HIGH-FREQUENCY RESISTANCE OF INDUCTANCE COILS

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ABSTRACT

The definition of the resistance of a coil is simple for direct current and may be given directly in terms of Ohm's Law. Similarly, the computation of the resistance of a coil made of a wire of known length, cross section, and material is quite easy for direct current. The same problem becomes difficult if an alternating current of high frequency is used instead of a direct current.

It is shown in this paper that a different definition of resistance must be given; that the value of the resistance thus defined depends upon the point in the coil with respect to which the resistance is defined; and it is shown how the resistance-variation method, described in Bureau of Standards Circular 74, may be employed to measure the resistance thus defined.

A formula is derived for the value of the current at any point of the coil when the emf induced in any portion of the coil is known. This formula is compared with the experimental results obtained on coil aerials. This formula presupposes the knowledge of the resistance measured by the resistance-variation method with respect to all points on the coil. In order to know this resistance, it is sufficient to know the resistance as measured with respect to one point and also to know the distribution of current in the coil. The value of the resistance with respect to one point of the coil is worked out and the result is verified experimentally.

CONTENTS

INTRODUCTORY

At low frequencies there is little difficulty in discussing the resistance of a coil. At sufficiently high frequencies, however, some difficulty is encountered. There are two effects which cause the difficulty. The first is the skin effect and the second is the capacity effect. It is the purpose of this discussion to treat the capacity effect. It is believed that the modifications introduced in this treatment by skin effect are only minor.

The capacity effect consists in the collection of charges on the wires of the coil and in the nonuniform distribution of current in the coil which these charges cause.

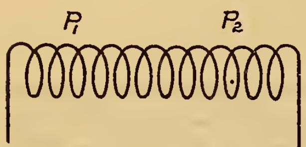

FIG. 1.—*Inductance coil*

If P_1, P_2 are two points of the coil (see Fig. 1), a certain charge collects on the wire between P_1 and P_2. If the current entering the section $P_1 P_2$ at P_1 is i_1, and if the current leaving the section at P_2 is i_2, and if Q should denote the charge in the section $P_1 P_2$ then

$$i_1 = \frac{dQ}{dt} + i_2$$

by the conservation of charge.

The quantity Q is determined mainly by the potential differences between the section $P_1 P_2$ and the remaining parts of the coil. These potential differences are roughly proportional to $\frac{di}{dt}$ where i is the average current in the coil.

This, of course, is true only provided there are no appreciable phase differences between the induced emf's due to various portions of the coil; i. e., provided the dimensions of the coil are small in comparison with the wave length used, and also provided the coil may be regarded as a perfect conductor. Thus, the difference between i_1 and i_2 is roughly proportional to i and the square of the frequency. For high frequencies, therefore, the current distribution is very nonuniform and for low frequencies it is practically uniform.

The nonuniformity of current affects the resistance of the coil. It is the purpose here to find the modification to be introduced in the ordinary treatment of resistance by the nonuniform current distribution.

I. MEANING OF THE TERM "RESISTANCE" AT HIGH FREQUENCIES

In alternating-current theory the resistance is taken to be the real part of a complex quantity z (called the complex impedance), which is such that if the current i is the real part of $I\epsilon^{j\omega t}$, and and if the emf e is the real part of $E\epsilon^{j\omega t}$, where I and E are real or complex quantities independent of the time t, where $j=\sqrt{-1}$, and where ϵ is the base of natural logarithms, then

$$E=zI.$$

This definition applies to circuits consisting of series combinations of inductances, resistances, and capacities independently of where the emf e is applied or where the current i is measured. If, however, the circuits consist of parallel combinations of resistance, inductance, and capacity, it is no longer immaterial where the emf e or the current i are measured.

In the case of the coil, also, it is not immaterial where e is inserted and where i is measured, because i is not the same at different points of the coil and because the effect of e on i at a given place may depend upon the place where e is inserted. This compels one to define the resistance not for the coil as a whole, but only for a particular point of the coil. One can imagine the wire cut at that point, the two faces of the cut separated by a very small gap and an emf inserted in that gap, and one can also imagine the current measured at that gap. Call the emf e and the current i and write

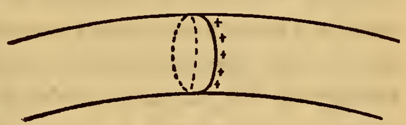

FIG. 2.—*Cross-section of wire with a double layer of charge*

$$e=E\epsilon^{j\omega t}, \qquad i=I\epsilon^{j\omega t}.$$

The impedance of the coil, together with the circuit connected across its terminals with reference to the point at which the gap was made, will be defined, as usual, by the equation

$$z=\frac{E}{I}.$$

The real part of the impedance will be termed the resistance and the coefficient of j in the imaginary part will be called the reactance.

It may be shown in the usual manner that the resistance R thus defined is such that $i^2 R$ is the power lost in the coil where i, R are measured at the point where the emf is applied.

II. SIMPLIFYING ASSUMPTION

In radiotelegraphy it is usually legitimate to assume that the coils are made of perfect conductors when it is desired to find the current distribution in the coils. A perfect conductor is an ideal substance whose properties are the limit of the properties approached by a conductor having a finite conductivity when the conductivity becomes infinite. For a perfect conductor the electric intensity at the surface is normal to the surface and the magnetic intensity at the surface is tangential to the surface.

The energy losses in a system consisting of perfect conductors are entirely energy losses due to radiation. In the case of a coil these energy losses are negligible, and so the coil itself, so far as its current distribution is concerned, can be looked at as a nondissipative system; i. e., a system in which no energy is dissipated.

It is well known that there is a very close analogy between electrical and dynamical systems, and that to most of the propositions of dynamics one can give an electrical interpretation. The analogy between dynamical and electrical systems becomes complete if charges are taken as coordinates. Then the energy stored up in the electric field is regarded as potential energy and the energy of the magnetic field of the currents is regarded as kinetic energy. Of course, in a consideration of a continuous system, such as a coil, the number of coordinates is infinite, so that the system should be considered as the limiting case of another system with a finite number of coordinates, as this number is increased indefinitely. In the case of the coil the wire may be subdivided into small portions along each length and the charges on these portions are then coordinates. As the number of sections is increased indefinitely the number of coordinates also increases indefinitely, and in the limit the coordinates represent the coil correctly. This is precisely analogous to the treatment of the vibrations of a string of finite and uniform density by replacing it by a weightless string with beads spaced along its length.

Nondissipative dynamical systems have been studied, their properties are known, and will be here applied to the case of the coil made out of a perfect conductor. A brief review of these properties will now be given.

III. PROPERTIES OF NONDISSIPATIVE SYSTEMS

Consider, then, a dynamical system whose kinetic energy T and whose potential energy V are given, respectively, by

$$T = \tfrac{1}{2}\,(a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \cdot\ \cdot\ \cdot + 2a_{12}\dot{q}_1\dot{q}_2 + \cdot\ \cdot)$$

$$V = \tfrac{1}{2}\,(b_{11}q_1^2 + b_{22}q_2^2 + \cdot\ \cdot\ \cdot + 2b_{12}q_1q_2 + \cdot\ \cdot)$$

where

$$q_1,\ q_2,\ q_3,\ \cdot\ \cdot\ \cdot,\ q_n$$

are the coordinates of the system and where

$$\dot{q}_1, \dot{q}_2, \cdot \cdot \cdot, \dot{q}_n$$

are the derivatives of the coordinates as to the time. The behavior of the system is given by Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = Q_r, \quad (r = 1, 2, \cdot \cdot \cdot, n)$$

where L is the kinetic potential defined by

$$L = T - V$$

and where Q_r is the component of generalized force corresponding to q_r. These equations when the values of T and V are substituted take the form

$$\left(a_{1r}\frac{d^2q_1}{dt^2} + b_{1r}q_1\right) + \left(a_{2r}\frac{d^2q_2}{dt^2} + b_{2r}q_2\right) + \cdot \cdot \cdot + \left(a_{nr}\frac{d^2q_n}{dt^2} + b_{nr}q_n\right) = Q_r$$

$$(r = 1, 2, \cdot \cdot \cdot n)$$

or symbolically

$$\left(a_{1r}\frac{d^2}{dt^2} + b_{1r}\right)q_1 + \left(a_{2r}\frac{d^2}{dt^2} + b_{2r}\right)q_2 + \cdot \cdot \cdot + \left(a_{nr}\frac{d^2}{dt^2} + b_{nr}\right)q_n = Q_r$$

If one deals with purely periodic motions of frequency ω in 2π seconds, then differentiation as to t when repeated twice amounts to multiplication by $-\omega^2$. Hence the equations of motion become

$$(-a_{1r}\omega^2 + b_{1r})q_1 + (-a_{2r}\omega^2 + b_{2r})\,q_2 + \cdot \cdot \cdot + (-a_{nr}\omega^2 + b_{nr})\,q_n = Q_r$$

The solution of this system of equations gives for the coordinates

$$q_s = \frac{\Delta_s}{\Delta}$$

where

$$\Delta = \begin{vmatrix} -a_{11}\omega^2 + b_{11}, & -a_{12}\omega^2 + b_{12}, & \cdot \cdot \cdot \cdot, & -a_{1n}\omega^2 + b_{1n} \\ -a_{12}\omega^2 + b_{12}, & -a_{22}\omega^2 + b_{22}, & \cdot \cdot \cdot \cdot, & -a_{2n}\omega^2 + b_{2n} \\ -a_{13}\omega^2 + b_{13}, & -a_{23}\omega^2 + b_{23}, & \cdot \cdot \cdot \cdot, & -a_{3n}\omega^2 + b_{3n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -a_{1n}\omega^2 + b_{1n}, & -a_{2n}\omega^2 + b_{2n}, & \cdot \cdot \cdot, & -a_{nn}\omega^2 + b_{nn} \end{vmatrix}$$

and where Δ_s is the result of changing in Δ the elements of the s^{th} column into $Q_1, Q_2, \cdot \cdot \cdot \cdot, Q_n$.

The condition for resonance is

$$\Delta = 0.$$

This is an equation in ω^2. It has n roots and gives the n natural periods of the system.

If $\Delta = O$, then provided neither of its first minors is zero, and provided neither of the quantities Q_s is zero, all of the quantities q_s become infinite. Similarly, their derivatives become infinite (provided $\omega \neq O$) and also all linear combinations of the coordinates or of their derivatives, which are not identically zero, become infinite.

Thus, for a nondissipative system resonance is independent of the relative values of the components of generalized force, and it is also independent of the quantity measured in detecting resonance as long as the quantity used increases when the coordinates increase. Thus, this criterion may be the maximum value of one coordinate or of the derivative of the coordinate as well as of a linear combination of several coordinates (not vanishing identically) or, finally, a linear nonvanishing combination of the several velocities.

IV. APPLICATION OF PROPERTIES OF NONDISSIPATIVE SYSTEMS TO COILS

A physical system is never a nondissipative system. However, if the dissipation of energy is sufficiently small its properties approach those of a nondissipative system.

The inductance coil made of copper wire has some energy losses. If, however, the resistance between any two points of it is negligible compared to the reactance between these two points, the coil may be considered for some purposes as a nondissipative system. The smaller the resistance is made in comparison with the reactance the more nearly is the coil nondissipative.

In the theory that follows it will be assumed that the reactance is sufficiently high and the resistance sufficiently low to make the coil nondissipative. For any actual coil this assumption must, of course, be justified. It is generally justified if the resistance of any part of the coil is negligible compared to its reactance. Under these conditions the presence of the resistance affects the current only to the second order of the resistance, as may be seen by expanding the impedance by Taylor's theorem.

The current distribution also is determined, primarily, by the reactance and is affected only to the second order by the resistance. To this order, then, the current distribution is the same as if the coil were nondissipative. Now, in a nondissipative system the ratios of the coordinates are independent of the distribution of generalized forces as resonance is approached.

To see the truth of this statement observe that the ratio of two coordinates, say, q_1 and q_2, is the ratio of corresponding determinants, say Δ_1, Δ_2. Here

$$\Delta_1 = Q_1\,\mathfrak{C}_{11} + Q_2\,\mathfrak{C}_{12} + \cdot \cdot \cdot \cdot + Q_n\,\mathfrak{C}_{1n}$$
$$\Delta_2 = Q_1\,\mathfrak{C}_{21} + Q_2\,\mathfrak{C}_{22} + \cdot \cdot \cdot \cdot + Q_n\,\mathfrak{C}_{2n}$$

where $\mathfrak{C}_{rs}$ is the minor of $-a_{rs}\omega^2 + b_{rs}$ in Δ.

However, Δ is a symmetrical determinant. Further, for resonance Δ vanishes. But if a symmetrical determinant vanishes then all the second minors of the complimentary determinant vanish. Hence

$$\frac{\mathfrak{C}_{11}}{\mathfrak{C}_{12}} = \frac{\mathfrak{C}_{12}}{\mathfrak{C}_{22}} = \frac{\mathfrak{C}_{13}}{\mathfrak{C}_{23}} = \cdot \cdot \cdot \cdot \cdot = \frac{\mathfrak{C}_{1n}}{\mathfrak{C}_{2n}} = \mathfrak{C}$$

say. And consequently

$$\frac{q_1}{q_2} = \frac{\Delta_1}{\Delta_2} = \mathfrak{C}$$

The ratio $\frac{q_1}{q_2}$ is therefore independent of the values of $Q_1, Q_2, \ldots,$ Q_n provided the resonance condition

$$\Delta(\omega) = O$$

is fulfilled, and also provided it is legitimate, for resonance, to compute the ratios of various q's on the assumption that the system is nondissipative. The latter proviso requires careful consideration, because for resonance the reactance of a circuit vanishes and consequently the current is determined mainly by the resistance and not by the reactance.

Nevertheless, in practice this does not give difficulty, because the only root of $\Delta(\omega)$ to be considered is the lowest, so that the reactance vanishes only for the whole circuit considered in series. Hence, even though the absolute value of the current due to an emf applied at a point P_1 (see Fig. 1) will depend on the resistance, still the distribution of current will be determined mainly by the very high reactances of the various parts of the coil. Thus, considering the portion of the coil between P_1 and P_2, the current may either be used up in charging that portion or else in passing through it and charging portions of the coil beyond $P_1 P_2$. The relative proportion of the parts of the current which are used in these ways is determined by the impedances of the part $P_1 P_2$ (used as a condenser) and the parts beyond $P_1 P_2$. If the conductivity of the material is high, the impedances are not distinguishable from the reactances, and therefore the proviso is satisfied. *It thus has been shown that the current distribution for resonance is independent of the distribution of emf's and that it is the same as if the system were nondissipative.*

These facts may also be extended to systems in which the total dissipation is not negligible, provided the emf is not induced in the parts which are not perfect conductors. For, the current distribution in the parts in which the conductors are nearly perfect is still the same. Thus, if a short resistance of negligible capacity be inserted in a cut made in the coil, the current distribution is not affected thereby, because the current entering the resistance is equal to the current leaving it and because everywhere outside the resistance the coil is nearly nondissipative.

1. RESISTANCE INSERTED IN COIL

As explained, a resistance inserted at some point in the coil does not affect the current distribution. It therefore cuts down the current in the same ratio at all points in the coil. Hence it follows that if the coil has been brought into resonance by adjusting the condenser it will still be in resonance when a resistance is inserted at any point.

In fact, let this point be P. Let an emf e be applied very near to P. Whether the condenser is tuned or not, if a resistance be inserted at P the absolute value of the current must be cut down because, with reference to the point, the coil and the condenser together have a certain impedance, say, $z=R+jX$. Whether or not R is equal to zero, an increase in R increases the absolute value of z, viz: $|z|=\sqrt{R^2+X^2}$
and therefore decreases the absolute value of the current.

2. RESISTANCE-VARIATION METHOD OF MEASURING RESISTANCE

This suggests a method of measuring the quantity R at any point in the coil. This method is known as the resistance-variation method. It is described for the case of the coil terminal in Bureau of Standards Circular 74, pages 180 to 185. It consists in bringing the coil into resonance by a proper adjustment of the tuning condenser and measuring the current I. Then, a resistance R_1 is inserted and a new current I_1 is observed.

As explained the ratio $\frac{I}{I_1}$ does not depend on the way in which the emf is induced. Therefore, one can imagine the emf induced very close to the point where the resistance R_1 is inserted. If the resistance with reference to this point is R and if the emf is E, then

$$I = \frac{E}{R}, \qquad I_1 = \frac{E}{R+R_1}$$

whence

$$R = \frac{R_1}{\frac{I}{I_1} - 1} \quad \ldots . \qquad (1)$$

Here I and I_1 are measured at the point where R_1 is inserted. However, their ratio was shown to be independent of the place where they are measured. Thus, I, I_1 may be measured at an arbitrarily chosen point in the coil. If the place where R_1 is inserted is varied, different values of R are obtained. If x be a parameter along the wire of the coil, R is then a function of x, say, $R\ (x)$.

3. COMPUTATION OF CURRENT AT ONE POINT OF THE COIL DUE TO EMF AT A DIFFERENT POINT

The function $R\ (x)$ is such that the current at x, due to an emf e applied at x, is $\frac{e}{R(x)}$ provided the coil is tuned. Suppose that an emf e is applied at x_1. It is desired to know the current at x_2. The currents at x_2 and x_1 are always in the same ratio, which may be conveniently written as $\frac{f(x_2)}{f(x_1)}$.

The current at x_1 due to e at x_1 is $\frac{e}{R(x_1)}$

Therefore, the current at x_2 due to e at x_1 is $\frac{e}{R(x_1)} \cdot \frac{f(x_2)}{f(x_1)}$. If, now, a resistance R_0 be inserted at x_2, the current at x_2 must be decreased to $\frac{R\ (x_2)}{R\ (x_2)+R_0}$ of its former value. Consequently, at x_1 also it has been decreased to that fraction of its former value. Therefore, at x_1 after the resistance R_0 has been inserted the current is

$$i_1' = \frac{e}{R(x_1)} \cdot \frac{R(x_2)}{R(x_2)+R_0} \tag{2}$$

Again, compare the result of inserting R_0 at x_1 with the result of inserting it at x_2 when the same current flows through the coil. In the second case the amount of energy lost per second is $\left(\frac{f(x_2)}{f(x_1)}\right)^2$ of what it is in the first case. Therefore, in terms of the current i_1 at x_1, the energy lost per second in R_0 when R_0 is inserted at x_2 is

$$R_0\left(\frac{f(x_2)}{f(x_1)}\right)^2 i_1^2$$

but the resistance $R(x_1)$ is such a quantity that the power lost in the coil is

$$R(x_1)\, i_1^2.$$

Therefore, the total power lost in both the coil and R_0 is

$$\left[R(x_1) + R_0\frac{f(x_2)^2}{f(x_1)^2}\right] i_1^2$$

Thus, when R_0 is inserted at x_2 the coil appears to have a resistance

$$R(x_1) + R_0\frac{f(x_2)^2}{f(x_1)^2}$$

with reference to x_1. Consequently, after R_0 is inserted the current at x_1 becomes

$$i_1' = \frac{e}{R(x_1) + R_0\frac{f(x_2)^2}{f(x_1)^2}} \tag{3}$$

Comparing (2) and (3)

$$R(x_1)\, f(x_1)^2 = R(x_2)\, f(x_2)^2 \tag{4}$$

Hence, also,

$$\frac{f(x_2)}{f(x_1)} = \sqrt{\frac{R(x_1)}{R(x_2)}} \quad . \; . \; . \; . \; . \; . \tag{5}$$

Further, the current at x_2 due to an emf e at x_1 is

$$i_2 = \frac{e}{R\ (x_1)}\ \frac{f(x_2)}{f(x_1)}$$

and therefore by (5) is

$$i_2 = \frac{e}{\sqrt{R\ (x_1)\ R\ (x_2)}} \tag{6}$$

If, then, x be a point of the coil and if in an element dx an emf $e(x)\ dx$ be induced, then the current at x_0 is

$$i_0 = \frac{1}{\sqrt{R\ (x_0)}} \int_{x_1}^{x_2} \frac{e\ (x)\ dx}{\sqrt{R\ (x)}} \tag{7}$$

where now x_1, x_2 denote the terminal values of x.

(A different and shorter proof of this may be given with the aid of Lord Rayleigh's reciprocal theorem.)

4. APPLICATION TO COIL AERIALS

Coil aerials are inductance coils used for receiving signals in radiotelegraphy. The emf due to the incident wave is distributed uniformly through the coil aerial. Thus, formula (7) may be used to compute the current at any point.

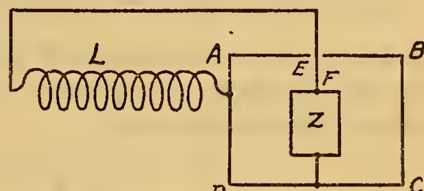

Fig. 3.—*Shield over impedance*

Qualitative checks of this theory are given by the experimental results on signal intensities obtained by means of coil aerials for the Signal Corps. The theory was also verified by some special experiments performed in the yard of the Bureau of Standards.

A coil aerial was wound on a wooden frame and placed between the two large rectangular coils of an electron tube generating set. The coils were so arranged as to give an approximately uniform field. The coil was tuned to the wave length of the generating set by means of a condenser across its terminals. An electrostatic voltmeter was connected across the condenser terminals. The tuning of the condenser was effected from a distance by a pulley and the voltmeter was read from a distance by means of a telescope. This eliminated the effect of the observer's body.

Leaving the wave length of the generating set unchanged, turns were cut off from the coil, alternately from both sides. It was retuned each time and the voltmeter reading was noted. Then, on the same frame a similar coil was wound and, using the same condenser, resistance measurements were made with reference to the condenser terminal and also with reference to a point exactly in the middle of the coil. The apparent inductance of the coil was also measured.

Now, it will be shown presently that if

N is the number of turns

L_a is the apparent inductance

and
$$R_e = \frac{R_o}{1+\frac{2}{3}\left(\sqrt{\frac{R_o}{R_m}}-1\right)}$$

R_o, R_m being respectively the resistance with reference to the terminal and with reference to the middle, then $\frac{NL_a}{R_e}$ must be proportional to the reading of the voltmeter. This was verified experimentally, as will be described presently. The relation can be derived as follows:

The reading is $\frac{i_o}{C\omega}$ where i_o is the current through the joint capacity of the condenser and voltmeter and where C is that joint capacity. By the definition of apparent inductance this is the same as $i_oL_a\omega$. By equation (7)

$$i_o = \frac{1}{R_o}\int_{x_1}^{x_2} e(x)\sqrt{\frac{R_o}{R(x)}}\,dx$$

But e (x) is constant if x denotes the length along the wire. Therefore, letting $e = \int_{x_1}^{x_2} e(x)dx$

$$i_o = \frac{e}{R_o}\,\frac{\int_{x_1}^{x_2}\sqrt{\frac{R_o}{R(x)}}\,dx}{x_2-x_1} = \frac{e}{R_o}\left(1+\theta\left(\sqrt{\frac{R_o}{R_m}}-1\right)\right)$$

where $\theta\left(\sqrt{\frac{R_o}{R_m}}-1\right)$ denotes the mean value of $\sqrt{\frac{R_o}{R(x)}}-1$

In the case of an ungrounded coil such as has been used R_m is the least value of $R(x)$. Thus, $\theta < 1$.
Further, by (5)

$$\sqrt{\frac{R_o}{R(x)}} - 1 = \frac{i(x)}{i_o} - 1$$

The quantity $\frac{i(x)}{i_o} - 1$ when plotted against x is nearly a parabola.[1]
Taking this approximation

$$\theta = \frac{2}{3}$$

This, of course, is only an approximation.
Thus,

$$i_o = \frac{e}{R_e}$$

where

$$R_e = \frac{R_o}{1 + \frac{2}{3}\left(\sqrt{\frac{R_o}{R_m}} - 1\right)}$$

Here e itself is proportional to the number of turns N. Thus, the voltmeter reading is proportional to

$$\frac{NL_a}{R_e}$$

The results of the experiment on a 3-foot coil wound with No. 18 wire, spaced 1 cm and used at 437 meters, are:

Number of turns	Ratio of experimental to theoretical voltage
6	0.98
7	1.00
8	1.01
9	1.04
10	1.12

Considering the fact that the resistances measured with reference to the middle and to the terminal may differ by a factor of 2, and that a slight dissymmetry was introduced in the resistance measurements by the thermocouple and galvanometer used in

[1] G. Breit, "The distributed capacity of inductance coils," Phys. Rev., 17, pp. 672, 673, Figs. 6, 7, 8; June, 1921.

measuring the current, that the value of θ is only an approximation, and that the experiment was performed outdoors, where the humidity has a variable effect on the resistance, this agreement may be regarded as satisfactory.

Formula (5) has been verified directly by means of thermo-galvanometers calibrated at 60 cycles.

5. LAW OF INCREASE OF RESISTANCE NEAR THE NATURAL PERIOD

So far measured values of resistance have been relied upon. It now becomes necessary to compute the resistance without measuring it. The whole coil may be imagined to be divided into a number of parts. The points of division may be so chosen that the mutual inductance of each part to the whole coil is the same.

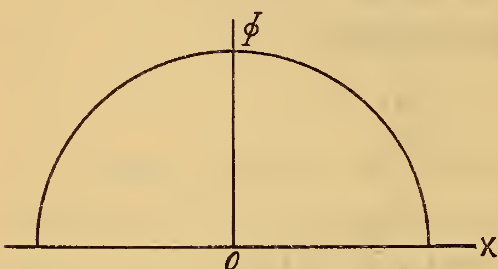

FIG. 4.—*Graphical representation of the function* ϕ

If the coil is wound closely, the current in any section, say the mth, is

$$\dot{i}_m = \dot{i}_0 + \phi(m)\omega^2 \bar{\dot{i}} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (8)$$

where

$$\bar{\dot{i}} = \frac{\dot{i}_1 + \dot{i}_2 + \cdot \cdot \cdot \cdot \cdot \cdot \cdot + \dot{i}_n}{n}$$

where n is the total number of parts.[2]

Let

$$\Phi = \frac{\sum_{n=1}^{m} \phi(m)}{n}$$

Then, summing (8) and dividing by n

$$\bar{\dot{i}} = \dot{i}_0 + \Phi\omega^2 \bar{\dot{i}}$$

that is,

$$\bar{\dot{i}} = \frac{\dot{i}_0}{1 - \Phi\omega^2}$$

and

$$\dot{i}_m = \dot{i}_0 \left[1 + \frac{\phi(m)\omega^2}{1 - \Phi\omega^2} \right]$$

[2] G. Breit, "The distributed capacity of inductance coils," Phys. Rev., **17**, pp. 669, 671, June, 1921; Section, explanation of constancy of C_0, equation (28*a*).

In the limit when the subdivision becomes very fine one can write

$$i(x) = i_0\left[1 + \frac{\phi(x)\omega^2}{1 - \Phi\omega^2}\right] \tag{9}$$

where x, as before, is a parameter along the wire. Again, for a closely wound coil the length of the elementary section is independent of its order. Hence, the parameter x must be taken as the length or else it must be a constant times the length along the wire.

Let $R(x)dx$ be the resistance of the section between x and $x+dx$ at the frequency in question with the particular current distribution which flows through the section.

The power lost in the section is

$$R(x)[i(x)]^2 dx$$

The power lost in the whole coil is

$$i_0^2 \int_{x_1}^{x_2} R(x)\left(\frac{i(x)}{i_0}\right)^2 dx$$

Therefore, the resistance with reference to the point x_0 is

$$R_0 = \int_{x_1}^{x_2} R(x)\left(\frac{i(x)}{i_0}\right)^2 dx$$

This by (9) becomes

$$R_0 = \int_{x_1}^{x_2} R(x)dx + \frac{2\omega^2}{1 - \Phi\omega^2}\int_{x_1}^{x_2} R(x)\phi(x)dx$$
$$+ \frac{\omega^4}{(1 - \Phi\omega^2)^2}\int_{x_1}^{x_2} R(x)[\phi(x)]^2 dx \cdots \tag{10}$$

In particular if $R(x)$ is independent of x

$$R_0 = \left[1 + \frac{2\omega^2\Phi}{1 - \Phi\omega^2} + \frac{\omega^4 \quad \overline{\phi^2}}{(1 - \Phi\omega^2)^2}\right]\int_{x_1}^{x_2} R(x)dx \tag{11}$$

where $\overline{\phi^2}$ is the average of ϕ^2 with respect to x.

If x_0 is a terminal point Φ is $\frac{1}{\omega_0^2}$ where ω_0 is 2π times the natural frequency of the coil, for

$$\bar{\text{i}} = \frac{i_0}{1 - \Phi\omega_0^2}$$

and since for the natural frequency the terminal current vanishes $i_o = o$ for a finite $\bar{I}$ which necessitates

$$\Phi = \frac{1}{\omega_o^2}$$

Thus, (11) may be also written as

$$R_o = \left[1 + \frac{2\frac{\omega^2}{\omega_o^2}}{1 - \frac{\omega^2}{\omega_o^2}} + \frac{\frac{\omega^4 \overline{\phi^2}}{\omega_o^4 \Phi^2}}{\left(1 - \frac{\omega^2}{\omega_o^2}\right)^2} \right] \int_{x_1}^{x_2} R(x) dx \quad \cdots \quad (12)$$

or, if it is preferred to use wave lengths

$$R_o = \left[1 + \frac{2}{\frac{\lambda^2}{\lambda_o^2} - 1} + \frac{\frac{\overline{\phi^2}}{\Phi^2}}{\left(\frac{\lambda^2}{\lambda_o^2} - 1\right)^2} \right] \int_{x_1}^{x_2} R(x) dx \qquad (13)$$

Now, $\int_1^{x_2} R(x)dx$ is the resistance which the coil would have if there were no capacity effect. The knowledge of skin effect is sufficient for its computation. Assuming the change in resistance due to skin effect to be known, it remains to find $\frac{\overline{\phi^2}}{\Phi^2}$ and λ_o in order to find the coil resistance.

If the current distribution in a coil is known the ratio $\frac{\overline{\phi^2}}{\Phi^2}$ may be calculated.

CALCULATION OF $\frac{\overline{\phi^2}}{\Phi^2}$ FOR SPECIAL CASES

The special case considered is that of a short, closely wound, solenoid when used ungrounded.

The quantity $\frac{\overline{\phi^2}}{\Phi^2}$ clearly depends only on the shape of the $\phi(x), x$ curve. It remains unchanged if $\phi(x)$ is multiplied by a constant factor.

It was shown in "The distributed capacity of inductance coils" (loc. cit.) that the quantity called there $\alpha(x)$ is

$$\alpha(x) = -\frac{10^{-11} KLl}{35.956 \pi a} \coth u_o \frac{\cos v}{|\sin v|}$$

It is therefore proportional to

$$\frac{\cos v}{|\sin v|}$$

The quantity $\alpha(x)$ is so chosen that $\left(\frac{di}{dt}\right)\alpha(x)dx$ is the charge in the segment of the coil between x and $x+dx$. Also, in the notation of the paper cited x is proportional to a length along the wire. The quantity v may be regarded as defined by

$$a \cos v = x$$

In order to obtain the current distribution from $\alpha(x)$ it suffices to note that

$$\frac{\partial i}{\partial x} = -\left(\frac{di}{dt}\right)\alpha(x)$$

Thus

$$i = -\frac{di}{dt}\int_a^x \alpha(x)dx$$

Hence

$$\phi \text{ is proportional to } \int_a^x \frac{xdx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$$

Thus, ϕ may be represented by a semicircular arc. The average value of $\sqrt{a^2-x^2}$ is then $\frac{\pi a}{4}$, and the square of the average value is $\frac{\pi^2 a^2}{16}$

The square of $\sqrt{a^2-x^2}$ is a^2-x^2, and its average is $\frac{2a^2}{3}$

Thus

$$\frac{\overline{\phi^2}}{\Phi^2} = \frac{32}{3\pi^2} = 1.080$$

With the aid of this expression, formula (13) for the case of a short single layer solenoid becomes

$$R_0 = \left[\left(\frac{1}{1-\frac{\lambda^2_0}{\lambda^2}}\right)^2 + \frac{0.080}{\left(\frac{\lambda^2}{\lambda_0^2}-1\right)^2}\right]\int_{x_1}^{x_2} R(x)dx \qquad (14)$$

It is worth while to call attention to the fact that this formula is true, independently of whether the solenoid is in an elliptical shield or not because the term coth u_0 occurring in $\alpha(x)$ does not affect the result.

(*a*) CASE OF GROUNDED COIL.—In the case of a grounded coil it is easiest to compute the resistance with reference to the ungrounded terminal, because if i_0 is the current at the terminal, $i_0 = o$ is the condition which must be satisfied for λ_0. From this resistance the resistance at any other point may be derived.

This remark is to serve as a caution not to take the grounded terminal as the terminal x_0. The formula (14) is then modified only inasmuch as the factor 0.080 is changed.

(*b*) EXPERIMENTAL TEST OF RESISTANCE FORMULAS.—Formula (14) is a convenient one for experimental verification.

If λ is not too near to λ_0 the second term in the parentheses may be neglected. Then (14) takes the simplified form

$$R_0 = \frac{\int_{x_1}^{x_2} R(x)\,dx}{\left(1 - \frac{\lambda_0^2}{\lambda^2}\right)^2} \qquad (14')$$

This is formula (35) of Lindemann (loc. cit.). It is seen, however, from the derivation that in the case of a shielded condenser it holds only for the resistance measured in the ungrounded lead. Also, it is apparent from formula (14) that (14′) is not an exact expression even for the case of a coil made of a perfect conductor and with constant skin effect throughout its length. Thus, for the case of perfect linkage, nearly perfect conductivity, but variable skin effect, formula (10) holds.

Since the influence of the place in which the resistance is inserted does not seem to have been realized in Lindemann's work, a new verification of formula (14′) was thought advisable. This verification was carried out on a coil designed to satisfy the assumptions made in deriving (14′). The coil was a single-layer coil wound with stranded wire (litz). The number of strands was 32, and each strand was insulated from the other.

According to a calculation made by Lindemann [3] the ratio of the increase of the resistance of a wire due to skin effect to its direct current resistance is

$$\frac{\pi^2\omega^2 z^2 r^6}{2\sigma^2 R^2}$$

where z = number of strands;
r = radius of a strand;
R = radius of the wire;
σ = resistivity in cgs electromagnetic units.

[3] R. Lindemann, Jahrbuch. d. drahtl. Tel., 4, p., 574 (28); 1910–1911.

Using this formula, and also the measured value of λ_o, which proved to be 200 meters, the ratios of increase of resistance in comparison with the direct-current resistance as well as the measured values are as below:

Wave-length (meters)	Skin effect factor	Capacity factor	Measured R	Apparent direct-current resistance
			Ohms	Ohms
717	1.142	1.176	6.65	4.94
781	1.120	1.146	6.42	4.99
800	1.114	1.139	6.25	4.93
908	1.089	1.103	5.94	4.94
1139	1.056	1.065	5.71	5.07
1255	1.046	1.051	5.47	4.97
Mean				4.97

In the last column is given the result of dividing the measured values recorded in the column before the last by the numbers given in the second and the third columns. For example—

$$4.94 = \frac{6.65}{1.142 \times 1.176}$$

The measured direct-current resistance is 4.84 ohms. It is less than the mean of the last column by 0.13 ohm, which is a deviation of 2.7 per cent. The values used are values measured with reference to the ungrounded lead.

V. SUMMARY

There has been discussed the meaning of the term "resistance" when applied to a coil in which high-frequency current is flowing. It was shown that the term must be applied only with reference to a particular point in the circuit.

The computation of the current at a given point in the circuit when an emf is arbitrarily induced in it and when the circuit is tuned has been carried through. The results have been checked experimentally.

The resistance of a coil has been computed on the assumption that the effect on it of skin effect is known. The formula has been checked experimentally.

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