GEOMETRICAL THEORY OF RADIATING SURFACES WITH DISCUSSION OF LIGHT TUBES.

By Edward P. Hyde.

Theoretical photometry assumes two general laws of radiation. (1) The law of variation of the intensity of illumination of a surface in inverse proportion to the square of the distance of the surface from the luminous source is merely a statement of a geometrical property, if the rectilinear propagation of light is assumed. (2) Lambert's law of variation of the intensity of a luminous surface in direct proportion to the cosine of the angle of emission is an empirical law based primarily on the observation that a uniformly bright sphere, when viewed at a distance, appears as a uniformly bright disk. It would seem to follow from Kirchhoff's law that Lambert's cosine law can be true only for a black body, but no satisfactory experiments have been made, so far as the writer knows, to test the law in its application to glowing surfaces. Numerous investigations of the cosine law as applied to diffusing screens have been undertaken, but the results of these are at considerable variance, due principally to the difficulty of obtaining perfect mat surfaces.

These two laws, the inverse square law and Lambert's cosine law, which are assumed as the basis of theoretical photometry, are applicable to infinitesimal sources. The inverse square law follows rigorously for a point source only, and the cosine law is always assumed as applicable primarily to the infinitesimal elements of a surface, since this is the inference from the observation that a uniformly bright sphere, when viewed at a distance, appears as a uniformly bright disk. These two laws may be stated mathematically as follows:

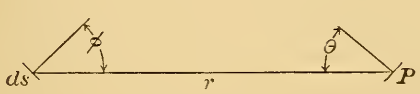

Fig. 1.

$$J = \frac{i\, dS \cos\phi \cos\theta}{r^2} \qquad (1)$$

where

J = intensity of illumination of the screen at P, i. e. the quantity of light per second falling normally on a unit area of the screen at P.

i = specific light intensity of the radiating element of surface dS, i. e. the quantity of light per second emitted normally by dS.

ϕ = angle of emission from dS.

θ = angle of incidence on the screen at P.

r = distance between dS and P.

Since these two laws are deduced for sources of infinitesimal dimensions, errors of considerable magnitude may result in applying them to extended sources. Particularly is this so in the case of the inverse square law. This law underlies the great majority of practical photometric measurements, and its applicability is seldom questioned. Indeed, when it is questioned, it is usually on the ground that since the source is extended it is impossible to determine the effective center of radiation. The source is considered as an aggregate of point sources, rather than as a continuous surface, to each element of which both the inverse square law and the cosine law apply, and which, therefore, considered as a whole, will have a complex law of its own, essentially different from the simple inverse square law.

In a previous paper[1] the writer solved the case for a finite cylinder in connection with a study of Talbot's law as applied to the rotating sectored disk. Since this case serves as an excellent illustration of the difference between a point source and a radiating surface, showing the errors resulting from an assumption of the inverse square law for such a surface, its solution will be repeated here with some additions.

Assuming Lambert's law and the inverse square law for infinitesimal elements of surface, as combined in equation (1), if we let ϕ (Fig. 2) be the angle of emission from any element of surface dS of the radiating cylinder; θ the angle of incidence of the ray from dS on a screen at P placed at right angles to OP, where OP lies in

[1] This Bulletin, 2, p. 1; 1906.

the plane perpendicular to the axis of the cylinder at its middle point, O; r the distance from the element dS to the screen at P; and i the specific light intensity, supposed to be constant over the surface of the cylinder: then the intensity of illumination of the screen at P is

$$J = \int\int \frac{i \cos\phi \cos\theta \, dS}{r^2}$$

taken over that part of the curved surface of the cylinder convex toward P.

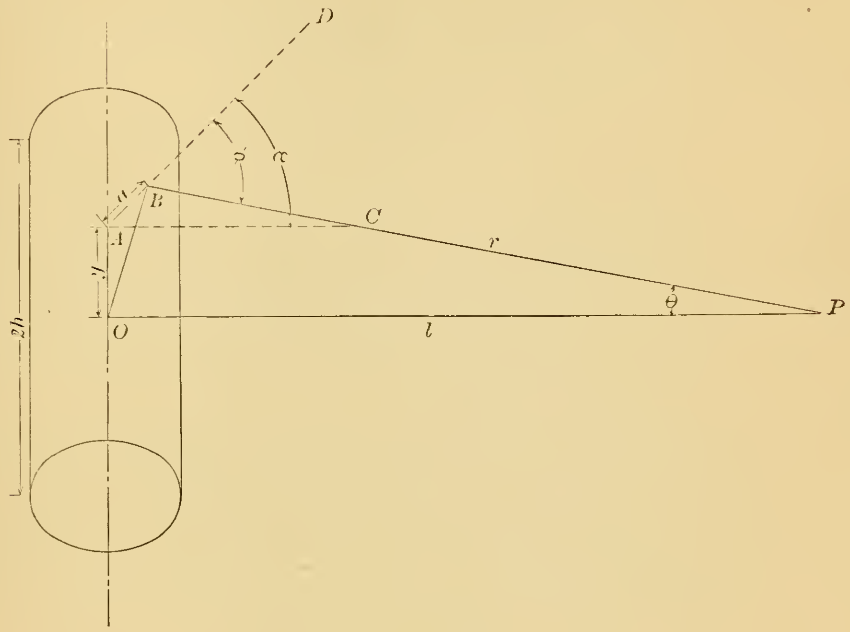

Fig. 2.

Expressing all the quantities involved in the above equation in terms of the two cylindrical coordinates α and y, and denoting the radius of the cylinder by a, and the height by $2h$, the intensity of illumination at P, at a distance, l, from the axis of the cylinder, is given by the equation,

$$J = ia\int\int \frac{(l\cos\alpha - a)(l - a\cos\alpha)}{(a^2 + l^2 - 2al\cos\alpha + y^2)^2} \, d\alpha \, dy \qquad (2)$$

in which the limits of y are $-h, +h$, and the limits of α are

$$-\cos^{-1}\frac{a}{l}, \ +\cos^{-1}\frac{a}{l}$$

The integral of the above expression is,

$$J=\frac{i}{l}\left\{a\cos^{-1}\frac{l^2-a^2-h^2}{l^2-a^2+h^2}+h\left[(q+1)\frac{1}{\sqrt{q}}\cot^{-1}\sqrt{\frac{p}{q}}-2\cot^{-1}\sqrt{p}\right]\right\} \quad (3)$$

in which

$$p=\frac{l+a}{l-a} \qquad q=\frac{(l+a)^2+h^2}{(l-a)^2+h^2} \quad (4)$$

Before substituting numerical values for a and h, it is interesting to note the form which the equation assumes when h approaches infinity. Under this condition q approaches unity, and in the limit equation (3) becomes

$$J=\frac{\pi i a}{l} \quad (5)$$

Thus the intensity of illumination due to an infinitely long uniformly radiating cylinder varies inversely as the first power of the distance from the axis of the cylinder, a result which also follows from purely physical considerations of the normal flow of energy across coaxial cylindrical surfaces.

Let us now give definite numerical values to a and h in equation (3), compute the intensity of illumination at different distances, l, and compare the relative values with those obtained on the assumption of the inverse square law. Let us make $h=10$ mm and $a=1$ mm,

TABLE I.

Deviation from the Inverse Square Law of the Radiation of a Cylinder 20 mm long and 1 mm radius.

Distances, l	J (Equation 3)	J (Inverse Square Law)	Deviation
3000 mm	1.0000 $\times J_{3000}$	1.0000 $\times J_{3000}$	±0.00 %
2000 "	2.2500 "	2.2500 "	+0.00 "
1000 "	9.0045 "	9.0000 "	+0.05 "
500 "	3.6040×10 "	3.6000×10 "	+0.11 "
200 "	2.2545×10^2 "	2.2500×10^2 "	+0.20 "
100 "	9.0081×10^2 "	9.0000×10^2 "	+0.09 "
80 "	1.4049×10^3 "	1.4062×10^3 "	−0.09 "
50 "	3.5593×10^3 "	3.6000×10^3 "	−1.13 "

since these are the approximate dimensions of an 88-watt Nernst glower, for which the case originally was solved. The results of substituting these values of a and h in equation (3) for different distances, l, are shown in Table I and Fig. 3.

In the first column of Table I are given the distances, l, for which the values of J were computed. The second column contains the

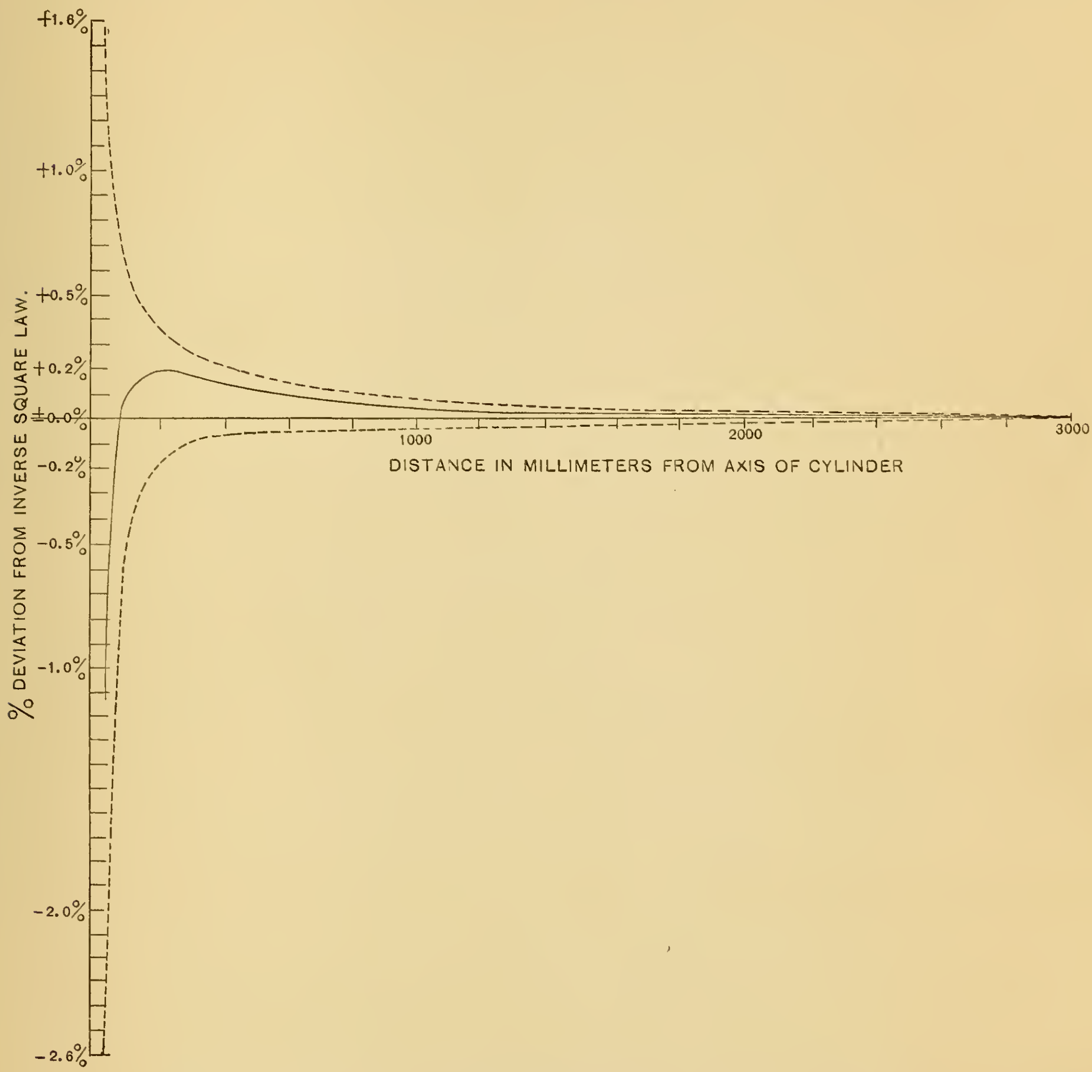

Fig. 3.—*Deviation of Radiation of Cylinder from Inverse Square Law.*

ratios of the intensities at the different distances, l, to that at $l = 3000$ mm, obtained by direct substitution in equation (3), and the third column gives the same ratios obtained by the inverse square law. The differences, expressed as percentage deviations from the inverse square law, are given in the fourth column. They are also shown in the form of a curve by the solid line in Fig. 3, in which abscissas are distances, and ordinates are percentage deviations from the inverse square law, as deduced from the value of the intensity at $l = 3000$ mm.

The value of J between $l = 3000$ mm and $l = 90$ mm is greater (as compared with J at $l = 3000$ mm) than the value of J deduced on the assumption of the inverse square law, the maximum deviation being about $+0.2$ per cent. At distances less than 90 mm the intensity becomes very much less than the values demanded by the inverse square law, the deviation at $l = 50$ mm being over 1 per cent. The reason for the peculiar form which the curve takes becomes evident when we consider the various elements that combine to determine the illumination at any distance. The greater part of the effective radiating surface is nearer the screen on which the illumination is calculated than the center of the cylinder, from which the distance, l, used in applying the inverse square law is measured. On approaching the cylinder, if this effect alone were considered, the illumination would increase more rapidly than that calculated on the assumption of the inverse square law. On the other hand, due to the increased inclination of the emitted light, both to the radiating surface elements and to the screen on which the rays impinge, the illumination falls off rapidly as the cylinder is approached. A third element, which, however, is negligible over the range of distance investigated, is the difference in the effective area of the cylindei at different distances. The combination of these various effects produces the peculiar form of the curve in Fig. 3. At great distances from the cylinder the difference in distance from the screen to the radiating surface elements and to the center of the cylinder is the most important element, and so the curve of deviation from the inverse square law rises at first. But as l becomes smaller the effect of the increased inclination of the emitted rays becomes more important, so that the curve crosses the axis at $l = 90$ mm, the deviations from the inverse square law becoming negative and increasing rapidly as the surface of the cylinder is approached.

Although it is impossible to separate completely the different effects, by considering the two extreme cases of (1) a cylinder with radius a but with an infinitesimal height, and (2) a cylinder with height $2h$ but with an infinitesimal radius, we can analyze the complex curve of Fig. 3 into two simple curves. For the first case, if we make h approach zero in equation (3) we obtain as the expression for the illumination due to the radiation from the edge of a flat circular disk,

$$J' = \frac{ih}{lp}\left[(p-1)\sqrt{p} + (p^2+1)\cot^{-1}\sqrt{\frac{1}{p}} - 2p\cot^{-1}\sqrt{p}\right] \qquad (6)$$

Substituting the numerical values used above, we get as the corresponding curve of deviations from the inverse square law the dotted curve in Fig. 3, lying entirely above the axis of abscissas.

For the second case, making a approach zero in equation (2), h remaining finite, we get as the expression for the illumination due to a radiating line of length $2h$,

$$J'' = 2ia\left[\frac{h}{l^2+h^2} + \frac{1}{l}\tan^{-1}\frac{h}{l}\right] \qquad (7)$$

Substituting numerical values in this equation, we get the dotted curve in Fig. 3 lying entirely below the axis of abscissas. The addition of the two dotted curves gives a curve coincident with the solid curve, which was plotted directly from equation (3).

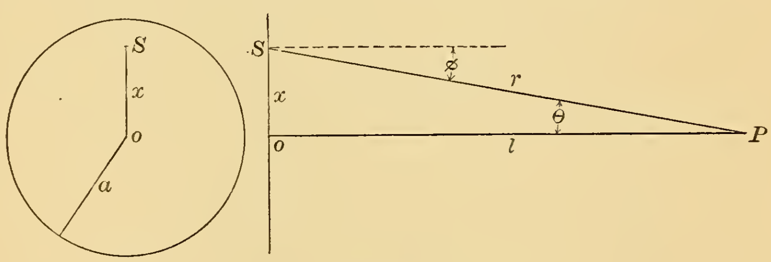

Fig. 4.

Another simple illustration of the treatment of a radiating surface, as distinguished from a point source, or aggregate of point sources, showing the errors resulting from the assumption of the applicability of the inverse square law to such a source, is had in the case of a finite plane of uniform specific light intensity, i. Suppose the plane to be a circle of radius a (Fig. 4). The illumination at P, of the screen normal to OP, where OP is perpendicular to the radiating circular disk at its center, O, is given by equation (1), integrated over the surface of the disk. Since, however, $\phi = \theta$, and all the variables can be expressed in terms of the single variable, x, the distance from the center of the disk to the element of surface dS, equation (1) becomes,

$$J = \int\int \frac{i \cos^2 \theta}{r^2} dS$$

$$= 2\pi i l^2 \int_0^a \frac{x\, dx}{(x^2+l^2)^2}$$

$$= \pi i \frac{a^2}{a^2+l^2} \qquad (8)$$

From this equation the illumination at different distances, l, can be calculated, and the relative values compared with those obtained from the inverse square law, starting from the illumination at some definite distance, say $l = 1000$ mm. The results of this comparison are shown in Table II and Fig. 5, analogous to Table I and Fig. 3 for the corresponding case of a radiating cylinder.

TABLE II.

Deviation from the Inverse Square Law of the Radiation of a Flat Circular Disk of 10 mm radius.

Distances, l	J (Equation 8)		J (Inverse Square Law)		Deviation
1000 mm	1.0000	$\times J_{1000}$	1.0000	$\times J_{1000}$	$\pm$ 0.00 %
800 "	1.5624	"	1.5625	"	− 0.01 "
500 "	3.9988	"	4.0000	"	− 0.03 "
300 "	1.1100×10	"	1.1111×10	"	− 0.10 "
100 "	9.9020×10	"	1.0000×10^2	"	− 0.98 "
50 "	3.8465×10^2	"	4.0000×10^2	"	− 3.8 "
30 "	1.0001×10^3	"	1.1111×10^3	"	− 10.0 "
10 "	5.0005×10^3	"	1.0000×10^4	"	− 50.0 "
5 "	8.0008×10^3	"	4.0000×10^4	"	− 80.0 "
1 "	9.9019×10^3	"	1.0000×10^6	"	− 99.0 "

In each of the above special cases the expression has been deduced for the illumination at different distances along a single line or in a single plane symmetrical with respect to the radiating surface. Thus in the case of the cylinder, equation (3) gives the illumination at points in the plane normal to the axis of the cylinder at its middle point; in the case of the disk, the illumination at points along the line normal to the disk at its center. We shall now derive the expression which will give the illumination at any point in space, though still only for a special case. Moreover, the idea of *tubes of light* will be introduced, and functions will be derived for the *light field*, analogous to *potential* and *intensity* in the electrostatic field.

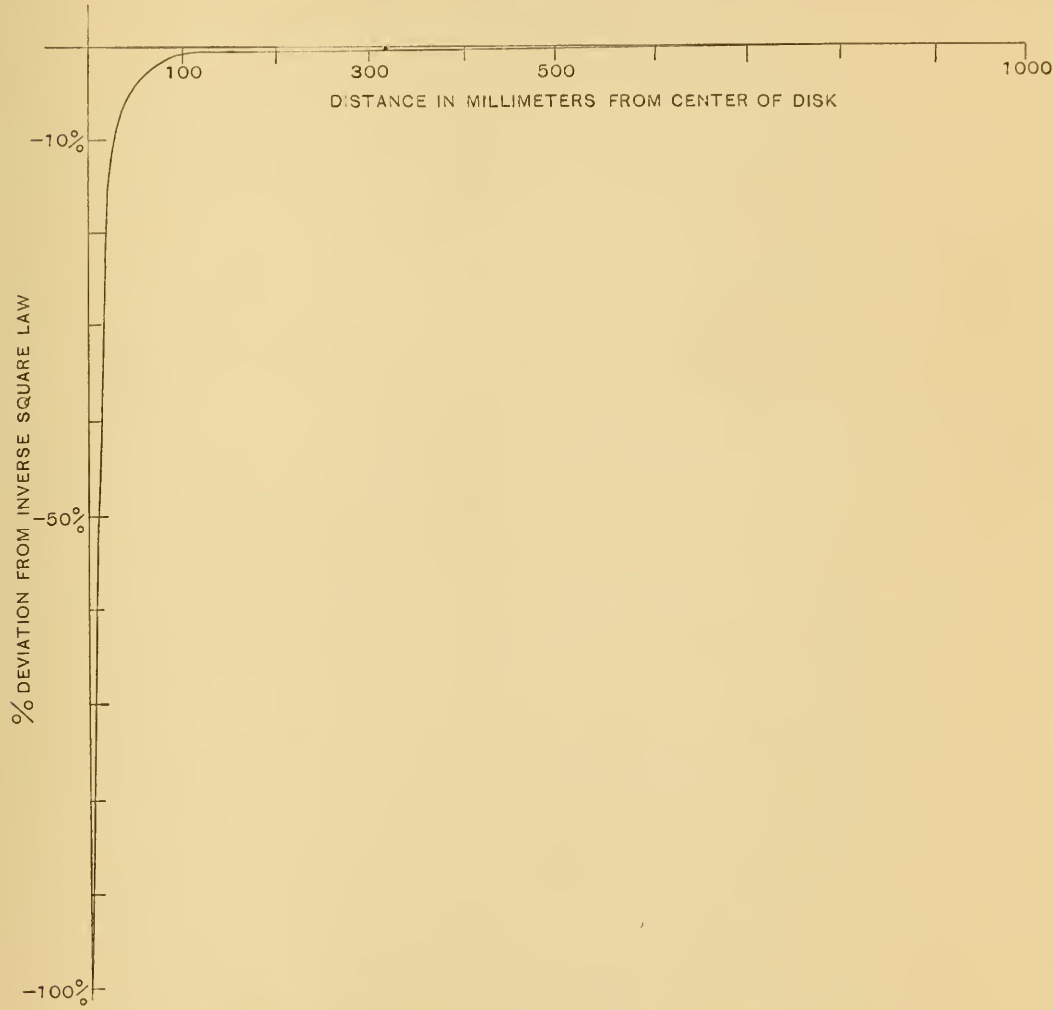

Fig. 5.—*Deviation of Radiation of Circular Disk from Inverse Square Law.*

Before proceeding, however, it is desirable to introduce the term *specific luminous flux* in place of the much less general term *illumination.* The *illumination* of a surface was defined by the International Congress of Electricians at Geneva in 1896 as the *luminous flux* across a given surface, divided by the area of the surface. The *luminous flux* was defined as the *intensity* of the source multiplied by the solid angle subtended at the source by the given surface.

This definition of *luminous flux* is applicable to point sources only, since the solid angle can have no meaning in the case of a source of finite dimensions. If, however, we define the *luminous flux* across a given surface as the quantity of luminous energy flowing normally across the surface in one second, we have a definition which is equally applicable to radiating surfaces and to point sources. Moreover, if the unit of *luminous flux* is defined as the quantity of luminous energy which in one second flows normally across a surface subtending a unit solid angle at a point source of unit *intensity*, the new definition of *luminous flux* is in perfect agreement with the existing definition for point sources, and is also applicable to radiating surfaces.

Intensity of illumination, which may be defined as

$$J=\frac{d\Phi}{dS} \tag{9}$$

where $d\Phi$ is the luminous flux across the surface dS, always presupposes the existence of a material screen. It is easily seen that under this condition the resultant flux per unit area across any imaginary surface may be zero, whereas the illumination on the two sides of a material screen coincident with the imaginary surface may be large, if the same on both sides of the screen. Since in the problem of the light field this unique property of the possible separation of the positive and negative flux across a surface is found, it is desirable, in considering the light field in its analogy to the electrostatic field to employ the term *luminous flux per unit area*, rather than the more special term *illumination.* At every point in space there is some definite direction in which the flux of luminous energy is a maximum. Let us define the quantity of luminous energy which in one second flows normally across a surface of unit area placed perpendicular to the direction of maximum flux, as the *specific luminous flux*, Φ_0 at the point. Φ_0 is a vector quantity, and the component in any direction equals numerically the difference in illumination on the two sides of an infinitely thin material screen placed perpendicular to the direction. If the illumination on one side is zero the *specific luminous flux* numerically equals the illumination on the other side of the screen. We shall employ the vector *specific luminous flux* in the following discussion, in which the rectilinear propagation of light is assumed.

Considering the case of an infinitely long radiating strip of uniform specific light intensity, i, let us determine the amount and direction of the specific luminous flux at every point in space. Since, however, the field will be the same in all planes perpendicular to the long dimension of the strip, it is only necessary to consider the distribution in any one of these parallel planes.

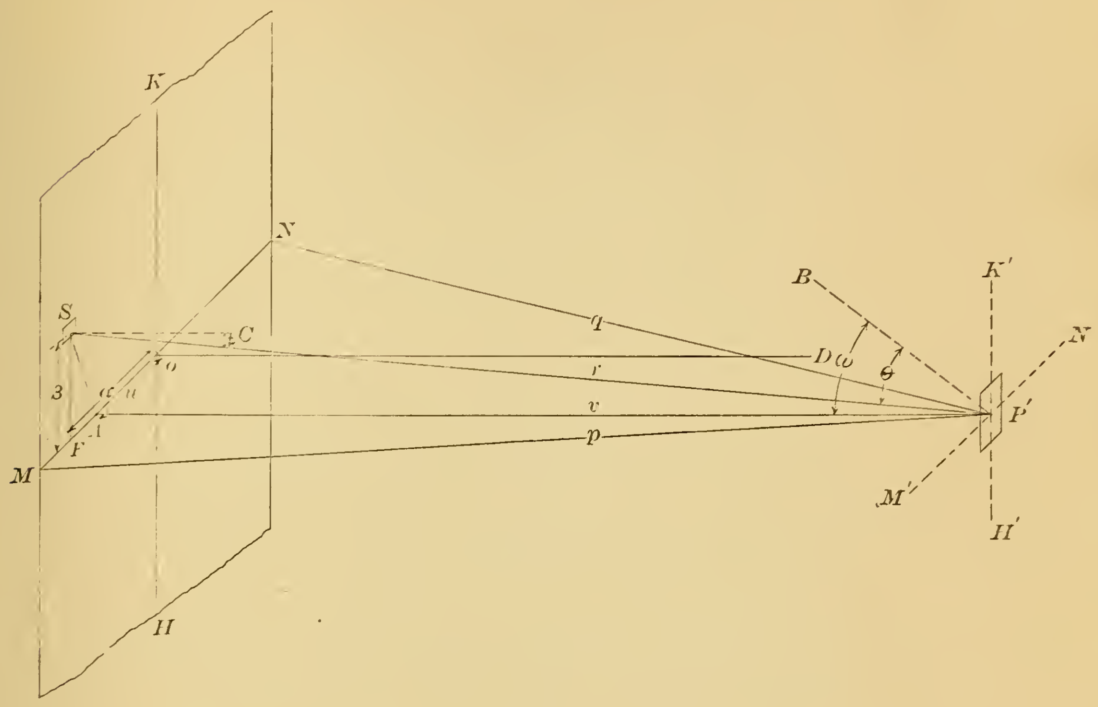

Fig. 6.

The component of the specific luminous flux at any point P (Fig. 6) across a surface, the normal to which lies in the same horizontal plane as the normal to the radiating strip, but inclined to it at an angle, ω, is (equation 1),

$$(\Phi_0)\omega = i\int\int\frac{\cos\phi\cos\theta}{r^2}dS \qquad (10)$$

taken over the entire strip. The distance from the element of surface dS at S to the point P is given by the equation,

$$r=\sqrt{\beta^2+(a-u)^2+v^2}$$

where a and β are the coordinates of the point S with reference to the rectangular axes KH and MN lying in the strip and having as origin the point O. u and v are the coordinates of the point P with reference to the axes MN and OD.

Moreover,

$$dS = d\alpha\, d\beta$$

and

$$\cos\phi = \frac{v}{\sqrt{\beta^2+(\alpha-u)^2+v^2}}.$$

Cos θ can be obtained as follows: θ is the angle between the ray PS and the normal PB to the surface at P. If the direction cosines of PB and PS with reference to the right-handed system of axes PA, PM′, and PK′ are, respectively, λ, μ, ν, and λ', μ', ν', then

$$\cos\theta = \lambda\lambda' + \mu\mu' + \nu\nu' \tag{11}$$

where

$$\lambda = \cos\omega \qquad \mu = -\sin\omega \qquad \nu = o$$

$$\lambda' = \cos\phi \qquad \mu' = \frac{\alpha-u}{\sqrt{\beta^2+(\alpha-u)^2+v^2}} \qquad \nu' = \text{———}$$

Equation (11) becomes, on substitution,

$$\cos\theta = \frac{v}{\sqrt{\beta^2+(\alpha-u)^2+v^2}}\cos\omega - \frac{\alpha-u}{\sqrt{\beta^2+(\alpha-u)^2+v^2}}\sin\omega$$

Substituting in equation (10) the above expressions for r, dS, $\cos\phi$ and $\cos\theta$,

$$(\Phi_0)_\omega = i\int\int\frac{v^2\cos\omega\, d\alpha\, d\beta}{[\beta^2+(\alpha-u)^2+v^2]^2} - i\int\int\frac{v(\alpha-u)\sin\omega\, d\alpha\, d\beta}{[\beta^2+(\alpha-u)^2+v^2]^2}$$

or

$$(\Phi_0)_\omega = iv^2\cos\omega\int\int\frac{d\alpha\, d\beta}{[\beta^2+(\alpha-u)^2+v^2]^2}$$

$$-\ iv\sin\omega\int\int\frac{(\alpha-u)\, d\alpha\, d\beta}{[\beta^2+(\alpha-u)^2+v^2]^2}$$

$$= iv^2 A\cos\omega - iv\, B\sin\omega \tag{12}$$

where

$$A = \int\int\frac{d\alpha\, d\beta}{[\beta^2+(\alpha-u)^2+v^2]^2} \qquad B = \int\int\frac{(\alpha-u)\, d\alpha\, d\beta}{[\beta^2+(\alpha-u)^2+v^2]^2} \tag{13}$$

the integrals extending from $\beta=-\infty$ to $\beta=+\infty$, and from $a=-1$ to $a=+1$, the width of the strip being taken as 2 units.

Evaluating the above integrals,

$$A=\int d\,a\int\frac{d\beta}{[\beta^2+(a-u)^2+v^2]^2}=\frac{\pi}{2}\int_{-1}^{+1}\frac{da}{[(a-u)^2+v^2]^{3/2}}$$

$$=\frac{\pi}{2\,v^2}\left[\frac{a-u}{\sqrt{(a-u)^2+v^2}}\right]_{-1}^{+1}=\frac{\pi}{2\,v^2}\left[\frac{1-u}{p}+\frac{1+u}{q}\right] \qquad (14)$$

where

$$p=\overline{PM}=\sqrt{(1-u)^2+v^2};\quad q=\overline{PN}=\sqrt{(1+u)^2+v^2} \qquad (15)$$

Similarly

$$B=\int(a-u)da\int_{-\infty}^{+\infty}\frac{d\beta}{[\beta^2+(a-u)^2+v^2]^2}$$

$$=\frac{\pi}{2}\int_{-1}^{+1}\frac{(a-u)da}{[(a-u)^2+v^2]^{3/2}}=\frac{\pi}{2}\left[\frac{1}{q}-\frac{1}{p}\right] \qquad (16)$$

Substituting these expressions for A and B in equation (12)

$$(\Phi_0)_\omega=\frac{\pi i}{2}\left[\frac{q(1-u)+p(1+u)}{pq}\right]\cos\omega+\frac{\pi iv}{2}\left[\frac{q-p}{pq}\right]\sin\omega \qquad (17)$$

In order to determine the direction of specific luminous flux, i. e. in order to find the value of ω, which will make $(\Phi_0)_\omega$ a maximum, put

$$\frac{\partial(\Phi_0)_\omega}{\partial\omega}=-\frac{\pi i}{2}\left[\frac{(q+p)-u(q-p)}{pq}\right]\sin\omega+\frac{\pi iv}{2}\left[\frac{q-p}{pq}\right]\cos\omega=0$$

from which

$$\tan\omega_0=\frac{v(q-p)}{(q+p)-u(q-p)} \qquad (18)$$

It can readily be shown that this angle ω_0 is the angle APE (Fig. 7) which the normal at the point (u, v) to the ellipse passing through the point P (u, v), and having as foci M and N, makes with the line PA parallel to the v-axis. (Compare Figs. 6 and 7 in which the same letters refer to corresponding points.) For if a and b are

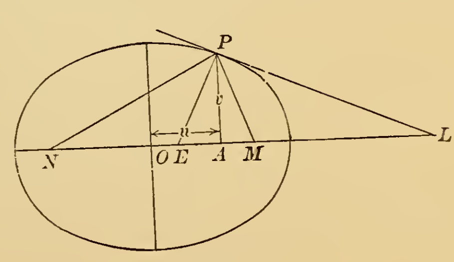

Fig. 7.

respectively the semimajor and semiminor axes of the ellipse, where $a^2-b^2=c^2=\overline{OM}^2=1$, the slope of the tangent at P is

$$\tan \text{OLP}=\tan \text{APE}=\frac{b^2u}{a^2v}=uv\frac{b^2}{a^2v^2} \tag{19}$$

But

$$\frac{u^2}{a^2}+\frac{v^2}{b^2}=1 \quad \text{or } a^2-u^2=\frac{a^2v^2}{b^2}$$

Therefore

$$\tan \text{APE}=\frac{uv}{a^2-u^2} \tag{20}$$

Similarly the expression for $\tan \omega_0$ reduces to the same form. From equation (18)

$$\tan \omega_0=\frac{v(q-p)}{(q+p)-u(q-p)}=\frac{v(q^2-p^2)}{(q+p)^2-u(q^2-p^2)} \tag{21}$$

For the ellipse with foci M, N,

$$q+p=\overline{MP}+\overline{NP}=2\,a \tag{22}$$

and from equations (15)

$$q^2-p^2=4\,u$$

Substituting these values in equation (21)

$$\tan \omega_0=\frac{uv}{a^2-u^2} \tag{23}$$

From equations (20) and (23)

$$\omega_0=\text{angle APE} \tag{24}$$

or the surface through any point (u,v) across which the specific luminous flux due to an infinitely long uniformly radiating strip of width MN=2 is a maximum, will be tangent to the ellipse passing through the point and having M and N as foci. Now, since the system of confocal hyperbolas having the foci M and N is orthogonal with respect to the system of confocal ellipses having the same foci, it follows that the direction of maximum specific luminous flux at any point in the plane (being the same for all parallel planes) is in the direction of the hyperbola through the point. In other words, the direction of the vector Φ_0 at all points in space is that of the hyperbola through the point.

Suppose now we draw the system of confocal hyperbolas (Fig. 8) in such a way that the distance between the intercepts on the axis of u of successive hyperbolas is constant and equal to $1/i$.

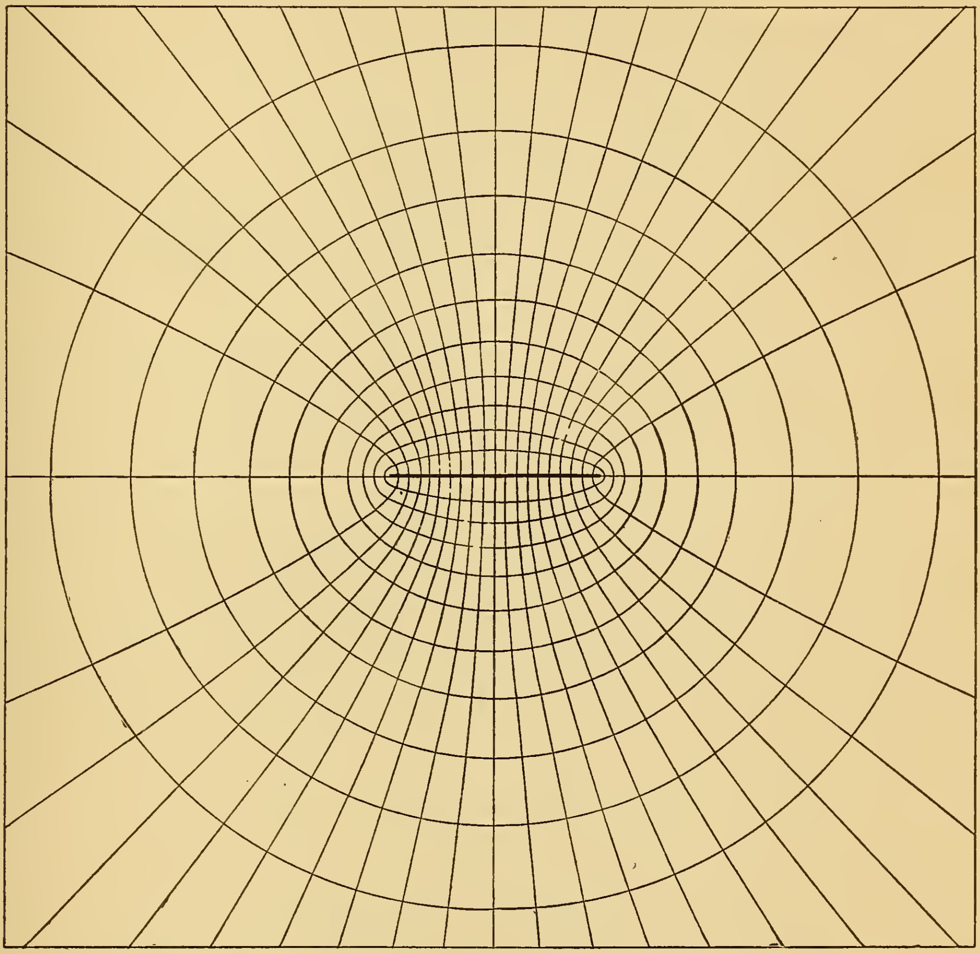

Fig. 8.—*Lines of Maximum Light Flux and Equipotential Surfaces for Uniformly Radiating Strip of Infinite Length.*

Since the specific luminous intensity, i, of a radiating surface is defined as the quantity of light per second radiated normally by a unit area of the surface, the tubes formed by the hyperbolic cylinders of unit height may be considered as unit light tubes, the flow of light being along the tube at all points, and equal to unity at the radiating surface.

22261—07——7

It will now be shown that the specific luminous flux at any point is proportional to the number of unit light tubes per unit area of a surface perpendicular to the direction of maximum flow. In other words, the specific luminous flux Φ_0 is solenoidal, the effect at any point being the same mathematically as if the energy starting out in a light tube continued in the same tube. Of course we know physically that an eye placed at any point can see the whole radiating surface, so that the solenoidal property of the specific luminous flux is merely a mathematical property, as is probably also true in the case of Faraday tubes. But the specific luminous flux Φ_0, and hence the intensity of illumination at any point, can be determined by the application of this property, which will now be proved, viz. that the quantity of luminous energy per second flowing across successive elliptic cylinders between two definite hyperbolic cylinders is constant.

Since the distance MN between the foci has been taken as equal to 2, the major and minor axes of the confocal ellipses and hyperbolas will be $2 \cosh x$, $2 \sinh x$, and $2 \cos y$, $2 \sin y$, respectively, where x and y are arbitrary parameters.

The u-v plane with the orthogonal system of confocal ellipses and hyperbolas corresponding to $x=$constant and $y=$constant, respectively, may be considered as the conformal representation of the x-y plane. The relations between the variables are expressed by the equations of the ellipses and hyberbolas:

$$\frac{u^2}{\cosh^2 x}+\frac{v^2}{\sinh^2 x}=1 \qquad \frac{u^2}{\cos^2 y}-\frac{v^2}{\sin^2 y}=1 \tag{25}$$

from which

$$u=\cosh x \ \cos y \qquad v=\sinh x \ \sin y \tag{26}$$

If in equation (19) we substitute for a and b, the semimajor and semiminor axes of the ellipse through any point, the expressions $\cosh x$ and $\sinh x$, and for u and v the values given in equation (26), we obtain for $\tan \omega_0$ (equation 24) the expression,

$$\tan \omega_0=\frac{b^2}{a^2}\frac{u}{v}=\frac{\sinh^2 x \cosh x \cos y}{\cosh^2 x \sinh x \sin y}$$

$$=\frac{\sinh x \cos y}{\cosh x \sin y} \tag{27}$$

from which

$$\sin \omega_0 = \frac{\sinh x \cos y}{\sqrt{\sinh^2 x + \sin^2 y}} \tag{28}$$

$$\cos \omega_0 = \frac{\cosh x \sin y}{\sqrt{\sinh^2 x + \sin^2 y}} \tag{29}$$

Substituting these values in equation (17)

$$\Phi_0 = \frac{\pi i}{2\,pq}\left\{\left[(q+p) - u(q-p)\right]\frac{\cosh x \sin y}{\sqrt{\sinh^2 x + \sin^2 y}} + (q-p)\sinh x \sin y \frac{\sinh x \cos y}{\sqrt{\sinh^2 x + \sin^2 y}}\right\} \tag{30}$$

For the ellipse (equation 22)

$$q+p = 2a = 2 \cosh x$$

and for the hyperbola,

$$q-p = 2a' = 2 \cos y$$

from which

$$\left.\begin{aligned} p &= \cosh x - \cos y \\ q &= \cosh x + \cos y \\ pq &= \cosh^2 x - \cos^2 y = \sinh^2 x + \sin^2 y \end{aligned}\right\} \tag{31}$$

Substituting these values in equation (30) and putting for u its value from equation (26)

$$\Phi_0 = \frac{\pi i \sin y}{\sqrt{\sinh^2 x + \sin^2 y}} \tag{32}$$

(The denominator $\sqrt{\sinh^2 x + \sin^2 y}$ which is equal to $\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$ is the linear magnification at the point (x, y) in the transformation from the x-y to the u-v plane).

It follows from equation (32) that the ratio of the two values of maximum specific luminous flux at the points of intersection of any definite hyperbola, y=constant, with two ellipses x' and x'' is

$$\frac{\Phi_0'}{\Phi_0''} = \frac{\sqrt{\sinh^2 x'' + \sin^2 y}}{\sqrt{\sinh^2 x' + \sin^2 y}} \tag{33}$$

Now the ratio of the distances on the two ellipses intercepted by the two consecutive hyperbolas y and $y+\Delta y$ is readily obtained. From equations (26), for any ellipse $x=$constant

$$\Delta u=-\cosh x \sin y \,\Delta y;\quad \Delta v=\sinh x \cos y \Delta y$$

and therefore

$$\Delta s=\sqrt{(\Delta u)^2+(\Delta v)^2}=\Delta y\sqrt{\sinh^2 x+\sin^2 y} \tag{34}$$

Hence the ratio of the distances on the two ellipses x' and x'', between the same two consecutive hyperbolas y and $y+\Delta y$, is

$$\frac{\Delta s'}{\Delta s''}=\frac{\sqrt{\sinh^2 x'+\sin^2 y}}{\sqrt{\sinh^2 x''+\sin^2 y}} \tag{35}$$

Comparing this equation with equation (33)

$$\frac{\Phi_0'}{\Phi_0''}=\frac{\Delta s''}{\Delta s'} \tag{36}$$

or

$$\Phi_0'\Delta s'=\Phi_0''\Delta s'' \tag{37}$$

which states that the total flux of light within a light tube remains constant. In other words Φ_0 is solenoidal, and therefore its divergence must be zero,

$$\frac{\partial}{\partial u}\Big(\Phi_0\Big)_u+\frac{\partial}{\partial v}\Big(\Phi_0\Big)_v=0 \tag{38}$$

At $x=\infty$, where the ellipses become circles, it follows from equation (32) that

$$\Phi_0=\frac{\pi\, i \sin y}{\sinh x} \tag{39}$$

and so is proportional to $\sin y$ at any circle $x=$constant. At $x=\infty$ the hyperbolas are radial, and it can easily be shown that the angle θ which the hyperbola at infinity makes with the axis of v is $\frac{\pi}{2}-y$. The slope of the tangent to the hyperbola is, in general,

$$\tan(90^\circ-\theta)=\frac{b'^2u}{a'^2v}=\frac{\cosh x \sin y}{\sinh x \cos y}$$

$$=\tan y \coth x$$

At $x=\infty$ this becomes

$$\tan(90^\circ-\theta)=\tan y$$

and therefore

$$\theta=90^\circ-y \tag{40}$$

Hence, since for x = constant, Φ_0 is proportional to $\sin y$ (equation 39) when x is so large compared with the width of the strip that the ellipses approximate circles, the number of the hyperbolic light tubes per unit arc of circle is proportional to $\cos\theta$, i. e. the distribution of light tubes follows the cosine law. In the analogous electrical problem of a charged strip of infinite length the tubes of force are uniformly distributed at $x=\infty$, but the charges are distributed over the surface of the conductor according to the cosine law, the density of charge at any point $u=u_1$ being proportional to the cosine of the angle which the tangent at infinity to the hyperbola that passes through the point $u=u_1$, $v=o$ makes with the axis of v, i. e. $\cos\theta$.

The fundamental difference between the electrical and optical problems for an infinite strip of finite width would seem to be that in the one case the condition is imposed that the field of force shall be uniform at infinity, and the distribution of the charge over the surface of the strip is determined by this condition. In the other case the surface charge is assumed to be uniform and the distribution of the light tubes at infinity is determined.

It is easy to obtain equations for the light field analogous to those of the electrostatic field. It has already been shown that Φ_0 is solenoidal, so that

$$\frac{\partial}{\partial u}\left(\Phi_0\right)_u+\frac{\partial}{\partial v}\left(\Phi_0\right)_v=o$$

Moreover, since the specific light flux along any ellipse (x=constant) is zero, the elliptic cylinders may be considered as level surfaces. Let Ψ be a function, holomorphic in a certain region of the x-y plane, and let it depend upon x only, so that for x constant Ψ is constant. Furthermore, let the function be of such a nature that for equal positive increments of x the changes in Ψ are equal in amount and negative in sign. Then

$$-\frac{\partial \Psi}{\partial x}=\text{constant.} \tag{41}$$

In the transformed plane Ψ becomes a function of u, v which is constant over the surface of the elliptic cylinders, and which decreases by equal amounts between successive cylinders, if the ellipses are drawn to correspond to equal increments of x. Since for x constant Ψ is constant

$$\frac{\partial \Psi}{\partial y}=o$$

Moreover, along any hyperbola from any ellipse to the next,

$$\int\left[\frac{\partial \Psi}{\partial x}\right]_{y\ \text{const.}} dx=\text{constant.} \tag{42}$$

If, now, dn be an infinitesimal distance along the normal to the ellipse at any point in the field,—i. e. along the tangent to the hyperbola through the point, since dn is proportional to dx multiplied by the coefficient of linear magnification in the transformation from the x-y to the u-v plane, it follows from equations (41) and (42) that for any point in the field

$$-\frac{\partial \Psi}{\partial n}=\frac{C}{\sqrt{\sinh^2 x+\sin^2 y}} \tag{43}$$

in which C is a constant.

Moreover, since (equation 32)

$$\Phi_0=\frac{\pi\, i \sin y}{\sqrt{\sinh^2 x+\sin^2 y}}$$

it follows that

$$\Phi_0=-C'\frac{\partial \Psi}{\partial n}\sin y \tag{44}$$

From this equation it is seen that, although in going from one hyperbola to another Φ_0 varies in proportion to sin y, along any one hyperbola Φ_0 at any point is proportional to the rate of decrease of Ψ along the hyperbola. Hence, Ψ may be considered as a special form of potential function, from which the specific luminous flux is derived according to equation (44).

In general, if we denote the specific luminous flux in any direction q by $(\Phi_0)_q$, and if the unit in terms of which Φ_0 is expressed be so chosen that C' in equation (44) becomes unity,

$$(\Phi_0)_q = \Phi_0 \cos(nq) = -\frac{\partial \Psi}{\partial n}\cos(nq)\sin y = -\frac{\partial \Psi}{\partial q}\sin y \quad (45)$$

Since Ψ is a function of x only, it follows from equation (41) that

$$\varDelta \Psi = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = o \quad (46)$$

In the transformation to the u-v plane

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = \left[\frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial v^2}\right]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

or

$$\varDelta \Psi = h^2 \varDelta' \Psi \quad (47)$$

in which h is the coefficient of linear magnification. Hence (equation 46)

$$\varDelta' \Psi = \frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial v^2} = o \quad (48)$$

In Fig. 8 the ellipses are drawn to correspond to equal increments of Ψ.

In the earlier paragraphs of the paper two examples were given of the numerical errors incident to the application of the inverse square law to radiating surfaces. A similar numerical computation might be made for the case we have just studied. In fact, the errors incident to the application of the inverse square law might be deduced for the variation of intensity with distance in any number of directions from the strip. But since two examples of the effect of distance have already been given, and since in the case of the radiating strip the specific luminous flux in any direction at any point in the field is known, an excellent opportunity is afforded to show numerically the errors incident to assuming for a finite radiating surface as a whole the cosine law which applies only to the infinitesimal elements of the surface.

In order to determine the errors incident to the application of the cosine law to the surface as a whole let us calculate the specific luminous flux at different distances in a direction normal to the strip at its middle point, and in a direction making an angle of 45° with the normal at the middle point. For any definite distance the former, multiplied by the cosine of 45°, would equal the latter if the cosine law, which has been assumed for the elements of surface, applied to the surface as a whole. The difference between the two values gives the error for the distance used.

From equations (26)

$$u = \cosh x \cos y \qquad v = \sinh x \sin y.$$

Therefore,

$$u^2 = (1 + \sinh^2 x)(1 - \sin^2 y) \qquad \sin^2 y = \frac{v^2}{\sinh^2 x} \tag{49}$$

$$u^2 \sinh^2 x = \sinh^4 x + (1 - v^2) \sinh^2 x - v^2 \tag{50}$$

In the direction making an angle of 45° with the normal to the strip at its middle point $v = u$, and so for distances measured in this direction x and y can be expressed in terms of the single coordinate u.

Substituting u for v in equation (50)

$$\sinh^4 x + (1 - 2u^2) \sinh^2 x = u^2$$

from which

$$\sinh x = \pm\sqrt{\tfrac{1}{2}(2u^2 - 1 + \sqrt{1 + 4u^4})} \tag{51}$$

Also from equations (49)

$$\sin y = \frac{v}{\sinh x} = \frac{u}{\sqrt{\tfrac{1}{2}(2u^2 - 1 + \sqrt{1 + 4u^4})}} \tag{52}$$

For distances measured in the direction of the normal to the strip at its middle point $u=o$ and therefore $\cos y=o$. Hence we have

$$\sin y = 1 \tag{53}$$

and from equations (49)

$$\sinh x = v \tag{54}$$

By substituting for $\sinh x$ and $\sin y$ in equation (32) the values given in equations (54) and (53), the specific luminous flux at different distances $d=v$ in the direction normal to the strip at its middle point is obtained. In a similar way, by substituting the values given in equations (51) and (52), the specific luminous flux at distances $d=\sqrt{2u^2}$ is obtained for points lying on the line making an angle of 45° with the normal to the strip at its middle point. The specific luminous flux determined in this way is not, however, that in the direction of 45°, but rather the maximum specific luminous flux at the point, which is in the direction of the hyperbola through the point. The value in the direction of 45° is obtained by multiplying the maximum value by the cosine of the angle between the given direction and the tangent to the hyperbola at the point. This angle will evidently be (Fig. 6) the difference between 45° and ω_0 where (equation 28)

$$\sin \omega_0 = \frac{\sinh x \cos y}{\sqrt{\sinh^2 x + \sin^2 y}} \tag{55}$$

The differences, expressed as percentage errors, between the true value of Φ_0 in the direction of 45° and that calculated from the value of Φ_0 in the direction of the normal by multiplying the latter by cos 45°, are shown in the form of a curve in Fig. 9. The abscissas are distances, d, expressed in the unit in terms of which the width of the strip is 2, and the ordinates are percentage errors between the true value and that computed on the assumption of the cosine law for the strip as a whole. For every distance the true value at 45° is greater than that deduced from the value at 0° by assuming the cosine law.

It is seen from a consideration of Fig. 9 that the curve approaches the axis of distances asymptotically, so that at $d=\infty$ the error would be zero. As d becomes smaller the error increases slowly at first, but at $d=6$ or 7 it begins to increase rapidly, reaching a maximum of 23.4 per cent at $d=1$. From this point it decreases rapidly to zero at $d=o$. That the error should be zero at $d=\infty$ and at $d=o$ might have been predicted, for at infinity the width of the strip is

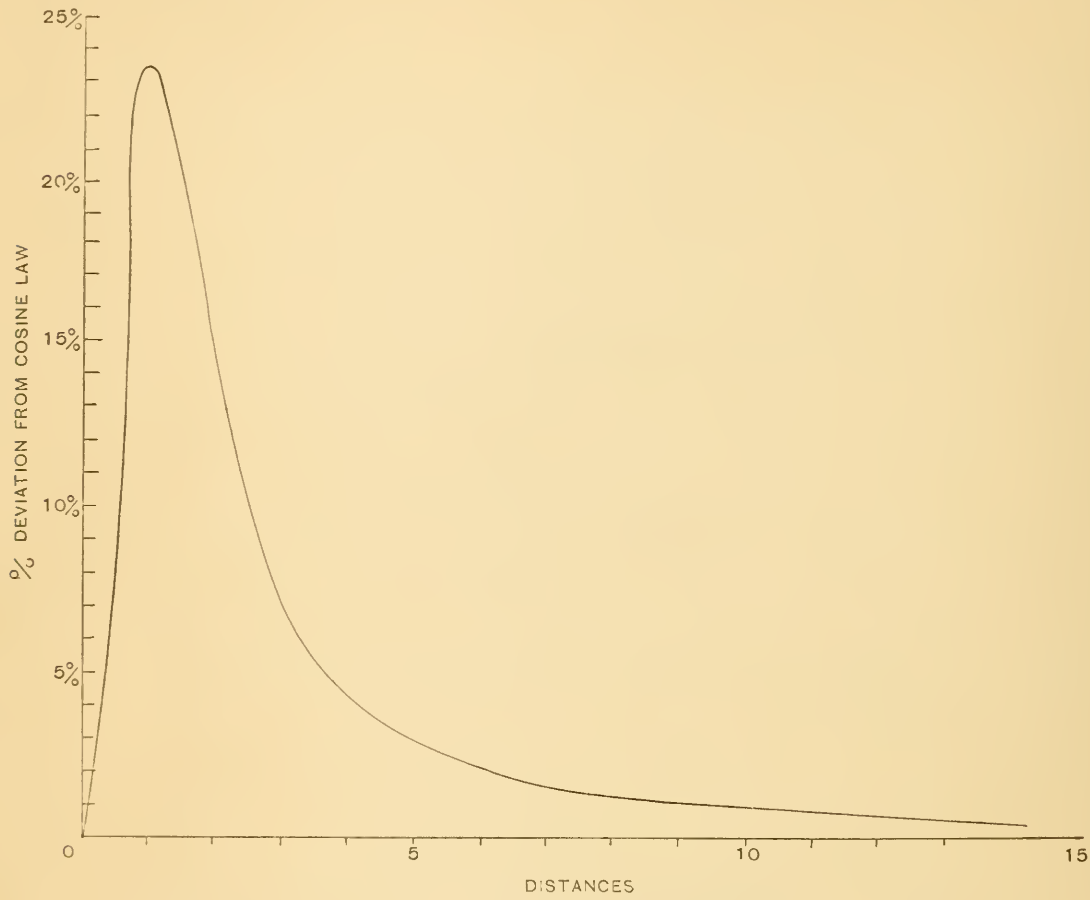

Fig. 9.—*Deviation of Radiation of Infinite Strip from Cosine Law.* (*45°*).

infinitesimal relative to the distance, and for a strip of infinitesimal width the cosine law by hypothesis would apply to the strip as a whole. At $d=o$ the total flux is due to the element of surface around the point $u=o$, and so the flux in any direction is equal to the maximum flux, which is normal to the surface, multiplied by the cosine of the angle of emission. In other words, at each of the two limits we are dealing with surfaces relatively infinitesimal in size, and for such surfaces we assumed the cosine law in the beginning.