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A GRAPHICAL METHOD FOR CALCULATING GROUND REFLECTION COEFFICIENTS

By Newbern Smith

In problems involving radiation from and reception on radio antennas it is usually necessary to obtain the plane-wave reflection coefficients of the earth for vertically and horizontally polarized waves. These are designated as R_V and R_H, respectively.

Since the earth is an imperfectly conducting dielectric the effective dielectric constant is a complex number. Hence the refractive index is also a complex number, the imaginary part of which is a measure of the absorption of the wave in theground, and of the phase change of the wave upon reflection from the ground. The evaluation of RV and RH is thus a rather tedious process, involving transformations of complex numbers. Under certain simplifying assumptions, however, which are justified to the degree of accuracy with which the ground constants are known, a graphical representation of RV and R_H can be made, which assists in their calculation.

The reflection coefficients of the earth, for a plane wave whose direction of propagation makes a vertical angle θ with the ground, are, for the vector directions of Fig. 1:

$$R_V = \frac{u\sqrt{1-u^2\cos^2\theta} - \sin\theta}{u\sqrt{1-u^2\cos^2\theta} + \sin\theta}$$ (for an electric vector in the plane of propagation)

and

$$R_H = \frac{u\sin\theta - \sqrt{1-u^2\cos^2\theta}}{u\sin\theta + \sqrt{1-u^2\cos^2\theta}}$$ (for an electric vector perpendicular to the plane of propagation).

For ground reflection these refer to vertically and horizontally polarized waves, respectively. In the formulas:

$$u^2 = \frac{1}{\epsilon - jx} = (\alpha + j\beta)^2$$

ϵ = specific inductive capacity of earth

$$x = \frac{18000\,\gamma}{f}$$

γ = conductivity of earth in mhos per meter.

(over)

For most types of soil u^2 is not greater then 0.2; less than 5% error is introduced into $1 \pm R_V$ and $1 \pm R_H$ by neglecting $u^2\cos^2\theta$ in comparison with 1. (A considerably greater error may occur in RV near Brewster's angle, however, for high frequencies and low conductivities).

Thus, to a good approximation, we may consider:

$$R_V = \frac{u - \sin\theta}{u + \sin\theta}, \quad 1 + R_V = \frac{2u}{u + \sin\theta} \quad 1 - R_V = \frac{2}{u\csc\theta + 1}$$

$$R_H = \frac{u\sin\theta - 1}{u\sin\theta + 1}, \quad 1 + R_H = \frac{2u}{u + \csc\theta} \quad 1 - R_H = \frac{2}{u\sin\theta + 1}$$

The graphical construction shown in Fig. 2 illustrates the relations between numerators and denominators of R_V and R_H. Let OA be unit, AF and AF' be equal to $\frac{1}{|u|}$ and ϕ = the phase angle of $\frac{1}{u}$. Lay off the length $n = \sin\phi$ perpendicular to FAF'; then $m = \cos\phi$. Draw FD and F'D' parallel to OA and equal to unity.

If $AB = AB' = \frac{1}{|u|}\sin\theta$, then $\frac{OB}{OB'} = |R_V|$ and angle BOB' = phase of R_V.

If $DC = D'C' = \sin\theta$, then $\frac{OC}{OC'} = |R_H|$ and angle COC' = phase of R_H.

Now $OB^2 = (AB - m)^2 + n^2$ and $OB'^2 = (AB + m)^2 + n^2$

Tofind the minimum value of $\frac{OB}{OB'}$ set

$$\frac{\partial}{\partial AB}\log\frac{OB^2}{OB'^2} = 0 = \frac{2(AB-m)}{(AB-m)^2+n^2} - \frac{2(AB+m)}{(AB+m)^2+n^2} = \frac{2(AB-m)}{AB^2-2AB + 1} - \frac{2(AB+m)}{AB^2 + 2ABm + 1}$$

This will be the case if:

$$2(AB-m)(AB^2+2ABm + 1) = 2(AB+m)(AB^2 - 2ABm + 1)$$

i.e., if AB = 1 and therefore = OA.

Now when AB = AB' = OA, angle BOB' = 90° and $|u| = \sin\theta$

Therefore R_V = minimum and its phase = 90° when $|u| = \sin\theta$

The minimum value of R_V is therefore given by:

$$R_{min} = \frac{\alpha - \sqrt{\alpha^2 + \beta^2} + j\beta}{\alpha + \sqrt{\alpha^2 + \beta^2} + j\beta} = \frac{\alpha^2 - (\alpha^2 + \beta^2) + 2j\beta\sqrt{\alpha^2 + \beta^2} + \beta^2}{\alpha^2 + (\alpha^2 + \beta^2) + 2\alpha\sqrt{\alpha^2 + \beta^2} + \beta^2}$$

$$= \frac{j\beta}{\alpha + \sqrt{\alpha^2 + \beta^2}} = \frac{\beta}{\alpha + \sqrt{\alpha^2 + \beta^2}} \angle 90^\circ$$

Now ϕ_0 the phase angle of $\frac{1}{u}$, $= -\tan^{-1}\frac{\beta}{\alpha} = -\sin^{-1}\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} = \cos^{-1}\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}$

$$\left|\tan \tfrac{1}{2}\phi\right| = \frac{\dfrac{\beta}{\sqrt{\alpha^2 + \beta^2}}}{1 + \dfrac{\alpha}{\sqrt{\alpha^2 + \beta^2}}} = \left|R_{min}\right|$$

Further,

$$1 - R_V = \frac{2\sin\theta}{u + \sin\theta} = \frac{2\sin\theta}{\sqrt{(\alpha + \sin\theta)^2 + \beta^2}} \angle\phi_1 \quad \text{where } \phi_1 = -\tan^{-1}\frac{\beta}{\alpha \sin\theta}$$

and, if R_{V0} = value of R_V for $\theta = 0$

$$1 - R_{V0} = 0 \angle\phi_0 \quad \text{where } \phi_0 = -\tan^{-1}\frac{\beta}{\alpha}$$

Therefore

$$\sin(\phi_1 - \phi_0) = \frac{\beta(\alpha + \sin\theta) - \alpha\beta}{\sqrt{(\alpha + \sin\theta)^2 + \beta^2}\sqrt{\alpha^2 + \beta^2}} = \frac{\sin\theta}{\sqrt{(\alpha + \sin\theta)^2 + \beta^2}} \cdot \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}$$

$$= \frac{\beta}{2\sqrt{\alpha^2 + \beta^2}} \left|1 - R_V\right|$$

and thus $1 - R_V$ lies on a circle in the complex plane, and therefore R_V itself lies on a circle.

Similarly,

$$1-R_H = \frac{2}{u \sin\theta + 1} = \frac{2}{\sqrt{(\alpha \sin\theta + 1)^2 + \beta^2 \sin^2\theta}} \angle \phi_2 \quad \text{where } \phi_2 = -\tan^{-1} \frac{\beta \sin\theta}{\alpha \sin\theta + 1}$$

and so it may be shown that

$$\sin(\phi_2 - \phi_0) = \frac{\beta}{2\sqrt{\alpha^2 + \beta^2}} \left| 1 - R_H \right|$$

so that $1 - R_H$, and hence also R_H, lies on the same circle as R_V.

This leads at once to the obvious graphical construction for R_V and R_H shown in Fig. 3. On the perpendicular bisector of a line of length 2 lay off the length $R_{min} = \tan \frac{1}{2} \phi$ upward. From the top of this length drop down a distance $\rho = \frac{1 + R^2_{min}}{2R_{min}}$ to locate the center of a circle of radius ρ, and draw the arc of this circle of which the base line of length 2 is the chord.

To find R_V lay off the angle $\psi_1 = 2\phi_1 = 2 \tan^{-1} \frac{\beta}{\alpha + \sin\theta}$ and draw R_V from the center of the base line to the circle as shown. To find R_H lay off the angle $\psi_2 = 2\phi_2 = 2 \tan^{-1} \frac{\beta \sin\theta}{\alpha \sin\theta + 1}$ and draw R_H in the above manner. This comes about from that the fact that the arc ψ is subtended by the phase angle of $1 - R$ and is therefore twice that angle. The nomograms of Figs. 4, 5, & 6 are attached for ready calculation of ψ_1 and ψ_2.

The quantity $1-R$ is found by drawing a line from A to the point where R intersects the circle, and the phase of $1-R$ is minus the angle measured clockwise from line AB to the line $1-R$. The quantity $1 + R$ is found by drawing a line from B to the point where R intersects the circle, and the phase angle is that measured counter-clockwise from line AB to the line $1 + R$.

It was mentioned above that the greatest error is likely to occur in the values of R_V near the Brewster angle θ, i.e., the value of θ for which R_V is a minimum. This error is largely in the angle θ_B itself rather than in the variation of R_V with θ. A better value of θ may be calculated from the formula

$$\theta_B = \sin^{-1} u \left[1 - \frac{u^2}{2(1 + |u|^2)} \right],$$

if more precise values of R_V near Brewster's angle are required.

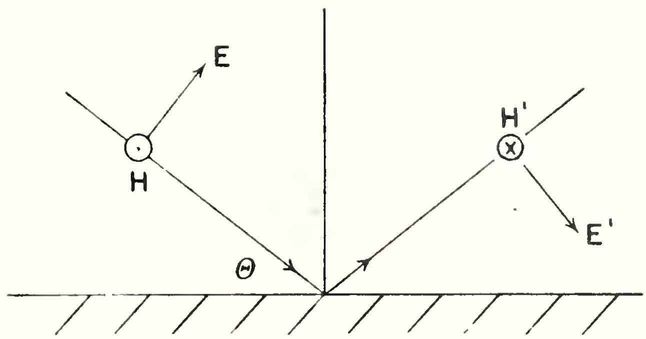
Although this method of determining R_H and R_V is quantitatively valid only for materials with dielectric constants and conductivities, the representation of R_H and R_V as vectors lying on a single curve, as a Fig. 3, is generally useful, for example in optics, in showing how the reflection coefficients vary with angle. In the case of a perfect reflector, both the R_H and R_V vectors lie at the left-hand end of the curve (point B) for all values of θ except that R_V is indeterminate at $\theta - 0$.

In the case of a perfect dielectric ($x = 0$)

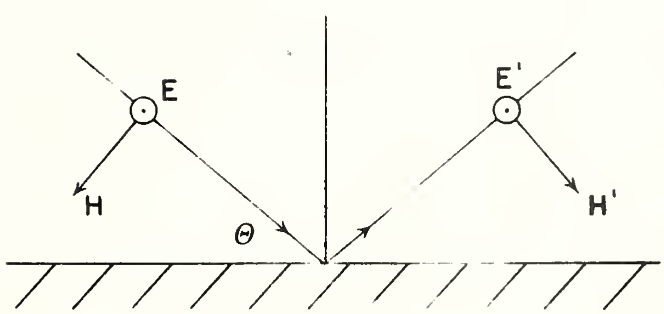
$$u = \sqrt{\frac{1}{\epsilon}}, \quad \alpha = \sqrt{\frac{1}{\epsilon}}, \quad \beta = 0, \quad R_{min} = \lim_{\epsilon \to 0} \frac{\epsilon}{2\alpha}, \quad \rho = \lim_{\epsilon \to 0} \frac{\alpha}{\epsilon}$$

and $1 = \lim_{\epsilon \to 0} \dfrac{2\epsilon}{\alpha + \sin\theta}$ $\quad 2 = \lim_{\epsilon \to 0} \dfrac{2\epsilon}{\alpha + \cos\theta}$ and the curve of Fig. 3 is a straight line between B and A and R_V and R_H have phases of either 0° or 180°.

Thus $\rho\Psi_1 = \dfrac{2\alpha}{\alpha + \sin\theta}$, and $\rho\Psi_2 = \dfrac{2\alpha}{\alpha + \csc\theta}$, represent the distances from B along the straight line of Fig. 2 to the ends of the vectors R_V and R_H respectively. At $\theta = 90^0$ $\rho\Psi_1 = \rho\Psi_2 = \dfrac{2\alpha}{1+\alpha}$ and R_V, $R_H - \dfrac{1-\alpha}{1+\alpha}$ $= \dfrac{\sqrt{\epsilon} - 1}{\sqrt{\epsilon} + 1}$ as is the usual formula for normal incidence. At Brewster's angle $\sin\theta = \alpha$ and $\rho\Psi_1 = 1$ giving $R_V = 0$. For θ less than Brewster's angle $\rho\Psi_1 > 1$ and so R_V lies to the right and has a phase 0°; for θ greater than Brewster's angle lies to the left, and for any angle R_H lies to the left, and so has a phase of 180°.



A. FOR VERTICALLY POLARIZED ELECTRIC FIELD.



B. FOR HORIZONTALLY POLARIZED ELECTRIC FIELD.

Fig. 1. DIRECTION CONVENTIONS FOR ELECTRIC AND MAGNETIC VECTORS.
⊙ = DIRECTED OUT OF PAPER.
⊗ = DIRECTED INTO PAPER.

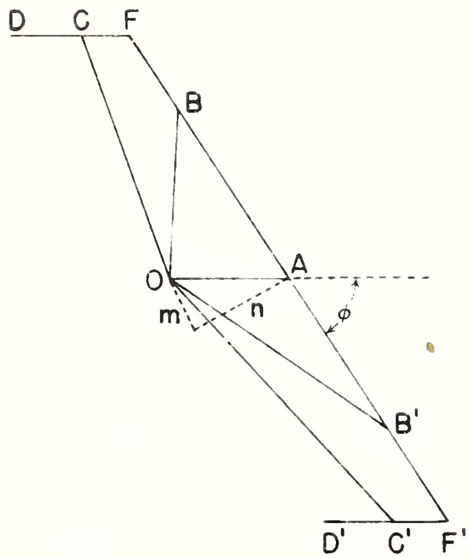


Fig. 2. GRAPHICAL REPRESENTATION OF NUMERATORS AND DENOMINATORS OF R_V AND R_H.

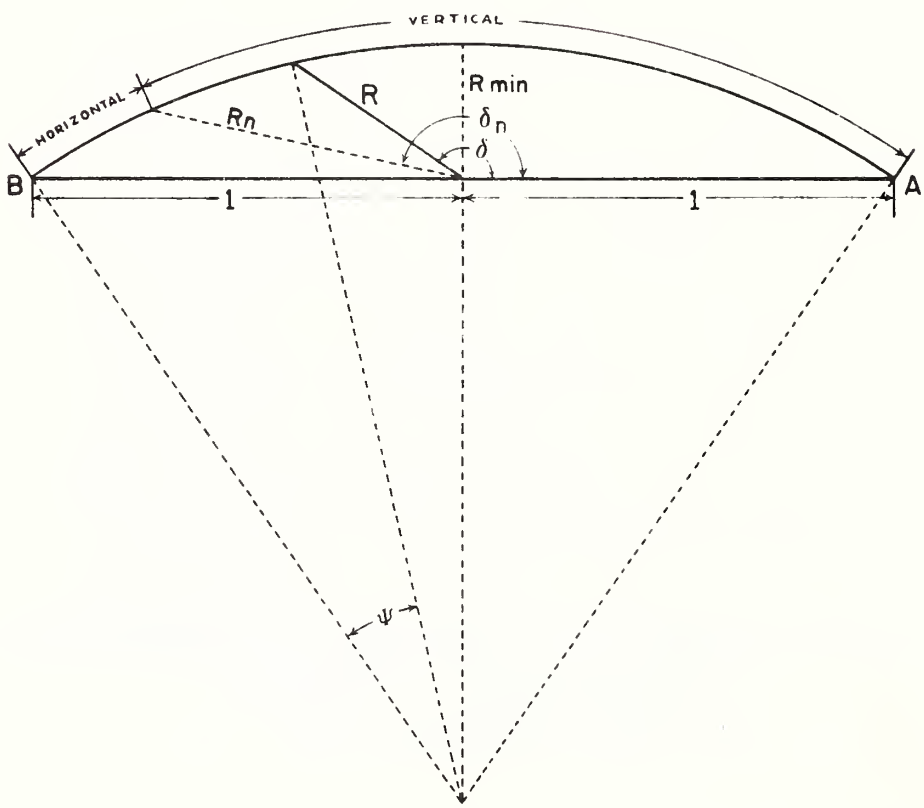
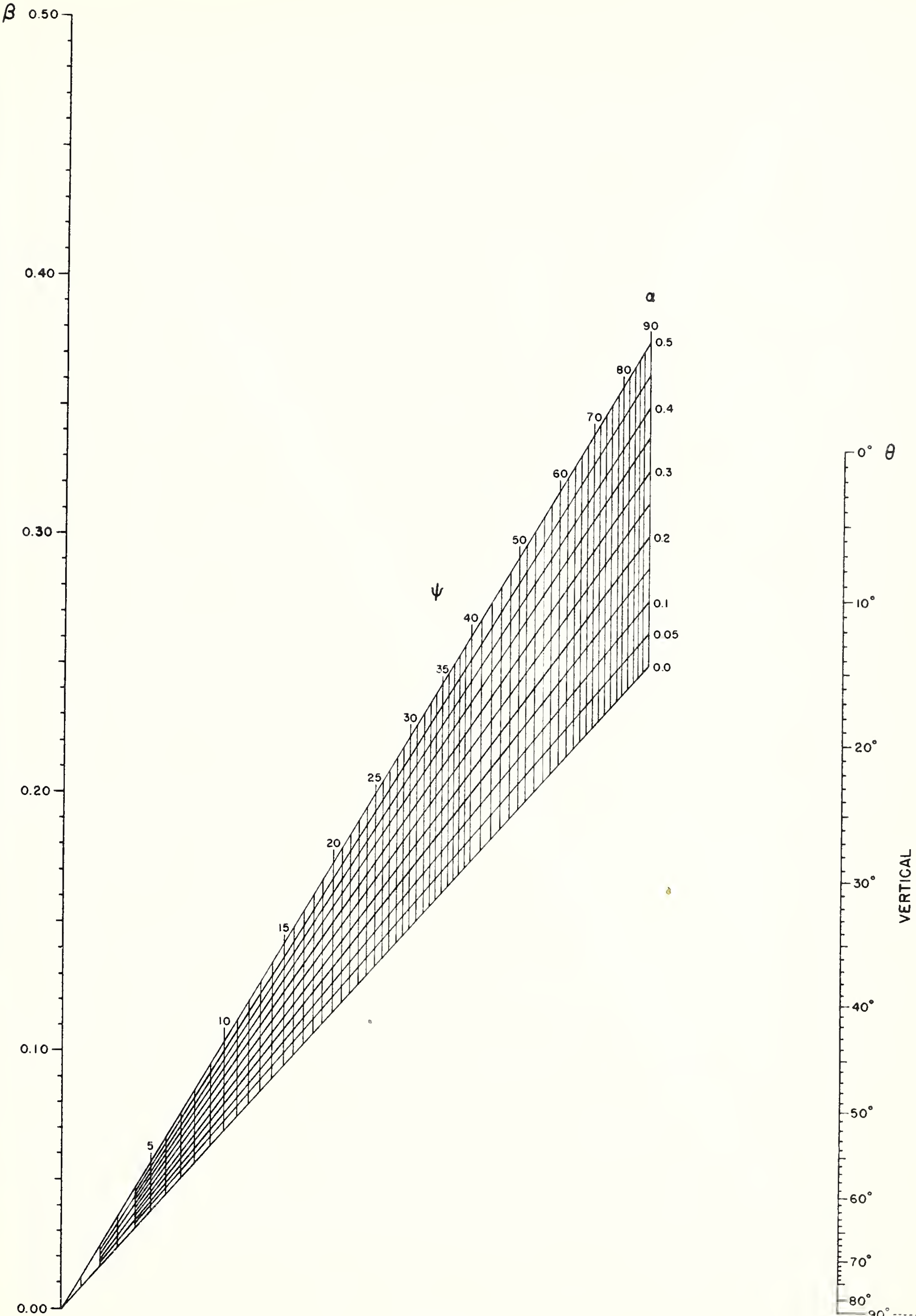


Fig. 3. GRAPHICAL CONSTRUCTION FOR R_V AND R_H.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80°
90°
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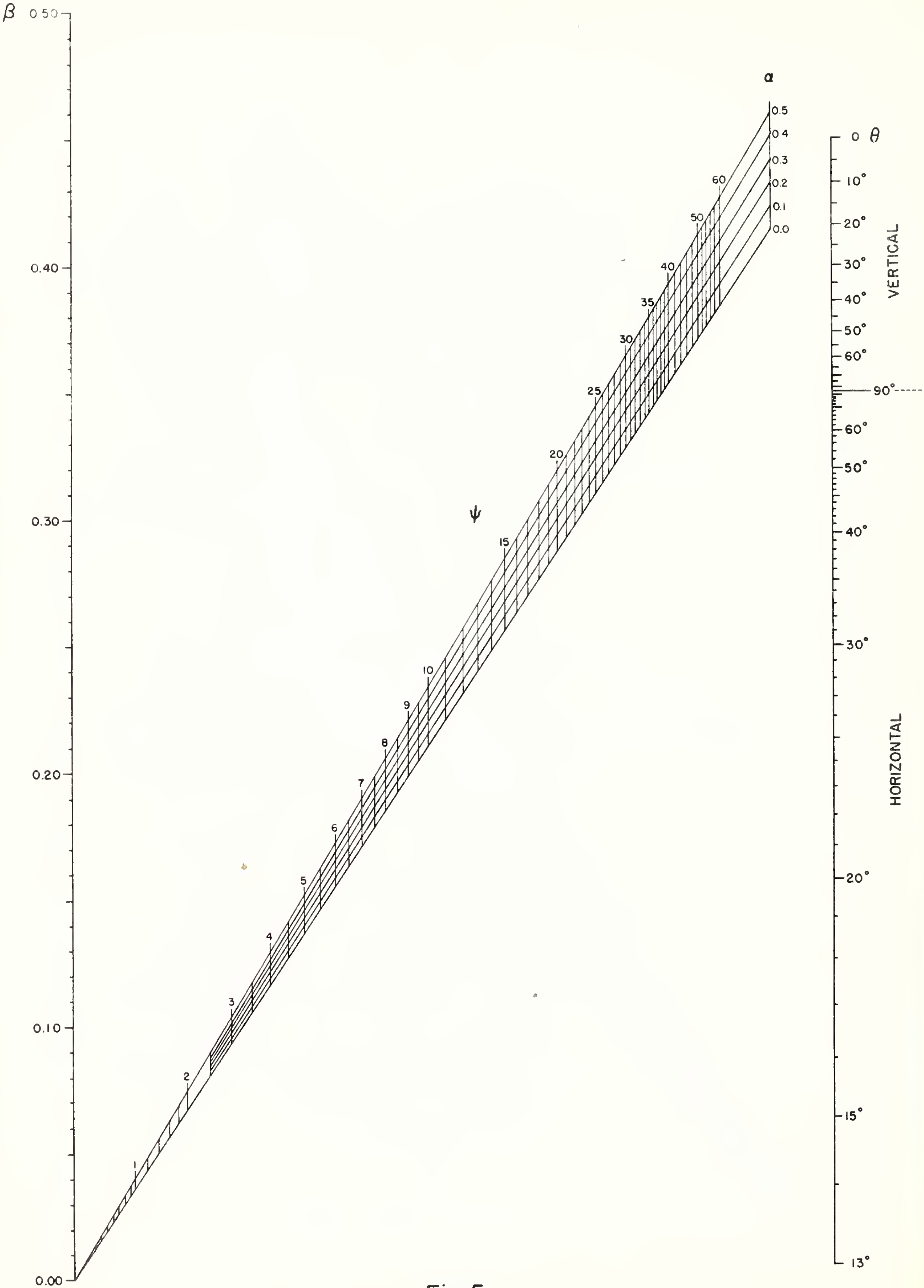
Fig. 4.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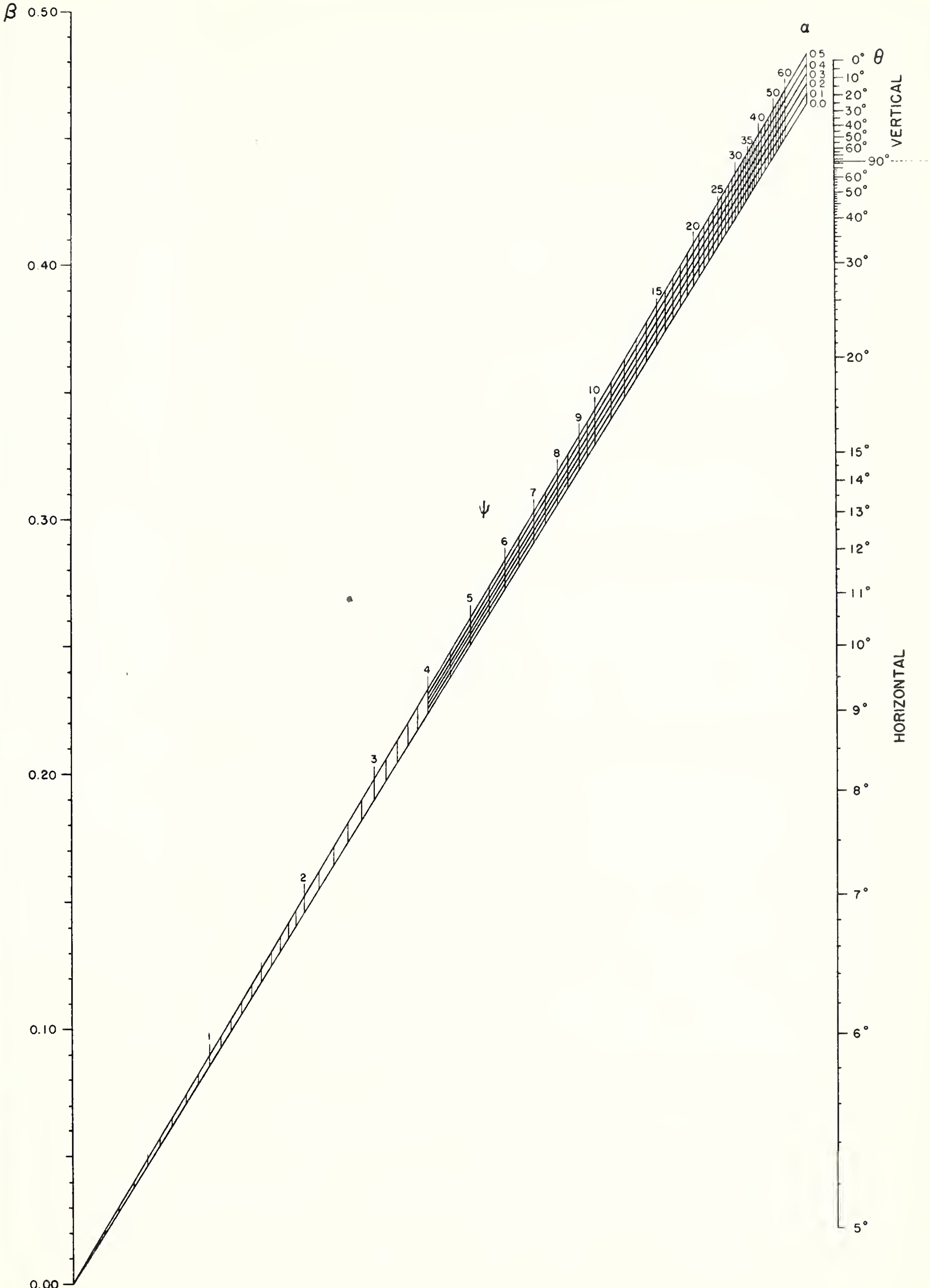
VERTICAL
HORIZONTAL

Fig. 5.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9°
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7°
6°
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VERTICAL
HORIZONTAL

Fig. 6.