NIST Technical Note 1911

Guidelines for the Registration of Two Coordinate Frames

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1 OVERVIEW

The purpose of registration is to obtain the transformation matrix between two coordinate frames. Typically, sensors record positional measurements in their own local coordinate frame. When positional data are acquired by two instruments or two datasets are acquired with the same instrument placed in two different locations, some of the points may be measured only in one frame but have to be accessed in the other frame. In order to register one frame to the other, a set of common points measured in both frames must be obtained. The quality of registration will usually depend on the location and noise of the points used for registration.

The procedures in this paper provide guidance on selecting points for registration and for calculating four metrics for determining the quality of registration. The numerical procedure for calculating the transformation matrix is not included in this paper as it is readily available. This paper is based on experiments and findings reported in Ref. [1].

2 DEFINITIONS AND NOTATION

1. Working frame – the coordinate frame of instrument A, which is the coordinate frame to be transformed.

2. Reference frame - the coordinate frame of instrument B, which is the frame into which the working frame is transformed.

   NOTE: *Instruments A and B may be the same instrument but placed in two different locations.*

3. Common points – a set of M points, in a quasi-regular grid or random pattern, that are measured by both instruments A and B, see Figure 1.

4. Fiducials or fiducial points – a subset of N common points ($N \leq M$) used to register the working frame to the reference frame, see Figure 1a.

5. Test points – a subset of ($M – N$) common points that are not used for registration and are used to evaluate the quality of registration.

6. Target points – a set of points, measured by instrument A but not by instrument B. These are points of interest to the user to which registration is applied, see Figure 1b.

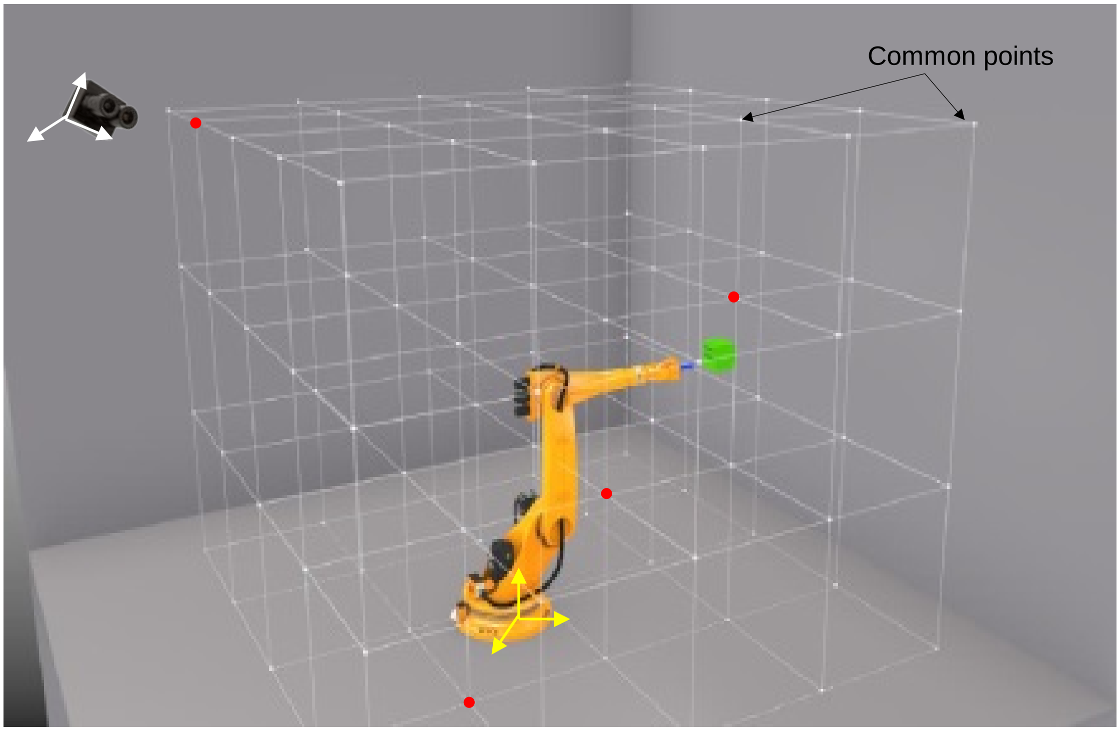
7. Index F – a scalar value used to determine the optimum set of N fiducials for registration based on placement of fiducials and the noise of the fiducials.

   NOTES:
   a. *A lower value of F indicates better registration*.
   b. *This index can be calculated quickly for millions of combinations of fiducials since no registration is performed; only basic arithmetic operations are involved*.

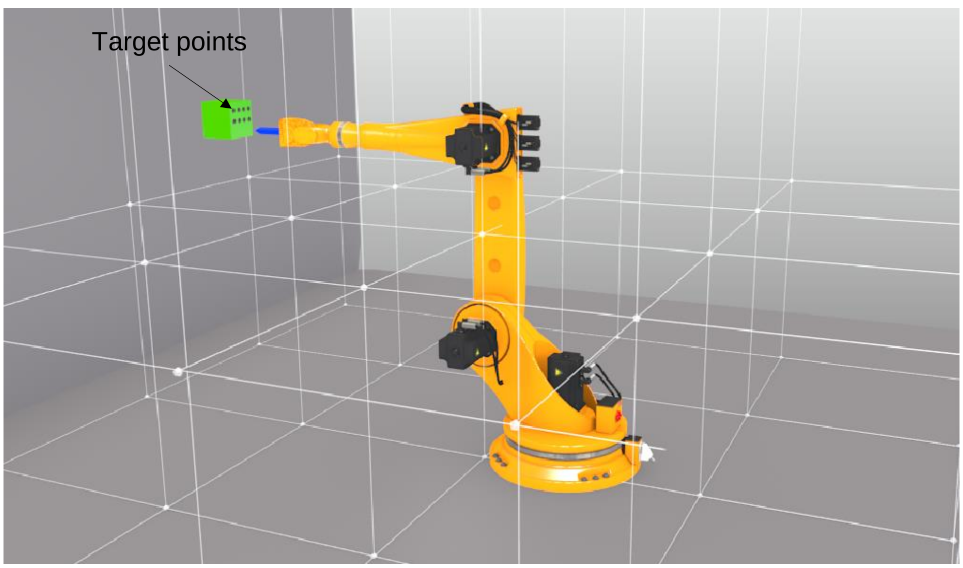
8. $\boldsymbol{\Omega}, \boldsymbol{\tau}$ – 3 x 3 rotation matrix, 1 x 3 translation vector from the registration transformation.

9. $\boldsymbol{p}_{ref\_k}$ – k^{th} test point measured in the reference frame.

10. $\boldsymbol{p}_{trans\_k}$ – k^{th} test point, $\boldsymbol{p}_k$, in the working frame transformed into the reference frame. $\boldsymbol{p}_{trans\_k} = \boldsymbol{\Omega}\, \boldsymbol{p}_k + \boldsymbol{\tau}$.



a.



b.

Figure 1. The entire cubic region represents the work volume. In (a) and (b), the white dots represent common points in a regular grid pattern for which the noise is characterized. The fiducials for registration are selected from this set of points. A potential set of 4 fiducials is represented by red dots in (a). In (b), the black dots on the green object represent the target points. Target points are points of interest to the user; as shown, these points could be points where the robot has to insert a pin into a hole and thus, it has to know the location of the hole within some tolerance.

3 NOISE CHARACTERIZATION

It is recommended that the instrument noise be characterized throughout the work volume for instruments A and B. If the noise characterization is unknown, the selection of fiducials may not yield the best possible registration.

3.1 PROCEDURE

1. Select locations of M common points throughout the work volume. M depends on the variability of the noise. If a large variation is observed between adjacent points, then a larger M may be needed. If little variation is observed between adjacent points, then a smaller M may suffice.

   NOTE:
   In Section 5.1, where index F is calculated, keep in mind that the number, M_N, of all combinations of N fiducials will depend on M and N as follows:

   $$M_N = \frac{M!}{N!\,(M-N)!} \qquad \text{Eq. 1}$$

   For example, for M = 125 and N = 4, M_N = 9,691,375.

2. Obtain repeated measurements at each common point with instruments A and B. It is recommended that at least 10 repeat measurements be obtained; more if feasible.

3. From the repeat measurements, calculate the mean position of the m^{th} common point in the working and reference frames, $\boldsymbol{p}_m$ and $\boldsymbol{p}_{ref\_m}$ for $m = 1,..., M$.

4. From the repeat measurements, calculate the variances [$var(x)$, $var(y)$, $var(z)$] of the coordinates for $\boldsymbol{p}_m$ and $\boldsymbol{p}_{ref\_m}$ for $m = 1,..., M$.. The variances are either the:
   a. the diagonal elements of the covariance matrix or
   b. the square of the standard deviation of the coordinate

5. Calculate the magnitude of the noise, σ_0, for each common point $\boldsymbol{p}_m$ and $\boldsymbol{p}_{ref\_m}$ using the following equation:

   $$\sigma_0(x, y, z) = \sqrt{var(x) + var(y) + var(z)} \qquad \text{Eq. 2}$$

4 BIAS

The presence of bias in the measurements will degrade the registration quality. Good practice dictates that instruments A and B be calibrated prior to use as this will eliminate most of the instrument bias. However, this may not always be possible or bias may be introduced into the measurements from other sources. Therefore, it is recommended that the existence of bias in the measurements be determined.

One way to determine if a bias exists is by comparing the relative distances between all pairs of common points, i and j, as calculated from measurements made by instrument A, d_{ij}, against those calculated from measurements made by the instrument B, D_{ij}. Even if only one instrument is used but in two different locations, d_{ij} may still be different from D_{ij} because bias may be a function of the distance of a point from the instrument.

4.1 PROCEDURE

1. For all possible pairs of common points, calculate d_{ij} and D_{ij}:

$$d_{ij} = \sqrt{\left(x_{A\_i} - x_{A\_j}\right)^2 + \left(y_{A\_i} - y_{A\_j}\right)^2 + \left(z_{A\_i} - z_{A\_j}\right)^2}\ , \qquad \text{Eq. 3a}$$

$$D_{ij} = \sqrt{\left(x_{B\_i} - x_{B\_j}\right)^2 + \left(y_{B\_i} - y_{B\_j}\right)^2 + \left(z_{B\_i} - z_{B\_j}\right)^2}\ , \qquad \text{Eq. 3b}$$

   where

   $\left(x_{A\_i}, y_{A\_i}, z_{A\_i}\right), \left(x_{A\_j}, y_{A\_j}, z_{A\_j}\right)$ are coordinates of common points $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$, as measured by instrument A, $1 \le i < j$ and $j = 2, \dots, M$. See Section 3.1, Step 3.

   $\left(x_{B\_i}, y_{B\_i}, z_{B\_i}\right)$, $\left(x_{B\_j}, y_{B\_j}, z_{B\_j}\right)$ are coordinates of common points $\boldsymbol{p}_{ref\_i}$ and $\boldsymbol{p}_{ref\_j}$, as measured by the instrument B for $1 \le i < j$ and $j = 2, \dots, M$. See Section 3.1, Step 3.

2. Calculate the error of the distance:

$$err_{ij} = d_{ij} - D_{ij} \qquad \text{Eq. 4}$$

3. Calculate the RMS of err_{ij}

$$RMS_{err} = \sqrt{\frac{2}{M(M-1)} \sum_{i=1}^{j-1} \sum_{j=2}^{M} err_{ij}^2}\ . \qquad \text{Eq. 5}$$

4. If $RMS_{err} \approx 0$ then there is little or no bias; otherwise there is a bias.

5 SELECTION OF FIDUCIALS FOR REGISTRATION

Find the fiducials for optimal registration using index F. It is recommended that a minimum of $N = 4$ fiducials be used for registration. Once the fiducials are selected, perform the registration.

NOTE: *The combination of fiducials that minimizes index F also yields small Q (see Appendix) and relatively small RMS_T (see Section 6).*

5.1 PROCEDURE

1. Calculate the index F for all possible M_N combinations.

$$F = \sum_{m=1}^{n-1} \sum_{n=2}^{N} \left(\frac{\sigma_0(\boldsymbol{Z}_n) + \sigma_0(\boldsymbol{Z}_m)}{d(\boldsymbol{Z}_n, \boldsymbol{Z}_m)} + \frac{\sigma_0(\boldsymbol{Y}_n) + \sigma_0(\boldsymbol{Y}_m)}{d(\boldsymbol{Y}_n, \boldsymbol{Y}_m)} \right), \qquad \text{Eq. 6}$$

   where

   $\boldsymbol{Z}_n$ and $\boldsymbol{Y}_n$ ($n = 1, \dots, N$) are two sets of corresponding fiducials in the working and reference frame, respectively, where $\boldsymbol{Z}_n$ and $\boldsymbol{Y}_n$ are mean points per Section 3.1, Step 3

   $d(\boldsymbol{Z}_n, \boldsymbol{Z}_m), d(\boldsymbol{Y}_n, \boldsymbol{Y}_m)$ are the distances between $\boldsymbol{Z}_n$ and $\boldsymbol{Z}_m$ and $\boldsymbol{Y}_n$ and $\boldsymbol{Y}_m$, respectively

   $\sigma_0(\boldsymbol{Z}_n)$ and $\sigma_0(\boldsymbol{Y}_n)$ are magnitude of noise (as defined in Eq. 2) at fiducial points $\boldsymbol{Z}_n$ and $\boldsymbol{Y}_n$

2. Select the set of fiducials that corresponds to the minimum F.

3. It was found in [1] that if all possible combinations M_N were investigated in Step 1of this section, then the addition of more fiducials may not be required as they only marginally improve the registration.

   However, if an additional fiducial is desired, then for N+1 fiducials:
   a. Start with the N fiducials selected in Step 2
   b. Add an additional fiducial from the remaining M–N fiducials
   c. Calculate index F for all possible M–N combinations
   d. Select the set of fiducials that correspond to the minimum F
   e. Repeat Step 3a-d if an additional fiducial is desired

4. Once the fiducials are selected, perform the registration and obtain rotation matrix $\boldsymbol{\Omega}$ and the translation vector $\boldsymbol{\tau}$.

6 REGISTRATION QUALITY

Since the target points are points of interest to the user, a useful metric to quantify registration performance would be the root mean square (RMS) of the target points. However, since target points are not measured by instrument B, this calculation is not possible.

We propose a procedure to estimate the RMS of target points using test points; this metric is RMS_T. This procedure involves calculating the distances, d_k, between transformed test points, $\boldsymbol{p}_{trans\_k}$, and the corresponding test points in the reference frame, $\boldsymbol{p}_{ref\_k}$, for k = 1 to K where K is the number of target points that are of interest. For reliable estimates of the RMS of the target points using test points, we recommend the selection of test points in the working frame that are closest to the target points in the working frame as measured by instrument A.

6.1 RMS_T

6.1.1 Procedure

1. In the working frame, select a set of K test points that are closest to the target points.

2. Calculate the distances between corresponding selected test points

$$\boldsymbol{d}_k = \left\|\boldsymbol{p}_{trans\_k} - \boldsymbol{p}_{ref\_k}\right\| \qquad \text{Eq. 7}$$

$$= \sqrt{\left(x_{trans\_k} - x_{ref\_k}\right)^2 + \left(y_{trans\_k} - y_{ref\_k}\right)^2 + \left(z_{trans\_k} - z_{ref\_k}\right)^2}$$

3. Calculate RMS_T:

$$RMS_T = \sqrt{\frac{\sum_{k=1}^{K} d_k^2}{K}} \qquad \text{Eq. 8}$$

REFERENCES

1. Franaszek, Marek and Cheok, Geraldine S., Optimization of Registration Performance Metrics, to be published as *NISTIR 8111*, Gaithersburg, MD, February 2016.

APPENDIX: Performance Metrics

Besides RMS_T, three other metrics may be used to evaluate the registration performance: Q, W, and RMS_F. Q and W are calculated using test points[1] and RMS_F is calculated using fiducials. The choice of metric depends on the application as the set of fiducials that minimizes one of the metric may not necessarily minimize the other metrics. The following sections describe the metrics and list the procedures to calculate them.

The computations of the performance metrics involve determinations of covariances and Jacobian matrices, and it is assumed that the reader is able to determine the required information.

A1 INDEX *W*

A1.1 Description

- W is a scalar value that characterizes the work volume.
- W is the median value of all w_k (Eq. 9) where w_k are dimensionless indices to check the distance between transformed test points and the corresponding test points in the reference frame.
- $w_k < 1$ indicates that the distance d_k between transformed test point $\boldsymbol{p}_{trans\_k}$ and its corresponding test point $\boldsymbol{p}_{ref\_k}$ is within one standard deviation of d_k.
- A large W ($W > 2$) indicates bad registration and the likely presence of bias.

A1.2 Procedure

1. For each test point, calculate the distance, d_k, between the transformed test point, $\boldsymbol{p}_{trans\_k}$, and the corresponding test point in the reference frame, $\boldsymbol{p}_{ref\_k}$ using Eq. 7.

2. Calculate w_k

$$w_k = \frac{d_k}{\sqrt{var(d_k)}}, \qquad \text{Eq. 9}$$

where

$$var(d_k) = \boldsymbol{J}_k \boldsymbol{C}_k \boldsymbol{J}_k^T \qquad \text{Eq. 10}$$

$$\boldsymbol{C}_k \text{is a 6 x 6 covariance matrix } = \begin{bmatrix} c(\boldsymbol{p}_{trans\_k}) & 0 \\ 0 & c(\boldsymbol{p}_{ref\_k}) \end{bmatrix} \qquad \text{Eq. 11}$$

$\boldsymbol{c}(\boldsymbol{p}_{trans\_k})$ is the covariance matrix of the transformed k^{th} test point

[1] Similar to RMS_T, Q and W would ideally be calculated using target points. However, for the same reason that RMS_T could not be calculated for target points, test points are used to calculate Q and W.

$$= \quad \boldsymbol{J}_t \begin{bmatrix} \boldsymbol{C}_{\boldsymbol{\Theta}} & 0 \\ 0 & \boldsymbol{c}(\boldsymbol{p}_{ref\_k}) \end{bmatrix} \boldsymbol{J}_t^T \qquad \text{Eq. 12}$$

$\boldsymbol{c}(\boldsymbol{p}_{ref\_k})$ is the covariance matrix of the k^{th} test point measured in the reference frame

$\boldsymbol{C}_{\Theta}$ is the 6 x 6 covariance matrix of $\boldsymbol{\Theta} = (\vartheta, \varphi, \rho, \tau_x, \tau_y, \tau_z)$ where the three angles $\{\vartheta, \varphi, \rho\}$ parameterize the rotation matrix $\boldsymbol{\Omega}$ and $\{\tau_x, \tau_y, \tau_z\}$ are the components of $\boldsymbol{\tau}$

$$\boldsymbol{J}_k \left(\boldsymbol{p}_{trans\_k}(\tilde{x}, \tilde{y}, \tilde{z}), \boldsymbol{p}_{ref\_k}(x, y, z)\right) = \frac{1}{d_k} [\tilde{x} - x, \tilde{y} - y, \tilde{z} - z, x - \tilde{x}, y - \tilde{y}, z - \tilde{z}]$$

$\boldsymbol{J}_t$ is the Jacobian matrix of the transformation from the working frame to the reference frame. See [1] for more detailed information.

3. W is the median of all w_k

A2 Index Q

A2.1 Description

- Q is a scalar value that characterizes the work volume.
- Q gives an indication of how much the uncertainties of the transformed test points increase due to the registration process relative to the uncertainties of the corresponding test points measured in the working frame.
- Q is the median of all q_k where q_k is the ratio of the radii (see Eq. 15) of two spheres with equivalent volumes $V(\boldsymbol{p}_{trans\_k})$ and $V(\boldsymbol{p}_k)$, see Eq. (13, 14).
- $Q \geq 1$; a smaller Q indicates better registration.

A2.2 Procedure

1. Calculate the covariance of the transformed test point, $\boldsymbol{c}(\boldsymbol{p}_{trans\_k})$, using Eq. 12 and then calculate its eigenvalues $\{\lambda_{trans\_1}, \lambda_{trans\_2}, \lambda_{trans\_3}\}$.

2. Calculate the volume of the uncertainty ellipsoid of the transformed point

$$V(\boldsymbol{p}_{trans\_k}) = \frac{4}{3}\pi \, \lambda_{trans\_1} \lambda_{trans\_2} \lambda_{trans\_3} \qquad \text{Eq. 13}$$

3. Calculate the covariance of the untransformed test point $\boldsymbol{c}(\boldsymbol{p}_k)$, in the working frame, and then calculate its eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$

4. Calculate the volume of the uncertainty ellipsoid of the test point in the working frame

$$V(\boldsymbol{p_k}) = \frac{4}{3}\pi\, \lambda_1\lambda_2\lambda_3 \qquad \text{Eq. 14}$$

5. Calculate q_k

$$q_k = \sqrt[3]{V(\boldsymbol{p}_{trans\_k})/V(\boldsymbol{p}_k)} \qquad \text{Eq. 15}$$

6. Q is the median of all q_k

A3 $\boldsymbol{RMS_F}$

A3.1 Description

- The RMS value of the distances between the transformed fiducial points and the corresponding fiducial points in the reference frame.
- A smaller RMS_F indicates better registration.
- Advantage: metric is readily available.
- Disadvantage: there is no correlation between RMS_T and RMS_F.

A3.2 Procedure

1. Calculate the distances between corresponding fiducial points

$$d_n = \sqrt{\left(x_{trans\_k} - x_{ref\_k}\right)^2 + \left(y_{trans\_k} - y_{ref\_k}\right)^2 + \left(z_{trans\_k} - z_{ref\_k}\right)^2} \qquad \text{Eq. 16}$$

2. Calculate RMS_F

$$RMS_F = \sqrt{\frac{\sum_{n=1}^{N} d_n^2}{N}} \qquad \text{Eq. 17}$$

<u>NOTE</u>:
For any set of N fiducials, $RMS_F > minRMS_F$ where

$$minRMS_F = \sqrt{\frac{1}{2N(N-1)}\sum_{i=1}^{j-1}\sum_{j=2}^{N} err_{ij}^2} \qquad \text{Eq. 18}$$

and err_{ij} is the error calculated in Eq. 4 for (i,j) pairs of fiducials (see [1] for derivation of Eq. 18).