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ON THE SELF-INDUCTANCE OF CIRCLES.

By Edward B. Rosa and Louis Cohen.

Various formulæ have been given by different authors for the self-inductance of circles; that is, for closed rings of circular cross section. Some of these formulæ are at once convenient and accurate, while others are both inconvenient and unreliable, and should be avoided in numerical calculations. We therefore propose in this paper to critically examine and test these various formulæ, and to show which of them are trustworthy and which are wrong. This seems the more necessary inasmuch as some of the latter have been given by writers of reputation and they have been quoted and used in the belief that they were correct.

The formula for the self-inductance of a circle was first given by Kirchhoff[1] in the following form:

$$L=2l\left\{\log\frac{l}{\rho}-1.508\right\} \tag{1}$$

where l is the circumference of the circular conductor and ρ is the radius of its cross section. This is equivalent to the following:

$$L=4\pi a\left\{\log\frac{8a}{\rho}-1.75\right\} \tag{2}$$

a being the radius of the circle. These formulæ are approximate, being more nearly correct as the ratio ρ/a is smaller.

A more accurate expression can be obtained from Maxwell's principle of the geometrical mean distance. The mutual inductance of two equal parallel circles near each other is, to a close approximation,

[1] Pogg. Annalen, **121**, p. 551; 1864.

$$M=4\pi a\left\{\left(1+\frac{3}{16}\frac{b^2}{a^2}\right)\log\frac{8a}{b}-\left(2+\frac{b^2}{16a^2}\right)\right\} \tag{3}$$

where a is the common radius of the circles and b their distance apart. The self-inductance of a single circular ring is equal to the mutual inductance of two equal and parallel circles whose distance apart is equal to the geometrical mean distance R of the cross section of the ring. Hence

$$L=4\pi a\left\{\left(1+\frac{3R^2}{16a^2}\right)\log\frac{8a}{R}-\left(2+\frac{R^2}{16a^2}\right)\right\} \tag{4}$$

Substituting in this equation the value of the geometrical mean distance for a circular area, $R=\rho\epsilon^{-\frac{1}{4}}=.7788\rho$, we obtain

$$L=4\pi a\left\{\left(1+0.1137\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-.0095\frac{\rho^2}{a^2}-1.75\right\} \tag{5}$$

This is a very accurate formula for circles in which the radius of section ρ is very small in comparison with the radius a of the circle. The geometrical mean distance R has, however, been computed on the supposition of a linear conductor, and can only be applied to circles of relatively small value of ρ/a. We must therefore expect an appreciable error in formula (5) when the ratio ρ/a is not very small. Formulæ 1, 2, and 5 have been deduced on the supposition of a uniform distribution of the current over the cross section of the ring.

If the ring is a hollow circular thin tube, or if the current in the ring is alternating and of extremely high frequency, so that it can be regarded as flowing on the surface of the ring, the geometrical mean distance for the section would be the radius ρ, and we should derive from (4) by substituting $R=\rho$,

$$L=4\pi a\left\{\left(1+\frac{3}{16}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-\frac{\rho^2}{16a^2}-2\right\} \tag{6}$$

In the case of solid rings carrying alternating currents of moderate frequency the value of L would be somewhere between the values given by (5) and (6).

WIEN'S FORMULÆ.

Max Wien[2] has given what is probably the most accurate formula yet derived for the self-inductance of a circle.

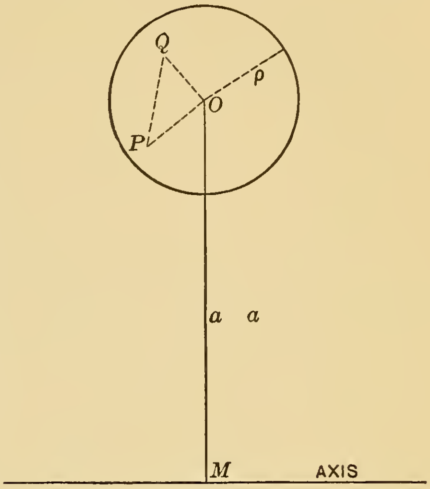

Fig. 1.

If we consider the ring of radius a and radius of section ρ, Fig. 1, to be made up of an indefinite number of elementary circular filaments, the self-inductance of the ring is equal to the mean value of the sum of the mutual inductances on each filament of all the others. If, therefore, we express the mutual inductance of an element at P on a second element at Q and integrate this over the entire area of the section, we obtain the mutual inductance of the single filament P on the entire ring. Integrating again over the ring we obtain the self-inductance of the ring. Wien's result is as follows:

$$L=4\pi a\left\{\left(1+\frac{1}{8}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-.0083\frac{\rho^2}{a^2}-1.75\right\} \qquad (7)$$

It will be noticed that the formula differs very slightly from the preceding (5). Neglecting the terms in ρ^2/a^2 we get from either (5) or (7) Kirchhoff's approximate formula.

If the current be not distributed uniformly over the section of the wire, but the current density at any point is proportional to the distance from the axis of the ring, Wien's formula for the self-inductance is

$$L=4\pi a\left\{\left(1+\frac{3}{8}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-.092\frac{\rho^2}{a^2}-1.75\right\} \qquad (8)$$

which differs very slightly from (7).

This would apply to the case of a ring revolving about a diameter in a uniform magnetic field.

As would be expected, (8) gives a greater value than (7).

Rayleigh and Niven gave[3] the following formula for a circular coil of n turns and of circular section:[4]

[2] Wied. Annalen, **53**, p. 928; 1894.

[3] Rayleigh's Collected Papers, II, p. 15.

[4] Neglecting the correction for effect of insulation and shape of section of the separate wires.

$$L = 4\pi n^2 a \left\{ \left(1 + \frac{\rho^2}{8a^2}\right) \log \frac{8a}{\rho} + \frac{\rho^2}{24a^2} - 1.75 \right\} \qquad (10)$$

When $n = 1$, this will be the self-inductance of a single circular ring. It agrees with Wien's, except as to one term, which is

$$+\frac{\rho^2}{24a^2} \text{ instead of } -0.0083 \frac{\rho^2}{a^2}.$$

If used for a coil of more than one turn, the expression for L (whether obtained from (10) or from one of the preceding more accurate expressions) must be corrected for the space occupied by the insulation between the wires and for the shape of the section.[5]

TESTS OF THE FOREGOING FORMULÆ FOR CIRCLES.

For a circle of radius $a = 25$ cm and $\rho = 0.05$ cm we obtain from the foregoing formulæ the following values of L:

By Wien's formula (7)	$L = 654.40537\ \pi$ cm
By Maxwell's formula (5)	$L = 654.40533\ \pi$ cm
By Rayleigh and Niven's (10)	$L = 654.40548\ \pi$ cm
By Kirchhoff's formula (2)	$L = 654.40496\ \pi$ cm
By Wien's second formula (8)	$L = 654.40617\ \pi$ cm

Thus, for so small a value of ρ/a as $\frac{1}{500}$ any of these formulæ is sufficiently accurate, the greatest difference being less than 1 in a million, except in the case of formula (8).

Take for further tests a circle for which $a = 25$, $\rho = 0.5$ cm, ρ/a being $\frac{1}{50}$, and another with $a = 10$, $\rho = 1.0$, ρ/a being $\frac{1}{10}$

	$\rho/a = \frac{1}{50}$	$\rho/a = \frac{1}{10}$
By Wien's formula	$L = 424.1761\ \pi$	$105.497\ \pi$
By Maxwell's formula	$L = 424.1734\ \pi$	$105.476\ \pi$
By Rayleigh and Niven's formula	$L = 424.1781\ \pi$	$105.517\ \pi$
By Kirchhoff's formula	$L = 424.1464\ \pi$	$105.281\ \pi$
By Wien's second formula	$L = 424.2326\ \pi$	$105.902\ \pi$

It will be seen that for the smallest ring of radius 10 cm and diameter of section 2 cm Maxwell's formula gives a result 1 part in 5,000 too small and Rayleigh and Niven's a value as much too large, while the simple approximate formula of Kirchhoff is in error by 1 in 500. For the larger ring the differences are much smaller.

Wien's second formula gives appreciably larger values, as it should do.

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[5] See Rosa, this Bulletin, **3**, p. 1; 1907.

RUSSELL'S FORMULAE.

In his recent paper in the Philosophical Magazine, Russell[6] derives the approximate expression (2) for the self-inductance of a circle by an original method. Assuming that the flux of magnetic force through the aperture of a ring is the same as though the current in

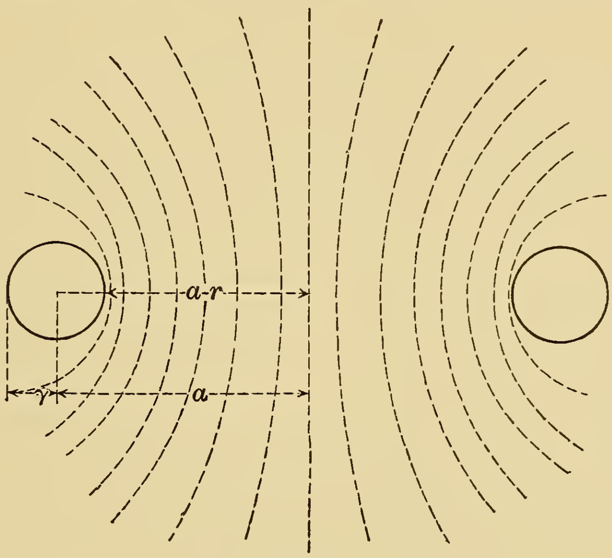

Fig. 2.

the ring were concentrated in the circular axis of the wire, he writes down the expression for the mutual inductance between the two coplanar circles, whose radii are a and $a-r$, which is the following:

$$M=\frac{8\pi a\sqrt{a(a-r)}}{\sqrt{k'}}(F'-E')=8\pi a(F-E)$$

where the modulus is $\dfrac{a-r}{a}$.

This will be *approximately* that part of the self-inductance of a ring due to the flux through the aperture.

He then derives the second part of the flux, namely, that inside the section of the ring, which he gives as

$$\phi_2=2\pi\int_0^r(a-\xi)Z\left(\frac{\xi^2}{r^2}\right)d\,\xi$$

[6] Phil. Mag. **13**, p. 428; 1907.

$$\text{where } Z = \frac{2i'}{\xi} + \frac{i'}{a} \log \frac{8a}{\xi}.$$

The integral of this expression for ϕ_2 is πa *approximately* (neglecting the second term in Z). The total flux inside the section of the ring is, however, $2\pi a$ nearly. The expression πa results from the consideration that these $2\pi a$ tubes cut only part of the section of the conductor. Hence, multiplying by the factor $\frac{\xi^2}{r^2}$ the quantity πa results, which is half the flux and is the second term of the self-inductance. Adding the two terms, Russell obtains his approximate expression for L, which he shows can be put in the form of (2). This is an interesting variation in the method of obtaining the approximate value of L, but in itself gives no indication of the degree of the approximation, as do Maxwell's and Wien's methods.

MINCHIN'S FORMULÆ.

Prof. Minchin[7] has undertaken to derive the self-inductance of a ring by finding the magnetic flux through the aperture of the ring

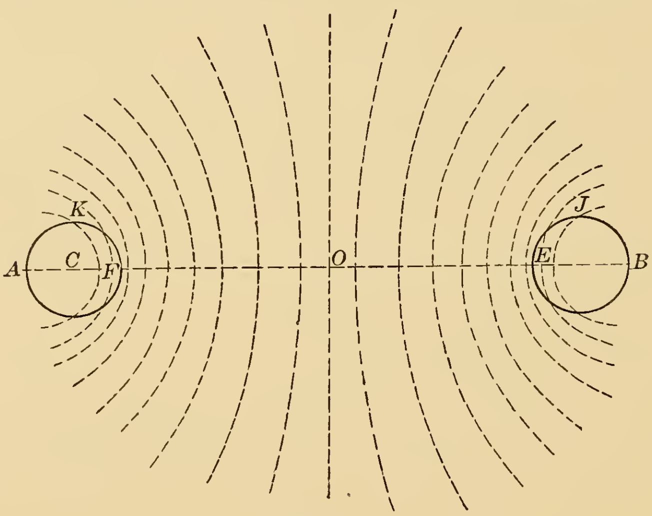

Fig. 3.

and adding the lines emanating from one side of the ring. Thus, suppose in Fig. 3, N_1 lines pass through the aperture FE and N_2 lines emerge from the upper surface EJB (taken throughout the circumference of the ring), then Minchin's statement is that $N_1 + N_2$

[7] Calculation of the coefficient of self-induction of a circular current of given aperture and cross section. Phil. Mag. **37**, p. 300; 1894.

is the total number of lines of force linked with and emanating from the current, and the number divided by i, the strength of the current, is the "coefficient of self-induction." *As this neglects entirely all the lines of force wholly within the ring, which also contribute to the self-inductance*, Minchin's expression for L is necessarily wrong. The large value found in the single example given by Minchin is mainly due to an error in calculation, as his formula gives *too small a value for L.* His expression (5) for the flux through the aperture is substantially correct, but his expression (8) which contains the factor c is very small and does not include the considerable number of lines wholly within the section of the wire. Hence the sum of the two (9) which he gives as the coefficient of self-induction is wrong. Minchin's expression for L slightly rearranged is as follows, where c means the same as ρ above,

$$L=4\pi a\left\{\left(1+\frac{c}{2a}-\frac{c^2}{32a^2}\right)\log\frac{8a}{c}-\left(2+\frac{5c}{8a}+\frac{19c^2}{64a^2}\right)\right\} \qquad (11)$$

This expression is derived on the assumption that the current is inversely proportional to the distance from the axis. When the wire is very small this reduces to

$$L=4\pi a\left(\log\frac{8a}{c}-2\right) \text{ approximately,} \qquad (12)$$

whereas we have seen above that the second term should be 1.75. The difference is due, as just stated, to neglecting the lines of force within the section of the wire; but changing 2 to 1.75 does not make (11) correct.

Minchin also finds the expression for the self-inductance for a superficial current in the circular ring. This is given in his expression (10) and is somewhat *greater* than the other. Of course the self-inductance is *less* for a superficial current than for a distribution through the section of the wire (as we have seen above), whether the latter is uniform or inversely as the distance from the axis of the ring, as assumed by Minchin.

Prof. Minchin says that Maxwell gives the approximate value of L for a circle the same as (12) above, agreeing with his result. This is, however, a mistake. Maxwell gives

$$L=4\pi a\left(\log\frac{8a}{R}-2\right)$$

but R is not the same as c above. R is the geometrical mean distance of the section of the wire, not the radius of section. As already shown, this leads to 1.75 for the absolute term.

HICKS'S FORMULÆ.

Prof. W. M. Hicks[8] has discussed this question from a different standpoint and has derived expressions for the self-inductance of a ring both for the cases of uniform distribution of current, and for current density inversely proportional to the distance from the axis. He also has misinterpreted Maxwell's approximate expression for the self-inductance of a ring, not noticing that r (as he writes it, R as Maxwell wrote it) is the geometrical mean distance of the section and not the radius of section.

Hicks derives two formulæ, one for uniform current density and the other for current density inversely as r, corresponding to (7) and (9) above. Hicks derived his formulæ by the use of toroidal functions and obtained the following expression:

$$L_u=4\pi a\left\{-\frac{7}{6}-\frac{2}{3}\,\frac{\cos^2 a+3\cos^3 a+4\cos^4 a}{(1+\cos a)^3}\right.$$
$$\left.+\frac{8\cos^5 a}{3(1+\cos a)^4}\left[6\log\frac{8a}{\rho}+\frac{1}{2}+39\left(\log\frac{8a}{\rho}-5\right)k^6\left(\frac{2115}{16}\log\frac{8a}{\rho}-\frac{981}{64}\right)\right]\right\} \tag{13}$$

where $\sin a=\frac{\rho}{a}$, $k^2=\frac{1-\cos a}{1+\cos a}$; a and ρ are as before the radii of the ring and of its section respectively.

When the current density is inversely as the radial distance from the axis,

$$L_v=4\pi a\left\{\frac{4\cos^3 a}{(1+\cos a)^2}\left(\left(1+\frac{7}{2}k^2\right)\log\frac{4}{k}-\frac{3}{2}-\frac{9}{2}k^2\right)-\frac{1+2\cos a}{12}\right\} \tag{14}$$

For ρ/a very small, the terms in k^2 may be neglected and $\cos a=1$ approximately, and we have as before for the approximate value of L

[8] Phil. Mag. **38**, p. 456; 1894.

$$L = 4\pi a\left(\log\frac{8a}{\rho} - 1.75\right)$$

Taking the three circles previously used to test the formulæ of Kirchhoff, Maxwell, Wien, and Rayleigh and Niven we have—

	$\rho/a = \frac{1}{500}$	$\rho/a = \frac{1}{50}$	$\rho/a = \frac{1}{10}$
Hicks (13) . .	$L_u = 654.3898\pi$	423.802π	102.7900π
Hicks (14) . .	$L_v = 654.4048\pi$	424.130π	105.2388π

It will be seen by comparing the above results by Hicks's formulæ with those previously given, that these values are in every case less than given by Wien's two formulæ, and in the case of uniform current density less than the values given by any of the formulæ, *even less than by Kirchhoff's approximate formula.* But the correction terms must always increase the value of the inductance. Hence, it appears that Hicks's formula for uniform density at least, and probably also for variable density, *is entirely untrustworthy, the correction terms making the error greater rather than less.* The approximate formula gives a result too small by $\frac{1}{500}$ in the case of the third ring, where $\rho/a = \frac{1}{10}$, while Hicks's elaborate formula gives a result too small by over 2.5% for this case.

It may be asked how we know the formulæ of Wien and Maxwell to be correct. The answer is that Maxwell's for large rings is derived directly from the expression for the mutual inductance of two parallel circles using the expression for the geometric mean distance of the circular section of the wire, which for a straight wire is an absolute expression, not an approximation. Hence, because they agree we know that for large circles (ρ/a small) both Maxwell's and Wien's expressions are correct to a very high degree, and since for the third circle they agree to 1 in 5000, we may safely assume they are quite accurate for that case also. The correction factor is positive for large circles and must always be positive.

Hicks's formulæ were derived by a very elaborate process, which involved successive approximations. It is evident that the errors occurring in these approximations exceeded the total value of the small correction terms which it was the object of the investigation to determine.

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BLÁTHY'S FORMULÆ.

Bláthy[9] gave an expression for the self-inductance of a circle which he supposed to be exact, and also expanded it into a more convenient form for calculation, the latter being presumably accurate to a very high degree.

The following are Bláthy's formulæ, the first being the so-called "exact expression:"

$$L=4\pi a\left\{\log_e \frac{4a-\rho+\sqrt{16a^2-8a\rho}}{\rho}+\frac{16a^2}{15\rho^2} - \left(\frac{16a^2}{15\rho^2}+\frac{4a}{15\rho}+\frac{8}{5}\right)\sqrt{1-\frac{\rho}{2a}}\right\} \qquad (15)$$

$$L=4\pi a\left\{0.57944+\log_e \frac{a}{\rho}-\frac{2\rho}{a}-\frac{\rho^2}{24a^2}-\frac{\rho^3}{48a^3}-\ldots\right\} \qquad (16)$$

Calculating the self-inductance of the above circles by the first of these formulæ, we have,

For largest circle,	$a=25$	$\rho=0.05$	$L=676.2056\pi$
For second circle,	$a=25$	$\rho=0.5$	$L=449.4835\pi$
For smallest circle,	$a=10$	$\rho=1.0$	$L=115.9656\pi$

Comparing these results with the values by the formulæ of Maxwell, Wien, and Kirchhoff we see that the first is in error by 3.5%, the second by 6%, and the third by 9%. The second formula gives substantially the same results.

Neglecting the three smallest terms in the second formula it may be written

$$L=4\pi a\left(\log \frac{8a}{\rho} - 1.50\right)$$

The absolute term should be 1.75 instead of 1.50, and this accounts for the principal part of the error in the results by Bláthy's formulæ.

Examining his method, we see that two assumptions have been made that are not permissible. The first is that one may integrate continuously up to the center of the wire in finding the total flux

[9] London Electrician, **24**, p. 630; April 25, 1890.

through the ring, and the second in assuming that all the lines within the ring cut the entire area of the section of the ring. This necessarily gives too large a value for L, and makes Bláthy's formula entirely unreliable.

Bláthy's formula is often given,[10] apparently because it is a simple formula and was supposed to be very exact. But Kirchhoff's is much simpler, and, as the three examples given show, is amply accurate for most cases.

We thus see that Kirchhoff's simple approximate formula (2) is not only very convenient, but for many cases amply accurate; that the formulæ (5) and (6) derived by means of Maxwell's principle of the geometrical mean distance, and formulæ (7), (8), and (9), derived by direct integration, are very accurate for all cases except where the cross section of the ring is very large in comparison with the radius a; and that the more complex formulæ of Minchin, Hicks, and Bláthy are wrong as well as inconvenient, and should be avoided.

WASHINGTON, August 10, 1907.

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[10] Heydweiller (Elektrische Messungen) gives only Bláthy's formula for circles.