

**NIST Technical Note
NIST TN 2324**

**Linnik point spread functions,
time-reversed logarithmic diffusion
equations, and blind deconvolution of
electron microscope imagery**

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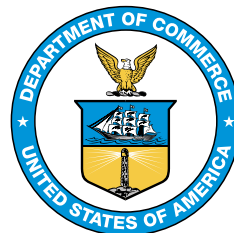
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Abstract

A non-iterative direct blind deconvolution procedure, previously used successfully to sharpen Hubble Space Telescope imagery, is now found useful in sharpening nanoscale scanning electron microscope (SEM) and helium ion microscope (HIM) images. The method is restricted to images $g(x, y)$, whose Fourier transforms $\hat{g}(\xi, \eta)$ are such that $\log |\hat{g}(\xi, 0)|$ is globally monotone decreasing and convex. The method is not applicable to defocus blurs. A point spread function in the form of a Linnik probability density function is postulated, with parameters obtained by least squares fitting the Fourier transform of the *preconditioned* microscopy image. Deconvolution is implemented in *slow motion* by marching backward in time, in Fourier space, from $t = 1$ to $t = 0$, in an associated logarithmic diffusion equation. Best results are usually found in a *partial deconvolution* at time $\bar{t}$, with $0 < \bar{t} < 1$, rather than in total deconvolution at $t = 0$. The method requires familiarity with microscopy images, as well as interactive search for optimal parameters.

Keywords

SEM images; HIM images; sharpening; denoising; deblurring; blind deconvolution; Linnik point spread functions; time-reversed logarithmic diffusion equations.

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1. Introduction

There are numerous processes for the extraction of meaningful information contained in various images. For proper identification, interpretation, and analysis, a human observer must first be able to perceive and recognize all of the information contained in a given image. Often, valuable pertinent information lies in faint details. A combination of local contrast enhancement and sharpening of fine image details, is especially helpful for scanning electron microscope (SEM) and helium ion microscope (HIM) images. Modern SEMs and HIMs can resolve sub-nanometer details [8], but their images often suffer from low contrast and appreciable noise. As a result, fine details can easily be overlooked. Denoising, contrast stretching, and sharpening, can significantly improve the results of quantitative analyses of these images.

This paper describes a new *non-iterative*, direct blind deconvolution procedure for sharpening images obtained from scanning electron microscopes, and Helium ion microscopes. As was the case in [3–6], the present method is only applicable to a restricted class of blurred images $g(x, y)$, with Fourier transforms $\hat{g}(\xi, \eta)$ such that $\log |\hat{g}(\xi, 0)|$ is globally monotone decreasing and convex. The method does not apply to defocus and motion blurs. We view the given image $g(x, y)$ as the convolution of the desired sharp image $f(x, y)$ with an unknown point spread function $h(x, y)$, together with an unknown amount of noise $n(x, y)$,

$$g(x, y) = \int_{R^2} h(x - u, y - v) f(u, v) du dv + n(x, y) = h \otimes f + n(x, y). \quad (1)$$

After an appropriate *preconditioning* of the microscopy image $g(x, y)$, the method is based on postulating a point spread function in the form of a Linnik probability density function $h(x, y)$, and then identifying the parameters $\gamma, \lambda > 0$, in the corresponding Linnik optical transfer function

$$\hat{h}(\xi, \eta) = \{1 + 4\pi^2\gamma(\xi^2 + \eta^2)\}^{-\lambda}, \quad (2)$$

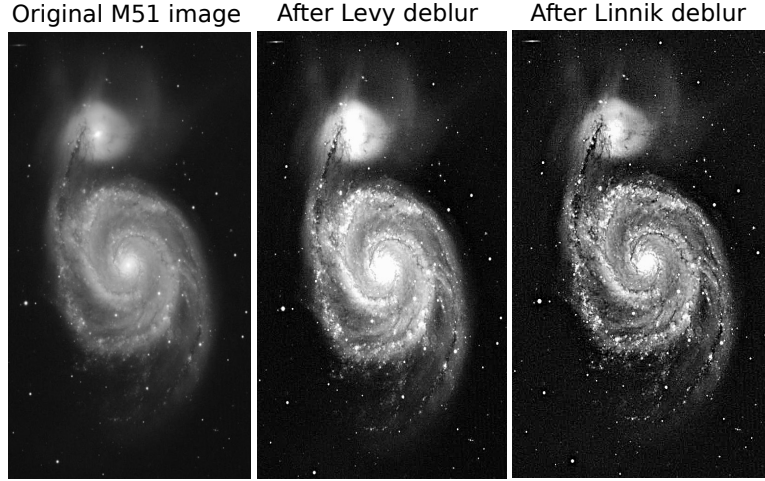
by least squares fitting the Fourier transform of the preconditioned $g(x, y)$. Here, for any function $p(x, y)$ in $L^1(R^2)$, we define its Fourier transform $\hat{p}(\xi, \eta)$ by

$$\hat{p}(\xi, \eta) = \int_{R^2} p(x, y) e^{-2\pi i(\xi x + \eta y)} dx dy. \quad (3)$$

With the Linnik optical transfer function $\hat{h}(\xi, \eta)$ in Eq. (2), it is useful to define $\hat{h}(\xi, \eta, t)$ as follows for $0 \leq t \leq 1$,

$$\hat{h}(\xi, \eta, t) = \{1 + 4\pi^2\gamma(\xi^2 + \eta^2)\}^{-\lambda t}, \quad 0 \leq t \leq 1. \quad (4)$$

Previous non-iterative direct blind deconvolution methods, based on candidate point spread functions in the form of heavy-tailed Lévy stable probability densities, were successfully used



As shown in [6], the use of Linnik probability densities, rather than Lévy stable densities, produces better results in deblurring astronomical imagery. Above Whirlpool Galaxy image was obtained at Kitt Peak National Observatory. Successful Linnik deblurring of several Hubble Space Telescope color images may also be found in [6]. Horizontal Field Width (HFW) in above images is approximately 53,000 light years.

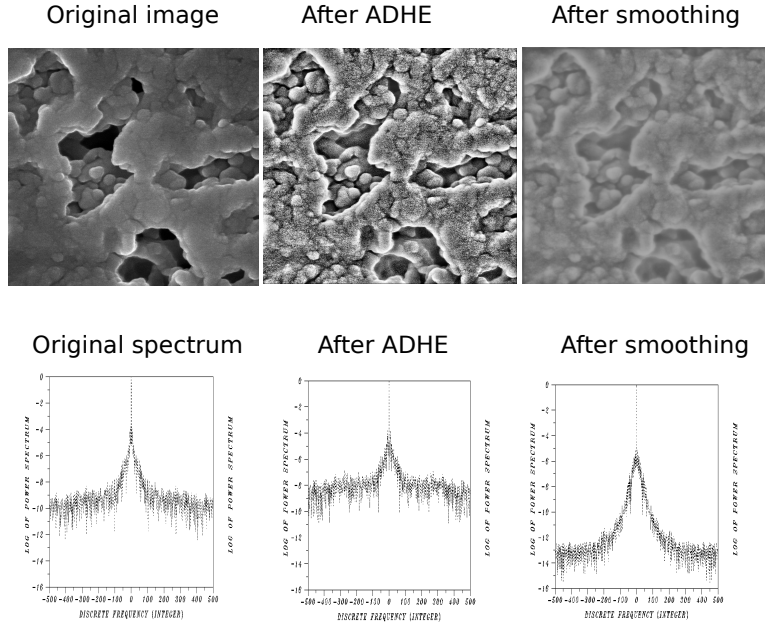
Figure 1. Kitt Peak National Observatory image of the Whirlpool Galaxy (M51)

in several applications [1–5]. In that approach, fast Fourier transform (FFT) algorithms are used to implement the deconvolution as a backward in time stepwise marching procedure, from $t = 1$ to $t = 0$, in a diffusion equation involving fractional powers of the negative Laplacian. Stopping the process prior to reaching $t = 0$, produces a *partial deconvolution* which is often beneficial. In [6], the use of Linnik probability densities, rather than Lévy stable densities, was found to produce significantly better results in deblurring Hubble Space Telescope and other astronomical imagery. This is illustrated in Figure 1, in the case of a Kitt Peak National Observatory image of the Whirlpool Galaxy (M51), obtained by Rector and Ramirez. In the Linnik blind deconvolution procedure discussed in [6], deconvolution unfolds as a backward in time marching procedure, from $t = 1$ to $t = 0$, in a diffusion equation involving the *logarithm* of the identity plus the negative Laplacian, $w_t = -\lambda\{\log(I + \gamma(-\Delta))\}w$. Using the Lipschitz exponent theory developed in [7], it can be shown *quantitatively* that the rightmost image in Figure 1 is significantly sharper than the other two images. The behavior of Linnik versus Lévy optical transfer functions at high and low frequencies, is discussed in [6, Sections 4-6], and that behavior is used to explain these improved sharpness results.

2. Preconditioning microscopy images

An important first step prior to Linnik blind deconvolution of microscopy images, consists of applying adaptive histogram equalization (ADHE) to the image. However, while this brings out useful information, it also produces significant noise. Smoothing that noisy

image by convolution with a low exponent Lévy probability density function is helpful. This is illustrated in Figure 2. In the smoothed ADHE image $g(x, y)$, least squares fitting $\log|\hat{g}(\xi, 0)|$ with the expression $-\lambda(\log(1 + 4\pi^2\gamma\xi^2) - b)$, where $b > 0$ is an appropriately chosen constant, leads to parameter values for λ and γ in $\hat{h}(\xi, \eta)$ defined in Eq. (2). This is shown in Figure 3.



Reveals valuable information while generating significant noise that must be smoothed out. This is reflected in the respective Fourier spectra. Above images are $1\mu m$ HFW secondary electron images.

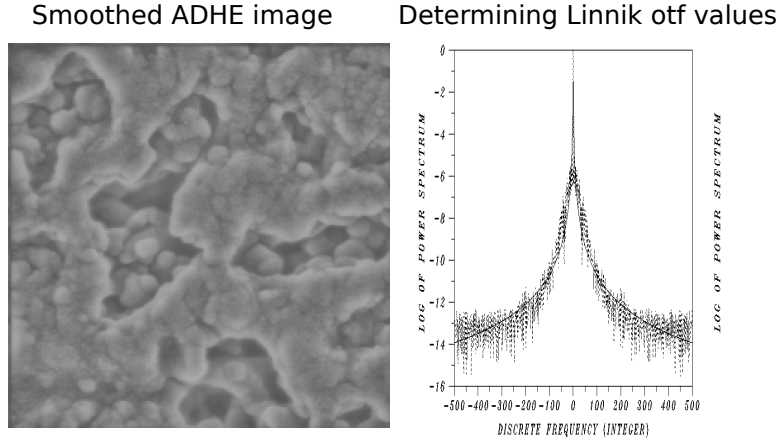
Figure 2. Adaptive Histogram Equalization (ADHE) image

3. Deconvolution by marching diffusion equations backward in time

As was the case in [4, Section 3], the Linnik blind deconvolution problem is solved by marching the logarithmic diffusion equation, $w_t = -\lambda\{\log(I + \gamma(-\Delta))\}w$, backward in time from $t = 1$ to $t = 0$, using the *preconditioned* microscopy image $g(x, y)$ as data at $t = 1$. The *slow evolution* (SECB) constraint, previously developed in [1], is applied to stabilize the ill-posed backward computation. A complete discussion given in [3, Section 3] leads to the partially deblurred Linnik SECB Fourier image $\hat{w}(\xi, \eta, t)$ defined as follows,

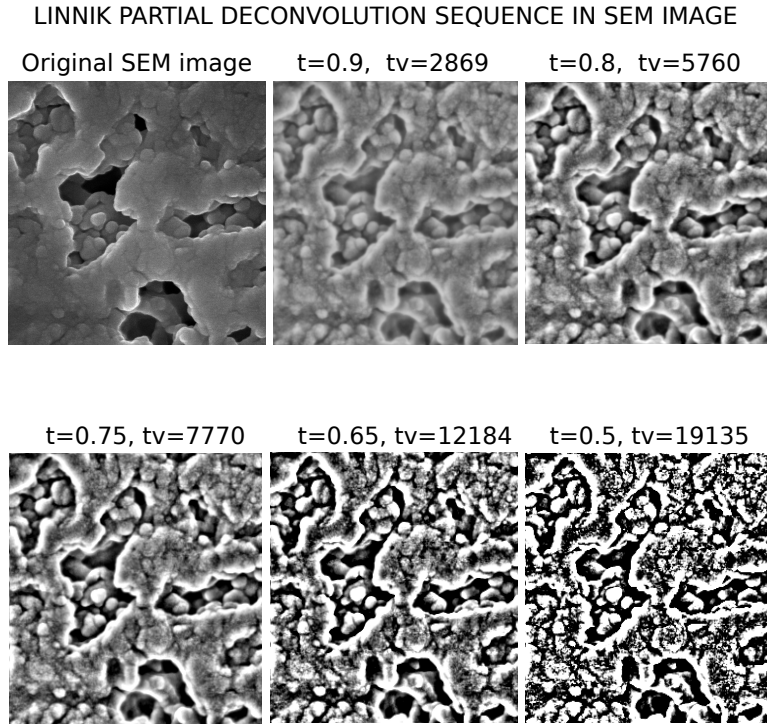
$$\hat{w}(\xi, \eta, t) = \frac{\hat{h}(\xi, \eta, t)\hat{h}(\xi, \eta)\hat{g}(\xi, \eta)}{\hat{h}^2(\xi, \eta) + K^{-2}(1 - \hat{h}(\xi, \eta, s))^2}, \quad 0 \leq t < 1, \quad (5)$$

with suitably chosen positive constants s , K . Typical values for these constants might be $s = 0.001$, $K = 100$. An inverse Fourier transform in Eq. (5) leads to $w(x, y, t)$,



In the smoothed ADHE image $g(x, y)$, least squares fitting $\log|\hat{g}(\xi, 0)|$ with the expression $-\lambda(\log(1 + 4\pi^2\gamma\xi^2) - 1.5)$, leads to parameter values $\lambda = 0.969$, $\gamma = 1.64$, for $\hat{h}(\xi, \eta)$ in Eq. (2)

Figure 3. Smoothed ADHE image



Partial deconvolution sequence with $s = 0.001$, $K = 300$. Best image is found at some $\bar{t}$ lying between $t = 0.75$ and $t = 0.8$. At smaller t values, $\|w(\cdot, t_n)\|_{tv}$ increases rapidly as serious noise develops. Above images are $1\mu m$ HFW secondary electron images.

Figure 4. Partial deconvolution sequence in SEM image

the partial deconvolution at time t . The above blind deconvolution procedure requires familiarity with microscopy images, as well as interactive search for useful values of s , K .

It may be helpful to produce a sequence of partially deconvolved images $w(x, y, t_n)$, for preselected decreasing values of t_n , with $1 > t_n > 0$. The image L^1 norm, $\|w(\cdot, t_n)\|_1$, as well as the image total variation norm $\|w(\cdot, t_n)\|_{tv}$, where

$$\|w(\cdot, t_n)\|_1 = \int_{R^2} |w(x, y, t_n)| dx dy, \quad \|w(\cdot, t_n)\|_{tv} = \int_{R^2} |\nabla w(x, y, t_n)| dx dy, \quad (6)$$

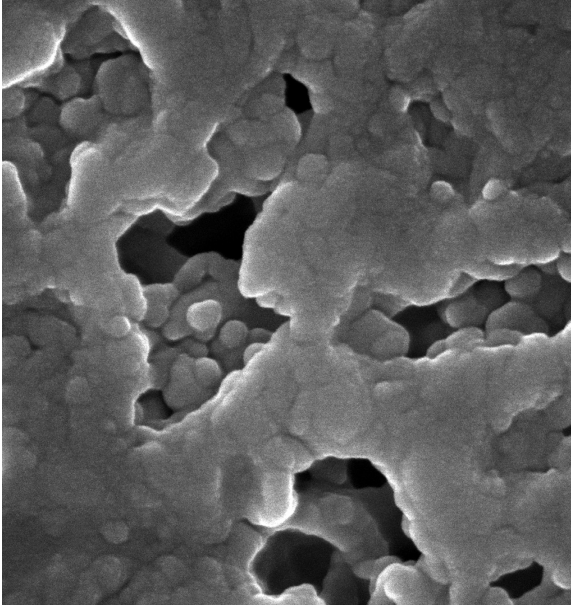
may also be computed at each t_n . With a good choice of s , K , it is typically found that $\|w(\cdot, t_n)\|_1$ remains constant as $t_n \downarrow 0$, while $\|w(\cdot, t_n)\|_{tv}$ increases systematically. However, rapidly increasing values of $\|w(\cdot, t_n)\|_1$ often indicate development of noise. A useful deblurred image might be found at $t_n = \bar{t} \geq 0.7$, while smaller values for t_n may produce images of lesser quality. This process is illustrated in Figure 4. There are several distinct triples $(s, K, \bar{t})$ that can produce distinct useful deblurred images $w(x, y, \bar{t})$. Examples of successful deblurred images, together with the parameters used in Eqs. (2) and (5) for each image, are given Figure 5 through Figure 12.

4. Concluding remarks

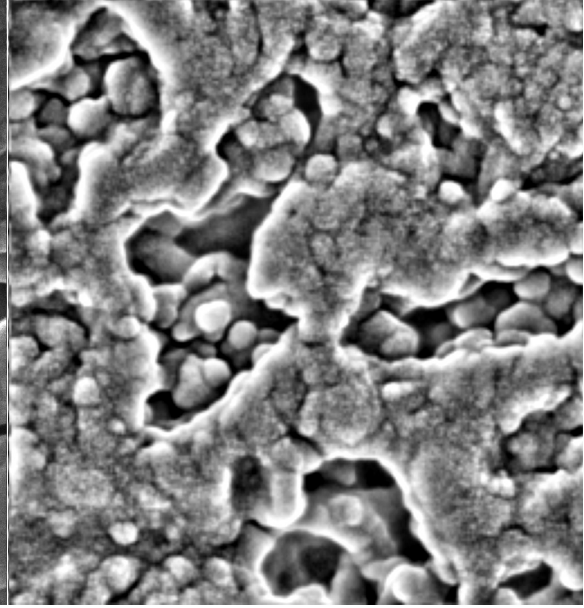
Extensive discussions and comparisons are given in [6], between Lévy and Linnik point spread functions in blind deconvolution of astronomical images. While preconditioning was not needed in the astronomical images considered in [6], such preconditioning plays a major role in electron microscopy. As illustrated in Figure 2, it is important to avoid oversmoothing the ADHE image.

Future applications of the above Linnik blind deconvolution procedure are contemplated for SEM imaging in biomedical, pharmaceutical, and semiconductor contexts. Familiarity with each of these contexts will likely be necessary to obtain useful results.

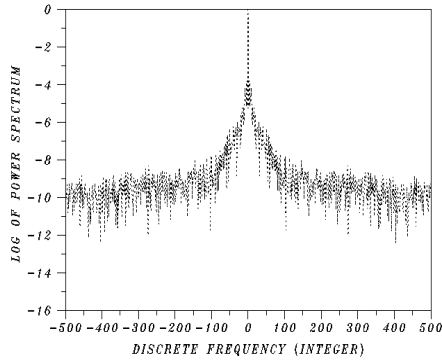
Original SEM image



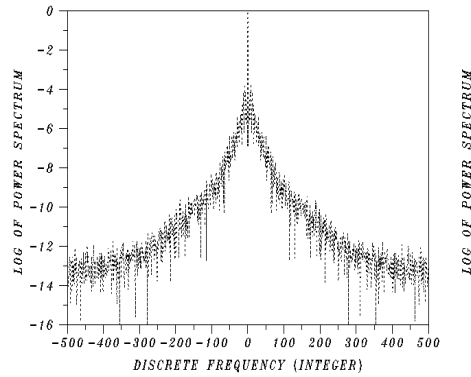
Deblurred SEM image



Original Fourier Spectrum



Deblurred Fourier Spectrum



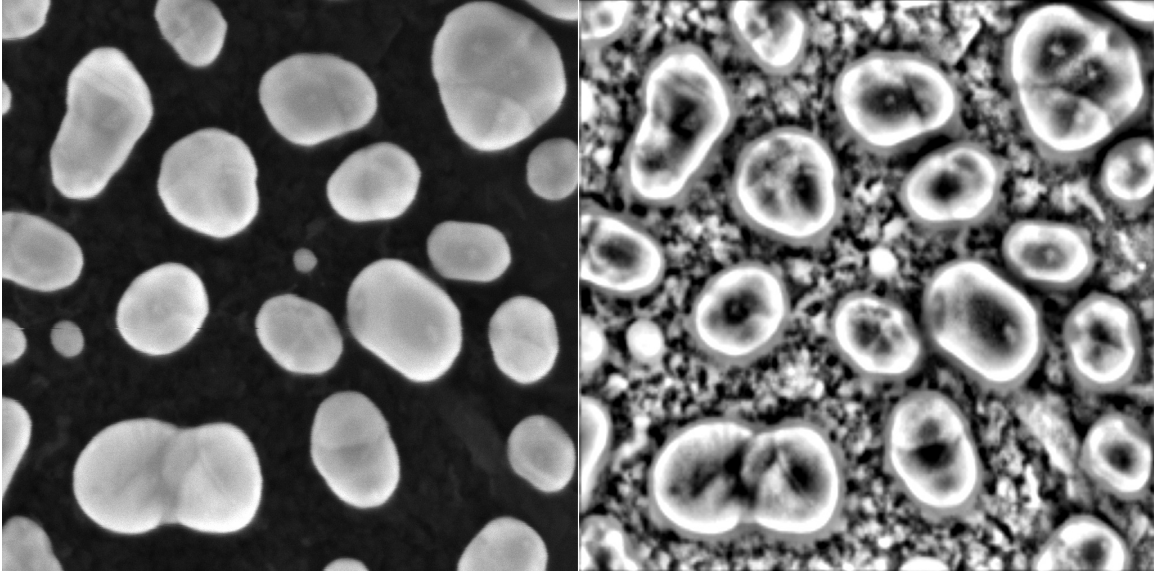
Left: Secondary electron image of an etched glass sample showing weak surface details. Right: rich details after Linnik processing.

Parameters in Eqs. (2) and (5): $\lambda = 0.969$, $\gamma = 1.64$, $s = 0.001$, $K = 300$, $\bar{t} = 0.77$.

Figure 5. $1\mu m$ HFW images

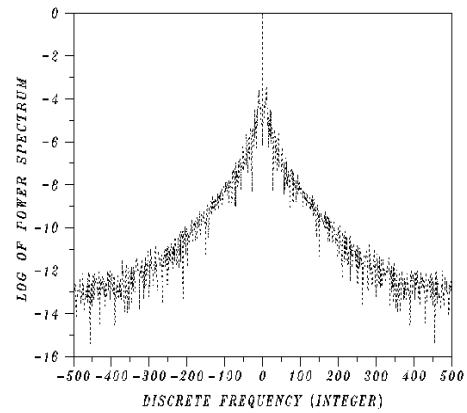
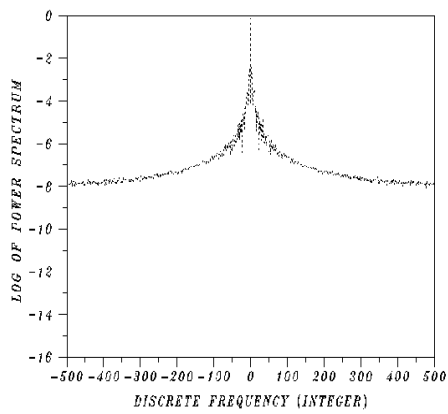
Original SEM image

Deblurred SEM image



Original Fourier Spectrum

Deblurred Fourier Spectrum

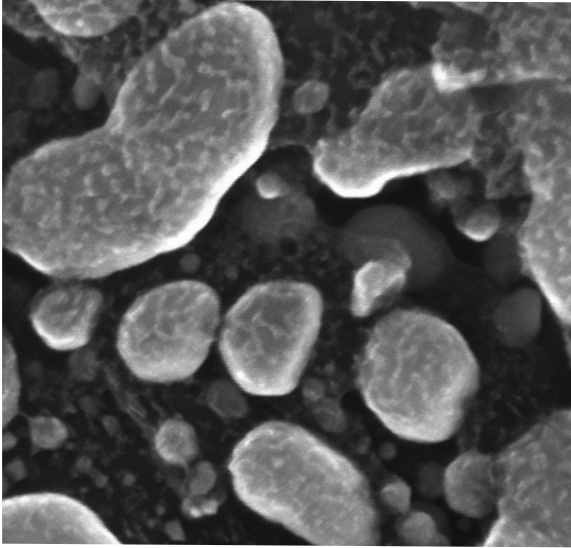


Left: $0.5nm$ resolution secondary electron image of Au nanoparticles on Si substrate with faint surface details. Right: rich details revealed after Linnik processing.

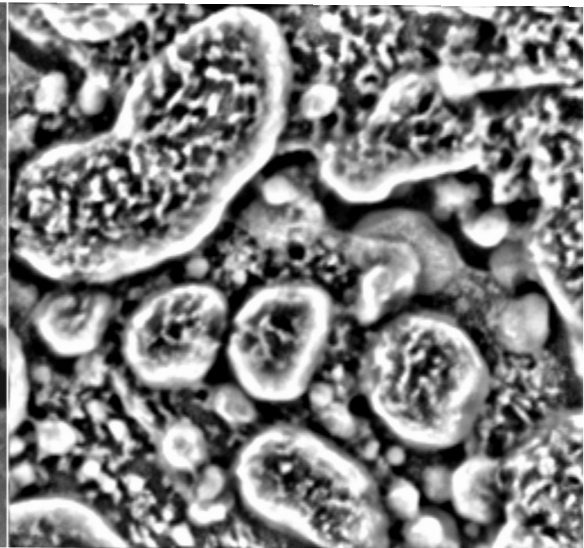
Parameters in Eqs. (2) and (5): $\lambda = 0.989$, $\gamma = 1.226$, $s = 0.001$, $K = 500$, $\bar{t} = 0.77$.

Figure 6. $0.25\mu m$ HFW images

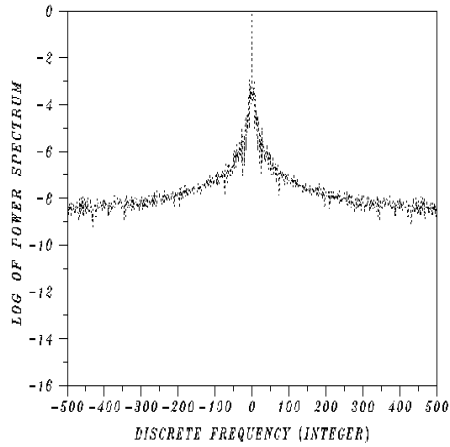
Original SEM image



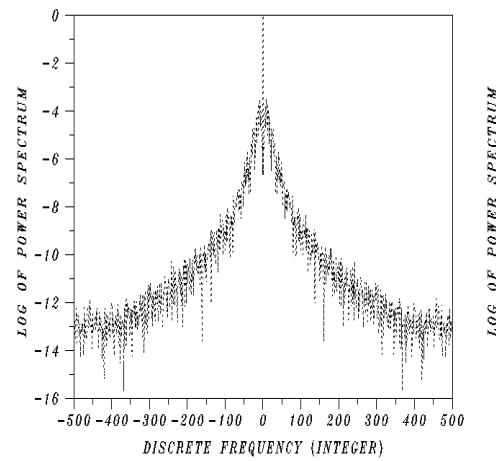
Deblurred SEM image



Original Fourier Spectrum



Deblurred Fourier Spectrum

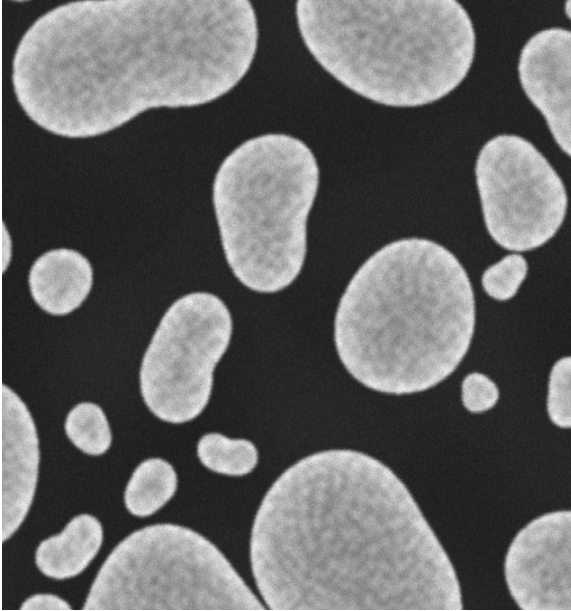


0.5 μm HFW images. Left: 0.7nm resolution secondary electron image of Au nanoparticles on C substrate, showing surface details only on the gold particles. Right: rich details revealed on the C areas after Linnik processing.

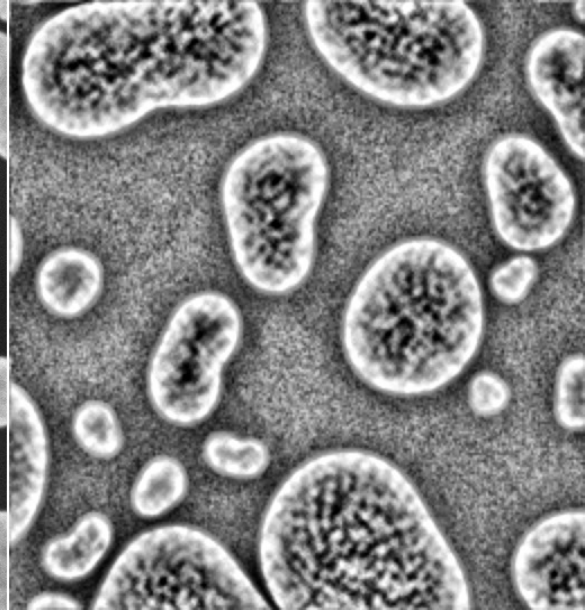
Parameters in Eqs. (2) and (5): $\lambda = 1.0$, $\gamma = 0.959$, $s = 0.001$, $K = 500$, $\bar{t} = 0.77$.

Figure 7. 0.5 μm HFW images

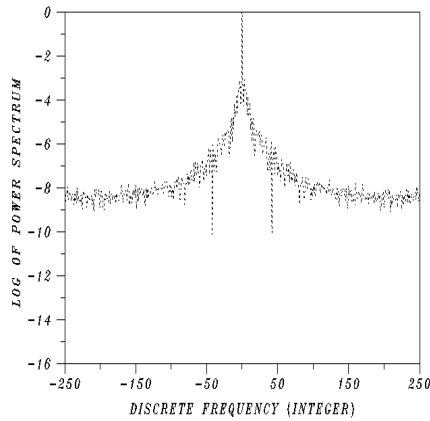
Original SEM image



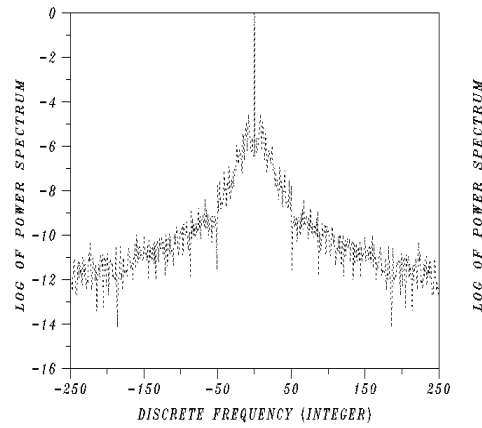
Deblurred SEM image



Original Fourier Spectrum



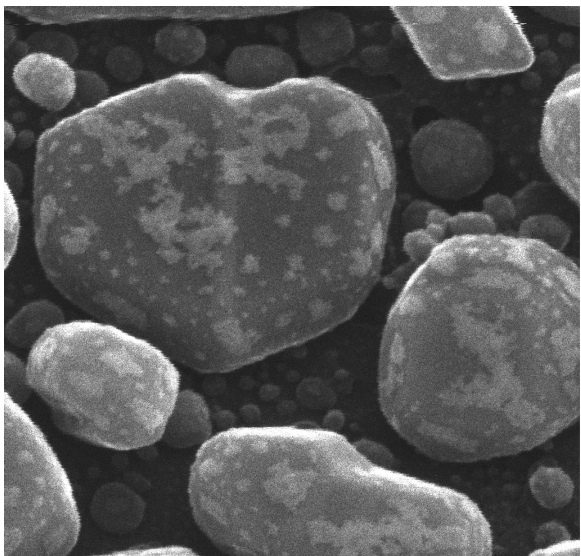
Deblurred Fourier Spectrum



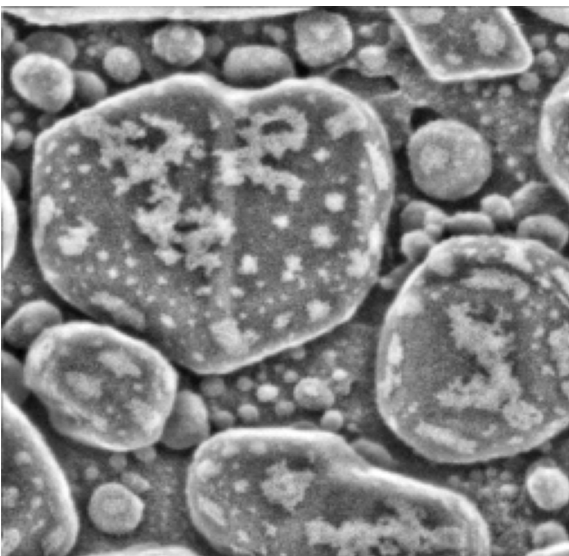
Left: simulated secondary electron image of Au nanoparticles on Si substrate, showing surface details only on the gold particles. Right: gray level variations revealed on the Si areas after Linnik processing.
Parameters in Eqs. (2) and (5): $\lambda = 0.993$, $\gamma = 1.06$, $s = 0.001$, $K = 500$, $\bar{t} = 0.8$.

Figure 8. $0.50\mu\text{m}$ HFW images

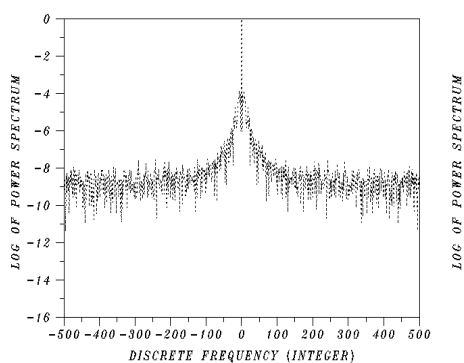
Original HIM image



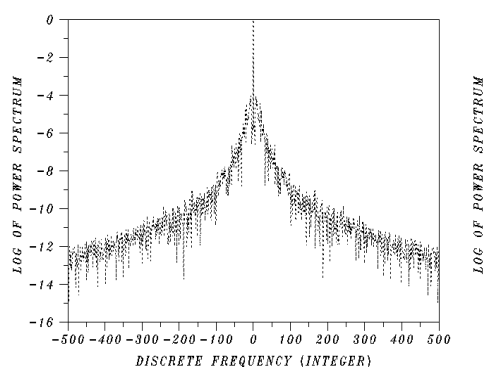
Deblurred HIM image



Original Fourier Spectrum



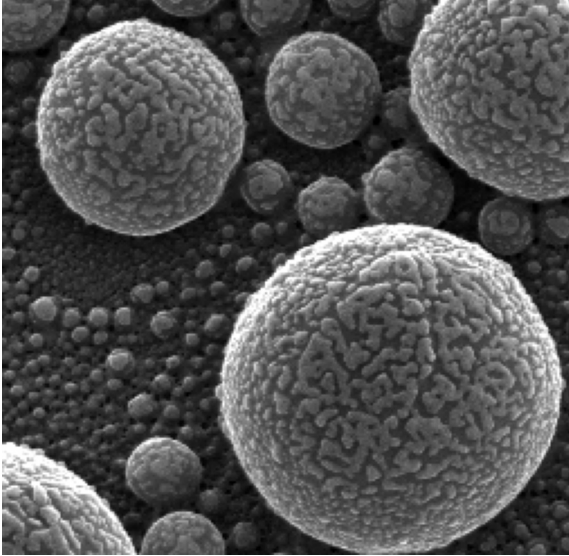
Deblurred Fourier Spectrum



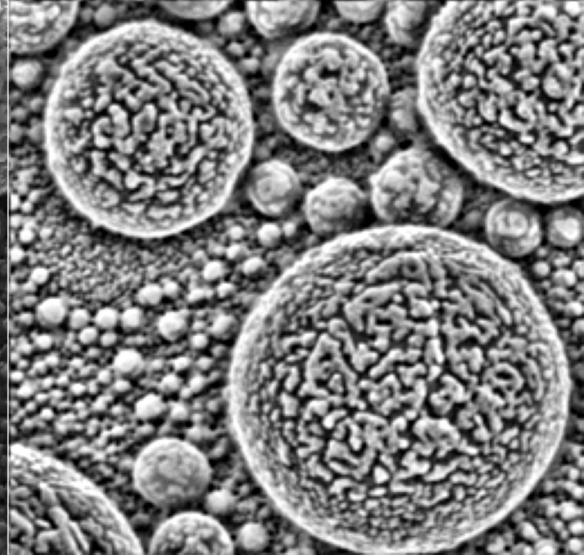
Left: 0.7nm resolution secondary electron image of Pt-decorated Au nanoparticles on C substrate, showing surface details on gold particles only. Right: rich details revealed on the C areas after Linnik processing.
Parameters in Eqs. (2) and (5): $\lambda = 1.0$, $\gamma = 0.367$, $s = 0.001$, $K = 500$, $\bar{t} = 0.77$.

Figure 9. 0.50 μ m HFW HIM images

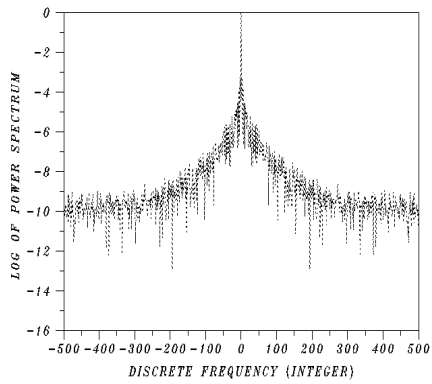
Original HIM image



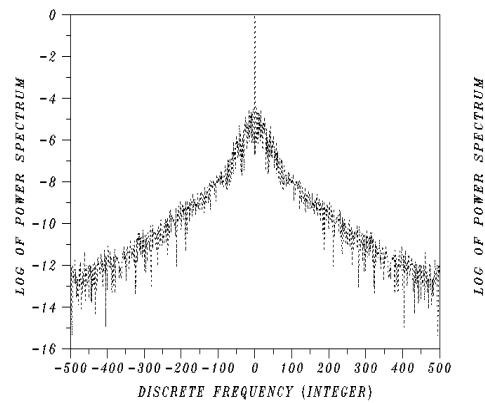
Deblurred HIM image



Original Fourier Spectrum



Deblurred Fourier Spectrum



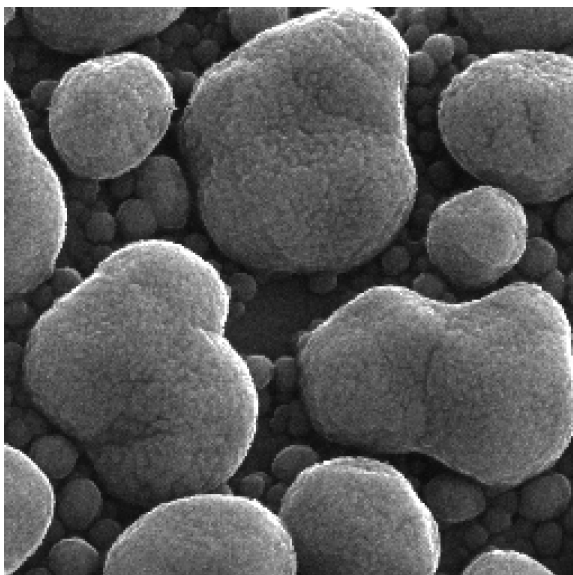
Left: high resolution Pt-decorated tinballs on Si substrate, showing surface details on the gold particles.

Right: rich details revealed in the dark areas after Linnik processing.

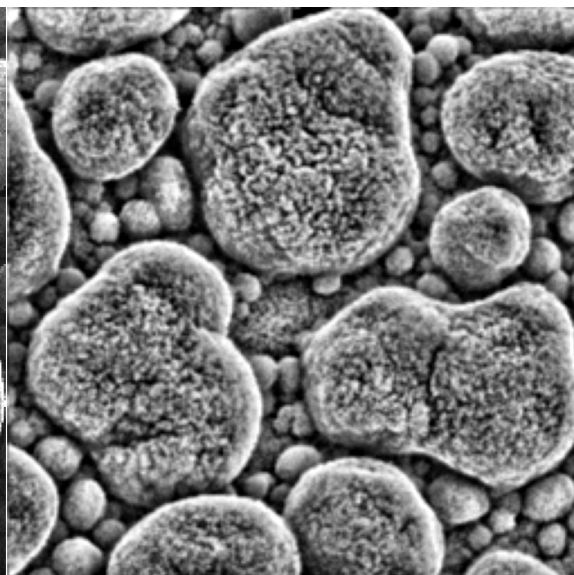
Parameters in Eqs. (2) and (5): $\lambda = 1.0$, $\gamma = 0.367$, $s = 0.001$, $K = 500$, $\bar{t} = 0.73$.

Figure 10. $5.0\mu\text{m}$ HFW HIM images

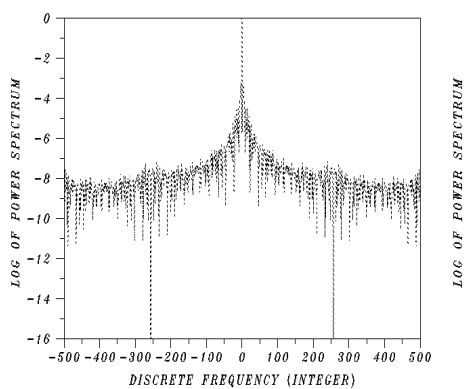
Original HIM image



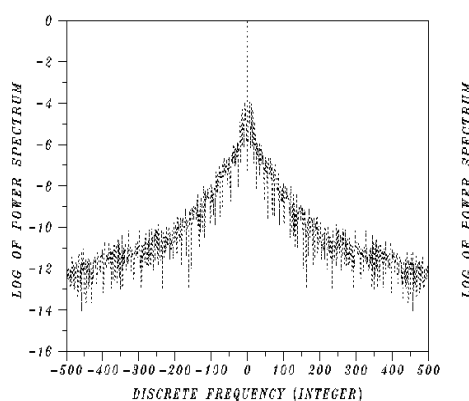
Deblurred HIM image



Original Fourier Spectrum



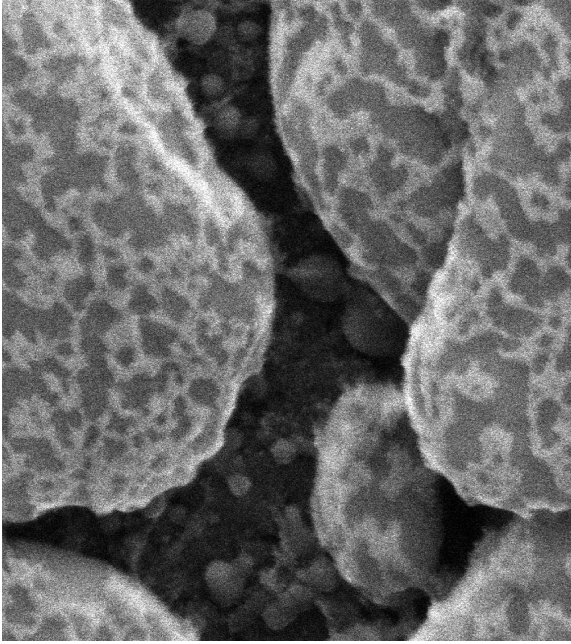
Deblurred Fourier Spectrum



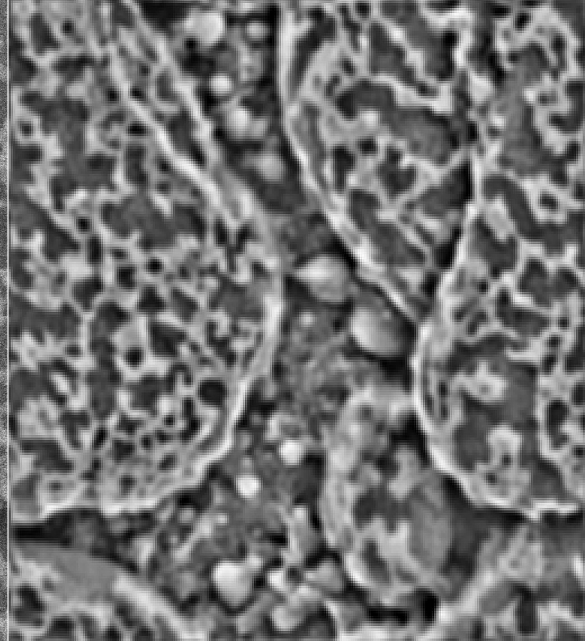
Left: 0.7nm resolution secondary electron image of Au nanoparticles on C substrate, showing surface details on the gold particles only. Right: rich details revealed after Linnik processing.
Parameters in Eqs. (2) and (5): $\lambda = 1.0$, $\gamma = 0.367$, $s = 0.001$, $K = 500$, $t = 0.77$.

Figure 11. 2.0 μm HFW HIM images

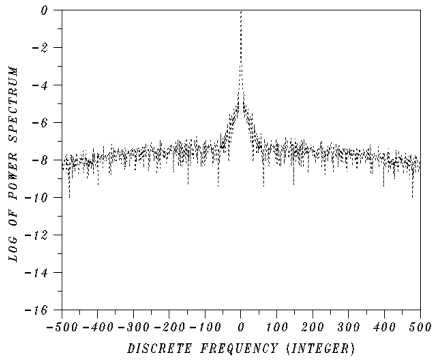
Original HIM image



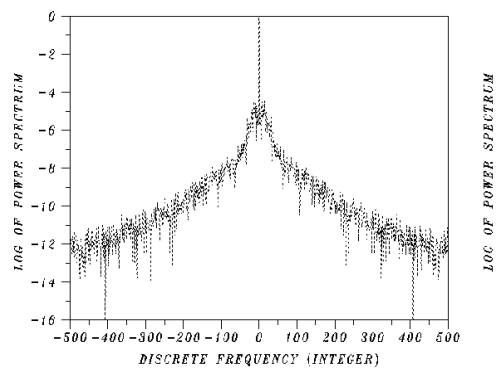
Deblurred HIM image



Original Fourier Spectrum



Deblurred Fourier Spectrum



Left: 0.7nm resolution secondary electron image of Pt-decorated Au nanoparticles on C substrate, showing surface details on the gold particles only. Right: rich details revealed after Linnik processing.
Parameters in Eqs. (2) and (5): $\lambda = 1.0$, $\gamma = 0.367$, $s = 0.001$, $K = 500$, $\bar{t} = 0.77$.

Figure 12. 0.5 μm HFW HIM images

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