pi \\
M_0 &= 107.4885\pi \\
\hline
 & \, 326.5774\pi \\
\therefore M &= 108.8591\pi \\
M_0 &= 107.4885\pi \\
\hline
\Delta M &= 1.3706\pi \text{ cm.}
\end{aligned}
$$

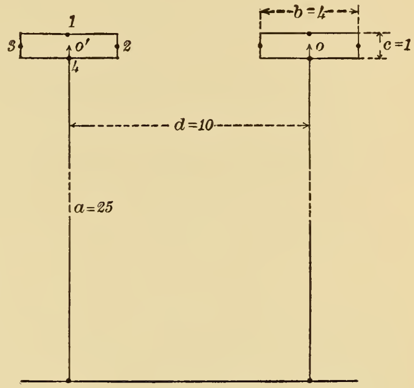

Fig. 42.

EXAMPLE 14. LYLE'S FORMULA (20). FOR COILS OF SQUARE SECTION.

$$A=a=25 \text{ cm}, \quad b=c=2 \text{ cm}, \quad d=10 \text{ cm}.$$

The equivalent radius $r=a\left(1+\dfrac{b^2}{24a^2}\right)$

$$r=25\left(1+\frac{4}{15000}\right)=25.00667 \text{ cm}.$$

M is now found by using formula 1, 5, or 10, employing r in place of a as the radius.

The result is $M=337.8475$, agreeing very closely with the result found under example 12.

$$M-M_0=\Delta M=.0517\pi$$

EXAMPLE 15. LYLE'S FORMULA (21). FOR COILS OF RECTANGULAR SECTION.

$$A=a=25, \quad b=4, \quad c=1, \quad d=10$$

$$r=25\left(1+\frac{1}{15000}\right)=25.00167$$

$\beta^2=\dfrac{b^2-c^2}{12}=\dfrac{15}{12}=1.25$, $2\beta=2.236$ cm, the distance apart of the two filaments which replace the coil. We now find by formula 1, 5, or 10 the mutual inductances of two circles 1, 2 on the two circles 3, 4, where $a=25.00167$ and d is 7.764, 10 and 12.236 cm, respectively. Thus:

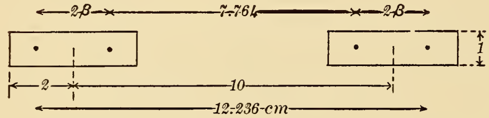

Fig. 43.

$$
\begin{aligned}
2\,M_{13} &= 215.00228\pi \\
M_{14} &= 90.31304\pi \\
M_{23} &= 130.14060\pi \\
\hline
4\,M &= 435.45592\pi \\
\therefore M &= 108.8640\ \pi \\
M_0 &= 107.4885\ \pi \\
\hline
\Delta M &= 1.3755\ \pi
\end{aligned}
$$

$\Delta M=$ the correction for section of the coils whose dimensions are given above. These values of M and ΔM agree nearly with the results obtained in example 13 above.

EXAMPLE 16. ROSA'S FORMULA (22). FOR COILS OF EQUAL RADII.

$$A=a=25, \quad b=4, \quad c=1, \quad d=10$$

(same coils as examples 13, 15).

$$\log_e\frac{8a}{d}=\log_e 20=2.9957$$

$$
\begin{aligned}
\frac{3b^2+c^2}{96a^2}\cdot\log_e\frac{8a}{d} &= \frac{49\times 2.9957}{60{,}000} = .0024465 \\
\frac{b^2-c^2}{12d^2} &= \frac{15}{1200} = .0125000 \\
\frac{2b^4+2c^4-5b^2c^2}{120d^4} &= \frac{434}{1{,}200{,}000} = .0003617 \\
\frac{3b^6-3c^6+14b^2c^4-14b^4c^2}{504d^6} &= \frac{8925}{504\times 10^6} = .0000177 \\
\frac{6b^4+6c^4+5b^2c^2}{5760a^2d^2} &= \frac{1622}{360\times 10^6} = .0000045 \\
\frac{7c^2d^2}{1024a^4}\left(\log_e\frac{8a}{d}-\frac{163}{84}\right) &= \underline{.0000018} \qquad .0153322 \\
-\frac{11b^2-3c^2}{192a^2} &= -\frac{173}{120{,}000} = -.0014417 \\
-\frac{15b^2d^2}{1024a^4}\left(\log_e\frac{8a}{d}-\frac{97}{60}\right) &= -\underline{.0000827} \qquad -\underline{.0015244} \\
& \qquad\qquad\qquad\qquad .0138078
\end{aligned}
$$

$$4a=100, \ \therefore \Delta M=1.3808\ \pi \text{ cm.}$$

This is a little larger value than found by formulæ (17) and (21), and we shall see later that it is more nearly correct than either of the other values.

EXAMPLE 17. ROSA'S FORMULÆ (23) AND (24). FOR COILS OF EQUAL RADII AND SQUARE SECTION.

$$A=a=25, \quad b=c=2, \quad d=10$$

$$\begin{array}{llll}
\log_e \frac{8a}{d} - 1 = & 2.9957 - 1 & = 1.9957 & \\
\frac{17b^2}{240d^2} = & \frac{68}{24{,}000} & = \underline{.0028} & 1.9985 \\
\frac{-a^2b^2}{5d^4} = & -\frac{2500}{50{,}000} & = - .0500 & \\
\frac{-3d^2}{16a^2}\left(\log_e \frac{8a}{d} - \frac{4}{3}\right) = & -\frac{300 \times 1.6624}{10{,}000} & = - \underline{.0499} & - \underline{.0999} \\
 & & & 1.8986
\end{array}$$

$$\frac{b^2}{6a} = \frac{4}{150} \quad \therefore \quad \varDelta M = .05063\pi$$

The approximate formula (24) would have given .0519 (agreeing with formulæ 14 and 20), which would be amply accurate for any experimental purpose. When the section is larger these small terms are, however, more important.

EXAMPLE 18. SECOND EXAMPLE BY FORMULA (23).

$$A=a=25, \quad b=c=5, \quad d=10$$

$$\begin{array}{lll}
\log_e \frac{8a}{d} - 1 & = 1.9957 & \\
\frac{17b^2}{240d^2} & = \underline{.0177} & 2.0134 \\
\frac{-a^2b^2}{5d^4} & = -.3125 & \\
-\frac{3d^2}{16a^2}\left(\log_e \frac{8a}{d} - \frac{4}{3}\right) & = -\underline{.0499} & -\underline{.3624} \\
 & & 1.6510
\end{array}$$

$$\frac{b^2}{6a} = \frac{25}{150}$$

$$\therefore \varDelta M = 0.2752\pi$$

$$\underline{M_0 = 107.4885\pi} \text{ (see example 12.)}$$

$$M = 107.7637\pi \text{ cm.}$$

This is a very simple formula for computing ΔM, and within a considerable range (i. e., d not larger than a and yet the coils not in contact) it is very accurate.

EXAMPLE 19. ROSA-WEINSTEIN FORMULA (25). FOR COILS OF EQUAL RADII AND EQUAL SECTION.

$$A=a=25, \quad b=4, \quad c=1, \quad d=10$$

$$a_1=15.0000533 \qquad \sin^2\gamma=\frac{2500}{2600}=\frac{25}{26}$$

$$a_2=\ 0.0020267 \qquad \cos^2\gamma=\frac{100}{2600}=\frac{1}{26}$$

$$a_3=\ 0.217 \qquad \frac{c^2}{24a^2}=.0000667$$

$$a_1-a_2-a_3+(2a_2-3a_3)\cos^2\gamma+8a_3\cos^4\gamma=14.7587120$$

$$a_1+\frac{a_2}{2}+2a_3+(2a_2+3a_3)\cos^2\gamma+8a_3\cos^4\gamma=15.4628292$$

$$A=0.0004730 \qquad \text{Also } F=3.0351168$$

$$B=0.0123901 \qquad E=1.0489686$$

$$\left(F-E\right)\left(A+\frac{c^2}{24a^2}\right)=0.0010719$$

$$EB=0.0129968$$

$$\text{Sum}=0.0140687$$

$$4\pi a\sin\gamma=100\pi\sqrt{\frac{25}{26}} \therefore \Delta M=1.3795\pi \text{ cm.}$$

This is not as simple to calculate as (22) and when d is less than $a/2$ is less accurate than (22). But for $d=a$ or greater it is more accurate than (22), and indeed the most accurate of all the formulæ.

EXAMPLE 20. FORMULA (26.) MUTUAL INDUCTANCE IN TERMS OF SELF-INDUCTANCE. FOR COILS RELATIVELY NEAR.

For $a=25$, $b_1=1$, $c=1$, we have, n being the number of turns in one of the two equal coils,

$$L_1=4\pi an^2\ (4.103816)$$

For $b=2$, $c=1$,

$$L_2=4\pi an^2\ (4\times 3.698695)$$

For $b=3$, $c=1$,

$$L=4\pi an^2\ (9\times 3.411766)$$

Then the mutual inductance of 1 on 3 is by formula (26)

$$M=4\pi a n^2\left[\frac{L+L_1-2L_2}{2}\right]$$
$$=4\pi a n^2\left[\frac{30.705894+4.103816-29.589560}{2}\right]$$
$$=4\pi a n^2\times 2.610075$$
$$=819.979\ n^2 \text{ cm.}$$

If $n=100$,

$$M=8.19979 \text{ millihenrys,}$$

as the mutual inductance of coil 1 on coil 3, Fig. 44.

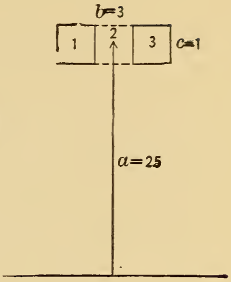

Fig. 44.

EXAMPLE 21. FORMULA (27). MUTUAL INDUCTANCE BY GEOMETRICAL MEAN DISTANCE.

$$A=25.1$$
$$a=25.0$$
$$b=c=0.1 \text{ cm}$$
$$d=0.1 \text{ cm}$$

The geometrical mean distance of two coils, corner to corner, as in Fig. 9, is 0.997701, and $\log\frac{r}{R}=0.002302$

$$\therefore\ \Delta M=100\times 0.002302\ (1.002)\pi$$
$$=0.2307\pi \text{ cm.}$$

3. MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS.

EXAMPLE 22. MAXWELL'S FORMULA (28) AND COHEN'S (35).

Two solenoids, Fig. 45, of equal length, 200 cm, each wound with a single layer coil.

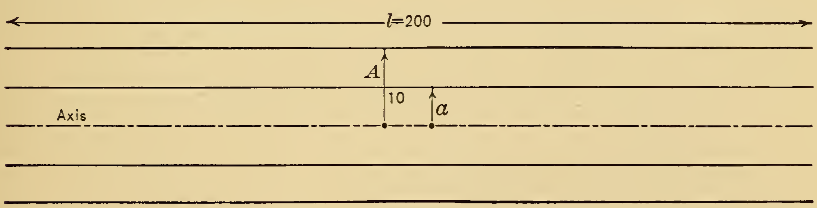

Fig. 45.

$$A=10=\text{radius of outer.}$$
$$a=\ \ 5=\text{radius of inner.}$$

Substituting in (28) for a we have the following:

$$a = 0.487508 - \frac{1}{16}\frac{a^2}{A^2}(0.999875) - \frac{1}{64}\frac{a^4}{A^4}(0.500001) - \frac{35}{2048}\frac{a^6}{A^6}\left(\frac{1}{7}\right)$$

$$= 0.487508 - .015610 - .000488 - .000038$$

$$= 0.471372$$

$$\therefore\ M = 4\pi^2 a^2 n^2 (200 - 9.42744)$$

$$M = 19057.25\pi^2 n^2$$

If $n = 10$ turns per cm, $M = \dfrac{100\ \pi^2 \times 19057.25}{10^9}$ henry

$$= 0.018809 \text{ henry.}$$

By the approximate formula of Maxwell (29)

$$2a = 1 - \frac{1}{8.4} - \frac{1}{64.16} - \frac{1}{1024.64} - \ldots$$

$$= 0.96773$$

$$\therefore\ M = 0.018784 \text{ henry.}$$

This example by Heaviside's extension of Maxwell's formula (see p. 23) has exactly the same value of M; that is, the added terms do not amount to as much as a millionth of a henry in this particular case.

To show that the result by Maxwell's formula (28) is very accurate for this case we may now calculate M by Cohen's absolute formula:

$$M = 4\pi n^2 (V - V_1)$$

Substituting in (35) for V we have the following terms:

$$V = 7863.79 + 4200532.04 - 4169106.25 - 23561.95$$
$$= 15727.63$$
$$V_1 = 1392.18 - 632.16 = 760.02$$
$$\therefore\ M = 4\pi n^2 (15727.63 - 760.02)$$
$$M = 0.0188088 \text{ henry.}$$

This agrees with the result by Maxwell's formula to within 1 part in 175000.

The example by Cohen's formula illustrates the disadvantage of that formula for numerical calculations. Aside from the fact that it is complicated, and involves the use of both complete and incomplete elliptic integrals, the value of M depends on the difference between very large positive and negative terms, so that in order to get a value of M correct to 1 part in 100000 it is necessary in the above example to calculate the large terms to 1 part in 25000000. As a means of testing other formulæ, however, this absolute formula is of great value.

EXAMPLE 23. RÒITI'S FORMULA (30) COMPARED WITH SEARLE AND AIREY'S (33).

We will now calculate the example, Fig. 46 (originally given by Searle and Airey[82]), by Ròiti's formula, and also by the formula of Searle and Airey.

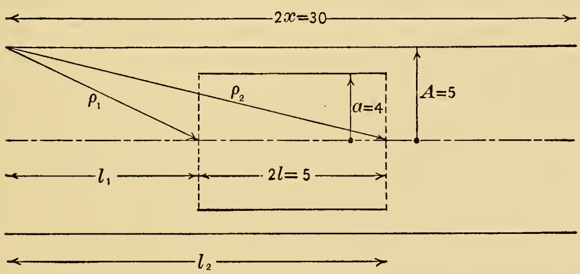

Fig. 46.

$2x = 30$ cm = length of outer solenoid.
$2l = 5$ " = " " inner "
$A = 5$ " = radius " outer "
$a = 4$ " = " " inner "

$$N_1 = 300 \text{ turns } \therefore n_1 = \frac{300}{30} = 10 \text{ per cm}$$

$$N_2 = 200 \text{ "} \qquad n_2 = \frac{200}{5} = 40 \text{ per cm}$$

$$l_1 = 12.5 \qquad \rho_1 = \sqrt{12.5^2 + 25} = 13.462912$$

$$l_2 = 17.5 \qquad \rho_2 = \sqrt{17.5^2 + 25} = 18.200275$$

$$\therefore \rho_2 - \rho_1 = 4.737363$$

[82] Electrician (London), **56**, p. 319; 1905.

$$\frac{A^2a^2}{8}\left(\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right)=+\ .012200$$

$$-\frac{A^4a^2}{16}\left(\frac{1}{\rho_1^5}-\frac{1}{\rho_2^5}\right)=-\ .000704$$

$$\left[\frac{5A^4a^4}{64}+\frac{5A^2a^6}{128}\right]\left(\frac{1}{\rho_1^7}-\frac{1}{\rho_2^7}\right)=+\ .000181$$

$$-\frac{35A^2a^6}{256}\left(\frac{1}{\rho_1^9}-\frac{1}{\rho_2^9}\right)=-\ .000022$$

$$+\frac{105A^6a^6}{1024}\left(\frac{1}{\rho_1^{11}}-\frac{1}{\rho_2^{11}}\right)=+\ .000002$$

$$\text{Sum}=\ 4.749020$$

$$4\pi^2a^2n_1n_2=25600\ \pi^2$$

$$\therefore\ M=\frac{25600\ \pi^2\times 4.749020}{10^9}\ \text{henry}$$

$$\text{or } M=0.001199896 \qquad \text{“}$$

Searle and Airey's formula (33) gives

$$M=1,198,480\ (1+0.001150+0.000034)$$
$$=1,199,900 \text{ cm}$$
$$=0.00119990 \text{ henry.}$$

The difference is inappreciable.

The same problem by Russell's formula (36) (extended to include six terms in each part of the formula) gives

$$M=0.00119989 \text{ henry.}$$

Thus these three formulæ all agree to within less than one part in 100,000. Searle and Airey's is the most rapidly convergent, and therefore most convenient. In other words, it is the most accurate for the same number of terms.

EXAMPLE 24. GRAY'S FORMULA (32) COMPARED WITH RÒITI'S (30).

Let the winding be 20 turns per cm on each coil; $n_1=n_2=20$.

$A=25$ cm $\qquad N_1=n_1\ A\sqrt{3}$

$a=10$ cm $\qquad N_2=n_2\ a\sqrt{3}$

$$\therefore\ N_1N_2=3n_1n_2Aa$$

$$d=\sqrt{x^2+A^2}=\frac{A}{2}\sqrt{7}$$

$$\therefore \quad M=\frac{2\pi^2a^2N_1N_2}{d}=4\pi^2a^2n_1n_2\left[\frac{3a}{\sqrt{7}}\right]$$

$$M=.0179057 \text{ henry.}$$

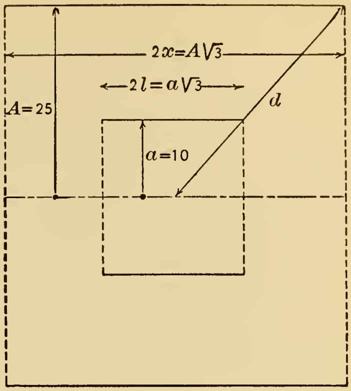

Fig. 47.

We have also calculated the mutual inductance for these coils by Ròiti's formula (30).

The value is, $M=.0179058$, which is practically identical with the value by Gray's formula.

When $A=25$ cm and $a=10$ cm, $N_1=20A\sqrt{3}=866.025$ and $N_2=20a\sqrt{3}=346.4$. As there must be an integral number of turns, let us suppose the winding is exactly 20 turns per cm on each coil and the lengths therefore 43.3 cm and 17.3 cm, respectively. Then $d=\sqrt{x^2+A^2}=\sqrt{625+\left(\frac{43.3}{2}\right)^2}=33.0715$ cm. This does not exactly conform to the condition imposed in deriving the simple formula (32) of Gray used above. Hence (32) will not be as exact with these slightly altered dimensions, and we must calculate at least one correction term to get an accurate value of M.

By Gray's formula (32), $M=\frac{2\pi^2 100\times 866\times 346}{33.0715\times 10^9}=.0178842$ henry.

The first correction term in (34) increases this value to .0178854 henry.

We will now calculate the mutual inductance of these coils by Ròiti's formula (30):

$A=25$	$2x=43.3$	$l_1=13.0$ cm	$\rho_1=$	28.17800
$a=10$	$2l=17.3$	$l_2=30.3$ "	$\rho_2=$	39.28218
			$\rho_2-\rho_1=$	11.10418
			2nd term $=+$	.22030
			3rd $=-$	.01781
			4th $=+$	.01952
			5th $=+$	.00156
			6th $=-$	.00453
			7th $=+$	.00274
			Sum $=$	11.32596

$$M=\frac{4\pi^2a^2n_1n_2\times 11.32596}{10^9}\text{ henry,}$$
$$=.0178853\text{ henry.}$$

This differs from the result by Gray's formula by only 1 part in 178000.

In taking the dimensions of coils where an accurate value of the mutual inductance is sought it should be borne in mind that the above formulæ have been derived on the supposition that the current is uniformly distributed over the length of the coaxial solenoids; in other words, these formulæ are all current-sheet formulæ. Hence, for coils made up of many turns of wire we must meet the conditions imposed by current-sheet formulæ. In calculating self-inductances, this requires an accurate determination of the size of the wire and of the distance between the axes of successive wires, from which we can calculate two correction terms to be combined with the value of L given by the current-sheet formulæ.[83]

In the case of mutual inductances, however, there are no correction terms to calculate; but we must take the dimensions of the current sheets that are equivalent to the coils of wire; that is, the radius of each coil will be the mean distance to the center of the wire, and the length of each will be the over-all length, including the insulation, when the coil is wound of insulated wire in contact, or the length from center to center of a winding of $n+1$ turns, where n is the whole number of turns used.[84] It is also very important that the winding on both coils shall be uniform,[85] and that the leads shall be brought out so that there shall be no mutual inductance due to them.

The mutual inductance will of course be different at high frequencies from its value at low frequencies. We assume, however that for all purposes for which an extremely accurate mutual inductance is desired the frequency of the current would be low, say, not more than a few hundred per second. If the value at very high frequency is desired the coil should be wound with stranded wire, each strand of which is separately insulated.

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[83] Rosa, this Bulletin, **2**, p. 181; 1906.

[84] Rosa, this Bulletin, **2**, p. 161, 1906; and vol. 3, p. 1, 1907.

[85] Searle and Airey, Electrician (London), **56**, p. 318; 1905.

4 MUTUAL INDUCTANCE OF A CIRCLE AND A COAXIAL SOLENOID.

EXAMPLE 25. ROSA'S FORMULA (42) COMPARED WITH JONES'S FORMULA (40).

Professor Jones gave the calculations by formula (40) of the constant of the Lorenz apparatus made for McGill University, obtaining the values given below, the second value being that obtained after the plate had been reground and again measured.

A calculation[86] of the same two cases by formula (42) gives very closely agreeing results.

	1st Value.	2nd Value, disc slightly smaller.
By formula (40) $M=$	18,056.36	18,042.52
" " (42) $M=$	18,056.34	18,042.62
Difference	.02	—.10

These differences, amounting to 1 part in a million in the first case and 5 parts in a million in the second case, are wholly negligible in the most refined experimental work.

EXAMPLE 26. FORMULA (42) COMPARED WITH JONES'S FIRST FORMULA.

Take as a second example the case given by Jones[87] to illustrate his first formula.

$A=10$ inches $\quad a=5$ inches $\quad x=2$ inches

$$d^2=104 \qquad \frac{a^2A^2}{d^4}=\frac{2500}{10816} \qquad \log\frac{a^2A^2}{d^4}=\bar{1}.3638733$$

	1st term	$=1.0000000$
	2 "	$=.0866771$
$X_2=2.8400$	3 "	$=.0118537$
$X_4=2.1064$	4 "	$=.0017781$
$X_6=1.5208$	5 "	$=.0002670$
$X_8=1.0173$	6 "	$=.0000379$
$X_{10}=0.5815$	7 "	$=.0000060$
	Sum	$=1.1006198$
	$\frac{2\pi^2a^2}{d}$	$=48.38972$

[86] This Bulletin, 3, p. 218; 1907.

[87] Phil. Mag., 27, p. 61; 1889. In this example, P_0 should be 0.654870 instead of 0.54870, as printed in Jones's article.

$\therefore M = 53.25868\ N$, N being the number of turns of wire on the coil.

Jones gives $M = 53.25879\ N$.

The difference between these values is 2 parts in a million.

EXAMPLE 27. CALCULATION OF CONSTANT OF AYRTON-JONES CURRENT BALANCE BY FORMULÆ (40) AND (42).

As a further test of the formulæ let us calculate the constant of an electro-dynamometer or current balance of the Ayrton-Jones type, of which AB, Fig. 48, is the upper fixed coil and ED is the moving coil, the circle S at the upper end lying in the plane through the middle of AB and the circle R at the lower end of ED lying in the middle plane of the lower fixed coil BC.

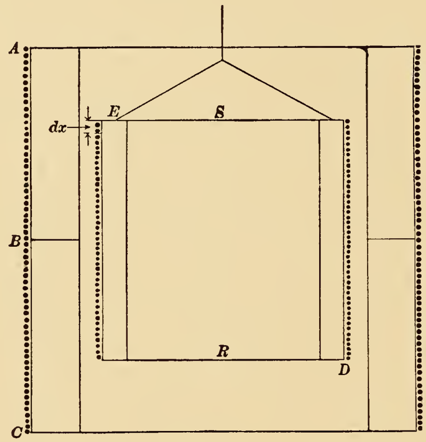

Fig. 48.

Assume the dimensions as follows:

$A = 16$ cm = radius of fixed coil, Fig. 16.

$a = 10$ cm = radius of moving coil.

$x_1 = 8$ cm = half length of $AB = O_1A$

$x_2 = 24$ cm = 1.5 times $AB = O_2A$

$n_1 = 10$ = number of turns per cm

$N_1 = 80$ = number of turns in distance $O_1A = x_1$, Fig. 16.

$N_2 = 240$ = number of turns in distance $O_2A = x_2$

$$d_1 = \sqrt{A^2 + x_1^2} = 8\sqrt{5} = \text{diagonal } AP_1, \text{ Fig. 16.}$$

$$d_2 = \sqrt{A^2 + x_2^2} = 8\sqrt{13} = \text{diagonal } AP_2$$

We have to determine two mutual inductances, namely, M_S between the coil O_1A of 80 turns on the circle S, and M_R between the coil O_2A of 240 turns on the circle R. In each case the circle is in the plane passing through the lower end of the coil.

Formula (12) will be used, taking N_1, x_1, and d_1 in the first case and N_2, x_2, and d_2 in the second case.

	For M_s	*For* M_R
A	16 cm	16 cm
a	10	10
x	8	24
A^2	256	256
x^2	64	576
$N=nx$	80	240
d^2	320	832
log d^2	2.5051500	2.9201233
$\frac{a^2A^2}{d^4}$	$\bar{1}$.3979400	$\bar{2}$.5679934
$\frac{x^2}{A^2}$	$\bar{1}$.3979400	0.1760913
X_2	+2.000	− 6.00
X_4	+0.250	+ 0.25
X_6	−0.9375	+23.5
X_8	−1.203	−45.7
X_{10}	−0.562	−49.0
1st term	1.0000000	1.0000000
2d "	+ .0937500	+ .0138683
3d "	+ .0097656	− .0006411
4th "	+ .0002670	+ .0000009
5th "	− .0002253	+ .0000027
6th "	− .0000662	− .0000002
7th "	- .0000036	.0000000
Sum = S	1.1034875	1.0132306

log S_1	=	0.0427674	log S_2	=	0.0057083
" $2\pi^2$	=	1.2953298	" $2\pi^2$	=	1.2953298
" $a^2(=100)$	=	2.0000000	" $a^2(=100)$	=	2.0000000
" $N_1(=80)$	=	1.9030900	" $N_2(=240)$	=	2.3802112
		5.2411872			5.6812493
" d_1	=	1.2525750	" d_2	=	1.4600616
log M_s	=	3.9886122	log M_R	=	4.2211877
$\therefore M_s$	=	9741.19	M_R	=	16641.32

THE SAME EXAMPLE BY JONES'S FORMULA.

We will now calculate M_S and M_R by Jones's second formula given above, using also the following equation to find $F-\Pi$:

$$\frac{k'^2 \sin\beta\cos\beta(F-\Pi)}{c} = F(k)E(k',\beta)+E(k)F(k',\beta)-F(k)F(k',\beta)-\frac{\pi}{2}$$

	For M_S	For M_R
A	16 cm	16 cm
a	10	10
x	8	24
$\Theta=2\pi N$	$160\,\pi$	$480\,\pi$
$c=\frac{2\sqrt{Aa}}{A+a}$	0.9730085	0.9730085
$c'=\sqrt{1-c^2}$	0.2307692	0.2307692
$k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+x^2}}$	0.9299812	0.7149701
$k'=\sqrt{1-k^2}$	0.3676073	0.6991550
$\log\sin\beta\left(\sin\beta=\frac{c'}{k'}\right)$	9.7977938	9.5186043
$F(k)$	2.4373371	1.8636661
$E(k)$	1.1323456	1.3449927
$\frac{F-E}{k^2}$	1.5088957	1.0146546
$F(k', \beta)$	0.6852557	0.3394833
$E(k', \beta)$	0.6721988	0.3333201
$\frac{k'^2\sin\beta\cos\beta\,(F-\Pi)}{c}$	−0.8266738	−1.1256799
$\frac{c'^2}{c^2}(F-\pi)$	−0.6851799	−0.4045298
$\log\left\{\frac{F-E}{k^2}+\frac{c'^2}{c^2}(F-\Pi)\right\}$	$\bar{1}.9157773$	$\bar{1}.7854187$
$\log\,(\Theta(A+a)ck)$	4.0728340	4.4357689
$\log M$	3.9886113	4.2211876
	$M_S=9741.17$ cm	$M_R=16641.32$

M_S differs from the value obtained by formula (12) by 2 parts in a million, M_R is identical.

M_S is the mutual inductance of the winding O_1A on S. The inductance M_1 of the whole coil AB on S is twice as much, that is

$$M_1 = 19482.34$$

The inductance of AB on R is M_R above, minus the inductance of O_2B on R which is the same as that of O_1A on S, that is, M_S. Therefore,

$$M_2 = 16641.32 - 9741.17 = 6900.15$$

Hence $M_1 - M_2 = 12582.19$ cm.

The force of attraction of the one winding AB in dynes is

$$\frac{1}{2} f = i_1 i_2 n_2 \ (M_1 - M_2).$$

The force due to the second winding BC is equal to this. Suppose $i_1 = i_2 = 1$ ampere $= 0.1$ c.g.s. unit of current and $n_2 = 10$ turns per cm. Then

$$i_1 i_2 n_2 = 0.10$$

$$\begin{aligned} \therefore f &= 0.20 \times 12582.19 \text{ dynes} \\ &= 2516.438 \text{ dynes} \\ 2f &= 5032.876 \text{ dynes} = \text{change of force on reversal of current} \\ &= 5.1356 \text{ gms where } g = 980. \end{aligned}$$

If there are two sets of coils, one on each side of the balance, as in the ampere balance built for the National Physical Laboratory, the force would be doubled again.

In calculating the mutual inductance of the disk and surrounding solenoid in the Lorenz apparatus the series (12) will be more convergent when the winding is long, and of course more convergent when the disk is not of too great diameter.

EXAMPLE 28. MUTUAL INDUCTANCE OF CAMPBELL'S FORM OF STANDARD BY FORMULAE (41) AND (42).

A cylinder 20 cm in diameter has two coils of 50 turns each wound as shown in Fig. 49, each covering 5 cm (=AB) with an interval of 10 cm between (=AA′). A secondary coil of 1000 turns of finer wire is wound in a channel S, with a mean radius of 14.5 cm. The magnetic field near S, due to the double solenoid, is very weak, and is zero at some point; at this place M will be a maximum, and variations in M due to small changes in A will be very small. To calculate M for the solenoid AB and the coil S, we take two cases, as in the preceding example. First, M for S and a winding O_2B of 100 turns; second, M for S and O_2A of 50 turns. The difference will be M for S and the actual winding AB. Or, supposing

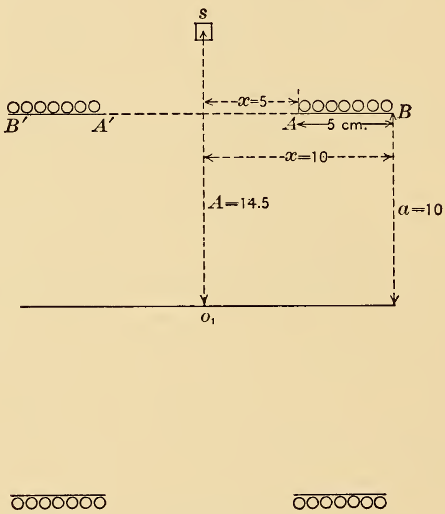

Fig. 49.

AB to have 100 turns, M will be the same as for AB of 50 and A′ B′ of 50. Using formula (41) we have the following values:

	For M_1		For M_2
$a=$	10	$=$	10
$A=$	14.5	$=$	14.5
$x=b=$	10	$=$	5.0
$\log k=$	1.9590874	$=$	1.98366715
$\gamma=$	$65^\circ 31' 7''.32$	$=$	$74^\circ 23' 38''.88$
$k'=$	$\sqrt{0.1717243}$	$=$	$\sqrt{0.0723711}$
$\beta=$	$26^\circ 18' 36''.85$	$=$	$43^\circ 3' 33''.02$
$\gamma'=$	$24^\circ 28' 52''.71$	$=$	$15^\circ 36' 21''.19$
$F=$	2.3267717	$=$	2.7312000
$E=$	1.1590043	$=$	1.0812388
$\frac{c}{k}(F-E)=$	1.2612955	$=$	1.6839704
$F(k,\beta)=$	0.4618972	$=$	0.7561693
$E(k,\beta)=$	0.4565314	$=$	0.7469284
$\psi=$	-1.0479406	$=$	-0.7784355

$$\frac{A-a}{b}\psi = -0.4715733 \qquad\qquad = -0.7005920$$

$$\frac{c}{k}(F-E)+\frac{A-a}{b}\psi = \quad 0.7897222 \qquad\qquad = \quad 0.9833784$$

$$n_1n_2 = \quad 200{,}000 \qquad\qquad = \quad 100{,}000$$

$$M_1 = \quad 24{,}313{,}660 \text{ cm} \qquad M_2 = \quad 15{,}137{,}960 \text{ cm}$$

$$= \quad 24.31366 \text{ millihenrys} = \quad 15.13796 \text{ millihenrys}$$

$$M = \quad M_1 - M_2 = 9.1757 \text{ millihenrys.}$$

Campbell gives[35] the value of M as 9.1762 millihenrys, but for want of any particulars of his calculation we do not know wherein the difference lies.

We have worked this problem out also by formula (42) with the following results:

$$M_1 = \quad 24.31369 \text{ millihenrys}$$

$$M_2 = \quad 15.13917 \quad \text{"} \quad \text{"}$$

$$M = \quad 9.1745 \quad \text{"} \quad \text{"}$$

The value of M_1 agrees with that found by (41) to about one part in a million. M_2 is, however, a little larger, making M smaller. This is due to the fact that formula (42) is not as convergent for $x=5$ in this problem as for $x=10$, and hence the terms neglected after the seventh are appreciable. Hence, for so short a coil as this, formula (40) or (41) will give a more accurate result than (42).

5. CIRCULAR RINGS OF CIRCULAR SECTION.

EXAMPLE 29. COMPARISON OF FIVE FORMULÆ FOR THE SELF INDUCTANCE OF CIRCLES.

For a circle of radius $a=25$ cm and $\rho=0.05$ cm we obtain from the five formulæ the following values of L:

By Kirchhoff's formula (45)	$L=654.40496\pi$ cm
By Maxwell's formula (47)	$L=654.40533\pi$ cm
By Wien's formula (49)	$L=654.40537\pi$ cm
By Rayleigh and Niven's (51)	$L=654.40548\pi$ cm
By Wien's second formula (50)	$L=654.40617\pi$ cm

Thus for so small a value of $\frac{\rho}{a}$ as 1/500 any of these formulæ is sufficiently accurate, the greatest difference being less than 1 in a million, except in the case of formula (50).

[35] A. Campbell, Proc. Roy. Soc., **79**, p. 428; 1907.

EXAMPLE 30. SECOND COMPARISON OF FIVE FORMULÆ FOR CIRCLES.

For a circle of radius $a=25$ cm, $\rho=0.5$ cm, $\frac{\rho}{a}$ being 1/50.

By Kirchhoff's formula (45)	$L=424.1464\pi$ cm
By Maxwell's formula (47)	$L=424.1734\pi$ cm
By Wien's formula (49)	$L=424.1761\pi$ cm
By Rayleigh and Niven's formula (51)	$L=424.1781\pi$ cm
By Wien's second formula (50)	$L=424.2326\pi$ cm

EXAMPLE 31. THIRD COMPARISON OF FIVE FORMULÆ FOR CIRCLES.

For a circle of radius $a=10$ cm, $\rho=1.0$, $\frac{\rho}{a}=1/10$.

By Kirchhoff's formula (45)	$L=105.281\pi$ cm
By Maxwell's formula (47)	$L=105.476\pi$ cm
By Wien's formula (49)	$L=105.497\pi$ cm
By Rayleigh and Niven's formula (51)	$L=105.517\pi$ cm
By Wien's second formula (50)	$L=105.902\pi$ cm

It will be seen that for the smallest ring of radius 10 cm and diameter of section 2 cm Maxwell's formula gives a result 1 part in 5,000 too small and Rayleigh and Niven's a value as much too large, while the simple approximate formula of Kirchhoff is in error by 1 in 500. For the larger ring the differences are much smaller.

Wien's second formula gives appreciably larger values than the others, as it should do.

6. SINGLE LAYER SOLENOIDS.

EXAMPLE 32. RAYLEIGH AND NIVEN'S FORMULA (54) AND CORRECTION FORMULA (59) COMPARED WITH THE SUMMATION FORMULA (60).

$a=25$ cm, $b=1$ cm, $n=10$ turns. Suppose the bare wire is 0.8 mm diameter, the covered wire 1.0 mm.

By formula (54)

$$L_s=4\pi\times 25\times 100\left\{\log_e 200-\frac{1}{2}+\frac{1}{20,000}\left(\log_e 200+\frac{1}{4}\right)\right\}$$

$$=10,000\pi\times 4.798595$$

$$=47,985.95\ \pi \text{ cm}$$

which is the value of L for a current sheet.

The correction ΔL by formula (59) is

$$\Delta L = 1000\ \pi\ (A+B)$$

Since $D=1.0$ mm and $d=0.8$ mm,

$$d/D = 0.8$$

By Table VII, $A=0.3337$

" " VIII, $B=0.2664$

$$A+B=0.6001$$

$$\therefore \quad \Delta L = 600.1\ \pi \text{ cm.}$$

$$\therefore \quad L = 47{,}985.95\ \pi - 600.1\ \pi$$

or, $$L = 47{,}385.85\ \pi \text{ cm.}$$

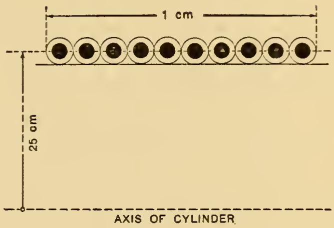

Fig. 50.

The value of L may also be calculated by the summation formula (60), using Wien's formula (49) for L_1 and Maxwell's formula (9) for the M's. The following are the values of the ten terms of (60) and the resulting value of L:

$$\begin{aligned}
10\ L &= 6767.20\ \pi \text{ cm.}\\
18\ M_{12} &= 10081.66\ \pi\\
16\ M_{13} &= 7852.54\ \pi\\
14\ M_{14} &= 6303.44\ \pi\\
12\ M_{15} &= 5057.87\ \pi\\
10\ M_{16} &= 3991.89\ \pi\\
8\ M_{17} &= 3047.79\ \pi\\
6\ M_{18} &= 2193.46\ \pi\\
4\ M_{19} &= 1408.98\ \pi\\
2\ M_{110} &= 680.99\ \pi\\
\text{Sum} = L &= 47385.82\ \pi \text{ cm.}
\end{aligned}$$

The difference of less than one in a million between the results obtained by formulæ (54) and (59) combined and formula (60) is a good check on the corrections of (59), which amount in this case to more than one per cent of the value of the self-inductance. Formula (54) for as short a coil as this is very accurate, the next term, the fourth term of (56), being inappreciable.

EXAMPLE 33.

As an extreme case to test the use of formulæ (54) and (59) we may calculate the self-inductance of a single turn of wire. Let us take the particular case already calculated by Maxwell's and Wien's formulæ, (47) and (49), example 29. The radius $a = 25$ cm, the diameter of the bare wire $= 1$ mm. We may now assume that the wire is covered and that the diameter D is 2 mm. Then $\frac{d}{D} = 0.5$. In using Rayleigh's current sheet formula we take the length of the equivalent current sheet as equal to D. We thus have

$$L_s = 4\pi a \left\{ \log_e \frac{200}{0.2} - \frac{1}{2} + \frac{0.04}{20,000} \left(\log_e \frac{200}{0.2} + \frac{1}{4} \right) \right\}$$

$$= 100\pi \left\{ 6.907755 - 0.5 + \frac{7.16}{500,000} \right\}$$

$$= 640.777\pi \text{ cm.}$$

From Tables VII and VIII $A = -0.1363$ and $B = 0$. Thus, since $n = 1$, $\Delta L = 4\pi a \times (-0.1363) = -13.63\pi$, and being negative is added to L_s. Hence

$$L = (640.777 + 13.63)\,\pi$$
$$= 654.407\,\pi.$$

This is practically identical with the value (654.405π cm) given by the other formulæ, example 29, the slight difference being due to the fact that the correction term A is carried only to four places of decimals.

If we had taken the bare wire of diameter 0.1 cm as equivalent to a current sheet 0.1 cm long in the above formulæ for L_s, we should have obtained a different value for L_s, but in that case $\frac{d}{D}$ would be unity and A would be $+.5568$. The resulting value of L would, however, be the same as above.

EXAMPLE 34. COFFIN'S FORMULA (56) COMPARED WITH LORENZ'S (58).

We will use for this case a single layer coil wound on an accurately measured marble cylinder belonging to the Bureau of Standards.

Length of winding, $l = 30.5510$ cm $= b$ in formula (58)
Radius " " $a = 27.0862$ cm
Number of turns, $n = 440$.

By (56)

$$L=4\pi\overline{440}^2\times 27.0862\left\{1.4590689+0.0878241-0.0020427\right.$$
$$\left.+.0001651-0.0000204\right\}$$

$$=4\pi\overline{440}^2\times 27.0862\times 1.5449950$$
$$=101810000 \text{ cm}=0.1018100 \text{ henry.}$$

By (58)

$$d^2=\ a^2+b^2=3868.0128$$
$$4a^2-b^2=2001.2858$$
$$\gamma=60°\ 34'\ 43.''61$$
$$\log F=0.3369388$$
$$\text{“}\ \ E=0.0811833$$

Then

$$L=\frac{4\pi.\overline{440}^2}{3\,(30.551)^2}.\left\{150050.14+126105.38-158977.00\right\}$$

or, $\qquad L=101810200 \text{ cm}=0.1018102$ henry.

The agreement is very close indeed, and a like agreement could be depended upon for all coils having the ratio of length to radius as small as in this case. For longer coils the difference rapidly increases.

EXAMPLE 35. STRASSER'S FORMULA (61) COMPARED WITH (54) AND (59) AND WITH (60.)

Take the coil of 10 turns used in example 32.

$a=25, \qquad d=0.10 \qquad \rho=0.04, \qquad n=10.$

From Table V, $\qquad A=97.92, \quad B=4187.55$

Substituting in (61),

$$L=100\pi\left[\ 10\left(\log_e\frac{200}{.04}-1.75\right)+90\left(\log_e\frac{200}{0.1}-2\right)-97.92\right.$$
$$\left.+\frac{100}{5000}\left\{\left(3\log_e\frac{200}{0.1}-1\right)\frac{9900}{12}-4187.55\right\}\right]$$

or, $\qquad L=100\pi\Big[\,473.8329+0.0276\,\Big]=47386.05\ \pi$ cm.

This very close agreement with the results by the other two methods (see example 32) is a confirmation of the accuracy of the constants A and B of Table V. Of course, a close agreement with (60) is to be expected, for (61) is derived directly from (60).

EXAMPLE 36. FORMULÆ (62) AND (63) FOR TOROIDAL COILS.

Professor Frölich's standard of self-inductance had the following dimensions:

$r_2 = 35.05377$ cm = outer mean radius.
$r_1 = 24.97478$ cm = inner mean radius.
$h = 20.08455$ cm = height, center to center of wire.
$\rho = 0.011147$ cm = radius of wire.
$n = 2738$ = whole number of turns.

These values substituted in (62) give

$$L_s = 0.1020893 \text{ henry.}$$

The correction $\varDelta L = -2nl\,(A+B)$ to be substituted in (63) to give the true value of L is found as follows:

The mean spacing of the winding is $D = \pi \frac{r_1 + r_2}{n} = 0.0689$

The diameter of the bare wire $d = 2\rho = .0223$

$\therefore \quad d/D = 0.324$

From Table VII,

$$A = -0.57$$
$$B = +0.33[88]$$
$$\therefore A + B = -0.24$$

$2nl = 2 \times 2738 \times 60.327 = 330300$ cm = whole length of wire in winding.

$$-2nl(A+B) = +79{,}300 \text{ cm}$$
$$= 0.0000793 \text{ henry}$$
$$L_s = 0.1020893 \quad \text{"}$$
$$L = 0.1021686 \quad \text{"}$$

Thus, the correction increases the value of the self-inductance. If the insulation were thinner and the wire thicker (with the same pitch) the correction might be of opposite sign. Thus, if ρ were .02

[88] This Bulletin, 4, p. 141; 1907. This value applies to any toroidal coils.

and hence d/D were 0.58, A would be $+0.012$ and ΔL would then be $-.0001130$ and $L = 0.1019763$ henry, considerably less than the preceding value.

7. CIRCULAR COILS OF RECTANGULAR SECTION.

EXAMPLE 37. MAXWELL'S APPROXIMATE FORMULÆ (64), (65) AND PERRY'S APPROXIMATE FORMULA (66) COMPARED WITH WEINSTEIN'S FORMULA (68).

Suppose a coil of mean radius 4 cm, with 100 turns of insulated wire, wound in a square channel 1×1 cm.

Substituting in (64) $a=4$, $n=10$, $R=0.44705$ (the g. m. d. of a square 1 cm on a side) we have

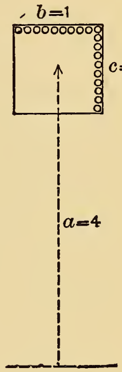

Fig. 51.

$$L = 4\pi\, 100 \left[\log_e \frac{32}{.44705} - 2 \right]$$

$$= 1.141 \text{ millihenrys.}$$

This is a first approximation to the self-inductance of the coil.

Formula (65) gives a second approximation as follows:

$$L = 4\pi\, 100 \left[\log_e \frac{32}{0.44705} \left(1 + \frac{3 \times \overline{0.447}^2}{256} \right) - \left(2 + \frac{\overline{0.447}^2}{256} \right) \right]$$

$$= 1.146 \text{ millihenrys.}$$

Perry's approximate formula, which applies only to relatively short coils, happens to give a very close approximation for this case. Substituting in (66), the above values, and also $b=c=1$,

$$L = \frac{4\pi\, 100 \times 16}{0.9268 + 0.44 + 0.39}$$

$$= 1.144 \text{ millihenrys.}$$

Substituting in the more accurate formula (68) of Weinstein we shall obtain a value with which to compare the above approximations.

$$L = 1600\pi \left[\left(1 + \frac{1}{384} \right) \log_e \frac{32}{1} + 0.03657 \times \frac{1}{16} - 1.194914 \right]$$

$$= 1.147 \text{ millihenrys.}$$

For $a=4$, $b=2$, $c=1$ $n=200$

Formula	(64)	gives	3.750	millihenrys
"	(65)	"	3.787	"
"	(66)	"	3.661	"
"	(68)	"	3.805	"

For $a=10$, $b=1$, $c=1$, $n=100$

Formula	(64)	gives	4.005	millihenrys
"	(65)	"	4.007	"
"	(66)	"	3.993	"
"	(68)	"	4.008	"

It will be seen that formula (66) does not give as close approximations as the others, except in the case of the first example, where it happens to give a value very close to that given by (68). All the values, those of (68) included, are subject to correction by (72) when the coil is wound with round insulated wire.

EXAMPLE 38. FORMULÆ (68) AND (69) COMPARED WITH CURRENT-SHEET FORMULÆ.

As a test of these formulæ we may calculate the self-inductance of a single turn of wire, using the case already calculated in example 33; that is, a circle of radius $a=25$ cm, and the diameter of the bare wire is 1 mm. Substituting these values in (68) we have

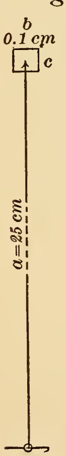

Fig. 52.

$$L=100\pi\left[\left(1+\frac{.01}{15000}\right)\log_e 2000+\frac{.03657}{(250)^2}-1.194914\right]$$

$$=640.5995\ \pi \text{ cm.}$$

Substituting in (69),

$$L=100\pi\left[\left(1+\frac{.01}{15000}\right)\log_e\frac{200}{\sqrt{.02}}-0.848340+\frac{.01\times .8162}{10000}\right]$$

$$=640.5995\ \pi \text{ cm,}$$

agreeing with the value by (68).

These values are for a conductor of square cross section. To reduce to a circular section of same diameter (0.1 cm) we must apply the second correction term of (72); that is, add to the above value

$$\Delta L = 4\pi a \times 0.138060$$

$$\text{Thus, } L = (640.5995 + 13.8060)\pi$$
$$= 654.4055\pi \text{ cm},$$

which agrees with the value found for the self-inductance of a round wire 0.1 cm diameter, bent into a circle of 25 cm radius, by formula (49), example 29, and formulæ (54) and (59), example 33.

EXAMPLE 39. STEFAN'S FORMULA (69) COMPARED WITH (54) BY MEANS OF ROSA'S CORRECTION FORMULA (70).

Suppose a coil of mean radius 10 cm, wound with 100 turns in a square channel 1 × 1 cm. Assuming the current uniformly distributed we obtain from (69), in which $y_1 = 0.848340$, $y_2 = 0.8162$,

$$\log_e \frac{8a}{\sqrt{b^2+c^2}} = \log_e \frac{80}{\sqrt{2}} = 4.03545$$

$$L_u = 4\pi \times 100{,}000 \left[\left(1 + \frac{4}{9600}\right) 4.03545 - 0.84834 + 0.00051 \right]$$
$$= 4\pi \times 318{,}930 \text{ cm}$$
$$= 4.00779 \text{ millihenrys.}$$

By formula (54) we have for the self-inductance of a current sheet for which $a = 10$, $b = 1$, $n = 1$,

$$L_s = 4\pi \times 38.83475 \text{ cm}$$

This is larger than the value for the coil of section 1 × 1 by ΔL, the value of the latter being given by formula (70).

By Table IX, $A = 0.6942$. More closely, it is 0.69415.[89]

By Table X, $B = 0$. In this case $n' = 1$. Hence,

$$\Delta_1 L = 4\pi \times 10 \times 0.69415 = 4\pi \times 6.9415 \text{ cm}$$
$$\therefore L_1 = 4\pi\ (38.83475 - 6.9415) = 400.782 \text{ cm.}$$

This is the value of the self-inductance for one turn only, the current being uniformly distributed. For 100 turns L is 10^4 times as great.

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[89] This Bulletin, 4, p. 369; 1907.

$$\therefore\ L_u = 4.00782 \text{ millihenrys.}$$

This value agrees with the above value by Stefan's formula within less than 1 part in 100,000.

For a coil of the same radius, but of length $b = 10$ cm, $c = 1$ cm, wound with 10 layers of 100 turns each, we have the following values:

By Stefan's formula, $y_1 = 0.59243$, $y_2 = 0.1325$

$$L_u = 4\pi \times 10 \times \overline{1000}^2 \times 1.55536$$
$$= 195.452 \text{ millihenrys.}$$

By (54) the current sheet value of L for 10 turns is

$$L_{10} = 4\pi \times 10 \times 100 \times 1.65095$$
$$= 4\pi \times 1650.95.$$

The correction for depth of section by (70) is, since by Tables IX and X, $A = 0.6942$, $B = 0.2792$, and therefore $A + B = 0.9734$

$$\varDelta_1 L = 4\pi 10 \times 10 \times 0.9734$$
$$= 4\pi \times 97.34$$
$$\therefore\ L_u = L_{10} - \varDelta_1 L = 4\pi(1650.95 - 97.34)$$
$$= 4\pi \times 1553.61 \text{ cm. for 10 turns.}$$

For $n = 1000$ turns the self-inductance will be $\overline{100}^2$ times as great.

$$L_u = 4\pi \times 15.5361 \times 10^6 \text{ cm}$$
$$= 195.232 \text{ millihenrys.}$$

This value is about 1 part in 900 smaller than the above value, showing that Stefan's formula gives too large results by that amount for a coil of this length. If the coil were twice as long, the error would be about ten times as great.

It is interesting to obtain by this method an estimate of the error by Stefan's formula for coils longer than those for which it is intended. For short coils it is seen to be very accurate, subject always to the corrections of formula (72), and for longer coils it gives a good approximation. The method of (70), however, applies to coils of any length.

EXAMPLE 40. STEFAN'S FORMULA (69) COMPARED WITH (60) AND WITH STRASSER'S (61) FOR COILS OF FEW TURNS, USING THE CORRECTION FORMULA (72).

Coil of 2 turns of wire, 0.4 mm diameter, wound in a circle of 1.46 cm radius with a pitch of 2 mm. Stefan's formula assumes a uniform distribution over a rectangular section. Suppose a section as shown in Fig. 53, 4×2 mm, with one turn of wire in the center of each square. For the rectangular section, with the current uniformly distributed, the self-inductance by Stefan's formula is with $a = 1.46$, $c/b = 0.5$, $y_1 = 0.7960$, $y_2 = 0.3066$, $L_u = 4\pi a n^2 \times 2.4763 = 4\pi a n \times 4.9526$, n being 2. To reduce this to the case of a winding of 2 turns of wire as shown we must apply the corrections given by (72) thus:

$$\begin{aligned} \log D/d = \log_e 5 &= 1.60944 \\ \text{second term} &= 0.13806 \\ \text{third term } E &= 0.00653 \\ &\overline{1.7540} \end{aligned}$$

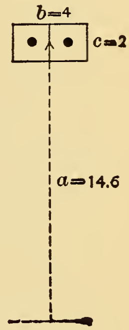

Fig. 53.

$$\therefore \Delta L = 4\pi a n \times 1.754$$

$$L = L_u + \Delta L = 4\pi a n \times 6.7066$$

$$= 246.1 \text{ cm.}$$

By the summation formula (60) we have in this case

$$\begin{aligned} L &= 2L_1 + 2M_{12} \\ &= 4\pi a[9.2400 + 4.1606] \\ &= 245.86 \text{ cm.} \end{aligned}$$

The value by Strasser's formula is the same as by the summation formula to which it is equivalent. We have also used formulæ (54) and (59) for this case and have obtained 246.0.

This is one of several problems calculated by Drude[90] by Stefan's formula. Drude concluded that Stefan's formula was inapplicable to such coils, as it gave results from 10 to 25 per cent too large. His trouble was, however, due to taking the length of the coil as the distance between the center of the first wire and the center of the last (instead of n times the pitch) and neglecting the correction

[90] Wied. Annal., 9, p. 601; 1902.

terms of formula (72). As we have seen above, Stefan's formula when properly used can be depended upon to give accurate results for short coils, and results within less than 1 per cent for coils of length equal to the radius of the coil.

We have calculated several other cases given by Drude and give below the results, together with his experimental values. The radius is the same in each case, and the numbers in the first column are the number of turns in the several coils.

n	By Stefan's Formula (69) and (72)	By Rayleigh's Formula (54) and (59)	By Strasser's Formula (61) or (60)	Drude's Observed Values (Values of L in Centimeters)
2	246.1	246.0	245.9	238.5
4	711.9	711.1	710.8	697.9
6	1298.7	1297.7	1297.8	1271.4
9	2318.0	2313.0	2315.7	2300.1

It will be seen that the values by the different formulæ agree very closely, and that the experimental values agree as closely as could be expected for such small inductances.

EXAMPLE 41. COHEN'S FORMULA (71) COMPARED WITH (70).

A solenoid of length $l=50$ cm, mean radius 5 cm, depth of winding 0.4 cm, is wound with 4 layers of wire of 500 turns each. Substituting these values in (68) we have ($n=10$)

$$L_s = 16\pi^2 n^2 (1144.3 + 3336.0 - 10.84 - 2.07)$$
$$= 70.551 \text{ millihenrys.}$$

By the second method we first find L_s by (54), then $\Delta_1 L$ by (70), and $\Delta_2 L$ by (72)

$$\begin{aligned} L_s &= 72.648 \text{ millihenrys} \\ -\Delta_1 L &= -2.167 \quad \text{"} \\ \Delta_2 L &= \underline{0.048} \quad \text{"} \\ L &= 70.529 \quad \text{"} \end{aligned}$$

This shows a very close agreement between (68) and (67).

In calculating L_s we may use Table IV. Since $d/l=0.2$

$$Q = 3.6324, \; an^2 = 5 \times \overline{2000}^2 = 20{,}000{,}000$$
$$L_s = 3.6324 \times 20{,}000{,}000 \text{ cm}$$

or,

$$L_s = 72.648 \text{ millihenrys.}$$

EXAMPLE 42. FORMULÆ (54) AND (59) COMPARED WITH (69) AND (72) FOR COIL OF 20 TURNS WOUND WITH A SINGLE LAYER.

$$a=25, \quad b=2 \text{ cm}, \quad c=0.1 \text{ cm}, \quad n=20.$$

Diameter of bare wire 0.6 mm, of covered wire 1.0 mm.

In the last case we obtained the self-inductance of the coil by two distinct methods, the first being the method of summation, the second by assuming the current uniformly distributed over the section, and then applying the three corrections C, F, E. In this problem we may first calculate L by use of the current sheet formula (54), and then apply the corrections for section, A and B formula (59); and, second, by Stefan's formula for uniform distribution, and apply the three corrections C, F, E, which give the value for a winding of round insulated wires.

Rayleigh's formula for this example gives:

$$L=4\pi a n^2\left\{\log_e 100-0.5+\frac{4}{20{,}000}\left(\log_e 100+\frac{1}{4}\right)\right\}$$

$$\log_e 100=4.605170$$

$$\frac{4}{20{,}000}\left(\log_e 100+\frac{1}{4}\right)=\underline{0.000971}$$

$$4.606141$$

$$\underline{-0.500000}$$

$$4.106141$$

$$4\pi a n^2=40{,}000\pi, \quad \therefore L_s=164{,}245.64\pi \text{ cm}.$$

This is the self-inductance of a winding of 20 turns of infinitely thin tape, each turn being 1 mm wide, with edges touching without making electrical contact, which arrangement fulfills the conditions of a current sheet. To reduce this to the case of round wires we must apply the corrections A and B for self and mutual induction.[91]

By Table VII, for $d/D=0.6$, $A=.0460$

By Table VIII, for $n=20$, $B=\underline{.2974}$

$$A+B=.3434$$

$$4\pi a n= \quad 2{,}000\pi$$

$$\Delta L=4\pi a n(A+B)= \quad 686.8\pi \text{ cm}$$

$$L=L_s-\Delta L=163{,}558.84\pi \text{ cm}.$$

By Stefan's formula we find, substituting the above values of a, n, b, c, and taking $y_1=.548990$ and $y_2=.1269$

$$L_u=162{,}234.60\pi \text{ cm}.$$

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[91] Rosa, This Bulletin, 2, p. 161; 1906.

The correction E for a single layer coil of 20 turns is given on page 49. The three corrections are then as follows:

$$C = .13806$$
$$F = .51082 = \log_e \frac{10}{6}$$
$$E = \underline{.01357}$$
$$.66245$$
$$\therefore \Delta L = 4\pi an(C+F+E) = 1324.90\pi \text{ cm.}$$
$$\therefore L = L_u + \Delta L = 163{,}559.50\pi \text{ cm.}$$

This value of L is greater than the value found by the other method by only four parts in a million. Thus we see that the method of calculating L_u by Stefan's or Weinstein's formula and applying the corrections C, F, E gives practically identical results with the method of summation and also with the current sheet method *for short coils.* When, however, the coils are longer, the agreement is not so good, for the reason that the formula of Weinstein (and Stefan's, derived from it) is not as accurate when the section of the coil is greater. Thus if the coil in the above problem had been 5 cm long and 2.5 mm deep and wound with 20 turns of heavier wire, the difference would have been 1 part in 25,000 (still very good agreement), and if it were 10 cm long and 0.5 cm deep (the radius being 25 cm) it would have been 1 part in 2,200. For most experimental work, therefore, Stefan's formula is amply accurate.

8. LINEAR CONDUCTORS.

EXAMPLE 43. FORMULÆ (73), (74), (75), AND (76).

A straight copper wire 200 cm long and 0.2 cm diameter will have a self-inductance by formula (74) of

$$L = 200\left(\log_e \frac{200}{0.1} - \frac{3}{4}\right) = 1370.18 \text{ cm.}$$

If it were twice as long

$$L = 400\left(\log_e \frac{400}{0.1} - \frac{3}{4}\right) = 3017.62 \text{ cm.}$$

The more exact formula (73) gives practically the same result where ρ is so small compared with l.

If the wire were of iron with a permeability of 1000, we should have in the first case for $l=100$

$$L=200\ (\log_e 2000-1+250)=51320 \text{ cm.}$$

For sufficiently rapid oscillations so that the current may be considered to be confined to the surface of the wire

$$L=200\ (\log_e 2000-1)=1320.18 \text{ cm.}$$

If the length of the conductor were 10 meters and the diameter 0.2 cm as before the self-inductance by (74) would be

$$L=2000\left(\log_e 20000-\frac{3}{4}\right)=18307.0 \text{ cm}$$
$$=18.307 \text{ microhenrys.}$$

EXAMPLE 44. FORMULÆ (77) AND (78).

Two parallel copper wires of length 100 cm and distance apart 200 cm will have a mutual inductance of

$$M=2\left[100 \log_e \frac{100+100\sqrt{5}}{200}-100\sqrt{5}+200\right]$$
$$=200\left[\log_e \frac{1+\sqrt{5}}{2}-\sqrt{5}+2\right]$$
$$=200\,(\log_e 1.61803-0.2361)$$
$$=49.02 \text{ cm.}$$

If the length of each conductor were 200 cm and the distance apart 100 cm, then

$$M=400\left[\log_e \frac{2+\sqrt{5}}{1}-\frac{\sqrt{5}}{2}+\frac{1}{2}\right]$$
$$=330.24 \text{ cm.}$$

The approximate formula (78) is only applicable when the length of the conductors is great compared with their distance apart. Suppose two conductors 10 meters long are 10 cm apart, then by (78)

$$M=2000\left[\log_e \frac{2000}{10}-1+\frac{10}{1000}\right]$$
$$=2000\ [5.2983-0.990]$$
$$=8616.6 \text{ cm}=8.6166 \text{ microhenrys.}$$

EXAMPLE 45. FORMULÆ (79) AND (80).

Suppose a return circuit of two parallel wires, each 10 meters long and 0.2 cm diameter, distant apart 10 cm, center to center. The self-inductance of the circuit, neglecting the ends, is by (80)

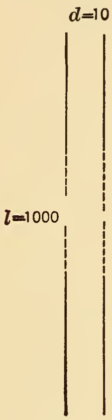

Fig. 54.

$$\begin{aligned} L &= 4000\left[\log_e \frac{10}{0.1} + \frac{1}{4} - \frac{10}{1000}\right] \\ &= 4000 \times 4.8452 \\ &= 19380.8 \text{ cm} = 19.3808 \text{ microhenrys.} \end{aligned}$$

We have already calculated (example 43) the self-inductance of one of these two wires by itself. Doubling the value we have 36.6140 microhenrys as the self-inductance of two wires in series. In example 44 we calculated the mutual inductance of these two wires. Doubling the value for M we have 17.2332 microhenrys. The resultant self-inductance of the circuit (neglecting the ends) is

$$\begin{aligned} L = 2L_1 - 2M &= 36.6140 - 17.2332 \\ &= 19.3808 \text{ microhenrys.} \end{aligned}$$

as found above by formula (78).

Taking account of the ends neglected above, we should find that $2L_1$ for the two ends by (74) is 181.9 cm and $2M$ by (77) is practically zero. Hence the self-inductance of the circuit is, including the ends,

$$L = 19.5627 \text{ microhenrys.}$$

EXAMPLE 46. FORMULA (81) FOR THE MUTUAL INDUCTANCE OF ADJACENT CONDUCTORS IN THE SAME STRAIGHT LINE.

When the two conductors are of equal length, $l = m$, and (81) becomes

$$M = 2\, l \log_e 2 = 2\, l \times 0.69315 \text{ cm.}$$

If $l = 1000$ cm, $M = 1386.3$ cm.
If $m = 1000\, l$, (81) gives

$$\begin{aligned} M &= l \log_e 1001 + 1000\, l \log 1.001 \\ &= l \log_e 1001 + l \text{ approximately.} \end{aligned}$$

If $l=1$ cm, we have

$$M=\log_e 1001+1000 \log_e 1.001$$
$$=6.909+0.999=7.908.$$

The self-inductance of the short wire AB, suppose 1 cm long and of 1 mm radius, is

$$L=2\left(\log_e \frac{2}{0.1}-.75\right)=2\ (2.9957-.75)=4.4915 \text{ cm},$$

which is a little more than one-half of the mutual inductance of AB and BC, BC being 1000 times the length of AB.

In closed circuits, all the magnetic lines due to a circuit are effective in producing self-inductance, and hence the self-inductance is always greater than the mutual inductance of that circuit with any other, assuming one turn in each. But with open circuits, as in this case, we may have a mutual inductance between two single conductors greater than the self-inductance of one of them.

EXAMPLE 47. FORMULA (83) FOR THE SELF-INDUCTANCE OF A RECTANGULAR BAR.

In formula (83), substituting $l=1000$, and $\alpha+\beta=2$ for a square bar 1000 cm long and 1 square cm section, we have, neglecting the small last term,

$$L=2000\left[\log_e \frac{2000}{2}+\frac{1}{2}\right]$$
$$=2000\ (6.908+0.5)=14816 \text{ cm}$$
$$=14.816 \text{ microhenrys.}$$

This would also be the self-inductance for any section having $\alpha+\beta=2$ cm.

EXAMPLE 48. FORMULÆ (84) AND (85) FOR THE SELF-INDUCTANCE OF A SQUARE MADE UP OF A ROUND WIRE.

If the side of the square is one meter, $a=100$ cm, $\rho=0.1$ cm, we have from (84)

$$L=800\ (\log_e 1000-0.524)$$
$$=5107 \text{ cm}=5.107 \text{ microhenrys.}$$

If $\rho=.05$ cm,

$$L=5662 \text{ cm}=5.662 \text{ microhenrys.}$$

That is, the self-inductance of such a rectangle of round wire is about 11 per cent greater for a wire 1 mm in diameter than for one 2 mm in diameter.

If l/ρ is constant, L is proportional to l, that is, if the thickness of the wire is proportional to the length of the wire in the square, the self-inductance of the square is proportional to its linear dimensions.

EXAMPLE 49. FORMULA (86) FOR THE SELF-INDUCTANCE OF A RECTANGLE OF ROUND WIRE.

Suppose a rectangle 2 meters long and 1 meter broad.

Substituting $a=200$ cm, $b=100$, $\rho=0.1$, in (86) we have

$$L=8017.1 \text{ cm}=8.017 \text{ microhenrys.}$$

We can obtain the same result from the values of self and mutual inductances calculated in examples 43 and 44. That is, the resultant self-inductance of the rectangle is the sum of the self-inductances of the four sides, minus twice the mutual inductances of the two pairs of opposite sides. Thus

$$L=(L_1+L_3)+(L_2+L_4)-2M_{13}-2M_{24}$$

$$\begin{array}{lrrr}
\text{By example 43,} & L_1+L_3 & =6035.24 & \\
 & L_2+L_4 & =2740.36 & 8775.60 \\
\text{By example 44,} & 2M_{13} & =660.48 & \\
 & 2M_{24} & =98.04 & 758.52 \\
 & & \therefore L= & 8017.08 \text{ cm} \\
 & & = & 8.0171 \text{ microhenrys.}
\end{array}$$

The agreement of this result with that obtained from formula (86) serves as a check on the latter formula, and also illustrates how the values of the self and mutual inductances of open circuits may be combined to give the self-inductance of a closed circuit.

EXAMPLE 50. FORMULÆ (87), (88), AND (89) FOR THE SELF-INDUCTANCE OF A RECTANGLE OR SQUARE MADE UP OF A BAR OF RECTANGULAR SECTION.

$$\text{Let } a=200,\ b=100,\ \alpha=\beta=1.0 \text{ cm.}$$

Substituting these values in (87) we obtain

$$\begin{aligned} L&=4\ (2971.05-1209.76-577.95-150+447.21+0.99)\\ &=5926.16 \text{ cm.}\end{aligned}$$

For a square 10 meters on a side, made of square bar 1 sq. cm cross section we have $a=1000$, $a=1$; substituting in (89)

$$L=8000\,(6.908+.033)$$
$$=8000\times 6.941 \text{ cm}=55.53 \text{ microhenrys.}$$

For a circular section, diameter 1 cm, $\rho=0.5$; substituting in (84)

$$L=8000\left(\log_e 2000+\frac{1}{2000}-0.524\right)$$
$$=8000\times 7.076 \text{ cm}=56.61 \text{ microhenrys,}$$

a little more than for a square section, as would be expected.

EXAMPLE 51. FORMULA (91) FOR THE MUTUAL INDUCTANCE OF PARALLEL SQUARES.

Suppose two parallel squares each 1 meter on a side, 10 centimeters distant from one another.

$a=100$, $d=10$. Substituting in (91),

$$M=8\left[100\log_e\left(\frac{1+\sqrt{1.01}}{1+\sqrt{2.01}}\cdot\frac{\sqrt{101}}{1}\right)+\sqrt{20100}-2\sqrt{10100}+10\right]$$
$$=800\left[\log_e\left(\frac{10.1+\sqrt{101}}{1+\sqrt{2.01}}\right)+\sqrt{2.01}-2\sqrt{1.01}+0.1\right]$$
$$=1142.5 \text{ cm}=1.1425 \text{ microhenrys.}$$

EXAMPLE 52. FORMULÆ (92), (93) AND (94) FOR THE SELF AND MUTUAL INDUCTANCE OF THIN STRAIGHT STRIPS OR TAPES.

Let the tape of thin copper be 10 meters long and 1 cm wide. Substituting $l=1000$ and $b=1$ in (92) we have

$$L=2000\left(\log_e 2000+\frac{1}{2}\right)$$
$$=2000\times 8.1009=16202 \text{ cm}$$
$$=16.202 \text{ microhenrys,}$$

as the self-inductance when the conducting strip is very thin. If the tape is 2 mm thick we may allow for the effect of the thickness by using (93) and we find

$$L=2000\times 7.9009 \text{ cm}=15.802 \text{ microhenrys,}$$

which differs slightly from the preceding value.

Two such tapes edge to edge in one plane will have a mutual inductance by (91) of

$$\begin{aligned} M &= 2000\ (\log_e 2000 - 0.8863) \\ &= 2000 \times 6.7146 \text{ cm} \\ &= 13.429 \text{ microhenrys.} \end{aligned}$$

EXAMPLE 53. FORMULA (96) FOR THE SELF-INDUCTANCE OF A RETURN CIRCUIT OF TWO PARALLEL SHEETS; NON-INDUCTIVE SHUNTS.

Suppose the dimensions of a thin manganin sheet which has been doubled on itself be as follows:

$$l = 30 \text{ cm}, \quad b = 10 \text{ cm}, \quad d = 1 \text{ cm}.$$

By (111) $\log R_2 = 1.0787$

$$\log R_1 = \log_e 10 - \frac{3}{2} = 0.8026$$

$$\begin{aligned} L &= 4l\ (\log R_2 - \log R_1) \\ &= 120 \times 0.2761 \\ &= 33.13 \text{ cm} \\ &= .0331 \text{ microhenrys.} \end{aligned}$$

EXAMPLE 54. FORMULA (101), 3 CONDUCTORS IN MULTIPLE.

Suppose three cylindrical conductors, each 10 meters long and 4 mm diameter, the distance apart of their centers being 1 cm. Substitute in (101) as follows:

$l = 1000$ cm, $\rho = 2$ mm, $d = 1$ cm. Then

$(r_g a^2)^{\frac{1}{3}} = 0.538$ cm and

$$L = 2000 \left(\log_e \frac{2000}{0.538} - 1 \right)$$

$$= 2000 \times 7.221 \text{ cm} = 14.442 \text{ microhenrys.}$$

If the whole current flowed through a single one of the three conductors the self-inductance would be

$$L = 2000 \left(\log_e \frac{2000}{0.2} - \frac{3}{4} \right) = 17.92 \text{ microhenrys,}$$

or about 25 per cent more than when divided among the three.

APPENDIX.

TABLES OF CONSTANTS AND FUNCTIONS USEFUL IN THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE.

TABLE I.

Maxwell's Table of Values of $\text{Log}\,\frac{M}{4\pi\sqrt{Aa}} = \left[\left(\frac{2}{k} - k\right) F - \frac{2}{k} E\right]$

(For use with Formula 1.)

	$\text{Log}\,\frac{M}{4\pi\sqrt{Aa}}$	Δ_1		$\text{Log}\,\frac{M}{4\pi\sqrt{Aa}}$	Δ_1
60° 0′	$\bar{1}$.499 4780	2 7868	65° 0′	$\bar{1}$.637 6633	2 7508
6′	$\bar{1}$.502 2648	2 7854	6′	$\bar{1}$.640 4141	2 7508
12′	$\bar{1}$.505 0502	2 7840	12′	$\bar{1}$.643 1649	2 7507
18′	$\bar{1}$.507 8342	2 7828	18′	$\bar{1}$.645 9156	2 7507
24′	$\bar{1}$.510 6170	2 7816	24′	$\bar{1}$.648 6663	2 7507
30′	$\bar{1}$.513 3986	2 7803	30′	$\bar{1}$.651 4170	2 7509
36′	$\bar{1}$.516 1789	2 7790	36′	$\bar{1}$.654 1679	2 7510
42′	$\bar{1}$.518 9579	2 7778	42′	$\bar{1}$.656 9189	2 7512
48′	$\bar{1}$.521 7357	2 7765	48′	$\bar{1}$.659 6701	2 7514
54′	$\bar{1}$.524 5122	2 7753	54′	$\bar{1}$.662 4215	2 7516
61° 0′	$\bar{1}$.527 2875	2 7743	66° 0′	$\bar{1}$.665 1731	2 7519
6′	$\bar{1}$.530 0618	2 7734	6′	$\bar{1}$.667 9250	2 7522
12′	$\bar{1}$.532 8352	2 7725	12′	$\bar{1}$.670 6772	2 7524
18′	$\bar{1}$.535 6077	2 7715	18′	$\bar{1}$.673 4296	2 7528
24′	$\bar{1}$.538 3792	2 7705	24′	$\bar{1}$.676 1824	2 7532
30′	$\bar{1}$.541 1497	2 7694	30′	$\bar{1}$.678 9356	2 7535
36′	$\bar{1}$.543 9191	2 7683	36′	$\bar{1}$.681 6891	2 7539
42′	$\bar{1}$.546 6874	2 7672	42′	$\bar{1}$.684 4430	2 7543
48′	$\bar{1}$.549 4546	2 7663	48′	$\bar{1}$.687 1973	2 7548
54′	$\bar{1}$.552 2209	2 7654	54′	$\bar{1}$.689 9521	2 7553
62° 0′	$\bar{1}$.554 9863	2 7645	67° 0′	$\bar{1}$.692 7074	2 7561
6′	$\bar{1}$.557 7508	2 7637	6′	$\bar{1}$.695 4635	2 7567
12′	$\bar{1}$.560 5145	2 7629	12′	$\bar{1}$.698 2202	2 7573
18′	$\bar{1}$.563 2774	2 7622	18′	$\bar{1}$.700 9775	2 7580
24′	$\bar{1}$.566 0396	2 7615	24′	$\bar{1}$.703 7355	2 7587
30′	$\bar{1}$.568 8011	2 7607	30′	$\bar{1}$.706 4942	2 7595
36′	$\bar{1}$.571 5618	2 7598	36′	$\bar{1}$.709 2537	2 7603
42′	$\bar{1}$.574 3216	2 7589	42′	$\bar{1}$.712 0140	2 7610
48′	$\bar{1}$.577 0805	2 7582	48′	$\bar{1}$.714 7750	2 7619
54′	$\bar{1}$.579 8387	2 7575	54′	$\bar{1}$.717 5369	2 7628
63° 0′	$\bar{1}$.582 5962	2 7570	68° 0′	$\bar{1}$.720 2997	2 7637
6′	$\bar{1}$.585 3532	2 7567	6′	$\bar{1}$.723 0634	2 7647
12′	$\bar{1}$.588 1099	2 7563	12′	$\bar{1}$.725 8281	2 7656
18′	$\bar{1}$.590 8662	2 7559	18′	$\bar{1}$.728 5937	2 7667
24′	$\bar{1}$.593 6221	2 7555	24′	$\bar{1}$.731 3604	2 7679
30′	$\bar{1}$.596 3776	2 7549	30′	$\bar{1}$.734 1283	2 7689
36′	$\bar{1}$.599 1325	2 7543	36′	$\bar{1}$.736 8972	2 7701
42′	$\bar{1}$.601 8868	2 7537	42′	$\bar{1}$.739 6673	2 7713
48′	$\bar{1}$.604 6405	2 7533	48′	$\bar{1}$.742 4386	2 7725
54′	$\bar{1}$.607 3938	2 7530	54′	$\bar{1}$.745 2111	2 7737
64° 0′	$\bar{1}$.610 1468	2 7527	69° 0′	$\bar{1}$.747 9848	2 7749
6′	$\bar{1}$.612 8995	2 7524	6′	$\bar{1}$.750 7597	2 7763
12′	$\bar{1}$.615 6519	2 7521	12′	$\bar{1}$.753 5360	2 7778
18′	$\bar{1}$.618 4040	2 7519	18′	$\bar{1}$.756 3138	2 7791
24′	$\bar{1}$.621 1559	2 7516	24′	$\bar{1}$.759 0929	2 7806
30′	$\bar{1}$.623 9075	2 7514	30′	$\bar{1}$.761 8735	2 7821
36′	$\bar{1}$.626 6589	2 7513	36′	$\bar{1}$.764 6556	2 7836
42′	$\bar{1}$.629 4102	2 7512	42′	$\bar{1}$.767 4392	2 7853
48′	$\bar{1}$.632 1614	2 7510	48′	$\bar{1}$.770 2245	2 7871
54′	$\bar{1}$.634 9124	2 7509	54′	$\bar{1}$.773 0116	2 7888
65° 0′	$\bar{1}$.637 6633	2 7508	70° 0′	$\bar{1}$.775 8004	2 7904

	$\log \frac{M}{4\pi\sqrt{Aa}}$	Δ_1		$\log \frac{M}{4\pi\sqrt{Aa}}$	Δ_1
70° 0′	$\bar{1}$.775 8004	2 7904	75° 0′	$\bar{1}$.918 5141	2 9472
6′	$\bar{1}$.778 5908	2 7920	6′	$\bar{1}$.921 4613	2 9522
12′	$\bar{1}$.781 3828	2 7938	12′	$\bar{1}$.924 4135	2 9572
18′	$\bar{1}$.784 1766	2 7956	18′	$\bar{1}$.927 3707	2 9623
24′	$\bar{1}$.786 9722	2 7975	24′	$\bar{1}$.930 3330	2 9676
30′	$\bar{1}$.789 7697	2 7995	30′	$\bar{1}$.933 3006	2 9729
36′	$\bar{1}$.792 5692	2 8017	36′	$\bar{1}$.936 2735	2 9783
42′	$\bar{1}$.795 3709	2 8037	42′	$\bar{1}$.939 2518	2 9838
48′	$\bar{1}$.798 1746	2 8056	48′	$\bar{1}$.942 2356	2 9895
54′	$\bar{1}$.800 9802	2 8078	54′	$\bar{1}$.945 2251	2 9951
71° 0′	$\bar{1}$.803 7880	2 8100	76° 0′	$\bar{1}$.948 2202	3 0007
6′	$\bar{1}$.806 5980	2 8124	6′	$\bar{1}$.951 2209	3 0066
12′	$\bar{1}$.809 4104	2 8148	12′	$\bar{1}$.954 2275	3 0127
18′	$\bar{1}$.812 2252	2 8172	18′	$\bar{1}$.957 2402	3 0188
24′	$\bar{1}$.815 0424	2 8195	24′	$\bar{1}$.960 2590	3 0251
30′	$\bar{1}$.817 8619	2 8220	30′	$\bar{1}$.963 2841	3 0316
36′	$\bar{1}$.820 6839	2 8245	36′	$\bar{1}$.966 3157	3 0380
42′	$\bar{1}$.823 5084	2 8270	42′	$\bar{1}$.969 3537	3 0446
48′	$\bar{1}$.826 3354	2 8297	48′	$\bar{1}$.972 3983	3 0514
54′	$\bar{1}$.829 1651	2 8323	54′	$\bar{1}$.975 4497	3 0583
72° 0′	$\bar{1}$.831 9974	2 8349	77° 0′	$\bar{1}$.978 5080	3 0652
6′	$\bar{1}$.834 8323	2 8377	6′	$\bar{1}$.981 5731	3 0723
12′	$\bar{1}$.837 6700	2 8406	12′	$\bar{1}$.984 6454	3 0795
18′	$\bar{1}$.840 5106	2 8435	18′	$\bar{1}$.987 7249	3 0869
24′	$\bar{1}$.843 3541	2 8464	24′	$\bar{1}$.990 8118	3 0944
30′	$\bar{1}$.846 2005	2 8494	30′	$\bar{1}$.993 9062	3 1020
36′	$\bar{1}$.849 0499	2 8525	36′	$\bar{1}$.997 0082	3 1099
42′	$\bar{1}$.851 9024	2 8556	42′	0.000 1181	3 1178
48′	$\bar{1}$.854 7580	2 8588	48′	0.003 2359	3 1259
54′	$\bar{1}$.857 6168	2 8620	54′	0.006 3618	3 1341
73° 0′	$\bar{1}$.860 4788	2 8653	78° 0′	0.009 4959	3 1426
6′	$\bar{1}$.863 3441	2 8688	6′	0.012 6385	3 1511
12′	$\bar{1}$.866 2129	2 8723	12′	0.015 7896	3 1598
18′	$\bar{1}$.869 0852	2 8759	18′	0.018 9494	3 1687
24′	$\bar{1}$.871 9611	2 8795	24′	0.022 1181	3 1778
30′	$\bar{1}$.874 8406	2 8831	30′	0.025 2959	3 1871
36′	$\bar{1}$.877 7237	2 8869	36′	0.028 4830	3 1964
42′	$\bar{1}$.880 6106	2 8907	42′	0.031 6794	3 2061
48′	$\bar{1}$.883 5013	2 8946	48′	0.034 8855	3 2159
54′	$\bar{1}$.886 3959	2 8986	54′	0.038 1014	3 2258
74° 0′	$\bar{1}$.889 2945	2 9025	79° 0′	0.041 3272	3 2360
6′	$\bar{1}$.892 1970	2 9066	6′	0.044 5633	3 2465
12′	$\bar{1}$.895 1036	2 9108	12′	0.047 8098	3 2570
18′	$\bar{1}$.898 0144	2 9151	18′	0.051 0668	3 2679
24′	$\bar{1}$.900 9295	2 9194	24′	0.054 3347	3 2789
30′	$\bar{1}$.903 8489	2 9239	30′	0.057 6136	3 2901
36′	$\bar{1}$.906 7728	2 9284	36′	0.060 9037	3 3016
42′	$\bar{1}$.909 7012	2 9329	42′	0.064 2053	3 3132
48′	$\bar{1}$.912 6341	2 9376	48′	0.067 5185	3 3252
54′	$\bar{1}$.915 5717	2 9424	54′	0.070 8437	3 3375
75° 0′	$\bar{1}$.918 5141	2 9472	80° 0′	0.074 1812	3 3500

	$\text{Log}\,\frac{M}{4\pi\sqrt{Aa}}$	Δ_1		$\text{Log}\,\frac{M}{4\pi\sqrt{Aa}}$	Δ_1
80° 0′	0.074 1812	3 3500	85° 0′	0.265 4154	4 6004
6′	0.077 5312	3 3628	6′	0.270 0156	4 6499
12′	0.080 8940	3 3760	12′	0.274 6655	4 7015
18′	0.084 2700	3 3892	18′	0.279 3670	4 7553
24′	0.087 6592	3 4027	24′	0.284 1223	4 8109
30′	0.091 0619	3 4165	30′	0.288 9332	4 8689
36′	0.094 4784	3 4307	36′	0.293 8021	4 9293
42′	0.097 9091	3 4452	42′	0.298 7314	4 9924
48′	0.101 3543	3 4601	48′	0.303 7238	5 0585
54′	0.104 8144	3 4752	54′	0.308 7823	5 1274
81° 0′	0.108 2896	3 4906	86° 0′	0.313 9097	5 1995
6′	0.111 7802	3 5064	6′	0.319 1092	5 2751
12′	0.115 2866	3 5226	12′	0.324 3843	5 3544
18′	0.118 8092	3 5392	18′	0.329 7387	5 4375
24′	0.122 3484	3 5561	24′	0.335 1762	5 5250
30′	0.125 9045	3 5735	30′	0.340 7012	5 6172
36′	0.129 4780	3 5912	36′	0.346 3184	5 7143
42′	0.133 0692	3 6094	42′	0.352 0327	5 8168
48′	0.136 6786	3 6280	48′	0.357 8495	5 9254
54′	0.140 3066	3 6470	54′	0.363 7749	6 0404
82° 0′	0.143 9536	3 6667	87° 0′	0.369 8154	6 1624
6′	0.147 6203	3 6869	6′	0.375 9777	6 2923
12′	0.151 3072	3 7076	12′	0.382 2700	6 4306
18′	0.155 0148	3 7287	18′	0.388 7006	6 5786
24′	0.158 7435	3 7503	24′	0.395 2792	6 7370
30′	0.162 4938	3 7722	30′	0.402 0162	6 9072
36′	0.166 2660	3 7949	36′	0.408 9234	7 0904
42′	0.170 0609	3 8183	42′	0.416 0138	7 2884
48′	0.173 8792	3 8425	48′	0.423 3022	7 5031
54′	0.177 7217	3 8673	54′	0.430 8053	7 7373
83° 0′	0.181 5890	3 8926	88° 0′	0.438 5417	7 9921
6′	0.185 4816	3 9185	6′	0.446 5341	8 2723
12′	0.189 4001	3 9452	12′	0.454 8064	8 5816
18′	0.193 3453	3 9728	18′	0.463 3880	8 9247
24′	0.197 3181	4 0013	24′	0.472 3127	9 3079
30′	0.201 3194	4 0308	30′	0.481 6206	9 7389
36′	0.205 3502	4 0606	36′	0.491 3595	10 2275
42′	0.209 4108	4 0915	42′	0.501 5870	10 7868
48′	0.213 5023	4 1236	48′	0.512 3738	11 4341
54′	0.217 6259	4 1565	54′	0.523 8079	12 1932
84° 0′	0.221 7824	4 1904	89° 0′	0.536 0011	13 0958
6′	0.225 9728	4 2255	6′	0.549 0969	14 1917
12′	0.230 1983	4 2617	12′	0.563 2886	15 5520
18′	0.234 4600	4 2991	18′	0.578 8406	17 2914
24′	0.238 7591	4 3379	24′	0.596 1320	19 6050
30′	0.243 0970	4 3778	30′	0.615 7370	22 8537
36′	0.247 4748	4 4192	36′	0.638 5907	27 7976
42′	0.251 8940	4 4621	42′	0.666 3883	36 3882
48′	0.256 3561	4 5065	48′	0.702 7765	55 9176
54′	0.260 8626	4 5526	54′	0.758 6941	
85° 0′	0.265 4154	4 6004			

The above table has been recalculated and some of the values corrected in the last place. The values given are sufficiently accurate to give M within one part in a million.

TABLE II.

Giving the Values of Log F and Log E as Functions of tan γ. (*See p. 41.*)

tan γ	Log F	F	Log E	E
0.1	0.1971 996	1.5747 065	0.1950 415	1.5669 007
0.2	0.2003 678	1.5862 361	0.1918 928	1.5555 817
0.3	0.2054 261	1.6048 192	0.1869 144	1.5378 514
0.4	0.2120 849	1.6296 146	0.1804 536	1.5151 429
0.5	0.2200 096	1.6596 236	0.1729 048	1.4890 346
0.6	0.2288 634	1.6938 051	0.1646 557	1.4610 185
0.7	0.2383 385	1.7311 652	0.1560 492	1.4323 502
0.8	0.2481 728	1.7708 135	0.1473 640	1.4039 900
0.9	0.2581 561	1.8119 912	0.1388 116	1.3766 121
1.0	0.2681 272	1.8540 745	0.1305 409	1.3506 441
1.5	0.3147 473	2.0641 787	0.0955 992	1.2462 329
2.0	0.3535 711	2.2572 057	0.0713 258	1.1784 897
2.5	0.3852 192	2.4278 352	0.0547 850	1.1344 491
3.0	0.4112 984	2.5780 917	0.0432 738	1.1047 748
4.0	0.4518 237	2.8302 429	0.0289 324	1.0688 885
5.0	0.4821 752	3.0351 154	0.0207 426	1.0489 205
7.5	0.5341 061	3.4206 300	0.0109 567	1.0255 497
10.0	0.5682 672	3.7005 581	0.0068 338	1.0158 598
12.5	0.5932 708	3.9198 622	0.0047 004	1.0108 819

TABLE III.

Values of the Constant K as Functions of x A and a A.

(For use in Formula 43.)

x A = / a A	.50	.75	1	1.25	1.50	1.75	2
0.50	9.39283	12.30385	14.27982	15.62795	16.56549	17.23299	17.71973
0.55	9.52044	12.40135	14.34594	15.67140	16.59411	17.25215	17.73283
0.60	9.66358	12.50816	14.41766	15.71837	16.62503	17.27286	17.74701
0.65	9.82296	12.62412	14.49474	15.76867	16.65813	17.29504	17.76221
0.70	9.99921	12.74897	14.57688	15.82212	16.69330	17.31865	17.77841
0.75	10.19272	12.88232	14.66377	15.87850	16.73039	17.34357	17.79554

TABLE IV.

Values of the Constant Q in Formula (58a), $L_s = n^2 a Q$.

For the self-inductance of a single-layer winding on a solenoid; n is the whole number of turns of wire in the winding and a is the mean radius. The corrections by Tables VII and VIII must be made to get L from L_s as usual. (See p. 42.)

In the following table $2a$ is the diameter, b is the length, and therefore $2a/b = \tan \gamma$. (See Fig. 21.)

$2\frac{a}{b} = \tan\gamma$	Q	$2\frac{a}{b} = \tan\gamma$	Q
0.20	3.63240	1.80	19.57938
0.30	5.23368	2.00	20.74631
0.40	6.71017	2.20	21.82049
0.50	8.07470	2.40	22.81496
0.60	9.33892	2.60	23.74013
0.70	10.51349	2.80	24.60482
0.80	11.60790	3.00	25.41613
0.90	12.63059	3.20	26.18009
1.00	13.58892	3.40	26.90177
1.20	15.33799	3.60	27.58548
1.40	16.89840	3.80	28.23494
1.60	18.30354	4.00	28.85335

For an explanation of the above formula see p. 41.

TABLE V.

Constants A and B for Strasser's Formula (61).

n	A	B	n	A	B
1	....	....	16	354.4	35 694
2	....	....	17	415.8	46 757
3	1.386	8.315	18	482.8	60 427
4	4.970	43.296	19	555.5	76 662
5	11.33	140.82	20	634.2	96 910
6	20.90	366.95	21	718.9	119 330
7	34.06	794.73	22	809.7	146 517
8	51.11	1499.55	23	906.6	178 140
9	72.32	2590.62	24	1009.8	217 338
10	97.92	4187.55	25	1119.4	259 868
11	128.17	6572.94	26	1235.4	305 044
12	163.14	9769.47	27	1357.9	359 767
13	202.1	14042.1	28	1487.0	421 783
14	248.2	19532.2	29	1618.1	491 819
15	298.6	26740.1	30	1765.4	570 515

TABLE VI.

Table of Constants for Stefan's Formula (69).

b/c or c/b	y_1	y_2	b/c or c/b	y_1	y_2
0.00	0.50000	0.1250	0.55	0.80815	0.3437
0.05	.54899	.1269	0.60	.81823	.3839
0.10	.59243	.1325	0.65	.82648	.4274
0.15	.63102	.1418	0.70	.83311	.4739
0.20	.66520	.1548	0.75	.83831	.5234
0.25	.69532	.1714	0.80	.84225	.5760
0.30	.72172	.1916	0.85	.84509	.6317
0.35	.74469	.2152	0.90	.84697	.6902
0.40	.76454	.2423	0.95	.84801	.7518
0.45	.78154	.2728	1.00	.84834	.8162
0.50	.79600	.3066			

TABLE VII.

Values of Correction Term A, Depending on the Ratio $\frac{d}{D}$ of the Diameters of Bare and Covered Wire on the Single Layer Coil.

(For use in Formula 59.)

$\frac{d}{D}$	A	Δ_1	$\frac{d}{D}$	A	Δ_1
1.00	0.5568	100	0.70	0.2001	144
.99	.5468	101	.69	.1857	146
.98	.5367	103	.68	.1711	148
.97	.5264	104	.67	.1563	150
.96	.5160	105	.66	.1413	152
.95	.5055	106	.65	.1261	155
.94	.4949	107	.64	.1106	157
.93	.4842	108	.63	.0949	160
.92	.4734	109	.62	.0789	163
.91	.4625	110	.61	.0626	166
.90	.4515	112	.60	.0460	168
.89	.4403	113	.59	.0292	171
.88	.4290	114	.58	.0121	174
.87	.4176	116	.57	— .0053	177
.86	.4060	117	.56	— .0230	180
.85	.3943	118	.55	— .0410	184
.84	.3825	120	.54	— .0594	187
.83	.3705	121	.53	— .0781	190
.82	.3584	123	.52	— .0971	194
.81	.3461	124	.51	— .1165	198
.80	.3337	126	.50	— .1363	
.79	.3211	127	.50	— .1363	1053
.78	.3084	129	.45	— .2416	1178
.77	.2955	131	.40	— .3594	1335
.76	.2824	133	.35	— .4928	1542
.75	.2691	134	.30	— .6471	1823
.74	.2557	136	.25	— .8294	2232
.73	.2421	138	.20	—1.0526	2877
.72	.2283	140	.15	—1.3403	4054
.71	.2143	142	.10	—1.7457	
.70	.2001				

TABLE VIII.

Values of the Correction Term B, Depending on the Number of Turns of Wire on the Single Layer Coil.

(For use in Formula 59.)

Number of Turns	B	Number of Turns	B
1	0.0000	50	0.3186
2	.1137	60	.3216
3	.1663	70	.3239
4	.1973	80	.3257
5	.2180	90	.3270
6	.2329	100	.3280
7	.2443	125	.3298
8	.2532	150	.3311
9	.2604	175	.3321
10	.2664	200	.3328
15	.2857	300	.3343
20	.2974	400	.3351
25	.3042	500	.3356
30	.3083	600	.3359
35	.3119	700	.3361
40	.3148	800	.3363
45	.3169	900	.3364
50	.3186	1000	.3365

TABLE IX.

Value of the Constant A_8 as a Function of t/a, t being the Depth of the Winding and a the Mean Radius.

(For use in Formula 70.)

t/a	A_8
0.	0.6949
0.10	0.6942
0.15	0.6933
0.20	0.6922
0.25	0.6909

TABLE X.

Values of the Correction Term B_8 depending on the Number of Turns of Square Conductor on Single Layer Coil.

(For use in Formula 70.)

Number of Turns	B_8	Number of Turns	B_8
1	0.0000	16	0.3017
2	.1202	17	.3041
3	.1753	18	.3062
4	.2076	19	.3082
5	.2292	20	.3099
6	.2446	21	.3116
7	.2563	22	.3131
8	.2656	23	.3145
9	.2730	24	.3157
10	.2792	25	.3169
11	.2844	26	.3180
12	.2888	27	.3190
13	.2927	28	.3200
14	.2961	29	.3209
15	.2991	30	.3218

TABLE XI.

Table of Napierian Logarithms to nine decimal places for Numbers from 1 to 100.

1	0.000 000 000	51	3.931 825 633
2	0.693 147 181	52	3.951 243 719
3	1.098 612 289	53	3.970 291 914
4	1.386 294 361	54	3.988 984 047
5	1.609 437 912	55	4.007 333 185
6	1.791 759 469	56	4.025 351 691
7	1.945 910 149	57	4.043 051 268
8	2 079 441 542	58	4.060 443 011
9	2.197 224 577	59	4.077 537 444
10	2.302 585 093	60	4.094 344 562
11	2.397 895 273	61	4.110 873 864
12	2.484 906 650	62	4.127 134 385
13	2.564 949 357	63	4.143 134 726
14	2.639 057 330	64	4.158 883 083
15	2.708 050 201	65	4.174 387 270
16	2.772 588 722	66	4.189 654 742
17	2.833 213 344	67	4.204 692 619
18	2.890 371 758	68	4.219 507 705
19	2.944 438 979	69	4.234 106 505
20	2.995 732 274	70	4.248 495 242
21	3.044 522 438	71	4.262 679 877
22	3.091 042 453	72	4.276 666 119
23	3.135 494 216	73	4.290 459 441
24	3.178 053 830	74	4.304 065 093
25	3.218 875 825	75	4.317 488 114
26	3.258 096 538	76	4.330 733 340
27	3.295 836 866	77	4.343 805 422
28	3.332 204 510	78	4.356 708 827
29	3.367 295 830	79	4.369 447 852
30	3.401 197 382	80	4.382 026 635
31	3.433 987 204	81	4.394 339 155
32	3.465 735 903	82	4.406 719 247
33	3.496 507 561	83	4.418 840 608
34	3.526 360 525	84	4.430 816 799
35	3.555 348 061	85	4.442 651 256
36	3.583 518 938	86	4.454 347 296
37	3.610 917 913	87	4.465 908 119
38	3.637 586 160	88	4.477 336 814
39	3.663 561 646	89	4.488 636 370
40	3.688 879 454	90	4.499 809 670
41	3.713 572 067	91	4.510 859 507
42	3.737 669 618	92	4.521 788 577
43	3.761 200 116	93	4.532 599 493
44	3.784 189 634	94	4.543 294 782
45	3.806 662 490	95	4.553 876 892
46	3.828 641 396	96	4.564 348 191
47	3.850 147 602	97	4.574 710 979
48	3.871 201 011	98	4.584 967 479
49	3.891 820 298	99	4.595 119 850
50	3.912 023 005	100	4.605 170 186

$\log 1525 = \log 25 + \log 61$
$\log 9.8 = \log 98 - \log 10$
etc.

TABLE XII.

Values of F and E.

The following table of elliptic integrals of the first and second kind is taken from Legendre's *Traité des Fonctions Elliptiques*, Vol. 2, Table VIII:

	F	Δ1	Δ2		E	Δ1	Δ2
0°	1.570 796	120	239	0°	1.570 796	— 120	—239
1	1.570 916	359	240	1	1.570 677	— 359	—239
2	1.571 275	599	240	2	1.570 318	— 598	—239
3	1.571 874	839	241	3	1.569 720	— 836	—238
4	1.572 712	1 080	22	4	1.568 884	—1 075	—238
5	1.573 792	1 321	243	5	1.567 809	—1 312	—237
6	1.575 114	1 564	244	6	1.566 497	—1 549	—236
7	1.576 678	1 808	246	7	1.564 948	—1 785	—235
8	1.578 486	2 054	247	8	1.563 162	—2 020	—234
9	1.580 541	2 302	249	9	1.561 142	—2 255	—233
10	1.582 843	2 551	252	10	1.558 887	—2 487	—232
11	1.585 394	2 803	254	11	1.556 400	—2 719	—230
12	1.588 197	3 057	257	12	1.553 681	—2 949	—228
13	1.591 254	3 314	260	13	1.550 732	—3 177	—227
14	1.594 568	3 574	263	14	1.547 554	—3 404	—225
15	1.598 142	3 836	266	15	1.544 150	—3 629	—223
16	1.601 978	4 103	270	16	1.540 521	—3 852	—221
17	1.606 081	4 373	274	17	1.536 670	—4 073	—218
18	1.610 454	4 647	278	18	1.532 597	—4 291	—216
19	1.615 101	4 925	283	19	1.528 306	—4 507	—214
20	1.620 026	5 208	288	20	1.523 799	—4 721	—211
21	1.625 234	5 495	293	21	1.519 079	—4 932	—208
22	1.630 729	5 788	298	22	1.514 147	—5 140	—205
23	1.636 517	6 087	304	23	1.509 007	—5 345	—202
24	1.642 604	6 391	311	24	1.503 662	—5 547	—199
25	1.648 995	6 702	317	25	1.498 115	—5 746	—196
26	1.655 697	7 019	324	26	1.492 368	—5 942	—192
27	1.662 716	7 343	332	27	1.486 427	—6 134	—189
28	1.670 059	7 675	340	28	1.480 293	—6 323	—185
29	1.677 735	8 015	349	29	1.473 970	—6 508	—181
30	1.685 750	8 364	358	30	1.467 462	—6 689	—177
31	1.694 114	8 722	367	31	1.460 774	—6 866	—173
32	1.702 836	9 089	377	32	1.453 908	—7 039	—168
33	1.711 925	9 466	388	33	1.446 869	—7 207	—164
34	1.721 391	9 854	400	34	1.439 662	—7 371	—159
35	1.731 245	10 254	412	35	1.432 291	—7 531	—155
36	1.741 499	10 666	425	36	1.424 760	—7 685	—150
37	1.752 165	11 091	439	37	1.417 075	—7 835	—145
38	1.763 256	11 530	453	38	1.409 240	—7 980	—140
39	1.774 786	11 983	469	39	1.401 260	—8 120	—134
40	1.786 770	12 452	486	40	1.393 140	—8 254	—129
41	1.799 222	12 938	504	41	1.384 886	—8 382	—123
42	1.812 160	13 442	523	42	1.376 504	—8 505	—117
43	1.825 602	13 965	543	43	1.367 999	—8 622	—111
44	1.839 567	14 508	565	44	1.359 377	—8 733	—105
45	1.854 075	15 073	588	45	1.350 644	—8 838	— 98

TABLE XII—Continued.

	F	Δ_1	Δ_2		E	Δ_1	Δ_2
45°	1.854 075	15 073	588	45°	1.350 644	−8 838	−98
46	1.869 148	15 661	613	46	1.341 806	−8 936	−92
47	1.884 809	16 274	640	47	1.332 870	−9 028	−85
48	1.901 083	16 914	669	48	1.323 842	−9 113	−78
49	1.917 997	17 584	700	49	1.314 729	−9 190	−71
50	1.935 581	18 284	735	50	1.305 539	−9 261	−63
51	1.953 865	19 017	770	51	1.296 278	−9 324	−56
52	1.972 882	19 787	809	52	1.286 954	−9 380	−48
53	1.992 670	20 597	852	53	1.277 574	−9 427	−40
54	2.013 266	21 449	898	54	1.268 147	−9 467	−31
55	2.034 715	22 347	949	55	1.258 680	−9 498	−22
56	2.057 062	23 296	1 004	56	1.249 182	−9 520	−14
57	2.080 358	24 300	1 064	57	1.239 661	−9 534	− 4
58	2.104 658	25 364	1 130	58	1.230 127	−9 538	+ 5
59	2.130 021	26 494	1 203	59	1.220 589	−9 533	+15
60	2.156 516	27 698	1 284	60	1.211 056	−9 518	+25
61	2.184 213	28 982	1 373	61	1.201 538	−9 492	36
62	2.213 195	30 355	1 472	62	1.192 046	−9 457	47
63	2.243 549	31 827	1 583	63	1.182 589	−9 410	58
64	2.275 376	33 410	1 708	64	1.173 180	−9 351	70
65	2.308 787	35 118	1 848	65	1.163 828	−9 281	82
66	2.343 905	36 965	2 006	66	1.154 547	−9 199	95
67	2.380 870	38 971	2 186	67	1.145 348	−9 104	109
68	2.419 842	41 158	2 393	68	1.136 244	−8 995	123
69	2.460 999	43 551	2 631	69	1.127 250	−8 872	138
70	2.504 550	46 181	2 907	70	1.118 378	−8 734	153
71	2.550 731	49 088	3 230	71	1.109 643	−8 581	169
72	2.599 820	52 318	3 611	72	1.101 062	−8 412	187
73	2.652 138	55 930	4 066	73	1.092 650	−8 225	205
74	2.708 068	59 996	4 614	74	1.084 425	−8 020	224
75	2.768 063	64 609	5 283	75	1.076 405	−7 796	245
76	2.832 673	69 892	6 112	76	1.068 610	−7 550	268
77	2.902 565	76 004	7 156	77	1.061 059	−7 282	292
78	2.978 569	83 160	8 497	78	1.053 777	−6 990	318
79	3.061 729	91 657	10 261	79	1.046 786	−6 672	347
80	3.153 385	101 918	12 647	80	1.040 114	−6 325	379
81	3.255 303	114 565	15 989	81	1.033 789	−5 946	415
82	3.369 868	130 554	20 879	82	1.027 844	−5 531	455
83	3.500 422	151 433	28 453	83	1.022 313	−5 076	502
84	3.651 856	179 886	41 130	84	1.017 237	−4 573	558
85	3.831 742	221 016	64 880	85	1.012 664	−4 016	626
86	4.052 758	285 896	118 167	86	1.008 648	−3 389	715
87	4.338 654	404 063	288 129	87	1.005 259	−2 675	842
88	4.742 717	692 193		88	1.002 584	−1 832	1081
89	5.434 910			89	1.000 752	− 752	
90				90	1.000 000		

TABLE XIII.

Values of log F and log E.

[See Note page 131.]

γ	Log F	Δ1	Δ2	Log E	Δ1	Δ2
45°.0	0.2681 2722	3 4688	105	0.1305 4086	2 8279	52
45.1	0.2684 7411	3 4793	105	0.1302 5807	2 8331	52
45.2	0.2688 2204	3 4898	105	0.1299 7476	2 8383	52
45.3	0.2691 7102	3 5004	106	0.1296 9094	2 8434	52
45.4	0.2695 2106	3 5110	106	0.1294 0659	2 8486	51
45.5	0.2698 7216	3 5216	106	0.1291 2174	2 8537	51
45.6	0.2702 2431	3 5322	106	0.1288 3636	2 8589	51
45.7	0.2705 7753	3 5428	107	0.1285 5048	2 8640	51
45.8	0.2709 3181	3 5535	107	0.1282 6408	2 8691	51
45.9	0.2712 8716	3 5642	107	0.1279 7717	2 8742	51
46.0	0.2716 4358	3 5749	108	0.1276 8975	2 8793	51
46.1	0.2720 0108	3 5857	108	0.1274 0182	2 8844	51
46.2	0.2723 5965	3 5965	108	0.1271 1338	2 8894	50
46.3	0.2727 1930	3 6073	108	0.1268 2444	2 8945	50
46.4	0.2730 8003	3 6181	109	0.1265 3499	2 8995	50
46.5	0.2734 4184	3 6290	109	0.1262 4504	2 9045	50
46.6	0.2738 0474	3 6399	109	0.1259 5459	2 9095	50
46.7	0.2741 6873	3 6508	110	0.1256 6364	2 9145	50
46.8	0.2745 3381	3 6618	110	0.1253 7218	2 9195	50
46.9	0.2748 9999	3 6728	110	0.1250 8023	2 9245	50
47.0	0.2752 6727	3 6838	110	0.1247 8778	2 9295	49
47.1	0.2756 3565	3 6948	111	0.1244 9483	2 9344	49
47.2	0.2760 0513	3 7059	111	0.1242 0139	2 9393	49
47.3	0.2763 7572	3 7170	111	0.1239 0746	2 9443	49
47.4	0.2767 4741	3 7281	112	0.1236 1303	2 9492	49
47.5	0.2771 2023	3 7393	112	0.1233 1811	2 9541	49
47.6	0.2774 9415	3 7505	112	0.1230 2271	2 9589	49
47.7	0.2778 6920	3 7617	112	0.1227 2681	2 9638	49
47.8	0.2782 4537	3 7729	113	0.1224 3043	2 9687	48
47.9	0.2786 2266	3 7842	113	0.1221 3357	2 9735	48
48.0	0.2790 0109	3 7955	113	0.1218 3622	2 9783	48
48.1	0.2793 8064	3 8069	114	0.1215 3838	2 9831	48
48.2	0.2797 6133	3 8183	114	0.1212 4007	2 9879	48
48.3	0.2801 4315	3 8297	114	0.1209 4128	2 9927	48
48.4	0.2805 2612	3 8411	115	0.1206 4201	2 9975	48
48.5	0.2809 1023	3 8526	115	0.1203 4226	3 0022	47
48.6	0.2812 9548	3 8641	115	0.1200 4204	3 0070	47
48.7	0.2816 8189	3 8756	116	0.1197 4134	3 0117	47
48.8	0.2820 6945	3 8872	116	0.1194 4017	3 0164	47
48.9	0.2824 5817	3 8988	116	0.1191 3854	3 0211	47
49.0	0.2828 4805	3 9104	117	0.1188 3643	3 0258	47
49.1	0.2832 3909	3 9221	117	0.1185 3385	3 0304	46
49.2	0.2836 3130	3 9338	117	0.1182 3081	3 0351	46
49.3	0.2840 2467	3 9455	118	0.1179 2730	3 0397	46
49.4	0.2844 1923	3 9573	118	0.1176 2333	3 0443	46
49.5	0.2848 1495	3 9691	118	0.1173 1890	3 0489	46
49.6	0.2852 1186	3 9809	119	0.1170 1401	3 0535	46
49.7	0.2856 0996	3 9928	119	0.1167 0866	3 0581	46
49.8	0.2860 0924	4 0047	119	0.1164 0286	3 0626	45
49.9	0.2864 0971	4 0167	120	0.1160 9660	3 0671	45
50.0	0.2868 1137	4 0286	120	0.1157 8988	3 0717	45

TABLE XIII—Continued.

γ	Log F	Δ1	Δ2	Log E	Δ1	Δ2
50°.0	0.2868 1137	4 0286	120	0.1157 8988	3 0717	45
50.1	0.2872 1424	4 0406	121	0.1154 8271	3 0762	45
50.2	0.2876 1830	4 0527	121	0.1151 7510	3 0807	45
50.3	0.2880 2357	4 0648	121	0.1148 6703	3 0851	45
50.4	0.2884 3005	4 0769	122	0.1145 5852	3 0896	44
50.5	0.2888 3774	4 0891	122	0.1142 4956	3 0940	44
50.6	0.2892 4665	4 1013	122	0.1139 4016	3 0985	44
50.7	0.2896 5677	4 1135	123	0.1136 3032	3 1028	44
50.8	0.2900 6812	4 1258	123	0.1133 2003	3 1072	44
50.9	0.2904 8070	4 1381	123	0.1130 0931	3 1116	43
51.0	0.2908 9451	4 1504	124	0.1126 9815	3 1159	43
51.1	0.2913 0955	4 1628	124	0.1123 8656	3 1203	43
51.2	0.2917 2584	4 1753	125	0.1120 7453	3 1246	43
51.3	0.2921 4336	4 1877	125	0.1117 6207	3 1289	43
51.4	0.2925 6214	4 2002	125	0.1114 4919	3 1332	43
51.5	0.2929 8216	4 2128	126	0.1111 3587	3 1374	42
51.6	0.2934 0344	4 2254	126	0.1108 2213	3 1417	42
51.7	0.2938 2597	4 2380	127	0.1105 0796	3 1459	42
51.8	0.2942 4977	4 2506	127	0.1101 9337	3 1501	42
51.9	0.2946 7483	4 2634	127	0.1098 7836	3 1543	42
52.0	0.2951 0117	4 2761	128	0.1095 6294	3 1584	41
52.1	0.2955 2878	4 2889	128	0.1092 4709	3 1626	41
52.2	0.2959 5767	4 3017	129	0.1089 3083	3 1667	41
52.3	0.2963 8784	4 3146	129	0.1086 1416	3 1708	41
52.4	0.2968 1930	4 3275	130	0.1082 9707	3 1749	41
52.5	0.2972 5205	4 3405	130	0.1079 7958	3 1790	41
52.6	0.2976 8610	4 3535	130	0.1076 6168	3 1831	40
52.7	0.2981 2144	4 3665	131	0.1073 4338	3 1871	40
52.8	0.2985 5810	4 3796	131	0.1070 2467	3 1911	40
52.9	0.2989 9606	4 3927	132	0.1067 0556	3 1951	40
53.0	0.2994 3533	4 4059	132	0.1063 8605	3 1991	40
53.1	0.2998 7592	4 4191	133	0.1060 6614	3 2030	39
53.2	0.3003 1783	4 4324	133	0.1057 4584	3 2070	39
53.3	0.3007 6107	4 4457	134	0.1054 2514	3 2109	39
53.4	0.3012 0564	4 4591	134	0.1051 0406	3 2148	39
53.5	0.3016 5155	4 4725	134	0.1047 8258	3 2186	38
53.6	0.3020 9880	4 4859	135	0.1044 6072	3 2225	38
53.7	0.3025 4739	4 4994	135	0.1041 3847	3 2263	38
53.8	0.3029 9733	4 5130	136	0.1038 1584	3 2301	38
53.9	0.3034 4863	4 5265	136	0.1034 9283	3 2339	38
54.0	0.3039 0128	4 5402	137	0.1031 6944	3 2377	37
54.1	0.3043 5530	4 5539	137	0.1028 4567	3 2414	37
54.2	0.3048 1069	4 5676	138	0.1025 2153	3 2451	37
54.3	0.3052 6745	4 5814	138	0.1021 9702	3 2488	37
54.4	0.3057 2559	4 5952	139	0.1018 7214	3 2525	37
54.5	0.3061 8511	4 6091	139	0.1015 4689	3 2562	36
54.6	0.3066 4602	4 6230	140	0.1012 2127	3 2598	36
54.7	0.3071 0833	4 6370	140	0.1008 9529	3 2634	36
54.8	0.3075 7203	4 6511	141	0.1005 6895	3 2670	36
54.9	0.3080 3714	4 6652	141	0.1002 4226	3 2705	35
55.0	0.3085 0365	4 6793	142	0.0999 1520	3 2741	35

TABLE XIII—Continued.

γ	Log F	Δ1	Δ2	Log E	Δ1	Δ2
55°.0	0.3085 0365	4 6793	142	0.0999 1520	3 2741	35
55.1	0.3089 7158	4 6935	142	0.0995 8779	3 2776	35
55.2	0.3094 4093	4 7077	143	0.0992 6003	3 2811	35
55.3	0.3099 1170	4 7220	143	0.0989 3193	3 2846	34
55.4	0.3103 8391	4 7364	144	0.0986 0347	3 2880	34
55.5	0.3108 5754	4 7508	145	0.0982 7467	3 2914	34
55.6	0.3113 3262	4 7652	145	0.0979 4553	3 2948	34
55.7	0.3118 0915	4 7798	146	0.0976 1605	3 2982	33
55.8	0.3122 8712	4 7943	146	0.0972 8623	3 3015	33
55.9	0.3127 6655	4 8089	147	0.0969 5607	3 3049	33
56.0	0.3132 4745	4 8236	147	0.0966 2559	3 3082	33
56.1	0.3137 2981	4 8384	148	0.0962 9477	3 3114	32
56.2	0.3142 1365	4 8532	149	0.0959 6363	3 3147	32
56.3	0.3146 9896	4 8680	149	0.0956 3216	3 3179	32
56.4	0.3151 8577	4 8829	150	0.0953 0037	3 3211	32
56.5	0.3156 7406	4 8979	150	0.0949 6826	3 3243	31
56.6	0.3161 6385	4 9129	151	0.0946 3583	3 3274	31
56.7	0.3166 5514	4 9280	151	0.0943 0309	3 3305	31
56.8	0.3171 4794	4 9432	152	0.0939 7003	3 3336	31
56.9	0.3176 4226	4 9584	153	0.0936 3667	3 3367	30
57.0	0.3181 3809	4 9736	153	0.0933 0300	3 3397	30
57.1	0.3186 3545	4 9890	154	0.0929 6903	3 3428	30
57.2	0.3191 3435	5 0044	155	0.0926 3475	3 3457	30
57.3	0.3196 3479	5 0198	155	0.0923 0018	3 3487	29
57.4	0.3201 3677	5 0353	156	0.0919 6531	3 3516	29
57.5	0.3206 4030	5 0509	156	0.0916 3014	3 3545	29
57.6	0.3211 4539	5 0666	157	0.0912 9469	3 3574	28
57.7	0.3216 5204	5 0823	158	0.0909 5895	3 3603	28
57.8	0.3221 6027	5 0980	158	0.0906 2292	3 3631	28
57.9	0.3226 7008	5 1139	159	0.0902 8662	3 3659	28
58.0	0.3231 8146	5 1298	160	0.0899 5003	3 3686	27
58.1	0.3236 9444	5 1458	160	0.0896 1317	3 3714	27
58.2	0.3242 0902	5 1618	161	0.0892 7603	3 3741	27
58.3	0.3247 2520	5 1779	162	0.0889 3862	3 3767	26
58.4	0.3252 4299	5 1941	162	0.0886 0095	3 3794	26
58.5	0.3257 6240	5 2104	163	0.0882 6301	3 3820	26
58.6	0.3262 8344	5 2267	164	0.0879 2481	3 3846	26
58.7	0.3268 0611	5 2431	165	0.0875 8635	3 3871	25
58.8	0.3273 3041	5 2595	165	0.0872 4764	3 3897	25
58.9	0.3278 5637	5 2761	166	0.0869 0867	3 3922	25
59.0	0.3283 8397	5 2927	167	0.0865 6945	3 3946	24
59.1	0.3289 1324	5 3094	168	0.0862 2999	3 3971	24
59.2	0.3294 4418	5 3261	168	0.0858 9028	3 3995	24
59.3	0.3299 7679	5 3429	169	0.0855 5033	3 4018	23
59.4	0.3305 1108	5 3598	170	0.0852 1015	3 4042	23
59.5	0.3310 4707	5 3768	171	0.0848 6973	3 4065	23
59.6	0.3315 8475	5 3939	171	0.0845 2908	3 4088	22
59.7	0.3321 2414	5 4110	172	0.0841 8820	3 4110	22
59.8	0.3326 6524	5 4282	173	0.0838 4710	3 4132	22
59.9	0.3332 0806	5 4455	174	0.0835 0578	3 4154	21
60.0	0.3337 5261	5 4629	175	0.0831 6424	3 4176	21

TABLE XIII—Continued.

γ	Log F	Δ1	Δ2	Log E	Δ1	Δ2
60°0	0.3337 5261	5 4629	175	0.0831 6424	3 4176	21
60.1	0.3342 9890	5 4803	175	0.0828 2248	3 4197	21
60.2	0.3348 4694	5 4979	176	0.0824 8051	3 4217	20
60.3	0.3353 9673	5 5155	177	0.0821 3834	3 4238	20
60.4	0.3359 4827	5 5332	178	0.0817 9596	3 4258	20
60.5	0.3365 0159	5 5510	179	0.0814 5338	3 4278	19
60.6	0.3370 5669	5 5688	179	0.0811 1060	3 4297	19
60.7	0.3376 1357	5 5868	180	0.0807 6763	3 4316	19
60.8	0.3381 7225	5 6048	181	0.0804 2446	3 4335	18
60.9	0.3387 3274	5 6229	182	0.0800 8111	3 4354	18
61.0	0.3392 9503	5 6412	183	0.0797 3758	3 4372	18
61.1	0.3398 5915	5 6595	184	0.0793 9386	3 4389	17
61.2	0.3404 2509	5 6778	185	0.0790 4997	3 4407	17
61.3	0.3409 9288	5 6963	186	0.0787 0590	3 4424	17
61.4	0.3415 6251	5 7149	187	0.0783 6167	3 4440	16
61.5	0.3421 3400	5 7336	188	0.0780 1727	3 4456	16
61.6	0.3427 0735	5 7523	188	0.0776 7270	3 4472	15
61.7	0.3432 8258	5 7712	189	0.0773 2798	3 4488	15
61.8	0.3438 5970	5 7901	190	0.0769 8310	3 4503	15
61.9	0.3444 3871	5 8091	191	0.0766 3807	3 4518	14
62.0	0.3450 1962	5 8283	192	0.0762 9290	3 4532	14
62.1	0.3456 0245	5 8475	193	0.0759 4758	3 4546	14
62.2	0.3461 8720	5 8668	194	0.0756 0212	3 4560	13
62.3	0.3467 7388	5 8863	195	0.0752 5652	3 4573	13
62.4	0.3473 6250	5 9058	196	0.0749 1079	3 4586	12
62.5	0.3479 5308	5 9254	197	0.0745 6494	3 4598	12
62.6	0.3485 4562	5 9451	198	0.0742 1895	3 4610	12
62.7	0.3491 4014	5 9650	199	0.0738 7285	3 4622	11
62.8	0.3497 3664	5 9849	200	0.0735 2664	3 4633	11
62.9	0.3503 3513	6 0050	202	0.0731 8030	3 4644	10
63.0	0.3509 3563	6 0251	203	0.0728 3387	3 4654	10
63.1	0.3515 3814	6 0454	204	0.0724 8732	3 4664	10
63.2	0.3521 4268	6 0658	205	0.0721 4068	3 4674	9
63.3	0.3527 4925	6 0862	206	0.0717 9394	3 4683	9
63.4	0.3533 5787	6 1068	207	0.0714 4711	3 4692	8
63.5	0.3539 6856	6 1275	208	0.0711 0019	3 4700	8
63.6	0.3545 8131	6 1483	209	0.0707 5319	3 4708	8
63.7	0.3551 9614	6 1693	210	0.0704 0610	3 4716	7
63.8	0.3558 1307	6 1903	212	0.0700 5895	3 4723	7
63.9	0.3564 3211	6 2115	213	0.0697 1172	3 4729	6
64.0	0.3570 5325	6 2328	214	0.0693 6442	3 4736	6
64.1	0.3576 7653	6 2542	215	0.0690 1706	3 4741	5
64.2	0.3583 0195	6 2757	216	0.0686 6965	3 4747	5
64.3	0.3589 2952	6 2974	218	0.0683 2218	3 4752	4
64.4	0.3595 5926	6 3191	219	0.0679 7466	3 4756	4
64.5	0.3601 9117	6 3410	220	0.0676 2710	3 4760	4
64.6	0.3608 2527	6 3630	221	0.0672 7950	3 4764	3
64.7	0.3614 6158	6 3852	223	0.0669 3186	3 4767	3
64.8	0.3621 0009	6 4075	224	0.0665 8420	3 4769	2
64.9	0.3627 4084	6 4299	225	0.0662 3650	3 4772	2
65.0	0.3633 8383	6 4524	227	0.0658 8879	3 4773	1

TABLE XIII—Continued.

γ	Log F	Δ1	Δ2	Log E	Δ1	Δ2
65°.0	0.3633 8383	6 4524	227	0.0658 8879	3 4773	1
65.1	0.3640 2907	6 4751	228	0.0655 4106	3 4774	1
65.2	0.3646 7658	6 4979	229	0.0651 9331	3 4775	+0
65.3	0.3653 2637	6 5209	231	0.0648 4556	3 4775	—0
65.4	0.3659 7846	6 5439	232	0.0644 9781	3 4775	1
65.5	0.3666 3286	6 5672	234	0.0641 5005	3 4775	1
65.6	0.3672 8957	6 5905	235	0.0638 0231	3 4773	2
65.7	0.3679 4863	6 6141	237	0.0634 5457	3 4772	2
65.8	0.3686 1003	6 6377	238	0.0631 0686	3 4769	3
65.9	0.3692 7380	6 6615	239	0.0627 5916	3 4767	3
66.0	0.3699 3995	6 6855	241	0.0624 1150	3 4764	4
66.1	0.3706 0850	6 7096	242	0.0620 6386	3 4760	4
66.2	0.3712 7946	6 7338	244	0.0617 1626	3 4756	5
66.3	0.3719 5284	6 7582	246	0.0613 6870	3 4751	5
66.4	0.3726 2866	6 7828	247	0.0610 2119	3 4746	6
66.5	0.3733 0694	6 8075	249	0.0606 7373	3 4740	6
66.6	0.3739 8768	6 8324	250	0.0603 2633	3 4734	7
66.7	0.3746 7092	6 8574	252	0.0599 7899	3 4727	7
66.8	0.3753 5666	6 8826	254	0.0596 3172	3 4720	8
66.9	0.3760 4492	6 9080	255	0.0592 8453	3 4712	8
67.0	0.3767 3572	6 9335	257	0.0589 3741	3 4703	9
67.1	0.3774 2907	6 9592	259	0.0585 9037	3 4695	9
67.2	0.3781 2499	6 9851	260	0.0582 4343	3 4685	10
67.3	0.3788 2349	7 0111	262	0.0578 9658	3 4675	11
67.4	0.3795 2460	7 0373	264	0.0575 4983	3 4664	11
67.5	0.3802 2833	7 0637	266	0.0572 0318	3 4653	12
67.6	0.3809 3471	7 0903	268	0.0568 5665	3 4642	12
67.7	0.3816 4373	7 1170	269	0.0565 1023	3 4629	13
67.8	0.3823 5544	7 1440	271	0.0561 6394	3 4617	13
67.9	0.3830 6984	7 1711	273	0.0558 1777	3 4603	14
68.0	0.3837 8695	7 1984	275	0.0554 7174	3 4589	15
68.1	0.3845 0679	7 2259	277	0.0551 2585	3 4575	15
68.2	0.3852 2938	7 2536	279	0.0547 8011	3 4559	16
68.3	0.3859 5475	7 2815	281	0.0544 3451	3 4544	16
68.4	0.3866 8290	7 3096	283	0.0540 8908	3 4527	17
68.5	0.3874 1386	7 3379	285	0.0537 4380	3 4510	18
68.6	0.3881 4765	7 3664	287	0.0533 9870	3 4493	18
68.7	0.3888 8429	7 3951	289	0.0530 5377	3 4475	19
68.8	0.3896 2380	7 4240	291	0.0527 0903	3 4456	19
68.9	0.3903 6620	7 4531	293	0.0523 6447	3 4436	20
69.0	0.3911 1152	7 4825	296	0.0520 2010	3 4416	21
69.1	0.3918 5977	7 5120	298	0.0516 7594	3 4396	21
69.2	0.3926 1097	7 5418	300	0.0513 3198	3 4375	22
69.3	0.3933 6515	7 5718	302	0.0509 8824	3 4353	23
69.4	0.3941 2234	7 6020	305	0.0506 4471	3 4330	23
69.5	0.3948 8254	7 6325	307	0.0503 0141	3 4307	24
69.6	0.3956 4579	7 6632	309	0.0499 5834	3 4283	24
69.7	0.3964 1211	7 6941	312	0.0496 1551	3 4259	25
69.8	0.3971 8152	7 7253	314	0.0492 7292	3 4233	26
69.9	0.3979 5405	7 7567	317	0.0489 3059	3 4208	26
70.0	0.3987 2972	7 7883	319	0.0485 8851	3 4181	27

TABLE XIII—Continued.

γ	Log F	△1	△2	Log E	△1	△2
70°.0	0.3987 2972	7 7883	319	0.0485 8851	3 4181	27
70.1	0.3995 0855	7 8202	322	0.0482 4670	3 4154	28
70.2	0.4002 9058	7 8524	324	0.0479 0516	3 4126	29
70.3	0.4010 7582	7 8848	327	0.0475 6390	3 4098	29
70 4	0.4018 6430	7 9175	329	0.0472 2292	3 4068	30
70.5	0.4026 5605	7 9504	332	0.0468 8224	3 4039	31
70.6	0.4034 5109	7 9836	335	0.0465 4185	3 4008	31
70.7	0.4042 4945	8 0171	337	0.0462 0177	3 3977	32
70.8	0.4050 5116	8 0508	340	0.0458 6201	3 3945	33
70.9	0.4058 5625	8 0849	343	0.0455 2256	3 3912	33
71.0	0.4066 6474	8 1192	346	0.0451 8344	3 3879	34
71.1	0.4074 7666	8 1538	349	0.0448 4465	3 3844	35
71.2	0.4082 9204	8 1887	352	0.0445 0621	3 3810	36
71.3	0.4091 1090	8 2239	355	0.0441 6812	3 3774	36
71.4	0.4099 3329	8 2594	358	0.0438 3038	3 3738	37
71.5	0.4107 5923	8 2952	361	0.0434 9300	3 3700	38
71.6	0.4115 8875	8 3313	364	0.0431 5600	3 3663	39
71.7	0.4124 2187	8 3677	367	0.0428 1937	3 3624	39
71.8	0.4132 5864	8 4044	371	0.0424 8313	3 3585	40
71.9	0.4140 9909	8 4415	374	0.0421 4729	3 3544	41
72.0	0.4149 4324	8 4789	377	0.0418 1184	3 3504	42
72.1	0.4157 9112	8 5166	381	0.0414 7681	3 3462	42
72.2	0.4166 4279	8 5547	384	0.0411 4219	3 3419	43
72.3	0.4174 9826	8 5931	388	0.0408 0799	3 3376	44
72.4	0.4183 5757	8 6319	391	0.0404 7423	3 3332	45
72.5	0.4192 2076	8 6710	395	0.0401 4091	3 3287	46
72.6	0.4200 8786	8 7105	399	0.0398 0804	3 3241	46
72.7	0.4209 5891	8 7503	402	0.0394 7563	3 3195	47
72.8	0.4218 3394	8 7906	406	0.0391 4368	3 3148	48
72.9	0.4227 1300	8 8312	410	0.0388 1220	3 3099	49
73.0	0.4235 9612	8 8722	414	0.0384 8121	3 3050	50
73.1	0.4244 8334	8 9136	418	0.0381 5070	3 3001	51
73.2	0.4253 7470	8 9554	422	0.0378 2070	3 2950	52
73.3	0.4262 7023	8 9976	426	0.0374 9120	3 2898	52
73.4	0.4271 6999	9 0402	430	0.0371 6221	3 2846	53
73.5	0.4280 7401	9 0832	435	0.0368 3375	3 2793	54
73.6	0.4289 8233	9 1267	439	0.0365 0582	3 2739	55
73.7	0.4298 9499	9 1706	443	0.0361 7843	3 2684	56
73.8	0.4308 1205	9 2149	448	0.0358 5160	3 2628	57
73.9	0.4317 3354	9 2597	452	0.0355 2532	3 2571	58
74.0	0.4326 5950	9 3049	457	0.0351 9961	3 2513	59
74.1	0.4335 9000	9 3506	462	0.0348 7448	3 2455	60
74.2	0.4345 2506	9 3968	467	0.0345 4993	3 2395	60
74.3	0.4354 6474	9 4435	472	0.0342 2598	3 2335	61
74.4	0.4364 0909	9 4906	477	0.0339 0263	3 2273	62
74.5	0.4373 5815	9 5583	482	0.0335 7989	3 2211	63
74.6	0.4383 1198	9 5865	487	0.0332 5778	3 2148	64
74.7	0.4392 7063	9 6352	492	0.0329 3630	3 2084	65
74.8	0.4402 3414	9 6844	498	0.0326 1546	3 2019	66
74.9	0.4412 0258	9 7341	503	0.0322 9528	3 1952	67
75.0	0.4421 7599	9 7844	509	0.0319 7575	3 1885	68

TABLE XIII—Continued.

γ	Log F	Δ1	Δ2	Log E	Δ1	Δ2
75°.0	0.4421 7599	9 7844	509	0.0319 7575	3 1885	68
75.1	0.4431 5444	9 8353	514	0.0316 5690	3 1817	69
75.2	0.4441 3797	9 8867	520	0.0313 3872	3 1748	70
75.3	0.4451 2664	9 9387	526	0.0310 2124	3 1678	71
75.4	0.4461 2051	9 9913	532	0.0307 0446	3 1607	72
75.5	0.4471 1965	10 0446	538	0.0303 8839	3 1535	73
75.6	0.4481 2410	10 0984	544	0.0300 7304	3 1462	74
75.7	0.4491 3394	10 1528	551	0.0297 5842	3 1388	75
75.8	0.4501 4922	10 2079	557	0.0294 4454	3 1313	76
75.9	0.4511 7001	10 2637	564	0.0291 3141	3 1237	77
76.0	0.4521 9638	10 3201	571	0.0288 1904	3 1159	78
76.1	0.4532 2839	10 3771	578	0.0285 0745	3 1081	79
76.2	0.4542 6610	10 4349	585	0.0281 9664	3 1002	80
76.3	0.4553 0959	10 4934	592	0.0278 8663	3 0921	82
76.4	0.4563 5893	10 5526	599	0.0275 7742	3 0839	83
76.5	0.4574 1419	10 6126	607	0.0272 6902	3 0757	84
76.6	0.4584 7545	10 6733	615	0.0269 6145	3 0673	85
76.7	0.4595 4278	10 7347	622	0.0266 5472	3 0588	86
76.8	0.4606 1625	10 7970	630	0.0263 4884	3 0502	87
76.9	0.4616 9594	10 8600	639	0.0260 4382	3 0415	88
77.0	0.4627 8195	10 9239	647	0.0257 3967	3 0327	89
77.1	0.4638 7433	10 9886	656	0.0254 3640	3 0237	91
77.2	0.4649 7319	11 0541	664	0.0251 3403	3 0147	92
77.3	0.4660 7860	11 1206	673	0.0248 3257	3 0055	93
77.4	0.4671 9066	11 1879	682	0.0245 3202	2 9962	94
77.5	0.4683 0945	11 2561	692	0.0242 3240	2 9868	95
77.6	0.4694 3506	11 3253	701	0.0239 3372	2 9772	97
77.7	0.4705 6760	11 3954	711	0.0236 3600	2 9676	98
77.8	0.4717 0714	11 4665	721	0.0233 3925	2 9578	99
77.9	0.4728 5379	11 5386	731	0.0230 4347	2 9479	100
78.0	0.4740 0766	11 6118	742	0.0227 4868	2 9378	102
78.1	0.4751 6884	11 6860	753	0.0224 5490	2 9277	103
78.2	0.4763 3743	11 7612	764	0.0221 6213	2 9174	104
78.3	0.4775 1355	11 8376	775	0.0218 7039	2 9070	105
78.4	0.4786 9731	11 9150	786	0.0215 7969	2 8964	107
78.5	0.4798 8881	11 9937	798	0.0212 9005	2 8858	108
78.6	0.4810 8818	12 0735	810	0.0210 0148	2 8750	109
78.7	0.4822 9553	12 1545	823	0.0207 1398	2 8640	111
78.8	0.4835 1098	12 2368	835	0.0204 2758	2 8529	112
78.9	0.4847 3466	12 3203	848	0.0201 4229	2 8417	113
79.0	0.4859 6669	12 4052	862	0.0198 5811	2 8304	115
79.1	0.4872 0721	12 4914	876	0.0195 7507	2 8189	116
79.2	0.4884 5635	12 5789	890	0.0192 9318	2 8073	118
79.3	0.4897 1424	12 6679	904	0.0190 1246	2 7955	119
79.4	0.4909 8103	12 7583	919	0.0187 3291	2 7836	120
79.5	0.4922 5687	12 8503	934	0.0184 5454	2 7716	122
79.6	0.4935 4189	12 9437	950	0.0181 7739	2 7594	123
79.7	0.4948 3626	13 0387	966	0.0179 0145	2 7470	125
79.8	0.4961 4013	13 1353	983	0.0176 2675	2 7345	126
79.9	0.4974 5367	13 2336	1000	0.0173 5330	2 7219	128
80.0	0.4987 7703	13 3336	1018	0.0170 8111	2 7091	129

TABLE XIII—Continued.

γ	Log F	Δ1	Δ2	Log E	Δ1	Δ2
80°0	0.4987 7703	13 3336	1018	0.0170 8111	2 7091	129
80.1	0.5001 1040	13 4354	1036	0.0168 1020	2 6962	131
80.2	0.5014 5394	13 5390	1054	0.0165 4058	2 6831	132
80.3	0.5028 0783	13 6444	1073	0.0162 7227	2 6698	134
80.4	0.5041 7227	13 7517	1093	0.0160 0529	2 6564	136
80.5	0.5055 4744	13 8610	1113	0.0157 3965	2 6429	137
80.6	0.5069 3354	13 9724	1134	0.0154 7536	2 6291	139
80.7	0.5083 3078	14 0858	1156	0.0152 1245	2 6153	140
80.8	0.5097 3936	14 2014	1178	0.0149 5092	2 6012	142
80.9	0.5111 5949	14 3192	1201	0.0146 9080	2 5870	144
81.0	0.5125 9141	14 4393	1225	0.0144 3210	2 5726	145
81.1	0.5140 3534	14 5617	1249	0.0141 7484	2 5581	147
81.2	0.5154 9151	14 6867	1274	0.0139 1903	2 5433	149
81.3	0.5169 6018	14 8141	1300	0.0136 6470	2 5285	151
81.4	0.5184 4159	14 9441	1327	0.0134 1185	2 5134	152
81.5	0.5199 3600	15 0769	1355	0.0131 6052	2 4981	154
81.6	0.5214 4369	15 2124	1384	0.0129 1070	2 4827	156
81.7	0.5229 6493	15 3508	1414	0.0126 6243	2 4671	158
81.8	0.5245 0001	15 4922	1445	0.0124 1572	2 4513	160
81.9	0.5260 4923	15 6366	1477	0.0121 7058	2 4354	162
82.0	0.5276 1289	15 7843	1510	0.0119 2704	2 4192	163
82.1	0.5291 9132	15 9352	1544	0.0116 8512	2 4029	165
82.2	0.5307 8485	16 0896	1579	0.0114 4483	2 3863	167
82.3	0.5323 9381	16 2476	1616	0.0112 0620	2 3696	169
82.4	0.5340 1857	16 4092	1655	0.0109 6924	2 3527	171
82.5	0.5356 5949	16 5747	1694	0.0107 3397	2 3356	173
82.6	0.5373 1696	16 7441	1736	0.0105 0041	2 3183	175
82.7	0.5389 9137	16 9177	1779	0.0102 6859	2 3007	177
82.8	0.5406 8313	17 0955	1823	0.0100 3851	2 2830	179
82.9	0.5423 9268	17 2778	1870	0.0098 1021	2 2651	181
83.0	0.5441 2047	17 4648	1918	0.0095 8371	2 2469	184
83.1	0.5458 6695	17 6566	1968	0.0093 5902	2 2285	186
83.2	0.5476 3260	17 8534	2021	0.0091 3616	2 2100	188
83.3	0.5494 1795	18 0555	2076	0.0089 1517	2 1912	190
83.4	0.5512 2350	18 2631	2133	0.0086 9605	2 1721	193
83.5	0.5530 4980	18 4764	2193	0.0084 7884	2 1529	195
83.6	0.5548 9744	18 6956	2255	0.0082 6355	2 1334	197
83.7	0.5567 6700	18 9211	2320	0.0080 5021	2 1137	199
83.8	0.5586 5912	19 1532	2389	0.0078 3884	2 0937	202
83.9	0.5605 7443	19 3921	2460	0.0076 2947	2 0735	204
84.0	0.5625 1364	19 6381	2535	0.0074 2211	2 0531	207
84.1	0.5644 7745	19 8916	2614	0.0072 1680	2 0324	209
84.2	0.5664 6661	20 1531	2697	0.0070 1356	2 0115	212
84.3	0.5684 8192	20 4228	2784	0.0068 1241	1 9903	214
84.4	0.5705 2420	20 7012	2875	0.0066 1338	1 9689	217
84.5	0.5725 9431	20 9887	2972	0.0064 1649	1 9472	220
84.6	0.5746 9318	21 2859	3073	0.0062 2177	1 9252	222
84.7	0.5768 2177	21 5932	3180	0.0060 2925	1 9029	225
84.8	0.5789 8109	21 9112	3293	0.0058 3896	1 8804	228
84.9	0.5811 7221	22 2405	3413	0.0056 5092	1 8576	231
85.0	0.5833 9626	22 5818	3539	0.0054 6516	1 8345	234

TABLE XIII—Continued.

γ	Log F	Δ1	Δ2	Log E	Δ1	Δ2
85°.0	0.5833 9626	22 5818	3539	0.0054 6516	1 8345	234
85.1	0.5856 5444	22 9357	3673	0.0052 8171	1 8111	237
85.2	0.5879 4801	23 3031	3816	0.0051 0060	1 7874	240
85.3	0.5902 7832	23 6846	3967	0.0049 2185	1 7634	243
85.4	0.5926 4679	24 0813	4127	0.0047 4551	1 7391	246
85.5	0.5950 5492	24 4940	4299	0.0045 7160	1 7145	249
85.6	0.5975 0432	24 9239	4481	0.0044 0015	1 6896	253
85.7	0.5999 9671	25 3720	4676	0.0042 3119	1 6643	256
85.8	0.6025 3391	25 8396	4885	0.0040 6476	1 6387	260
85.9	0.6051 1788	26 3281	5109	0.0039 0089	1 6127	263
86.0	0.6077 5069	26 8390	5349	0.0037 3962	1 5864	267
86.1	0.6104 3459	27 3739	5607	0.0035 8097	1 5598	270
86.2	0.6131 7198	27 9346	5886	0.0034 2499	1 5327	274
86.3	0.6159 6543	28 5231	6186	0.0032 7172	1 5053	278
86.4	0.6188 1775	29 1418	6512	0.0031 2118	1 4775	282
86.5	0.6217 3193	29 7929	6865	0.0029 7343	1 4493	286
86.6	0.6247 1122	30 4794	7248	0.0028 2850	1 4207	290
36.7	0.6277 5916	31 2042	7667	0.0026 8642	1 3917	295
86.8	0.6308 7958	31 9709	8124	0.0025 4725	1 3622	299
86.9	0.6340 7668	32 7834	8626	0.0024 1103	1 3323	304
87.0	0.6373 5501	33 6459	9177	0.0022 7779	1 3020	308
87.1	0.6407 1961	34 5636	9785	0.0021 4759	1 2712	313
87.2	0.6441 7597	35 5422	10459	0.0020 2048	1 2398	318
87.3	0.6477 3019	36 5881	11208	0.0018 9649	1 2080	324
87.4	0.6513 8900	37 7089	12043	0.0017 7569	1 1757	329
87.5	0.6551 5989	38 9132	12980	0.0016 5813	1 1428	335
87.6	0.6590 5121	40 2112	14035	0.0015 4385	1 1093	340
87.7	0.6630 7233	41 6147	15230	0.0014 3292	1 0753	347
87.8	0.6672 3380	43 1377	16590	0.0013 2540	1 0406	353
87.9	0.6715 4757	44 7967	18149	0.0012 2134	1 0053	360
88.0	0.6760 2724	46 6116	19948	0.0011 2081	9693	367
88.1	0.6806 8840	48 6064	22040	0.0010 2387	9327	374
88.2	0.6855 4904	50 8104	24492	0.0009 3060	8953	382
88.3	0.6906 3009	53 2597	27396	0.0008 4107	8571	390
88.4	0.6959 5605	55 9993	30870	0.0007 5536	8181	399
88.5	0.7015 5598	59 0862	35077	0.0006 7355	7782	408
88.6	0.7074 6460	62 5940	40245	0.0005 9573	7374	418
88.7	0.7137 2400	66 6184	46693	0.0005 2199	6956	429
88.8	0.7203 8584	71 2878	54895	0.0004 5242	6527	441
88.9	0.7275 1462	76 7773	65561	0.0003 8715	6087	453
89.0	0.7351 9234	83 3334	79812	0.0003 2628	5633	467
89.1	0.7435 2568	91 3146	99496	0.0002 6995	5166	483
89.2	0.7526 5714	101 2642	127847	0.0002 1829	4683	501
89.3	0.7627 8356	114 0489	170975	0.0001 7146	4181	522
89.4	0.7741 8844	131 1464	241655	0.0001 2965	3660	546
89.5	0.7873 0308	155 3119	370693	0.0000 9305	3114	576
89.6	0.8028 3427	192 3813	650756	0.0000 6192	2538	615
89.7	0.8220 7240	257 4569	1501510	0.0000 3654	1923	670
89.8	0.8478 1809	407 6079		0.0000 1731	1253	774
89.9	0.8885 7889			0.0000 0479	479	
90.0	Inf.			0.0000 0000		

The preceding table of logarithms of the elliptic integrals of the first and second kinds is taken from Legendre's *Traité des Fonctions Elliptiques*, Vol. 2, Table I. The values from 45° to 90° are given for intervals of 0.1°. The values from 0° to 45°, which are comparatively seldom required, have been omitted. For formula and table to be used in interpolation, see page 132.

TABLE XIV.

Binomial Coefficients for Interpolation by Differences.

k	Coefficients of Δ_2 and Δ_3		k	Coefficients of Δ_2 and Δ_3		k	Coefficients of Δ_2 and Δ_3		k	Coefficients of Δ_2 and Δ_3	
	K_2	K_3		K_2	K_3		K_2	K_3		K_2	K_3
.01	—.005	+.003	.26	—.096	+.056	.51	—.125	+.062	.76	—.091	+.038
.02	—.010	+.006	.27	—.099	+.057	.52	—.125	+.062	.77	—.089	+.036
.03	—.015	+.010	.28	—.101	+.058	.53	—.125	+.061	.78	—.086	+.035
.04	—.019	+.013	.29	—.103	+.059	.54	—.124	+.060	.79	—.083	+.033
.05	—.024	+.015	.30	—.105	+.060	.55	—.124	+.060	.80	—.080	+.032
.06	—.028	+.018	.31	—.107	+.060	.56	—.124	+.059	.81	—.077	+.031
.07	—.033	+.021	.32	—.109	+.061	.57	—.123	+.058	.82	—.074	+.029
.08	—.037	+.024	.33	—.111	+.062	.58	—.122	+.058	.83	—.071	+.028
.09	—.041	+.026	.34	—.112	+.062	.59	—.121	+.057	.84	—.067	+.026
.10	—.045	+.028	.35	—.114	+.063	.60	—.120	+.056	.85	—.064	+.024
.11	—.049	+.031	.36	—.115	+.063	.61	—.119	+.055	.86	—.060	+.023
.12	—.053	+.033	.37	—.117	+.063	.62	—.118	+.054	.87	—.057	+.021
.13	—.057	+.035	.38	—.118	+.064	.63	—.117	+.053	.88	—.053	+.020
.14	—.060	+.037	.39	—.119	+.064	.64	—.115	+.052	.89	—.049	+.018
.15	—.064	+.039	.40	—.120	+.064	.65	—.114	+.051	.90	—.045	+.016
.16	—.067	+.041	.41	—.121	+.064	.66	—.112	+.050	.91	—.041	+.015
.17	—.071	+.043	.42	—.122	+.064	.67	—.111	+.049	.92	—.037	+.013
.18	—.074	+.045	.43	—.123	+.064	.68	—.109	+.048	.93	—.033	+.012
.19	—.077	+.046	.44	—.123	+.064	.69	—.107	+.047	.94	—.028	+.010
.20	—.080	+.048	.45	—.124	+.064	.70	—.105	+.045	.95	—.024	+.008
.21	—.083	+.049	.46	—.124	+.064	.71	—.103	+.044	.96	—.019	+.007
.22	—.086	+.051	.47	—.125	+.064	.72	—.101	+.043	.97	—.015	+.005
.23	—.089	+.052	.48	—.125	+.063	.73	—.099	+.042	.98	—.010	+.003
.24	—.091	+.053	.49	—.125	+.063	.74	—.096	+.040	.99	—.005	+.002
.25	—.094	+.055	.50	—.125	+.063	.75	—.094	+.039	1.00	—.000	+.000

INTERPOLATION FORMULA.

$$f(a+h)=f(a)+k\triangle_1+\frac{k(k-1)}{2!}\triangle_2+\frac{k(k-1)(k-2)}{3!}\triangle_3$$
$$+\frac{k(k-1)(k-2)(k-3)}{4!}\triangle_4+\ldots \quad (a)$$

$$\text{or, } f(a+h)=f(a)+k\triangle_1+K_2\triangle_2+K_3\triangle_3+\ldots \quad (b)$$

where the constants K_2 and K_3 are given in the above table as functions of k and

$$k=\frac{h}{\delta}$$

where h is the remainder above the value of a for which the function is given in the table, and δ is the increment of a in the table.

ILLUSTRATION.

To find the value of log F for $49°\ 15'\ 36''=49°.260$

For $49.2°$ log $F=0.2836\,3130=f(a)$

$h=.06$, $\delta=0.1$ $k=0.6$

From Table XIV, $K_2=-.120$

$K_3=+.056$

From Table XIII,

$\triangle_1=39338$

$\triangle_2=117$

$\triangle_3=1$

Substituting these values of K_2, K_3, $\triangle_1$, $\triangle_2$, $\triangle_3$ in formula (b) above we have as the value of log F for the given angle

$$\log F=0.2836\,3130+.00023603-.00000014=0.28386719.$$

WASHINGTON, December 17, 1907.