AN INTEGRATION METHOD OF DERIVING THE ALTERNATING-CURRENT RESISTANCE AND INDUCTANCE OF CONDUCTORS

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CONTENTS

I. INTRODUCTION

The method heretofore used in deriving formulas for the computing of the alternating-current resistance and inductance of conductors requires the determination of the differential equations of the magnetic field and the solution of these equations under the conditions imposed by the shape of the conductor. Formulas have been derived for only a limited number of forms.[1]

The method outlined in this paper is one of integration. It requires that the conductor be divided into infinitesimal filaments by surfaces which coincide with the lines of current flow. The magnetic field at any point is the sum of the magnetic fields of all of these filaments. The counter-electromotive force in a filament is determined by the rate at which the magnetic fields of all the other filaments cut this filament.

[1] The most important of these have been collected by Rosa and Grover, B. S. Bulletin, 8, p. 172; 1911; Scientific Paper No. 169.

II. OUTLINE OF THE METHOD

If small wires are placed side by side and an alternating electromotive force applied to their terminals, the current which flows in any one of the wires will be determined not only by its resistance and inductance, but also by the counter-electromotive force caused by the magnetic field of the others cutting this wire. If there are only two identical wires, the current will be the same in each. However, if there are three or more, they may be so arranged that the counter-electromotive force in some is different from that in others, thereby causing the current to be different in different wires.

If the resistance and self-inductance of each wire are known and the mutual inductance between the wires taken in pairs is known, equations in sufficient number can be set up to solve for the current and the phase of the current in each wire. For example, if there are three conductors whose resistances are r_1, r_2, and r_3, whose self-inductances are l_1, l_2, and l_3, whose mutual inductances are m_{12}, m_{23}, and m_{13}, and whose instantaneous currents are I'_1, I'_2, and I'_3:

$$E' = I'_1 r_1 + l_1 \frac{dI'_1}{dt} + m_{12} \frac{dI'_2}{dt} + m_{13} \frac{dI'_3}{dt}$$

$$E' = I'_2 r_2 + l_2 \frac{dI'_2}{dt} + m_{12} \frac{dI'_1}{dt} + m_{23} \frac{dI'_3}{dt}$$

$$E' = I'_3 r_3 + l_3 \frac{dI'_3}{dt} + m_{13} \frac{dI'_1}{dt} + m_{23} \frac{dI'_2}{dt}$$

where

$$E' = E \cos \omega t.$$

To solve, assume that

$$I'_1 = I_1 \cos (\omega t - \phi_1)$$

$$I'_2 = I_2 \cos (\omega t - \phi_2)$$

$$I'_3 = I_3 \cos (\omega t - \phi_3).$$

Substituting these values in the equation, there result three equations when $\omega t = 0$ and three independent ones when $\omega t = \frac{\pi}{2}$, making six equations from which to determine the six unknown quantities; viz, the magnitude and phase of the current in each of the three wires.

A solid conductor of infinite length may be considered as made up of an infinite number of filaments, and each of these behaves as though it were an infinitesimal wire carrying a current. The instantaneous current in any one filament is determined from the equation

$$E' = r\,\delta I'_x + l_0 \frac{d}{dt}\delta I'_x + \iint M_{xy} \frac{d}{dt}\delta I'_y, \tag{1}$$

where E' is the instantaneous electromotive force, r and l_0 the resistance and self-inductance of the filament, $\delta I'_x$ the current in the filament, M_{xy} the mutual inductance between this filament and another filament at Y in which the current is $\delta I'_y$. If there are other conductors in the field, the integration must include these also. Having determined $\delta I'_x$, the total current I' through the conductor is the sum of those through the filaments, or

$$I' = \iint \delta I'_x = \iint U'_x\,dS \tag{2}$$

where U'_x is the current density at x and dS an element of the cross section of the conductor.

In equation (1), several of the quantities approach either zero or infinity as the area of the element approaches zero. Hence it is necessary to examine each term of the equation and to retain only those which have a finite value.

In the term $r\,\delta I'_x$, $r = \frac{\sigma l}{dS}$ and $\delta I'_x = U'_x\,dS$, where σ is the resistivity of the material, l the length and dS the area of the filament, and U'_x the current density. Hence $r\,\delta I'_x = U'_x\,\sigma l$, which shows that this term is a constant and independent of the area of the filament.

The term $l_0 \frac{d}{dt}\delta I'_x$ is the counter electromotive force caused by the cutting of the filament by the magnetic field which the current $\delta I'_x$ sets up around and in the filament. If the filament becomes infinitesimal, the current, and hence the magnetic field, is infinitesimal, so that in the limit this term becomes zero.

The term $\iint M_{xy} \frac{d}{dt}\delta I'_y$ is the counter electromotive force in the filament at X caused by the cutting of this filament by the magnetic field which is set up by the currents in all the other filaments. This is a finite quantity, although M_{xy} becomes infinite when Y approaches X. If analyzed mathematically, it is found that this is a case where the integral of a function which has one infinite point is a finite quantity.

It follows that equation (1) may, by substituting for $\delta I'_y$, $U'_y dS$, be written in the form

$$E' = U'_x \sigma l + \int\int M_{xy} \frac{d U'_y}{dt} d S. \tag{3}$$

This equation may be used to determine the value of U'_x in conductors of known form, provided the necessary integrations can be performed. Having determined U'_x, equation (2)—viz, $I' = \int\int U'_x \, d S$—gives the total current in the conductor.

Equation (3) applies only to conductors which are so long that the end effects may be neglected. To apply (3) to a particular case, it is necessary to be able to express M_{xy} as a function of the distance between the filaments, and this can be done only when the permeability of the conductor is unity.

III. ALTERNATING-CURRENT RESISTANCE AND INDUCTANCE OF A STRAIGHT CYLINDRICAL CONDUCTOR

Formulas for the alternating-current resistance and inductance of a straight cylindrical conductor of infinite length have been developed by several investigators.[2] In all cases they have started from the differential equation of the magnetic field.

The method of integration outlined above has been applied to this case to determine whether it will readily give useful results. After the complete derivation using real power series had been completed, it was found that the work might be much simplified by the use of complex power series. However, the comparison of the two methods is a matter of some interest, so that both are given.

1. DERIVATION OF FORMULAS USING REAL POWER SERIES

Let P_x and P_y be any two points in the circular cross section of a conductor of length l, and let $\rho_x \, \theta_x$ and $\rho_y \, \theta_y$ be the polar coordinates of these points.

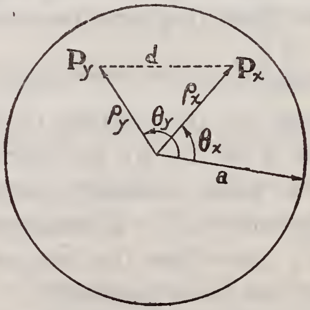

Fig. 1.—*The cross section of a cylindrical conductor, showing coordinates*

[2] The different formulas and the methods of reducing from one to another are given in the paper by Rosa and Grover already referred to.

If d is the distance between P_x and P_y,

$$d = \sqrt{\rho_x^2 + \rho_y^2 - 2\rho_x\rho_y \cos(\theta_x - \theta_y)};$$

also,[3]

$$M_{xy} = 2l\left(\log \frac{2l}{d} - 1\right)$$

$$= 2l\ (\log 2l - 1) - l \log d^2$$

$$= 2l\ (\log 2l - 1) - l \log [\rho_x^2 + \rho_y^2 - 2\rho_x\rho_y \cos(\theta_x - \theta_y)]$$

Assume that

$$E' = E \cos \omega t, \therefore U'_x = U_x \cos(\omega t - \phi_x)$$

and

$$U'_y = U_y \cos(\omega t - \phi_y)$$

Substituting these values and the value for M_{xy} in equation (3), the following equation is obtained:

$$E \cos \omega t = \sigma l\, U_x \cos(\omega t - \phi_x) + \int_0^a \rho_y d\rho_y \int_0^{2\pi} \Big\{ 2l\ (\log 2l - 1)$$

$$- l \log \Big[\rho_x^2 + \rho_y^2 - 2\rho_x\rho_y \cos(\theta_x - \theta_y) \Big] \Big\} \Big\{ -\omega U_y \sin(\omega t - \phi_y) \Big\} d\theta_y$$

$$= \sigma l\, U_x \cos\phi_x \cos \omega t + \sigma l\, U_x \sin \phi_x \sin \omega t$$

$$+ \omega \int_0^a \rho_y d\rho_y \int_0^{2\pi} \Big\{ 2l\ (\log 2l - 1) - l \log \Big[\rho_x^2 + \rho_y^2 - 2\rho_x\rho_y \cos(\theta_x - \theta_y) \Big] \Big\}$$

$$\Big\{ U_y \sin\phi_y \cos \omega t - U_y \cos \phi_y \sin \omega t \Big\} d\theta_y. \qquad (4)$$

This equation determines the value of U_x and ϕ_x.

To solve the equation, assume that $U_x \sin \phi_x$ and $U_x \cos \phi_x$ (and hence corresponding values of U_y and ϕ_y) can be developed in a power series with undetermined coefficients. By substituting this series in equation (4) the value of the coefficients can be determined.

Since the values of U and ϕ are symmetrical around the center, θ does not enter into the value of either, hence the assumed power series may be written

$$U \cos \phi = a_0 + b_0\rho + c_0\rho^2 + d_0\rho^3 + e_0\rho^4 + f_0\rho^5 + g_0\rho^6 + h_0\rho^7 + i_0\rho^8 + \cdots$$

$$U \sin \phi = A_0 + B_0\rho + C_0\rho^2 + D_0\rho^3 + E_0\rho^4 + F_0\rho^5 + G_0\rho^6 + H_0\rho^7 + I_0\rho^8 + \cdots$$

Substituting these values in (4), each term is either an integral containing only some power of ρ_y or an integral of the type I_1, given in the appendix. Making the integrations and putting $2l\ (\log 2l - 1 - \log a) = L_\infty$ (the inductance due to the field outside

[3] B. S. Bulletin, 8, p. 151 (Scientific Papers No. 169).

the conductor, which is the inductance at infinite frequency), equation (4) becomes

$$
\begin{aligned}
E \cos \omega t = \sigma l \cos \omega t\{a_0 + b_0\rho_x + c_0\rho_x^2 + \cdots\cdots\} \\
+\sigma l \sin \omega t\{A_0 + B_0\rho_x + C_0\rho_x^2 + \cdots\cdots\} \\
+2\pi\omega L_\infty \cos\omega t\left(\frac{A_0a^2}{2}+\frac{B_0a^3}{3}+\frac{C_0a^4}{4}+\cdots\cdots\right) \\
-2\pi\omega L_\infty \sin\omega t\left(\frac{a_0a^2}{2}+\frac{b_0a^3}{3}+\frac{c_0a^4}{4}+\cdots\cdots\right) \\
+4\pi\omega l \cos \omega t\left(\frac{A_0a^2}{4}+\frac{B_0a^3}{9}+\frac{C_0a^4}{16}+\cdots\cdots\right) \\
-4\pi\omega l \cos \omega t\left(\frac{A_0\rho_x^2}{4}+\frac{B_0\rho_x^3}{9}+\frac{C_0\rho_x^4}{16}+\cdots\cdots\right) \\
-4\pi\omega l \sin \omega t\left(\frac{a_0a^2}{4}+\frac{b_0a^3}{9}+\frac{c_0a^4}{16}+\cdots\cdots\right) \\
+4\pi\omega l \sin \omega t\left(\frac{a_0\rho_x^2}{4}+\frac{b_0\rho_x^3}{9}+\frac{c_0\rho_x^4}{16}+\cdots\cdots\right) \qquad (5)
\end{aligned}
$$

If $\omega t = 0$, $\cos \omega t = 1$, and $\sin \omega t = 0$, then putting $\eta = \frac{\pi\omega}{\sigma}$

$$
\begin{aligned}
\frac{E}{\sigma l} = a_0 + b_0\rho_x + c_0\rho_x^2 + \cdots + \frac{2\eta L_\infty}{l}\left(\frac{A_0a^2}{2}+\frac{B_0a^3}{3}+\frac{C_0a^4}{4}+\cdots\right) \\
+4\eta\left(\frac{A_0a^2}{4}+\frac{B_0a^3}{9}+\frac{C_0a^4}{16}+\cdots\right) \\
-4\eta\left(\frac{A_0\rho_x^2}{4}+\frac{B_0\rho_x^3}{9}+\frac{C_0\rho_x^4}{16}+\cdots\right) \qquad (6)
\end{aligned}
$$

If $\omega t = \frac{\pi}{2}$, $\cos \omega t = 0$ and $\sin \omega t = 1$

$$
\begin{aligned}
0 = A_0 + B_0\rho_x + C_0\rho_x^2 + \cdots - \frac{2\eta L_\infty}{l}\left(\frac{a_0a^2}{2}+\frac{b_0a^3}{3}+\frac{c_0a^4}{4}+\cdots\right) \\
-4\eta\left(\frac{a_0a^2}{4}+\frac{b_0a^3}{9}+\frac{c_0a^4}{16}+\cdots\right) \\
+4\eta\left(\frac{a_0\rho_x^2}{4}+\frac{b_0\rho_x^3}{9}+\frac{c_0\rho_x^4}{16}+\cdots\right) \qquad (7)
\end{aligned}
$$

The coefficients of like powers of ρ_x may be equated in both (6) and (7). The equations thus formed are sufficient for the determination of the values of A_0, B_0, C_0 . . . and a_0, b_0, c_0 . . . .

Since both B_0 and b_0 are zero, all coefficients of odd terms of ρ_x are zero. The remaining values are given in Table 1.

TABLE 1.—Coefficients of Powers of ρ_x Taken from Equations (6) and (7)

Power of ρ	Coefficient from (6)	Coefficient from (7)
ρ_x^0	$\frac{E}{\sigma l}=a_o+\frac{2\eta L_\infty}{l}\left(\frac{A_oa^2}{2}+\frac{C_oa^4}{4}+\cdot\cdot\right)+4\eta\left(\frac{A_oa^2}{4}+\frac{C_oa^4}{16}+\cdot\cdot\cdot\cdot\right)$	$0=A_o-\frac{2\eta L_\infty}{l}\left(\frac{a_oa^2}{2}+\frac{c_oa^4}{4}+\cdot\cdot\right)-4\eta\left(\frac{a_oa^2}{4}+\frac{c_oa^4}{16}+\cdot\cdot\cdot\cdot\cdot\right)$
ρ_x^2	$c_o=A_o\eta$	$C_o=-a_o\eta$
ρ_x^4	$e_o=\frac{C_o\eta}{4}=-\frac{a_o\eta^2}{4}$	$E_o=-\frac{c_o\eta}{4}=-\frac{A_o\eta^2}{4}$
ρ_x^6	$g_o=\frac{E_o\eta}{9}=-\frac{A_o\eta^3}{36}$	$G_o=-\frac{e_o\eta}{9}=\frac{a_o\eta^3}{36}$
ρ_x^8	$i_o=\frac{G_o\eta}{16}=\frac{a_o\eta^4}{576}$	$I_o=-\frac{g_o\eta}{16}=\frac{A_o\eta^4}{576}$
ρ_x^{4n}	$x_o=(-1)^n\frac{a_o(4\eta)^{2n}}{2^2\cdot4^2\cdot6^2\cdot\cdot(4n)^2}$	$X_o=(-1)^n\frac{A_o(4\eta)^{2n}}{2^2\cdot4^2\cdot6^2\cdot\cdot(4n)^2}$
ρ_x^{4n+2}	$y_o=(-1)^n\frac{A_o(4\eta)^{2n+1}}{2^2\cdot4^2\cdot6^2\cdot\cdot\cdot(4n+2)^2}$	$Y_o=-(-1)^n\frac{a_o(4\eta)^{2n+1}}{2^2\cdot4^2\cdot6^2\cdot\cdot(4n+2)^2}$

NOTE.—x_o and y_o are used, respectively, as the $4n$ and the $4n+2$ letters of the alphabet.

Substituting the values of the coefficients obtained from the higher powers of ρ_x in the equations for the coefficients of ρ_x^0, the following equations result:

$$\begin{aligned}\frac{E}{\sigma l}=a_o+\frac{2\eta L_\infty}{l}\Bigg[&\frac{A_oa^2}{2}-\frac{a_o\eta a^4}{4}-\frac{A_o\eta^2a^6}{24}+\frac{a_o\eta^3a^8}{288}+\cdot\cdot\cdot\cdot\cdot\cdot\\&+(-1)^n\frac{4^{2n}}{4n+2}\frac{A_o\eta^{2n}a^{4n+2}}{2^2\cdot4^2\cdot6^2\cdot\cdot(4n)^2}-(-1)^n\frac{4^{2n+1}}{4n+4}\frac{a_o\eta^{2n+1}a^{4n+4}}{2^2\cdot4^2\cdot6^2\cdot\cdot(4n+2)^2}+\cdot\cdot\cdot\cdot\Bigg]\\+4\eta\Bigg[&\frac{A_oa^2}{4}-\frac{a_o\eta a^4}{16}-\frac{A_o\eta^2a^6}{144}+\frac{a_o\eta^3a^8}{2304}+\cdot\cdot\cdot\cdot\cdot\\&+(-1)^n\frac{4^{2n}}{(4n+2)^2}\frac{A_o\eta^{2n}a^{4n+2}}{2^2\cdot4^2\cdot6^2\cdot\cdot(4n)^2}\\&-(-1)^n\frac{4^{2n+1}}{(4n+4)^2}\frac{a_o\eta^{2n+1}a^{4n+4}}{2^2\cdot4^2\cdot6^2\cdot\cdot(4n+2)^2}+\cdot\cdot\cdot\cdot\Bigg]\end{aligned}\tag{8}$$

$$\begin{aligned}0=A_o-\frac{2\eta L_\infty}{l}\Bigg[&\frac{a_oa^2}{2}+\frac{A_o\eta a^4}{4}+\frac{a_o\eta^2a^6}{24}+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\&+(-1)^n\frac{4^{2n}}{4n+2}\frac{a_o\eta^{2n}a^{4n+2}}{2^2\cdot4^2\cdot6^2\cdot\cdot(4n)^2}\\&+(-1)^n\frac{4^{2n+1}}{4n+4}\frac{A_o\eta^{2n+1}a^{4n+4}}{2^2\cdot4^2\cdot6^2\cdot\cdot(4n+2)^2}+\cdot\cdot\cdot\cdot\Bigg]\\-4\eta\Bigg[&\frac{a_oa^2}{4}+\frac{A_o\eta a^4}{16}-\frac{a_o\eta^2a^6}{144}-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\&+(-1)^n\frac{4^{2n}}{(4n+2)^2}\frac{a_o\eta^{2n}a^{4n+2}}{2^2\cdot4^2\cdot6^2\cdot\cdot(4n)^2}\\&+(-1)^n\frac{4^{2n+1}}{(4n+4)^2}\frac{A_o\eta^{2n+1}a^{4n+4}}{2^2\cdot4^2\cdot6^2\cdot\cdot(4n+2)^2}+\cdot\cdot\cdot\cdot\cdot\Bigg]\end{aligned}\tag{9}$$

In order to simplify these equations, the following substitutions may be made:

$$1-\frac{\eta^2 a^4}{72}+\cdots+(-1)^n \frac{4}{n+1}\frac{(4\eta a^2)^{2n}}{2^2\cdot 4^2\cdot 6^2\cdots(4n+2)^2}+\cdots\cdots=W_1$$

$$1-\frac{\eta^2 a^4}{144}+\cdots+(-1)^n \frac{4}{(n+1)^2}\frac{(4\eta a^2)^{2n}}{2^2\cdot 4^2\cdot 6^2\cdots(4n+2)^2}+\cdots\cdots=X_1$$

$$1-\frac{\eta^2 a^4}{12}+\cdots+(-1)^n \frac{(4\eta a^2)^{2n}}{(2n+1)\ 2^2\cdot 4^2\cdot 6^2\cdots(4n)^2}+\cdots\cdots=Y_1$$

$$1-\frac{\eta^2 a^4}{36}+\cdots+(-1)^n \frac{(4\eta a^2)^{2n}}{(2n+1)^2\ 2^2\cdot 4^2\cdot 6^2\cdots(4n)^2}+\cdots\cdots=Z_1$$

Then equations (8) and (9) become

$$a_0\left(1-\frac{\eta^2 a^4 L_\infty}{2l}W_1-\frac{\eta^2 a^4}{4}X_1\right)+A_0\left(\frac{\eta a^2 L_\infty}{l}Y_1+\eta a^2 Z_1\right)=\frac{E}{\sigma l} \quad (10)$$

$$a_0\left(\frac{\eta a^2 L_\infty}{l}Y_1+\eta a^2 Z_1\right)-A_0\left(1-\frac{\eta^2 a^4 L_\infty}{2l}W_1-\frac{\eta^2 a^4}{4}X_1\right)=0 \quad (11)$$

Solving for a_0 and A_0

$$a_0=\frac{E}{\sigma l}\frac{1-\frac{\eta^2 a^4 L_\infty W_1}{2l}-\frac{\eta^2 a^4 X_1}{4}}{\left(1-\frac{\eta^2 a^4 L_\infty W_1}{2l}-\frac{\eta^2 a^4 X_1}{4}\right)^2+\left(\frac{\eta a^2 L_\infty Y_1}{l}+\eta a^2 Z_1\right)^2}$$

$$A_0=\frac{E}{\sigma l}\frac{\frac{\eta a^2 L_\infty Y_1}{l}+\eta a^2 Z_1}{\left(1-\frac{\eta^2 a^4 L_\infty W_1}{2l}-\frac{\eta^2 a^4 X_1}{4}\right)^2+\left(\frac{\eta a^2 L_\infty Y_1}{l}+\eta a^2 Z_1\right)^2}$$

Since the current, $\delta I_x'$, through the filament at P_x is $U_x'\rho_x d\rho_x d\theta_x$, the total current I' through the conductor at any instant is given by the integral

$$I'=\int\int \delta I_x'=\int\int U_x'\ \rho_x\ d\rho_x\ d\theta_x$$

$$=\int_0^a \rho_x\ d\rho_x \int_0^{2\pi}\Big[(a_0+c_0\ \rho_x^2+\cdots\cdots)\cos\omega t$$

$$+(A_0+C_0\rho_x^2+\cdots\cdots)\sin\omega t\Big]d\theta_x$$

Integrating

$$I'=I\cos(\omega t-\phi)=2\pi\left(\frac{a_0 a^2}{2}+\frac{c_0 a^4}{4}+\cdots\cdots\right)\cos\omega t$$

$$+2\pi\left(\frac{A_0 a^2}{2}+\frac{C_0 a^4}{4}+\cdots\cdots\right)\sin\omega t \quad (12)$$

Hence putting successively $\omega t = 0$ and $\omega t = \frac{\pi}{2}$

$$I \cos \phi = 2\pi\left(\frac{a_0 a^2}{2} + \frac{c_0 a^4}{4} + \cdots\right) = 2\pi\left(\frac{a_0 a^2}{2} Y_1 + \frac{A_0 \eta a^4}{4} W_1\right) \quad (13)$$

$$I \sin \phi = 2\pi\left(\frac{A_0 a^2}{2} + \frac{C_0 a^4}{4} + \cdots\right) = 2\pi\left(\frac{A_0 a^2}{2} Y_1 - \frac{a_0 \eta a^4}{4} W_1\right) \quad (14)$$

$$I^2 = 4\pi^2\left\{\left(\frac{a_0 a^2}{2} + \frac{c_0 a^4}{4} + \cdots\right)^2 + \left(\frac{A_0 a^2}{2} + \frac{C_0 a^4}{4} + \cdots\right)^2\right\}$$

$$= 4\pi^2 (a_0^2 + A_0^2)\left(\frac{a^4}{4} Y_1^2 + \frac{\eta^2 a^8}{16} W_1^2\right) \quad (15)$$

In an alternating-current circuit the effective resistance R is determined by the energy loss, or

$$I^2 R = E I \cos \phi$$

$$\therefore R = \frac{E I \cos \phi}{I^2} \quad (16)$$

Likewise the inductance is defined by the equation

$$\omega L = \frac{E I \sin \phi}{I^2} \quad (17)$$

Substituting values from (13), (14), and (15) in (16) and (17)

$$R = \frac{E\left(a_0 Y_1 + \frac{A_0 \eta a^2}{2} W_1\right)}{\pi a^2 (a_0^2 + A_0^2)\left(Y_1^2 + \frac{\eta^2 a^4}{4} W_1^2\right)} \quad (18)$$

$$\omega L = \frac{E\left(A_0 Y_1 - \frac{a_0 \eta a^2}{2} W_1\right)}{\pi a^2 (a_0^2 + A_0^2)\left(Y_1^2 + \frac{\eta^2 a^4}{4} W_1^2\right)} \quad (19)$$

Substituting the values of A_0 and a_0

$$R = \frac{\sigma l}{\pi a^2} \frac{Y - \frac{\eta^2 a^4 L_\infty}{2l} W_1 Y_1 - \frac{\eta^2 a^4 X_1 Y_1}{4} + \frac{\eta^2 a^4 L_\infty}{2l} W_1 Y_1 + \frac{\eta^2 a^4}{2} W_1 Z_1}{Y_1^2 + \frac{\eta^2 a^4 W_1^2}{4}} \quad (20)$$

$$\omega L = \frac{\sigma l}{\pi a^2} \frac{\frac{\eta a^2 L_\infty}{l} Y_1^2 + \eta a^2 Y_1 Z_1 - \frac{\eta a^2}{2} W_1 + \frac{\eta^3 a^6 L_\infty}{4l} W_1^2 + \frac{\eta^3 a^6 X_1 W_1}{8}}{Y_1^2 + \frac{\eta^2 a^4 W_1^2}{4}} \quad (21)$$

$$R = R_0 \frac{Y_1 + \frac{\eta^2 a^4}{2}\left(W_1 Z_1 - \frac{X_1 Y_1}{2}\right)}{Y_1^2 + \frac{\eta^2 a^4 W_1^2}{4}} \quad (22)$$

Where R_0, representing the direct current resistance, equals $\frac{\sigma l}{\pi a^2}$

$$L = L_\infty + \frac{l}{2} \frac{2\, Y_1 Z_1 - W_1 + \frac{\eta^2 a^4}{4} X_1 W_1}{Y_1^2 + \frac{\eta^2 a^4\, W_1^2}{4}} \cdot \qquad (23)$$

Where L_∞, representing the inductance at infinite frequency, equals $2l$ ($\log 2l - 1 - \log a$) and ω is given by the equation

$$\frac{\sigma\eta}{\pi} = \omega.$$

Substituting the values of W_1, X_1, Y_1, and Z_1

$$R = R_0 \frac{1 + \frac{\eta^2 a^4}{6} + \frac{\eta^4 a^8}{480} + \frac{\eta^6 a^{12}}{181440} + \cdots\cdots}{1 + \frac{\eta^2 a^4}{12} + \frac{\eta^4 a^8}{1440} + \frac{\eta^6 a^{12}}{725760} + \cdots\cdots} \qquad (24)$$

$$L = L_\infty + \frac{l}{2} \frac{1 + \frac{\eta^2 a^4}{24} + \frac{\eta^4 a^8}{4320} + \frac{\eta^6 a^{12}}{2903040} + \cdots\cdots}{1 + \frac{\eta^2 a^4}{12} + \frac{\eta^4 a^8}{1440} + \frac{\eta^6 a^{12}}{725760} + \cdots\cdots} \qquad (25)$$

These correspond exactly with the asymptotic formulas of Russell.[4] Tables to facilitate the computation have been published by Savidge [5] and by Rosa and Grover.[6]

The series W_1, X_1, Y_1, and Z_1 may be expressed in terms of the ber and bei functions and their derivatives. By putting $q^2 = 4\eta a^2$, and comparing them term by term, the following relationships hold:

$$W_1 = -\frac{16}{q^3} \operatorname{ber}' q \qquad (26)$$

$$X_1 = \frac{64}{q^4} (1 - \operatorname{ber} q) \qquad (27)$$

$$Y_1 = \frac{2}{q} \operatorname{bei}' q \qquad (28)$$

$$Z_1 = \frac{4}{q^2} \operatorname{bei} q \qquad (29)$$

Substituting these in equations (22) and (23), the well-known solution in terms of the ber and bei functions result.[7]

[4] Phil. Mag., **17**, p. 524; 1909.

[5] Phil. Mag., **19**, p. 49; 1910.

[6] B. S. Bulletin, **8**, pp. 173-226 (Scientific Paper 169).

[7] B. S. Bulletin, **8**, p. 175.

Therefore, for a circular conductor, the method of integration gives results identical with those obtained by previous investigators.

2. DERIVATION OF FORMULAS USING COMPLEX POWER SERIES

The derivation of the formulas may be much simplified by the use of complex quantities.

Let

$$E' = E\epsilon^{i\omega t},\ U' = U\epsilon^{i(\omega t-\phi)},\ \frac{dU'}{dt} = i\omega U\epsilon^{i(\omega t-\phi)}$$

Where

$$i = \sqrt{-1}$$

Substituting in (3), viz:

$$E' = U'_x \sigma l + \int\int M_{xy} \frac{dU_y'}{dt} dS$$

$$E\epsilon^{i\omega t} = \sigma l U_x \epsilon^{i\omega t} \epsilon^{-i\phi_x} \int\int M_{xy} i\omega U_y \epsilon^{i\omega t} \epsilon^{-i\phi_y} dS \tag{30}$$

Take out $\epsilon^{i\omega t}$ and insert the value of

$$M_{xy} = 2l\left(\log\frac{2l}{d} - 1\right) = 2l\ (\log 2l - 1) - l \log d^2$$

$$E = \sigma l U_x \epsilon^{-i\phi_x} + i\omega \int_0^a \rho_y d\rho_y \int_0^{2\pi} \Big\{ 2l\ (\log 2l - 1)$$

$$- l \log\left[\rho_x^2 + \rho_y^2 - 2\rho_x\rho_y \cos(\theta_x - \theta_y)\right]\Big\} U_y \epsilon^{-i\phi_y} d\theta_y \tag{31}$$

Assume

$$U_x \epsilon^{-i\phi_x} = A + B\rho_x + C\rho_x^2 + D\rho_x^3 + \cdots\cdot$$

Where A, B, C, D, etc., are complex numbers.
Substituting in (31) and putting

$$L_\infty = 2l\left(\log\frac{2l}{a} - 1\right)$$

$$E = \sigma l(A + B\rho_x + C\rho_x^2 + D\rho_x^3 + \cdots\cdot)$$

$$+ 2\pi i\omega L_\infty \left[\frac{Aa^2}{2} + \frac{Ba^3}{3} + \frac{Ca^4}{4} + \cdots\cdot\right]$$

$$+ 4\pi i\omega l \left[\frac{Aa^2}{4} + \frac{Ba^3}{9} + \frac{Ca^4}{16} + \cdots\cdot\right]$$

$$- 4\pi i\omega l \left[\frac{A\rho_x^2}{4} + \frac{B\rho_x^3}{9} + \frac{C\rho_x^4}{16} + \cdots\cdot\right] \tag{32}$$

When $\rho_x = 0$

$$E = \sigma l A + 2\pi i \omega L_\infty \left[\frac{Aa^2}{2} + \frac{Ba^3}{3} + \frac{Ca^4}{4} + \cdots \right]$$

$$+ 4\pi i \omega l \left[\frac{Aa^2}{4} + \frac{Ba^3}{9} + \frac{Ca^4}{16} + \cdots \right] \qquad (33)$$

Equating the like powers of ρ_x and letting $\frac{\pi\omega}{\sigma} = \eta$, the values of all the coefficients may be expressed in terms of A as indicated in Table 2:

TABLE 2.—Values of Coefficients of Powers of ρ_x in Terms of A from Equation (32)

Powers of ρ	Coefficients	
ρ_x	$B=0$	
ρ_x^2	$\sigma l C = A i \omega \pi l$	$C = \frac{i\omega\pi A}{\sigma} = i\eta A$
ρ_x^3	$D=0$	
ρ_x^4	$\sigma l E = \frac{C i \omega \pi l}{4}$	$E = -\frac{\omega^2\pi^2 A}{4\sigma^2} = -\frac{\eta^2 A}{(2!)^2}$
ρ_x^5	$F=0$	
ρ_x^6	$\sigma l G = \frac{E i \omega \pi l}{9}$	$G = -\frac{i\omega^3\pi^3 A}{4 \cdot 9\sigma^3} = -\frac{i\eta^3 A}{(3!)^2}$
ρ_x^7	$H=0$	
ρ_x^8	$\sigma l I = \frac{G i \omega \pi l}{16}$	$I = \frac{\omega^4\pi^4 A}{4 \cdot 9 \cdot 16\sigma^4} = \frac{\eta^4 A}{(4!)^2}$
ρ_x^{2n-1}	$Q=0$	
ρ_x^{2n}		$R = \frac{i^n \eta^n A}{(\eta !)^2}$

NOTE.—Q and R are used, respectively, as the 2n−1 and 2n letters of the alphabet.

Substituting these values in (33):

$$E = \sigma l A + 2\pi i \omega L_\infty A \left[\frac{a^2}{2} + \frac{i\eta a^4}{4} - \frac{\eta}{24} - \frac{i\eta^3 a^8}{288} + \cdots + \frac{i^n \eta^n a^{2n+2}}{(n!)^2(2n+2)} + \cdots \right]$$

$$+ 4\pi i \omega l A \left[\frac{a^2}{4} + \frac{i\eta a^4}{16} - \frac{\eta^2 a^6}{144} - \frac{i\eta^3 a^8}{2304} + \cdots + \frac{i^n \eta^n a^{2n+2}}{[n!(2n+2)]^2} + \cdots \right] \qquad (34)$$

To determine the total current through the conductor

$$I' = \iint \delta I'_x = \iint U'_x \rho_x d\rho_x d\theta_x$$

$$= \int_0^a \rho_x d\rho_x \int_0^{2\pi} \epsilon^{i\omega t} (A + C\rho_x^2 + E\rho_x^4 + \cdots) d\theta_x$$

Integrating:

$$I\epsilon^{(i\omega t - \phi)} = 2\pi \epsilon^{i\omega t} \left(\frac{Aa^2}{2} + \frac{Ca^4}{4} + \frac{Ea^6}{6} + \cdots \right)$$

Simplifying and substituting values of C, E, etc., from Table 2:

$$I\epsilon^{-i\phi}=2\pi A\left(\frac{a^2}{2}+\frac{i\eta a^4}{4}-\frac{\eta^2a^6}{24}-\frac{i\eta^3a^8}{288}+\frac{\eta^4a^{10}}{5760}+\cdot\cdot\right) \qquad (35)$$

Also $\frac{E'}{I'}=R+i\omega L$, or $\frac{E}{I\epsilon^{-i\phi}}=R+i\omega L$ (36)

Letting W_1, X_1, Y_1, and Z_1 represent the same values as before, then substituting from equations (34) and (35) in equation (36):

$$R+i\omega L=\frac{\sigma l+\pi i\omega a^2L_\infty\left(Y_1+\frac{i\eta a^2W_1}{2}\right)+\pi a^2 i\omega l\left(Z_1+\frac{i\eta a^2X_1}{4}\right)}{\pi a^2\left(Y_1+\frac{i\eta a^2W_1}{2}\right)} \qquad (37)$$

Rationalizing the denominator by multiplying by $Y_1-\frac{i\eta a^2W_1}{2}$ and taking the real parts

$$R=\frac{\sigma lY_1-\pi a^2\omega l\frac{na^2X_1Y_1}{4}+\pi a^2\omega l\frac{na^2W_1Z_1}{2}}{\pi a^2\left(Y_1^2+\frac{n^2a^4W_1^2}{4}\right)} \qquad (38)$$

Letting $\frac{\sigma l}{\pi a^2}=R_0$

$$R=R_0\frac{Y_1+\frac{\eta^2a^4W_1Z_1}{2}-\frac{\eta^2a^4X_1Y_1}{4}}{\left(Y_1^2+\frac{\eta^2a^4W_1^2}{4}\right)} \qquad (39)$$

$$R=R_0\frac{Y_1+\frac{\eta^2a^4}{2}\left(W_1Z_1-\frac{X_1Y_1}{2}\right)}{Y_1^2+\frac{\eta^2a^4W_1^2}{4}} \qquad (40)$$

In like manner

$$L=L_\infty+\frac{l}{2}\frac{2Y_1Z_1-W_1+\frac{\eta^2a^4X_1W_1}{4}}{Y_1^2+\frac{\eta^2a^4W_1^2}{4}} \qquad (41)$$

These equations are identical with (22) and (23) showing that the result using complex power series is the same as with real power series.

IV. ALTERNATING-CURRENT RESISTANCE AND INDUCTANCE OF A RETURN CIRCUIT

If a return circuit consists of two parallel cylindrical conductors, whose length is great compared to the diameters of the wires and to the distance between them, the method of integration can be applied to the determination of the alternating current resistance and inductance of the circuit.[8] If the wires have the same diameter, the current distribution in one wire is symmetrical about the line joining the centers of the two wires. If the equation of current distribution is given in polar coordinates, it will be identical for the two wires if the angles are measured from the line joining the centers of the wires.

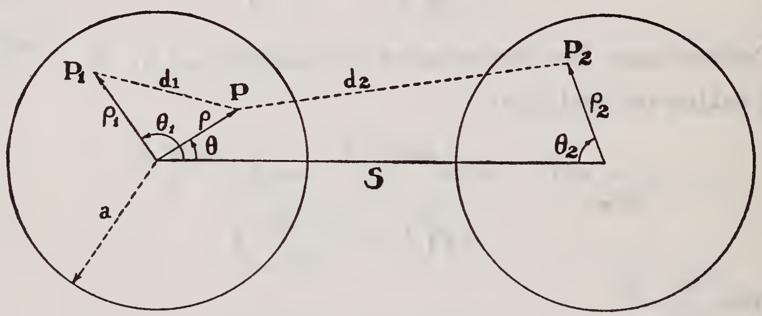

FIG. 2.—*The cross section of a return circuit, showing coordinates*

Let P with coordinates ρ, θ be a point in one conductor at which the current density is to be determined, P_1 another point in the same conductor, and P_2 a point in the return conductor whose center is at a distance s from the center of the first conductor. Let U', U'_1, and U'_2 designate the instantaneous current density and U, U_1, and U_2 the maximum current density at P, P_1, and P_2, respectively. Also let m_1 represent the mutual inductance between the two filaments at P and P_1, whose distance apart is d_1, and m_2 that between filaments at P and P_2, whose distance apart is d_2.

Equation (3) for this case becomes

$$E' = U'\sigma l + \int\int m_1 \frac{dU'_1}{dt} dS_1 - \int\int m_2 \frac{dU_2'}{dt} dS_2 \qquad (42)$$

where dS_1 and dS_2 are elements of area at P_1 and P_2.

The mutual inductance between two long filaments[9] is

$$m = 2l\ (\log 2l - 1) - l \log d^2$$

[8] Nicholson has published a formula covering this case (Phil. Mag., 18, p. 417; 1909). However, the author has not found any record of its application to experimental results, and efforts to compute by it indicate that the correction factor, due to the proximity of the conductors, is not only much too small, but has the wrong sign.

[9] B. S. Bulletin, 8, p. 151 (Scientific Paper No. 169).

Substituting values of m_1 and m_2 in (42)

$$E' = U'\sigma l + 2l\,(\log\ 2l - 1)\int\!\!\int \frac{dU_1'}{dt}\ dS_1 - l\int\!\!\int \log\ d_1^2 \frac{dU_1'}{dt}\ dS_1$$

$$- 2l\,(\log\ 2l - 1)\int\!\!\int \frac{dU_2'}{dt}\ dS_2 + l\int\!\!\int \log\ d_2^2 \frac{dU_2'}{dt}\ dS_2 \qquad (43)$$

Since the current in the two conductors is the same,

$$\int\!\!\int U_1' dS_1 = \int\!\!\int U_2'\ dS_2.$$

So that terms two and four of the second member of (43) cancel.

But $E' = E\epsilon^{i\omega t}$, $U' = U\epsilon^{i(\omega t - \phi)}$, $U_1' = U_1\epsilon^{i(\omega t - \phi_1)}$, etc.

$$d_1^2 = \rho^2 + \rho_1^2 - 2\rho\rho_1 \cos\ (\theta - \theta_1)$$

$$d_2^2 = (s - \rho \cos\theta - \rho_2 \cos\theta_2)^2 + (\rho \sin\theta - \rho_2 \sin\theta_2)^2$$

$$= q^2 + \rho_2^2 - 2q\rho_2 \cos\ (\alpha - \theta_2),$$

where $q \cos\alpha = s - \rho \cos\theta$

and $q \sin\alpha = \rho \sin\theta$

so that $q^2 = s^2 - 2s\ \rho \cos\theta + \rho^2$.

Hence equation (43) becomes

$$E = \sigma l\ U\epsilon^{-i\phi}$$

$$- i\omega l\int_0^a \rho_1 d\rho_1 \int_0^{2\pi} \log\ [\rho^2 + \rho_1^2 - 2\rho\rho_1 \cos\ (\theta - \theta_1)]\ U_1\epsilon^{-i\phi_1} d\theta_1$$

$$+ i\omega l\int_0^a \rho_2 d\rho_2 \int_0^{2\pi} \log\ [q^2 + \rho_2^2 - 2q\rho_2 \cos\ (\alpha - \theta_2)]\ U_2\epsilon^{-i\phi_2} d\theta_2 \qquad (44)$$

Both U and ϕ are functions not only of ρ but also of θ. But as U and ϕ have the same values for $-\theta$ as for θ, the functions of θ when expressed in a Fourier series will be in terms of the cosine series only. Hence, assume

$$\begin{aligned} U\epsilon^{-i\phi} = {} & a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \cdots\cdots \\ & + \rho(b_0 + b_1 \cos\theta + b_2 \cos 2\theta + \cdots\cdot) \\ & + \rho_2\ (c_0 + c_1 \cos\theta + c_2 \cos 2\theta + \cdots\cdot) \\ & + \cdot\cdot(\cdots\cdots\cdots) \end{aligned} \qquad (45)$$

If these values are substituted in (44) and the integration performed, a comparison of coefficients shows that many vanish. Equation (45) reduces to the following:

$$\begin{aligned} U\epsilon^{-i\phi} = {} & a_0 + c_0\rho^2 + e_0\rho^4 + g_0\rho^6 + \cdots\cdot \\ & + (b_1\rho + d_1\rho^3 + f_1\rho^5 + \cdot\cdot) \cos\theta \\ & + (c_2\rho^2 + e_2\rho^4 + g_2\rho^6 + \cdot\cdot) \cos 2\theta \\ & + (d_3\rho^3 + f_3\rho^5 + h_3\rho^7 + \cdot\cdot) \cos 3\theta \\ & + (e_4\rho^4 + g_4\rho^6 + i_4\rho^8 + \cdot\cdot) \cos 4\theta \\ & + (\cdots\cdots\cdots\cdot) \end{aligned} \qquad (46)$$

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Substituting (46) in (44) and integrating by using the integrals given in the Appendix,

$$
\begin{aligned}
E = \sigma l \{ & a_0 + c_0\rho^2 + e_0\rho^4 + g_0\rho^6 + \cdots\cdots \\
& + [b_1\rho + d_1\rho^3 + f_1\rho^5 + h_1\rho^7 + \cdots\cdot] \cos\theta \\
& + [c_2\rho^2 + e_2\rho^4 + g_2\rho^6 + i_2\rho^8 + \cdots\cdot] \cos 2\theta \\
& + [d_3\rho^3 + f_3\rho^5 + h_3\rho^7 + j_3\rho^9 + \cdots\cdot] \cos 3\theta \\
& + \cdots\cdots\cdots\cdots \} \\
& - i\omega l \left\{ 4\pi \log a \left(\frac{a_0 a^2}{2} + \frac{c_0 a^4}{4} + \frac{e_0 a^6}{6} + \frac{g_0 a^8}{8} + \cdots\cdot \right) \right\} \\
& + i\omega l \left\{ 4\pi \left(\frac{a_0 a^2}{4} + \frac{c_0 a^4}{16} + \frac{e_0 a^6}{36} + \frac{g_0 a^8}{64} + \cdots\cdot \right) \right\} \\
& - i\omega l \left\{ 4\pi \left(\frac{a_0 \rho^2}{4} + \frac{c_0 \rho^4}{16} + \frac{e_0 \rho^6}{36} + \frac{g_0 \rho^8}{64} + \cdots\cdot \right) \right\} \\
& - i\omega l \left\{ 4\pi \left[\frac{b_1\rho^3}{8} + \frac{d_1\rho^5}{24} + \frac{f_1\rho^7}{48} + \frac{h_1\rho^9}{80} + \cdots + \frac{x_1\rho^{2m+1}}{(2m+1)^2 - 1} + \cdots\cdot \right] \right. \\
& \left. - 2\pi\rho \left[\frac{b_1 a^2}{2} + \frac{d_1 a^4}{4} + \frac{f_1 a^6}{6} + \frac{h_1 a^8}{8} + \cdots + \frac{x_1 a^{2m}}{2m} + \cdots\cdot \right] \right\} \cos\theta \\
& - i\omega l \left\{ 4\pi \left[\frac{c_2\rho^4}{12} + \frac{e_2\rho^6}{32} + \frac{g_2\rho^8}{60} + \frac{i_2\rho^{10}}{96} + \cdots + \frac{x_2\rho^{2m}}{4m^2 - 4} + \cdots\cdot \right] \right. \\
& \left. - \pi\rho^2 \left[\frac{c_2 a^2}{2} + \frac{e_2 a^4}{4} + \frac{g_2 a^6}{6} + \frac{i_2 a^8}{8} + \cdots + \frac{x_2 \rho^{2m-2}}{2m-2} + \cdots\cdot \right] \right\} \cos 2\theta \\
& - i\omega l \left\{ 4\pi \left[\frac{d_3\rho^5}{16} + \frac{f_3\rho^7}{40} + \frac{h_3\rho^9}{72} + \frac{j_3\rho^{11}}{112} + \cdots + \frac{x_3\rho^{2m-1}}{(2m-1)^2 - 9} + \cdots \right] \right. \\
& \left. - \frac{2\pi}{3}\rho^3 \left[\frac{d_3 a^2}{2} + \frac{f_3 a^4}{4} + \frac{h_3 a^6}{6} + \frac{j_3 a^8}{8} + \cdots + \frac{x_3 \rho^{2m-4}}{2m-4} + \cdots\cdot \right] \right\} \cos 3\theta \\
& - i\omega l \left\{ \text{terms in } \cos 4\theta,\ \cos 5\theta, \text{ etc.} \right\} \\
& + i\omega l \left\{ 4\pi \log q \left[\frac{a_0 a^2}{2} + \frac{c_0 a^4}{4} + \frac{e_0 a^6}{6} + \frac{g_0 a^8}{8} + \cdots\cdot \right] \right\} \\
& - i\omega l \left\{ 2\pi \left[\frac{b_1 a^4}{4} + \frac{d_1 a^6}{6} + \frac{f_1 a^8}{8} + \frac{h_1 a^{10}}{10} + \cdots\cdot \right] \frac{\cos\alpha}{q} \right\} \\
& - i\omega l \left\{ \frac{2\pi}{2} \left[\frac{c_2 a^6}{6} + \frac{e_2 a^8}{8} + \frac{g_2 a^{10}}{10} + \frac{i_2 a^{12}}{12} + \cdots\cdot \right] \frac{\cos 2\alpha}{q^2} \right\} \\
& - i\omega l \left\{ \frac{2\pi}{3} \left[\frac{d_3 a^8}{8} + \frac{f_3 a^{10}}{10} + \frac{h_3 a^{12}}{12} + \frac{j_3 a^{14}}{14} + \cdots\cdot \right] \frac{\cos 3\alpha}{q^3} \right\} \\
& - i\omega l \left\{ \text{terms in } \cos\frac{4\alpha}{q^4}, \cos\frac{5\alpha}{q^5}, \text{ etc.} \right\}
\end{aligned}
\qquad (47)
$$

By equating to zero the coefficients of like terms of ρ and θ, many of the undetermined coefficients are easily evaluated. The most important of these are given in Table 3:

TABLE 3.—Coefficients of Terms Involving ρ and θ from Equation (47)

Let $\frac{\pi\omega}{\sigma}=\eta$

Coefficients of—	Values obtained	Coefficients of—	Values obtained
ρ^2	$c_0=i\eta a_0$	$\rho^4\cos 2\theta$	$e_2=\frac{i\eta c_2}{3}$
ρ^4	$e_0=\frac{i\eta c_0}{4}=-\frac{\eta^2 a_0}{4}$	$\rho^6\cos 2\theta$	$g_2=\frac{i\eta e_2}{8}=-\frac{\eta^2 c_2}{24}$
ρ^6	$g_0=\frac{i\eta e_0}{9}=-\frac{i\eta^3 a_0}{36}$	$\rho^8\cos 2\theta$	$i_2=\frac{i\eta g_2}{15}=-\frac{i\eta^3 c_2}{360}$
ρ^8	$i_0=\frac{i\eta g_0}{16}=\frac{\eta^4 a_0}{576}$	$\rho^{10}\cos 2\theta$	$k_2=\frac{i\eta i_2}{24}=\frac{\eta^4 c_2}{8640}$
ρ^{10}	$k_0=\frac{i\eta i_0}{25}=\frac{i\eta^5 a_0}{14400}$	$\rho^{12}\cos 2\theta$	$m_2=\frac{i\eta k_2}{35}=\frac{i\eta^5 c_2}{302400}$
ρ^{12}	$m_0=\frac{i\eta k_0}{36}=-\frac{\eta^6 a_0}{518400}$	$\rho^{14}\cos 2\theta$	$o_2=\frac{i\eta m_2}{48}=-\frac{\eta^6 c_2}{14515200}$
$\rho^3\cos\theta$	$d_1=\frac{i\eta b_1}{2}$	$\rho^5\cos 3\theta$	$f_3=\frac{i\eta d_3}{4}$
$\rho^5\cos\theta$	$f_1=\frac{i\eta d_1}{6}=-\frac{\eta^2 b_1}{12}$	$\rho^7\cos 3\theta$	$h_3=\frac{i\eta f_3}{10}=-\frac{\eta^2 d_3}{40}$
$\rho^7\cos\theta$	$h_1=\frac{i\eta f_1}{12}=-\frac{i\eta^3 b_1}{144}$	$\rho^9\cos 3\theta$	$j_3=\frac{i\eta h_3}{18}=-\frac{i\eta^3 d_3}{720}$
$\rho^9\cos\theta$	$j_1=\frac{i\eta h_1}{20}=\frac{\eta^4 b_1}{2880}$	$\rho^{11}\cos 3\theta$	$l_3=\frac{i\eta j_3}{28}=\frac{\eta^4 d_3}{20160}$
$\rho^{11}\cos\theta$	$l_1=\frac{i\eta j_1}{30}=\frac{i\eta^5 b_1}{86400}$	$\rho^{13}\cos 3\theta$	$n_3=\frac{i\eta l_3}{40}=\frac{i\eta^5 d_3}{806400}$
$\rho^{13}\cos\theta$	$n_1=\frac{i\eta l_1}{42}=-\frac{\eta^6 b_1}{3628800}$	$\rho^{15}\cos 3\theta$	$p_3=\frac{i\eta n_3}{54}=-\frac{\eta^6 d_3}{43545600}$

Substituting the values from Table 3, the series in equation (47) containing powers of the radius, a, reduce as follows:

$$\frac{a_0a^2}{2}+\frac{c_0a^4}{4}+\frac{e_0a^6}{6}+\frac{g_0a^8}{8}+\cdots=\frac{a_0a^2}{2}\Big[1+\frac{i\eta a^2}{2}-\frac{\eta^2a^4}{12}-\frac{i\eta^3a^4}{144}+\frac{\eta^4a^8}{2880}+\frac{i\eta^5a^{10}}{86400}+\cdots\Big]\equiv\frac{a_0a^2}{2}A_0$$

$$\frac{a_0a^2}{4}+\frac{c_0a^4}{16}+\frac{e_0a^6}{36}+\frac{g_0a^8}{64}+\cdots=\frac{a_0a^2}{4}\Big[1+\frac{i\eta a^2}{4}-\frac{\eta^2a^4}{36}-\frac{i\eta^3a^6}{576}+\frac{\eta^4a^8}{14400}+\frac{i\eta^5a^{10}}{518400}+\cdots\Big]\equiv\frac{a_0a^2}{4}A$$

$$\frac{b_1a^2}{2}+\frac{d_1a^4}{4}+\frac{f_1a^6}{6}+\frac{g_1a^8}{8}+\cdots=\frac{b_1a^2}{2}\Big[1+\frac{i\eta a^2}{4}-\frac{\eta^2a^4}{36}-\frac{i\eta^3a^6}{576}+\frac{\eta^4a^8}{14400}+\frac{i\eta^5a^{10}}{518400}+\cdots\Big]\equiv\frac{b_1a^2}{2}A_1$$

$$\frac{c_2a^2}{2}+\frac{e_2a^4}{4}+\frac{g_2a^6}{6}+\frac{i_2a^8}{8}+\cdots=\frac{c_2a^2}{2}\left[1+\frac{i\eta a^2}{6}-\frac{\eta^2a^4}{72}\right.$$
$$\left.-\frac{i\eta^3a^6}{1440}+\frac{\eta^4a^8}{43200}+\frac{i\eta^5a^{10}}{1814400}+\cdots\right]\equiv\frac{c_2a^2}{2}C_0$$

$$\frac{d_3a^2}{2}+\frac{f_3a^4}{4}+\frac{h_3a^6}{6}+\frac{j_3a^8}{8}+\cdots=\frac{d_3a^2}{2}\left[1+\frac{i\eta a^2}{8}-\frac{\eta^2a^4}{120}\right.$$
$$\left.-\frac{i\eta^3a^6}{2880}+\frac{\eta^4a^8}{100800}+\frac{i\eta^5a^{10}}{4838400}+\cdots\right]\equiv\frac{d_3a^2}{2}D_0$$

$$\frac{b_1a^4}{4}+\frac{d_1a^6}{6}+\frac{f_1a^8}{8}+\frac{h_1a^{10}}{10}+\cdots=\frac{b_1a^4}{4}\left[1+\frac{i\eta a^2}{3}-\frac{\eta^2a^4}{24}\right.$$
$$\left.-\frac{i\eta^3a^6}{360}+\frac{\eta^4a^8}{8640}+\frac{i\eta^5a^{10}}{302400}+\cdots\right]\equiv\frac{b_1a^4}{4}B_1$$

$$\frac{c_2a^6}{6}+\frac{e_2a^8}{8}+\frac{g_2a^{10}}{10}+\frac{i_2a^{12}}{12}+\cdots=\frac{c_2a^6}{6}\left[1+\frac{i\eta a^2}{4}-\frac{\eta^2a^4}{40}\right.$$
$$\left.-\frac{i\eta^3a^6}{720}+\frac{\eta^4a^8}{20160}+\frac{i\eta^5a^{10}}{806400}+\cdots\right]\equiv\frac{c_2a^6}{6}C_1$$

$$\frac{d_3a^8}{8}+\frac{f_3a^{10}}{10}+\frac{h_3a^{12}}{12}+\frac{j_3a^{14}}{14}+\cdots=\frac{d_3a^8}{8}\left[1+\frac{i\eta a^2}{5}-\frac{\eta^2a^4}{60}\right.$$
$$\left.-\frac{i\eta^3a^6}{1260}+\frac{\eta^4a^8}{40320}+\frac{i\eta^5a^{10}}{1814400}+\cdots\right]\equiv\frac{d_3a^8}{8}D_1$$

It will be noted that the series are all arranged so that the first term is unity. It can be shown that they are all converging series.

Substitute the series given above in (47) and retain only the constant terms and those of the form $\rho^n \cos n\theta$, since the coefficients of all other terms are zero. Also insert expansions of $\log q$, of $\frac{\cos \alpha}{q}$, of $\frac{\cos 2\alpha}{q^2}$, etc., as given in the appendix. Put $\eta=\frac{\pi\omega}{\sigma}$. Then

$$\frac{E}{\sigma l}=a_0+b_1\rho\cos\theta+c_2\rho^2\cos 2\theta+d_3\rho^3\cos 3\theta+\cdots$$
$$-i\eta[2a_0a^2A_0\log a-a_0a^2A_1-b_1a^2A_1\rho\cos\theta-\frac{1}{2}c_2a^2C_0\rho^2\cos 2\theta$$
$$-\frac{1}{3}d_3a^2D_0\rho^3\cos 3\theta+\cdots]$$
$$+i\eta[2a_0a^2A_0(\log s-\frac{\rho}{s}\cos\theta-\frac{\rho^2}{2s^2}\cos 2\theta-\frac{\rho^3}{3s^3}\cos 3\theta-\cdots)]$$
$$-i\eta\left[\frac{b_1a^4B_1}{2s}(1+\frac{\rho}{s}\cos\theta+\frac{\rho^2}{s^2}\cos 2\theta+\frac{\rho^3}{s^3}\cos 3\theta+\cdots)\right]$$
$$-i\eta\left[\frac{c_2a^6C_1}{6s^2}(1+\frac{2\rho}{s}\cos\theta+\frac{3\rho^2}{s^2}\cos 2\theta+\frac{4\rho^3}{s^3}\cos 3\theta+\cdots)\right]$$
$$-i\eta\left[\frac{d_3a^8D_1}{12s^3}(1+\frac{3\rho}{s}\cos\theta+\frac{6\rho^2}{s^2}\cos 2\theta+\frac{10\rho^3}{s^3}\cos 3\theta+\cdots)\right] \quad (48)$$

The total current through the conductor is the sum of the currents through the filaments, or

$$I' = \int\int \delta I' = \int\int U' \rho d\rho d\theta$$

$$= \epsilon^{i\omega t} \int_0^a \rho d\rho \int_0^{2\pi} \{a_0 + c_0\rho^2 + e_0\rho^4 + \cdots$$

$$+ (b_1\rho + d_1\rho^3 + f_1\rho^5 + \cdots) \cos\theta$$

$$+ (\text{terms in } \cos 2\theta, \cos 3\theta, \text{etc.})\} d\theta$$

$$= 2\pi\epsilon^{i\omega t}\left[\frac{a_0a^2}{2} + \frac{c_0a^4}{4} + \frac{e_0a^6}{6} + \cdots\right]$$

$$= \pi a^2 a_0 A_0 \epsilon^{i\omega t}$$

since the integrals of the cosine terms are each equal to zero.

But

$$I' = I\epsilon^{i(\omega t - \phi)}$$

$$\therefore\ I\epsilon^{-i\phi} = \pi a^2 a_0 A_0 \tag{49}$$

The alternating-current resistance and inductance of a circuit is given by the formula

$$\frac{E}{I\epsilon^{-i\phi}} = R + i\omega L \tag{50}$$

By substituting (48) and (49) in (50) and equating to zero the coefficients of each power of ρ, the following equations result:

$$R + i\omega L = \frac{\sigma l}{\pi a^2 a_0 A_0}\left\{a_0 + 2i\eta a^2 a_0 A_0 \log\frac{s}{a} + i\eta a^2 a_5 A_1 - i\eta\frac{a^4}{s}\left(\frac{b_1B_1}{2} + \frac{c_2C_1a^2}{6s} + \frac{d_3D_1a^4}{12s^2} + \cdots\right)\right\} \tag{51}$$

$$0 = b_1 + i\eta\left(b_1a^2A_1 - \frac{2a_0a^2A_0}{s} - \frac{b_1a^4B_1}{2s^2} - \frac{c_2a^6C_1}{3s^3} - \frac{d_3a^8D_1}{4s^4} - \cdots\right) \tag{52}$$

$$0 = c_2 + i\eta\left(\frac{c_2a^2C_0}{2} - \frac{a_0a^2A_0}{s^2} - \frac{b_1a^4B_1}{2s^3} - \frac{c_2a^6C_1}{2s^4} - \frac{d_3a^8D_1}{2s^5} - \cdots\right) \tag{53}$$

$$0 = d_3 + i\eta\left(\frac{d_3a^2D_0}{3} - \frac{2a_0a^2A_0}{3s^3} - \frac{b_1a^4B_1}{2s^4} - \frac{2c_2a^6C_1}{3s^5} - \frac{5d_3a^8D_1}{6s^6} - \cdots\right) \tag{54}$$

In order to evaluate R and L, it is necessary to eliminate a_0, b_1, c_2, and d_3 from equation (51). To accomplish this,solve for d_3 in equation (54) and substitute in the other three. Then solve for c_2 in the third of the resulting equations and substitute in the other two. In this way the elimination may be accomplished. This is outlined below.

Solving (54) for d_3

$$d_3=\frac{i\eta\left(\frac{2a_0a^2A_0}{3s^3}+\frac{b_1a^4B_1}{2s^4}+\frac{2c_2a^6C_1}{3s^5}\right)}{1+i\eta a^2\left(\frac{D_0}{3}-\frac{5a^6D_1}{6s^6}\right)}$$

$$=\frac{i\eta\left(\frac{2a_0a^2A_0}{3s^3}+\frac{b_1a^4B_1}{2s^4}+\frac{2c_2a^6C_1}{3s^5}\right)}{B_1-\frac{5i\eta a^8D_1}{6s^6}} \tag{55}$$

Let $D_2=B_1-\frac{5i\eta a^8D_1}{6s^6}$

then $d_3=i\eta a^2\left(\frac{2a_0A_0}{3s^3D_2}+\frac{b_1a^2B_1}{2s^4D_2}+\frac{2c_1a^4C_2}{3s^5D_2}\right)$ (56)

Substituting this value of d_3 in (51), (52), and (53) and representing the resistance to direct current by $R_0=\frac{\sigma l}{\pi a^2}$, equations (57), (58), and (59) result.

$$\frac{R+i\omega L}{R_0}=\frac{1}{A_0a_0}\Bigg\{a_0+2i\eta a^2a_0A_0\log\frac{s}{a}+i\eta a^2a_0A_1$$
$$-\frac{i\eta a^4}{s}\left[\frac{b_1B_1}{2}+\frac{a^2c_2C_2}{6s}+\frac{i\eta a^6D_1}{12s^2}\left(\frac{2a_0A_0}{3s^3D_2}+\frac{b_1a^2B_1}{2s^4D_2}\right)\right]\Bigg\} \tag{57}$$

$$0=b_1+i\eta\left\{a^2b_1A_1-\frac{2a_0a^2A_0}{s}-\frac{b_1a^4B_1}{2s^2}-\frac{c_2a^6C_1}{3s^3}-\frac{i\eta a_0a^{10}D_1A_0}{6s^7D_2}\right\} \tag{58}$$

$$0=c_2+i\eta\left\{-\frac{a_0a^2A_0}{s^2}-\frac{b_1a^4B_1}{2s^3}+\frac{c_2a^2C_0}{2}-\frac{c_2a^6C_1}{2s^4}\right\} \tag{59}$$

Solving (59) for c_2

$$c_2=\frac{\frac{i\eta a^2}{s^2}\left(a_0A_0+\frac{b_1a^2B_1}{2s}\right)}{1+i\eta a^2\left(\frac{C_0}{2}-\frac{a^4C_1}{2s^4}\right)} \tag{60}$$

Let $C_2=1+i\eta a^2\left(\frac{C_0}{2}-\frac{a^4C_1}{2s^4}\right)\equiv A_0-\frac{i\eta a^6C_1}{2s^4}$

then $c_2=\frac{i\eta a^2}{s^2C_2}\left(a_0A_0+\frac{b_1a^2B_1}{2s}\right)$ (61)

Substituting the value of c_2 in equations (57) and (58).

$$\frac{R+i\eta L}{R_0}=\frac{1}{A_0}+\frac{i\eta a^2A_1}{A_0}+2i\eta a^2\log\frac{s}{a}$$
$$-\frac{i\eta a^4}{sa_0A_0}\left\{\frac{i\eta a_0a^6A_0D_1}{18s^5\mathrm{D}_2}+\frac{i\eta b_1a^8B_1D_1}{24s^6\mathrm{D}_2}\right.$$
$$\left.+\frac{b_1B_1}{2}\left(1+\frac{i\eta a^8D_1}{4s^6D_2}\right)+\frac{a^2C_1}{6s}\left(\frac{i\eta a_0a^2A_0}{s^2C_2}+\frac{i\eta b_1a^4B_1}{2s^3C_2}\right)\right\} \quad (62)$$

$$0=b_1+i\eta b_1a^2A_1-\frac{i\eta b_1a^4B_1}{2s^2}-\frac{2i\eta a_0a^2A_0}{s}$$
$$-\frac{i\eta a^6C_1}{3s^3}\left(\frac{i\eta a_0a^2A_0}{s^2C_2}+\frac{i\eta b_1a^4B_1}{2s^3C_2}\right)+\frac{\eta^2a_0a^{10}D_1A_0}{6s^7D_2} \quad (63)$$

Solving (63) for b_1,

$$b_1=\frac{\dfrac{2i\eta a_0a^2A_0}{s}-\dfrac{\eta^2a_0a^8A_0C_1}{3s^5C_2}-\dfrac{\eta^2a_0a^{10}A_0D_1}{6s^7D_2}}{1+i\eta a^2A_1-\dfrac{i\eta a^4B_1}{2s^2}+\dfrac{\eta^2a^{10}B_1C_1}{6s^6C_2}}. \quad (64)$$

Let $B_2=1+i\eta a^2A_1-\dfrac{i\eta a^4B_1}{2s^2}+\dfrac{n^2a^{10}B_1C_1}{6s^6C_2}$

then

$$b_1=\frac{2i\eta a_0a^2A_0}{sB_2}-\frac{\eta^2a_0a^8A_0C_1}{3s^5B_2C_2}-\frac{\eta^2a_0a^{10}A_0D_1}{6s^7B_2D_2} \quad (65)$$

substituting the value of b_1 in equation (62)

$$\frac{R+i\omega L}{R_0}=\frac{1+i\eta a^2A_1}{A_0}+2i\eta a^2\log\frac{s}{a}$$
$$-\frac{i\eta a^4}{s^2}\left\{\left[\frac{2i\eta a^2}{B_2}-\frac{\eta^2a^8C_1}{3s^4B_2C_2}-\frac{\eta^2a^{10}D_1}{6s^6B_2D_2}\right]\right.$$
$$\left[\frac{B_1}{2}+\frac{i\eta a^6B_1C_1}{12s^4C_2}+\frac{i\eta a^8B_1D_1}{8s^6D_2}+\frac{i\eta a^8B_1D_1}{24s^6D_2}\right]$$
$$\left.+\frac{i\eta a^4C_1}{6s^2C_2}+\frac{i\eta a^6D_1}{18s^4D_2}\right\} \quad (66)$$

$$=\frac{1+i\eta a^2A_1}{A_0}+2i\eta a^2\log\frac{s}{a}$$
$$-\frac{i\eta a^4}{s^2}\left\{\frac{i\eta a^6D_1}{18s^4D_2}+\frac{i\eta a^2B_1}{B_2}-\frac{\eta^2a^8B_1C_1}{3s^4B_2C_2}+\frac{i\eta a^4C_1}{6s^2C_2}\right\} \quad (67)$$

$$=1+\frac{i\eta a^2A_1}{A_0}+2i\eta a^2\log\frac{s}{a}+\frac{\eta^2a^6B_1}{s^2B_2}$$
$$+\frac{i\eta^2a^8C_1}{6s^4C_2}+\frac{i\eta^3a^{12}B_1C_1}{3s^6B_2C_2}+\frac{\eta^2a^{10}D_1}{18s^6D_2} \quad (68)$$

Let $\eta a^2 = \lambda$
then equation (68) becomes

$$\frac{R+i\omega L}{R_0} = \frac{1+i\lambda A_1}{A_0} + 2i\lambda \log\left(\frac{s}{a}\right) + \frac{\lambda^2 B_1}{B_2}\left(\frac{a}{s}\right)^2 + \frac{i\lambda^2 C_1}{6C_2}\left(\frac{a}{s}\right)^4 + \frac{i\lambda^3 B_1 C_1}{3B_2 C_2}\left(\frac{a}{s}\right)^6 + \frac{\lambda^2 D_1}{18 D_2}\left(\frac{a}{s}\right)^6 \quad (69)$$

Where

$$A_0 = 1 + \frac{i\lambda}{2} - \frac{\lambda^2}{12} - \frac{i\lambda^3}{144} + \frac{\lambda^4}{2880} + \frac{i\lambda^5}{86400} + \cdots \quad (70)$$

$$A_1 = 1 + \frac{i\lambda}{4} - \frac{\lambda^2}{36} - \frac{i\lambda^3}{576} + \frac{\lambda^4}{14400} + \frac{i\lambda^5}{518400} + \cdots \quad (71)$$

$$B_1 = 1 + \frac{i\lambda}{3} - \frac{\lambda^2}{24} - \frac{i\lambda^3}{360} + \frac{\lambda^4}{8640} + \frac{i\lambda^5}{302400} + \cdots \quad (72)$$

$$C_1 = 1 + \frac{i\lambda}{4} - \frac{\lambda^2}{40} - \frac{i\lambda^3}{720} + \frac{\lambda^4}{20160} + \frac{i\lambda^5}{806400} + \cdots \quad (73)$$

$$D_1 = 1 + \frac{i\lambda}{5} - \frac{\lambda^2}{60} - \frac{i\lambda^3}{1260} + \frac{\lambda^4}{40320} + \frac{i\lambda^5}{1814400} + \cdots \quad (74)$$

$$B_2 = 1 + i\lambda A_1 - \frac{i\lambda B_1}{2}\left(\frac{a}{s}\right)^2 + \frac{\lambda^2 B_1 C_1}{6C_2}\left(\frac{a}{s}\right)^6 \quad (75)$$

$$C_2 = A_0 - \frac{i\lambda C_1}{2}\left(\frac{a}{s}\right)^4 \quad (76)$$

$$D_2 = B_1 - \frac{5i\lambda D_1}{6}\left(\frac{a}{s}\right)^6 \quad (77)$$

Separating the preceding series into real and imaginary parts, and representing the real part by the corresponding capital Greek letters, and the imaginary part by the corresponding lower case Greek letters as:

$$A_0 = \mathrm{A}_0 + \alpha_0 i \qquad C_1 = \Gamma_1 + \gamma_1 i$$
$$A_1 = \mathrm{A}_1 + \alpha_1 i \qquad C_2 = \Gamma_2 + \gamma_2 i$$
$$B_1 = \mathrm{B}_1 + \beta_1 i \qquad D_1 = \Delta_1 + \delta_1 i$$
$$B_2 = \mathrm{B}_2 + \beta_2 i \qquad D_2 = \Delta_2 + \delta_2 i$$

then the real part of equation (69) gives the resistance at a frequency of $2\pi\omega$, as:

$$\frac{R}{R_0} = \frac{\mathrm{A}_0 - \lambda(\mathrm{A}_0\alpha_1 - \mathrm{A}_1\alpha_0)}{\mathrm{A}_0^2 + \alpha_0^2} + \lambda^2\left(\frac{a}{s}\right)^2 \frac{\mathrm{B}_1\mathrm{B}_2 + \beta_1\beta_2}{\mathrm{B}_2^2 + \beta_2^2} + \frac{\lambda^2}{6}\left(\frac{a}{s}\right)^4 \frac{\Gamma_1\Gamma_2 + \gamma_1\gamma_2}{\Gamma_2^2 + \gamma_2^2} + \frac{\lambda^3}{3}\left(\frac{a}{s}\right)^6 \frac{(\mathrm{B}_2\gamma_2 + \beta_2\Gamma_2)(\mathrm{B}_1\Gamma_1 - \beta_1\gamma_1) - (\mathrm{B}_1\gamma_1 + \beta_1\Gamma_1)(\mathrm{B}_2\Gamma_2 - \beta_2\gamma_2)}{(\mathrm{B}_2\gamma_2 + \beta_2\Gamma_2)^2 + (\mathrm{B}_2\Gamma_2 - \beta_2\gamma_2)^2} + \frac{\lambda^2}{18}\left(\frac{a}{s}\right)^6 \frac{\Delta_1\Delta_2 + \delta_1\delta_2}{\Delta_2^2 + \delta_2^2} \quad (78)$$

Since

$$\frac{\omega L}{R_0}=\frac{L\pi\omega a^2}{\sigma l}=\frac{L\eta a^2}{l}=\frac{L\lambda}{l}$$

the imaginary part of (69) gives the inductance at a frequency of $2\pi\omega$, as

$$L=2l\log\frac{s}{a}+l\frac{A_0A_1+\alpha_0\left(\alpha_1-\frac{1}{\lambda}\right)}{A_0^2+\alpha_0^2}+l\lambda\left(\frac{a}{s}\right)^2\frac{\beta_1B_2-B_1\beta_2}{B_2^2+\beta_2^2}$$
$$+\frac{l\lambda}{6}\left(\frac{a}{s}\right)^4\frac{\gamma_1\Gamma_2-\Gamma_1\gamma_2}{\Gamma_2^2+\gamma_2^2}$$
$$+\frac{l\lambda^2}{3}\left(\frac{a}{s}\right)^6\frac{(B_1\Gamma_1-\beta_1\gamma_1)(B_2\Gamma_2-\beta_2\gamma_2)+(B_1\gamma_1+\beta_1\Gamma_1)(B_2\gamma_2+\beta_2\Gamma_2)}{(B_2\gamma_2+\beta_2\Gamma_2)^2+(B_2\Gamma_2-\beta_2\gamma_2)^2}$$
$$+\frac{l\lambda}{18}\left(\frac{a}{s}\right)^6\frac{\delta_1\Delta_2-\Delta_1\delta_2}{\Delta_2^2+\delta_2^2} \qquad (79)$$

It should be noted that the values of R and L as given in equations (78) and (79) are for one of the conductors of a return circuit. The total resistance and inductance are twice these values.

Although equations (78) and (79) give the alternating current resistance and inductance of a return circuit at any spacing of the wires, yet the series involved are not convergent for high frequency with large wires very close together.

The following table gives the highest frequency for which the series are convergent when the wires are as close together as possible:

TABLE 4.—Highest Frequency at Which Formula Will Hold for Different Diameters of Wire as Close Together as Possible

A. W. G. No.	Diameter in mils	Diameter in centimeters	Highest frequency	A. W. G. No.	Diameter in mils	Diameter in centimeters	Highest frequency
0000	460	1.168	1000	18	40	0.102	110 000
000	410	1.04	1300	20	32	.0812	200 000
00	365	.927	1600	22	25.3	.0642	340 000
0	325	.825	2000	24	20.1	.0510	520 000
2	258	.655	3400	26	15.9	.0404	830 000
4	204	.518	5100	28	12.6	.0320	1 300 000
6	162	.411	8000	30	10.0	.0254	2 100 000
8	128	.325	13 000	32	8.0	.0203	3 300 000
10	102	.259	20 000	34	6.3	.0160	5 300 000
12	81	.206	32 000	36	5.0	.0127	8 500 000
14	64	.162	52 000	38	4.0	.0101	13 000 000
16	51	.129	82 000	40	3.1	.00786	22 000 000

V. APPLICATION OF FORMULAS TO EXPERIMENTAL RESULTS

The alternating current resistance of a return circuit as computed by formula (78) has been compared with the experimental results which were obtained by Kennelly, Laws, and Pierce.[10] This comparison is given in Table 5. The two values agree within the experimental error, except at 1000 cycles with the wires close together. While the series are convergent at these values of spacing and frequency, yet it will be necessary to develop a large number of terms to obtain an accurate result.

TABLE 5.—Resistance of a No. 0000 Solid Copper Wire as Measured by Kennelly, Laws, and Pierce and as Calculated by Formula (78)

	Spacing	0.03 cm	0.8 cm	6.4 cm	20 cm	60 cm
60 cycles	Calculated	1.019	1.010	1.005	1.005	1.005
	Measured	1.017	1.012	1.008	1.006	1.004
	Per cent difference	− .2	+ .2	+ .3	+ .1	− .1
400 cycles	Calculated	1.572	1.307	1.183	1.176	1.175
	Measured	1.590	1.295	1.184	1.180	1.175
	Per cent difference	+1.1	− .6	+ .08	+ .3	0
1000 cycles	Calculated	2.908	1.982	1.680	1.662	1.660
	Measured	2.688	1.928	1.700	1.690	1.670
	Per cent difference	−8.2	−2.8	+1.1	+1.7	+ .6

Measurements of the inductance and resistance of a No. 2 copper wire in the form of a return circuit have been made in the inductance and capacity laboratory of the Bureau of Standards by C. N. Hickman and Miss C. Matilda Sparks. Measurements were made at four frequencies for each of seven spacings. The measured and computed values are given in Tables 6 and 7. They show an agreement within experimental error at all spacings with the possible exception of the inductance at 3000 cycles with the wires very close together. This is probably caused by the slow convergence of the series.

[10] Trans. A. I. E. E., 34. Part II, p. 1953.

TABLE 6.—Ratio of Alternating-Current Resistance to Direct-Current Resistance of a Return Circuit

[Length of circuit=1716.3 cm. No. 2 Cu. wire. Diam., 0.651 cm]

Spacing between wires in centimeters	Frequency	R/R_0 measured	R/R_0 computed	Per cent difference	Value of each term in equation (78)				
					First term	Second term	Third term	Fourth term	Fifth term
0.039	500	1.116	1.106	0.9	1.030	0.073	0.003	....	....
	1000	1.350	1.350	0	1.113	.223	.011	0.004	....
	2000	1.883	1.898	.3	1.355	.479	.037	.024	0.003
	3000	2.403	2.390	.5	1.608	.653	.065	.058	.007
0.175	500	1.083	1.082	.1	1.030	.051	.002	....	....
	1000	1.255	1.274	1.5	1.113	.155	.005	.001	....
	2000	1.740	1.715	1.4	1.355	.333	.018	.008	.001
	3000	2.111	2.114	.1	1.608	.453	.031	.020	.002
0.67	500	1.050	1.050	0	1.030	.020	....	....	....
	1000	1.172	1.174	.2	1.113	.061	.001	....	....
	2000	1.472	1.489	1.1	1.355	.131	.003	....	....
	3000	1.789	1.792	1.7	1.608	.178	.005	.001	....
1.30	500	1.034	1.039	.5	1.030	.009	....	....	....
	1000	1.143	1.141	.2	1.113	.028	....	....	....
	2000	1.402	1.416	1.	1.355	.059	.001	....	....
	3000	1.705	1.692	.8	1.608	.082	.001	.001	....
2.42	500	1.032	1.034	.2	1.030	.004	....	....	....
	1000	1.133	1.124	.8	1.113	.011	....	....	....
	2000	1.383	1.379	.3	1.355	.024	....	....	....
	3000	1.635	1.641	.4	1.608	.033	....	....	....
5.15	500	1.032	1.031	.1	1.030	.001	....	....	....
	1000	1.120	1.115	.4	1.113	.002	....	....	....
	2000	1.361	1.362	.1	1.355	.007	....	....	....
	3000	1.619	1.618	.1	1.608	.010	....	....	....

The values given in the table are the ratios of the alternating-current resistance at the indicated frequency and spacing to the direct-current resistance. With large spacings the alternating-current resistance of one wire is not affected by the presence of the other wire. At 5 cm an appreciable effect is noticeable, while when the wires are very close together the increase in resistance is several times the increase at the larger spacings.

The results given in Table 6 are shown graphically in the curves of Fig. 3, in which the relative increase of resistance with increasing frequency is shown. At the higher frequencies the resistance increases rapidly as the spacing decreases.

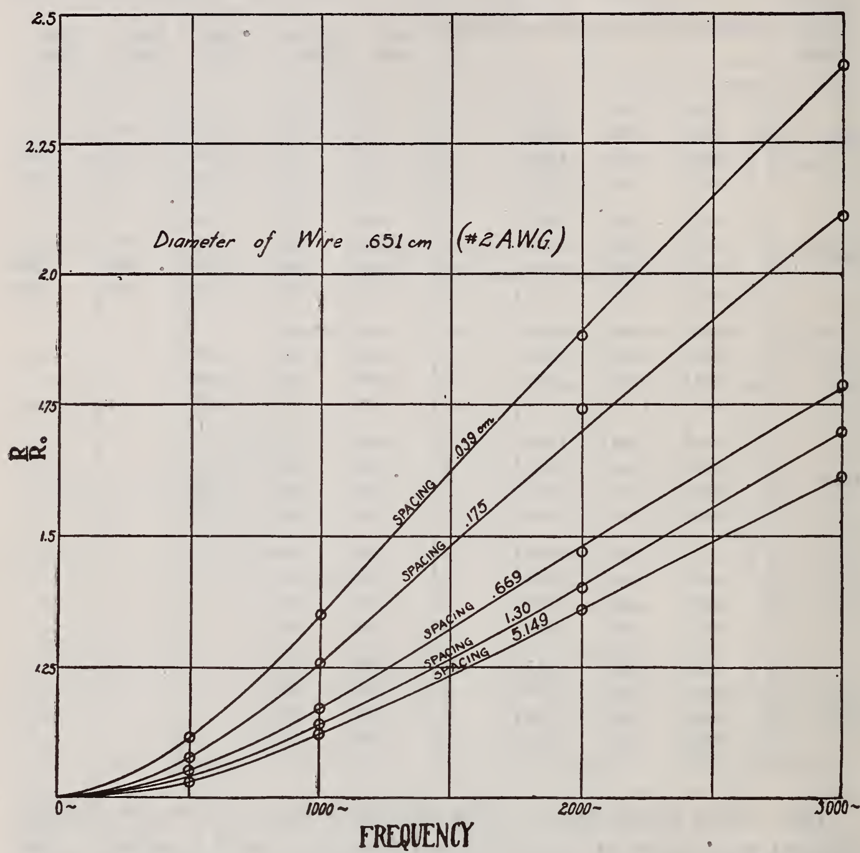

FIG. 3.—*Curves showing the change of resistance of a return circuit at different frequencies*

TABLE 7.—Alternating-Current Inductance of a Return Circuit

[Length of circuit=1716.3 cm. No. 2 Cu. wire. Diam., 0.651 cm]

Spacing between wires in centimeters	Frequency	Measured inductance microhenrys	Computed inductance microhenrys	Per cent difference	Inductance; current distribution of infinite spacing	Values of each term in equation (79)			
						First or log.term	Second term	Third term	Fourth term
0.039	0	....	6.88	....	6.88	5.16	1.72	....	....
	500	6.64	6.70	.9	6.85	5.16	1.69	−0.15	....
	1000	6.30	6.34	.6	6.77	5.16	1.61	− .41	−0.02
	2000	5.73	5.71	.3	6.58	5.16	1.42	− .84	− .03
	3000	5.18	5.29	2.0	6.39	5.16	1.23	−1.05	− .05
0.175	0	....	8.11	....	8.11	6.39	1.72	....	....
	500	7.99	7.98	.1	8.08	6.39	1.69	− .10	....
	1000	7.71	7.70	.1	8.00	6.39	1.61	− .29	− .01
	2000	7.20	7.21	.1	7.81	6.39	1.42	− .59	− .01
	3000	6.86	6.86	0	7.62	6.39	1.23	− .73	− .02
0.67	0	....	11.34	....	11.34	9.62	1.72	....	....
	500	11.24	11.27	.2	11.31	9.62	1.69	− .04	....
	1000	11.10	11.11	.1	11.22	9.62	1.61	− .11	....
	2000	10.79	10.81	.2	11.03	9.62	1.42	− .23	....
	3000	10.52	10.55	.3	10.84	9.62	1.23	− .29	....
1.30	0	....	14.02	....	14.02	12.30	1.72	....	....
	500	13.90	13.97	.5	13.99	12.30	1.69	− .02	....
	1000	13.78	13.85	.5	13.91	12.30	1.61	− .05	....
	2000	13.58	13.61	.2	13.72	12.30	1.42	− .10	....
	3000	13.27	13.39	.9	13.53	12.30	1.23	− .13	....
2.42	0	....	17.13	....	17.13	15.41	1.72	....	....
	500	17.03	17.10	.4	17.10	15.41	1.69	− .01	....
	1000	16.94	17.00	.4	17.00	15.41	1.61	− .02	....
	2000	16.70	16.79	.5	16.83	15.41	1.42	− .04	....
	3000	16.47	16.59	.7	16.64	15.41	1.23	− .05	....
5.15	0	....	21.50	....	21.50	19.78	1.72	....	....
	500	21.42	21.47	.2	21.47	19.78	1.69	....	....
	1000	21.33	21.38	.2	21.39	19.78	1.61	− .01	....
	2000	21.12	21.19	.3	21.20	19.78	1.42	− .01	....
	3000	20.89	20.99	.5	21.01	19.78	1.23	− .02	....

The first term of (79) is the inductance of two tubes. Hence this term may be considered as the inductance caused by the magnetic field external to the conductor. The second term, identical with the last term of (23), gives the inductance caused by the field inside a single conductor. Hence the sum of terms one and two gives the inductance of the circuit, assuming that the current distribution is the same as for infinite spacing. The other terms show the effect on the inductance of the change in current distribution caused by the field of the adjacent wire.

In Fig. 4 the relative decrease of inductance with increasing frequency is shown. The inductance, L_0, at zero frequency is always greater than the inductance, L, at any other frequency, so that function $\frac{L-L_0}{L_0}$ is in all cases negative. As the spacing is decreased, the relative inductance decreases rapidly.

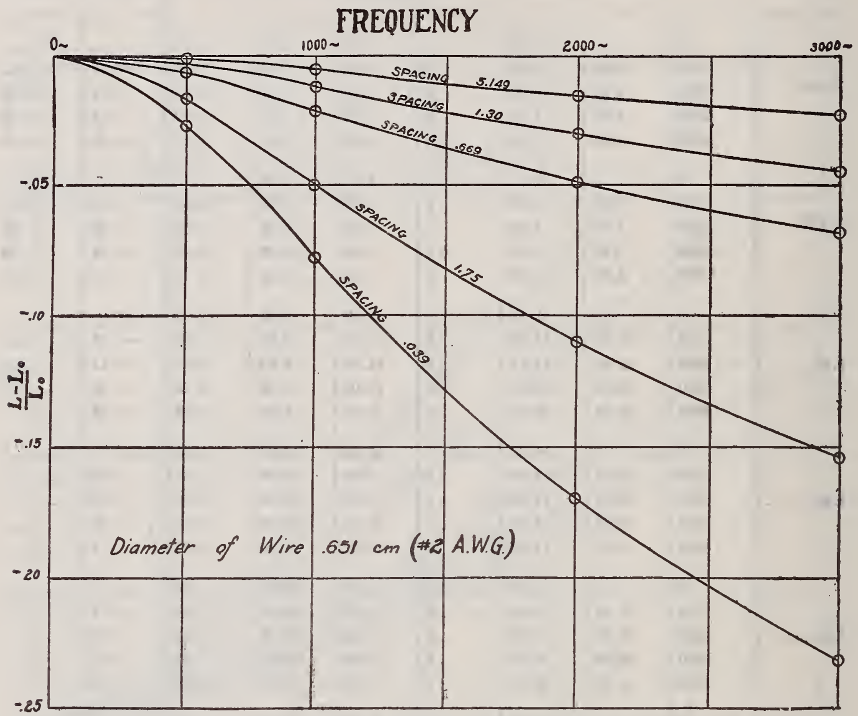

Fig. 4.—*Curves showing the effect of frequency on the inductance of a return circuit*

I have received valuable suggestions from a number of my colleagues at the Bureau of Standards. Also, Dr. F. W. Grover, of Colby College, has read the manuscript with care and corrected several errors in the numerical coefficients. Dr. T. J. I'a. Bromwich, of Cambridge University, England, has suggested methods for simplifying the integration in certain cases.

VI. APPENDIX.—EVALUATION OF INTEGRALS AND DEVELOPMENT OF SERIES

In this appendix are given the evaluation of some of the integrals and the expansion of some of the series which are necessary for the development of the formulas of this paper. In each case the nomenclature is that used in the body of the paper. Only those formulas which are not readily found in text-books of mathematics are included.

1. TO EXPAND LOG q IN A FOURIER SERIES

The quantity q is one side of a triangle of which the other two sides are ρ and s, having an included angle of θ. Hence

$$q^2 = s^2 + \rho^2 - 2\ s\rho \cos\theta$$

Expressing $\cos\theta$ in terms of exponentials

$$q^2 = s^2\left[1 + \frac{\rho^2}{s^2} - \frac{\rho}{s}\left(\epsilon^{i\theta} + \epsilon^{-i\theta}\right)\right]$$

$$= s^2\left[1 - \frac{\rho}{s}\epsilon^{i\theta}\right]\left[1 - \frac{\rho}{s}\epsilon^{-i\theta}\right]$$

Taking the logarithm of both sides of this equation

$$\log q^2 = \log s^2 + \log\left[1 - \frac{\rho\epsilon^{i\theta}}{s}\right] + \log\left[1 - \frac{\rho\epsilon^{-i\theta}}{s}\right]$$

Expanding the last two terms and taking their sum

$$2\log q = \log s^2 - \left[\frac{\rho}{s}\left(\epsilon^{i\theta} + \epsilon^{-i\theta}\right) + \frac{\rho^2}{2s^2}\left(\epsilon^{2i\theta} + \epsilon^{-2i\theta}\right) + \cdots\cdot\right]$$

$$= 2\ \log s - 2\left[\frac{\rho}{s}\cos\theta + \frac{\rho^2}{2s^2}\cos 2\theta + \frac{\rho^3}{3s^3}\cos 3\theta + \cdots\cdots\right]$$

$$\log q = \log s - \frac{\rho}{s}\cos\theta - \frac{\rho^2}{2s^2}\cos 2\theta - \frac{\rho^3}{3s^3}\cos 3\theta - \cdots\cdot$$

$$= \log s - \sum_1^\infty \frac{1}{r}\left(\frac{\rho}{s}\right)^r \cos r\theta$$

2. TO DEVELOP $\frac{\cos n\alpha}{q^n}$ IN A FOURIER SERIES OF θ

Since $\quad q\cos\alpha = s - \rho\cos\theta$

and $\quad q\sin\alpha = \rho\sin\theta$

it follows that $\cos\alpha + i\sin\alpha = \dfrac{s - \rho\cos\theta + i\rho\sin\theta}{q} = \dfrac{s - \rho\epsilon^{-i\theta}}{q}$

and $\quad q^2 = s^2 + \rho^2 - s\rho\left(\epsilon^{i\theta} + \epsilon^{-i\theta}\right)$

But $\quad \cos n\alpha + i\sin n\alpha = (\cos\alpha + i\sin\alpha)^n$

$$\therefore \frac{\cos n\alpha + i\sin n\alpha}{q^n} = \left[\frac{s - \rho\epsilon^{-i\theta}}{s^2 + \rho^2 - s\rho\left(\epsilon^{i\theta} + \epsilon^{-i\theta}\right)}\right]^n = \frac{1}{\left(s - \rho\epsilon^{i\theta}\right)^n}$$

$$= \frac{1}{s^n}\left(1 - \frac{\rho\epsilon^{i\theta}}{s}\right)^{-n}$$

expanding by the binomial theorem and taking the real part

$$\frac{\cos n\alpha}{q^n}=\frac{1}{s^n}\left[1+\frac{n\rho\cos\theta}{s}+\frac{n(n+1)}{2}\frac{\rho^2}{s^2}\cos 2\theta\right.$$
$$\left.+\frac{n(n+1)(n+2)}{3!}\frac{\rho^3}{s^3}\cos 3\theta+\cdots\right]$$

3. EVALUATION OF THE INTEGRAL I_1

The integral I_1 is given by the following equation where m and n are integers.

$$I_1=\int_0^a \rho_1 d\rho_1\int_0^{2\pi}\rho_1^m\left\{\log\left[\rho^2+\rho_1^2-2\rho\rho_1\cos(\theta-\theta_1)\right]\right\}\cos n\theta_1\,d\theta_1$$

As shown above, the logarithm term may be expanded as follows:

$$\log\left[\rho^2+\rho_1^2-2\rho\rho_1\cos(\theta-\theta_1)\right]$$

$$=2\log\rho-2\sum_1^\infty\frac{1}{r}\left(\frac{\rho_1}{\rho}\right)^r\cos r(\theta-\theta_1)\ \text{if}\ \rho>\rho_1$$

$$=2\log\rho_1-2\sum_1^\infty\frac{1}{r}\left(\frac{\rho}{\rho_1}\right)^r\cos r(\theta-\theta_1)\ \text{if}\ \rho_1>\rho$$

When this is substituted in I_1, there result two types of integrals to be evaluated. The first gives the following values:

$$\int_0^{2\pi}2\log\rho\cos n\theta_1\,d\theta_1=0\quad\text{when}\ n>0$$
$$=4\pi\log\rho\quad\text{when}\ n=0$$

The second type of integral, viz:

$$\int_0^{2\pi}-2\sum_1^\infty\frac{1}{r}\left(\frac{\rho_1}{\rho}\right)^r\cos r(\theta-\theta_1)\cos n\theta_1\,d\theta_1$$

may be integrated term by term as follows:

$$\int_0^{2\pi}\cos r(\theta-\theta_1)\cos n\theta_1\,d\theta_1$$

$$=1/2\int_0^{2\pi}\left\{\cos\left[r(\theta-\theta_1)+n\theta_1\right]+\cos\left[r(\theta-\theta_1)-n\theta_1\right]\right\}d\theta_1$$

$$=0\ \text{when}\ r\neq n$$
$$=\pi\cos n\theta\ \text{when}\ r=n$$

Dividing the integration with respect to ρ_1, in the original integral, into two parts, viz, from o to ρ and from ρ to a:

$$I_1=\int_0^\rho \rho_1^{m+1}\,d\rho_1\int_0^{2\pi}\left\{2\log\rho-2\sum_1^\infty\frac{1}{r}\left(\frac{\rho_1}{\rho}\right)^r\cos r\,(\theta-\theta_1)\right\}\cos n\theta_1\,d\theta_1$$

$$+\int_\rho^a \rho_1^{m+1}\,d\rho_1\int_0^{2\pi}\left\{2\log\rho_1-2\sum_1^\infty\frac{1}{r}\left(\frac{\rho}{\rho_1}\right)^r\cos r\,(\theta-\theta_1)\right\}\cos n\theta_1\,d\theta_1$$

$$=-\frac{2\pi\cos n\theta}{n}\left[\int_0^\rho\frac{\rho_1^{m+n+1}}{\rho^n}d\rho_1+\int_\rho^a\rho^n\rho_1^{m-n+1}d\rho_1\right]\quad\text{when } n>o$$

$$=4\pi\left[\int_0^\rho\rho_1^{m+1}\log\rho\,d\rho_1+\int_\rho^a\rho_1^{m+1}\log\rho_1\,d\rho_1\right]\quad\text{when } n=o$$

All of these are readily evaluated.

Below are given the values of I_1 under all the conditions which may arise:

When $n>o$ and $n-m\neq 2$.

$$I_1=\frac{2\pi\cos n\theta}{n}\left[\rho^{m+2}\left(\frac{1}{m-n+2}-\frac{1}{m+n+2}\right)-\frac{\rho^n a^{m-n+2}}{m-n+2}\right]$$

$$=2\pi\cos n\theta\left[\frac{2\rho^{m+2}}{(m+2)^2-n^2}-\frac{\rho^n a^{m-n+2}}{(m-n+2)\,n}\right]$$

When $n>o$ and $n-m=2$.

$$I_1=\frac{2\pi\rho^n\cos n\theta}{n}\left[\log\rho-\log a-\frac{1}{2n}\right]$$

when $n=o$

$$I_1=\frac{4\pi\rho^{m+2}\log a}{m+2}+\frac{4\pi\,(a^{m+2}-\rho^{m+2})}{(m+2)^2}$$

4. EVALUATION OF THE INTEGRAL I_2

The integral I_2 is given by the following equation where $q \gtreqless \rho_2$

$$I_2=\int_0^a\rho_2 d\rho_2\int_0^{2\pi}\log\left[q^2+\rho_2^2-2q\rho_2\cos(\alpha-\theta_2)\right]\rho_2^m\cos n\,\theta_2 d\theta_2$$

Applying the same methods as under I_1

$$I_2=\int_0^a\rho_2^{m+1}d\rho_2\left[-\frac{2\pi}{n}\left(\frac{\rho_2}{q}\right)^n\cos n\alpha\right],\ \text{when } n>o$$

$$=\int_0^a\rho_2^{m+1}d\rho_2(4\,\pi\log q),\ \text{when } n=o$$

Substituting the values of $\frac{\cos n\alpha}{q^n}$ and of $\log q$ as given above:

$$I_2 = -\frac{2\pi}{n}\int_0^a \rho_2^{m+n+1} d\rho_2 \left[\frac{1}{s^n}\left\{1 + n\frac{\rho}{s}\cos\theta + \frac{n(n+1)}{2!}\left(\frac{\rho}{s}\right)^2 \cos 2\theta + \frac{n(n+1)(n+2)}{3!}\left(\frac{\rho}{s}\right)^3 \cos 3\theta + \cdots\right\}\right]$$

$$= -\frac{2\pi a^{m+n+2}}{n(m+n+2)s^n}\left[1 + n\left(\frac{\rho}{s}\right)\cos\theta + \frac{n(n+1)}{2!}\left(\frac{\rho}{s}\right)^2 \cos 2\theta + \cdots\right]$$

when $n > 0$

$$I_2 = 4\pi\int_0^a \rho_2^{m+1} d\rho_2 \left[\log s - \frac{\rho}{s}\cos\theta - \frac{\rho^2}{2s^2}\cos 2\theta - \frac{\rho^3}{3s^3}\cos 3\theta - \cdots\right]$$

$$= \frac{4\pi a^{m+2}}{m+2}\left[\log s - \frac{\rho}{s}\cos\theta - \frac{\rho^2}{2s^2}\cos 2\theta - \frac{\rho^3}{3s^3}\cos 3\theta - \cdots\right]$$

when $n = 0$.

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