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On the Theory of Fading Properties of a Fluctuating Signal Imposed on a Constant Signal

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On the Theory of Fading Properties of a Fluctuating Signal Imposed on a Constant Signal

H. Bremmer *

This paper deals with a theoretical investigation of the fading properties of a signal composed of a fluctuating contribution, and another steady contribution with fixed amplitude and phase. It is assumed that the central-limit theorem may be applied to two proper quantities describing the fading signal as a quasi-monochromatic function of the time. The results are applicable to any autocorrelation function for the fluctuating contribution.

The first part of the paper (sections 1 to 16) is mainly restricted to the idealized case in which any two components of the fluctuating part of the complete signal that are in quadrature with respect to their phase do have identical statistical properties; the fluctuating part is then termed a "random" signal. This idealized case is shown to constitute but an approximation if applied to the fluctuating field due to first order scattering in a turbulent atmosphere. Therefore, in the second part of the paper (sections 17 to 26) the theory has been extended to fluctuating contributions (then termed "quasi-random" contributions) not satisfying the above condition of isotropy. All results then depend on two complex correlation functions $a(\tau)$ and $b(\tau)$ instead of on the single function $a(\tau)$ governing the simplified theory. In contrast to $a(\tau)$, the function $b(\tau)$ does not exclusively depend on the energy spectrum of the fluctuating contribution.

The fading properties investigated for the composed signal are the distribution functions of both the amplitude and phase, as well as the average number of crossings of each of them (per unit time interval) through any given level. The complicated general formulas reduce to simple expressions in the two extreme cases of (a) absence of the steady signal (e. g., tropospheric scatter propagation to distances far beyond the transmitter's horizon), and (b) predominance of the steady signal (e. g., line of sight propagation to distances well within the horizon). The first limiting case leads, when neglecting the influence of $b(\tau)$ (then being very small), to the well-known Rayleigh distribution for the amplitude, to a homogeneous distribution for the phase, and to a fading rapidity (defined as the average number of crossings per unit time interval through the level most frequently passed) which is for the amplitude faster than for the phase by a factor of 3.04. The second limiting case amounts to normal distributions for both amplitude and phase, and to fading rapidities of these quantities which are only identical insofar as the different behaviour of the in-phase and in-quadrature component (with respect to the steady signal) does have no numerical importance. On the other hand, measurements of the difference in the amplitude-fading rapidity and the phase-fading rapidity will reveal the effect of the asymmetry with respect to the two mentioned components of the scattered signal.

1. Introduction

The simplest statistical property of a fluctuating quantity is its *distribution function* $F(a)$. It is defined as follows in the case of the amplitude a of a hf signal

$$P(a>a_0)=F(a_0).$$

The left-hand side represents the probability for an amplitude $a>a_0$. The total range of amplitudes being $0<a_0<\infty$, we have the extreme values $F(0)=1$ and $F(\infty)=0$. The derivative of $F(a)$ amounts to the *probability density* $p(a)=F'(a)$; the quantity $p(a)\,da$ then constitutes the probability for an amplitude situated in the interval between a and $a+da$.

Neither $F(a)$ nor $p(a)$ yields any information about the rapidity of fading. The latter is characterized, e. g., by the average number of crossings per second, $N(a)$ say, of the amplitude through a specified level a. Such a quantity is more complicated since a time interval enters as a new parameter. Other yet more intricate quantities, e. g., the average number of extreme values per second, will not be considered in this paper. Moreover, our analysis will be restricted to cases involving the two following idealized circumstances:

(A) Signals the amplitude of which satisfies a *Rayleigh distribution* (random signal). The latter is given by

$$p(a)=\frac{2}{<a^2>}\,a\exp\left\{-\frac{a^2}{<a^2>}\right\}, \tag{1}$$

where the symbol $<\quad>$ henceforth denotes average values; in the case under consideration we have

$$<a^2>=\int_0^\infty da\,p(a)\,a^2.$$

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The Rayleigh distribution constitutes the asymptotic approximation of the distribution for the distance from the starting point in the two-dimensional random-walk problem. It is therefore to be expected, if the field originates from the incoherent addition of individual contributions having well-defined amplitudes, but mutual phase differences $>>2\pi$ which may be considered as random. In fact, each individual (monochromatic) contribution can here be represented in a plane by an arrow the length and orientation of which correspond to its modulus and phase, respectively. The distribution of the modulus of the sum of n contributions (represented by n arrows of fixed lengths but random directions) then approaches (1) for large values of n.

Practical realizations of this model are:

(A1) Tropospheric signals over distances well beyond the transmitter's horizon that can be ascribed to forward scattering by turbulent atmospheric blobs. The nonfluctuating background signal due to diffraction is negligible here. The contributions from the various incoherently scattering blobs here constitute the elements having random phases;

(A2) Ionospheric signals at frequencies above the muf (frequencies not propagated by ordinary reflections). The field to be considered originates exclusively from forward scattering by turbulent blobs in the ionosphere;

(B) Signals composed of the sum of a steady component and another quasi-random component the distribution of which (see eq 92) constitutes an extension of a Rayleigh distribution. We mention the following practical realizations:

(B1) Tropospheric line-of-sight propagation. The steady component results from the ideally stratified troposphere, and usually consists of the vectorial sum of the fields associated with the direct ray and a ray reflected against the earth. The quasi-random component arises from the extra amount produced by the scattering turbulent blobs, this amount being the only one present in the above case (A1);

(B2) Single-mode ionospheric one-hop transmission. The steady signal is produced by ordinary reflection against an ideally stratified ionosphere, the Rayleigh-distributed signal by the additional scattering due to ionospheric turbulence.

List of Symbols

$<\alpha>$ = average value of α

$P(a>a_0)$ = probability for a signal a to surpass a level a_0,

$p(a)$ = probability density of variable a,

$h(t)$ = fluctuating signal,

ω_0 = angular frequency of carrier,

$x(t)$ = component of fluctuating signal in phase with $\cos(\omega_0 t)$,

$y(t)$ = component of fluctuating signal in quadrature with $\cos(\omega_0 t)$,

$f(x_1, \ldots, x_n)$ = joint probability density for orthogonal variables,

$F(\omega)$ = Fourier transform of fluctuating signal $h(t)$,

$W(\omega)$ = energy spectrum of fluctuating signal $h(t)$,

$\tau = t_2 - t_1$ = separation in time between two moments of observation t_1 and t_2,

A = matrix of the coefficients of the bilinear form in exponent of central-limit theorem,

Λ = covariance matrix,

Λ^{-1} = reciprocal matrix of Λ,

$|\Lambda|$ = determinant of matrix Λ,

$a(\tau)$ = complex correlation function defined by second term of (12), or by (69),

$b(\tau)$ = complex correlation function defined by first term of (12), or by (69),

z = modulus of quantity represented by (17),

ψ = phase of quantity represented by (17),

R_0 = amplitude of steady signal proportional to $\cos(\omega_0 t)$,

$G(R, \dot{R}, \Phi, \ldots)$ = joint probability density for polar variables (R = amplitude; Φ = phase),

$\bar{\omega}$ = average frequency with respect to energy spectrum of the fluctuating signal,

σ_ω = rms value of frequency with respect to energy spectrum of the fluctuating signal,

$\dot{x}, \ddot{R}$, etc. = time derivatives of x, R, etc.,

I_0 = Bessel function of order zero and imaginary argument,

$N_A(a)$=average number of crossings (per unit time interval) of amplitude through level a,
$N_\Phi(\varphi)$=average number of crossings (per unit time interval) of phase through level φ,
TQ, QP, TP=mutual distances between T (transmitter), Q (arbitrary, scattering point) and P (receiver),
$\delta n(Q, t)$=deviation of refractive index at point Q and at the time t,
l=scale of turbulence,
M_{ik}=cofactor of Λ with respect to its element Λ_{ik},
l_{ik}=coefficient of τ^2 term in expansion of M_{ik} with respect to τ,
m_{ik}=coefficient of τ^4 term in expansion of M_{ik} with respect to τ,
$H^2=\langle h^2\rangle$=variance of fluctuating signal,
$H(t)$=sum of steady signal and fluctuating signal.

2. Representation of a Quasi-Monochromatic Signal

The scattering effects described above produce, if energized by an originally monochromatic hf signal, a quasi-monochromatic signal. The latter can be represented as follows,

$$h(t)=x(t)\cos(\omega_0 t)+y(t)\sin(\omega_0 t)=\mathrm{Re}[\{x(t)+iy(t)\}\exp(-i\omega_0 t)], \tag{2}$$

in which $x(t)$ and $y(t)$ change little over 1 cycle of duration $2\pi/\omega_0$.

We introduce the Fourier transform $F(\omega)$ of $h(t)$ according to

$$h(t)=\int_{-\infty}^{\infty} d\omega\, F(\omega)\, e^{i\omega t}. \tag{3}$$

The main statistical properties to be derived are connected with the so-called *energy spectrum* given by

$$W(\omega)=|F(\omega)|^2=F(\omega)F^*(\omega)=F(\omega)F(-\omega). \tag{4}$$

The fluctuating signal $h(t)$ constitutes the *complete* fluctuating function in case A which itself is characterized moreover, by the absence of any significant difference between the statistical properties of $x(t)$ and $y(t)$; the corresponding function $h(t)$ is henceforth termed random. On the other hand, the function $h(t)$ only constitutes a fluctuating component imposed on another steady component in case B; the anisotropy with respect to $x(t)$ and $y(t)$, which may exist in this latter case, leads to a quasi-random function $h(t)$ which is more general than the corresponding random function of case A.

The negative frequencies occurring in the Fourier synthesis (3) can be reduced to positive ones by the substitution $\omega=-\omega'$ for $\omega<0$. The contribution from the negative frequencies then proves to be the complex conjugate of that from the originally positive ones. This results in the following alternative representations of our fluctuating component:

$$h(t)=2\,\mathrm{Re}\left\{\int_0^\infty d\omega F^*(\omega)e^{-i\omega t}\right\}$$
$$=\mathrm{Re}\left\{2\int_0^\infty d\omega F^*(\omega)e^{-i(\omega-\omega_0)t}\cdot e^{-i\omega_0 t}\right\}$$

A comparison of the last expression and (2) shows that the two *real* functions $x(t)$ and $y(t)$ could be defined by the real and imaginary part of the following complex relation,

$$x(t)+iy(t)=2\int_0^\infty d\omega F^*(\omega)e^{-i(\omega-\omega_0)t}. \tag{5}$$

This leads to reasonable definitions (to be maintained henceforth) for $x(t)$ and $y(t)$, though not to the only possible ones. In fact, the single original function $h(t)$ can not involve unique definitions for both $x(t)$ and $y(t)$ without introducing additional conditions.

At any frequency ω, the time t being fixed, we consider the ensemble of configurations that differ only with respect to the phase of $F(\omega)$. We then obtain zero values for the ensemble averages $\langle x\rangle$ and $\langle y\rangle$, provided we merely assume equal probabilities for any two transforms $F(\omega)$ for which arg $F(\omega)$ differs by π.

3. Application of the Central-Limit Theorem

The "orthogonal" quantities $x(t)$ and $y(t)$ may be compared with the corresponding "polar" quantities $\rho(t)$ and $\Phi(t)$ connected with the former according to

$$x(t)+iy(t)=\rho(t)\cdot\exp\{i\Phi(t)\}.$$

The relation (2) then transforms into

$$h(t)=\mathrm{Re}[\rho\exp\{-i(\omega_0 t-\Phi)\}]. \tag{6}$$

The amplitude $\rho(t)$ and the phase $\Phi(t)$ have direct practical importance. However, our analysis will be worked out for x and y first, since these quantities are simpler and have zero averages (as mentioned above) at any time t, both for the random and the quasi-random fluctuating component.

All four quantities

$$\left.\begin{matrix} x_1=x(t_1), & x_2=x(t_2)=x(t_1+\tau), \\ y_1=y(t_1), & y_2=y(t_2)=y(t_1+\tau) \end{matrix}\right\} \tag{7}$$

therefore also have zero averages, which is important when applying the so-called central limit theorem to the probability density of the quantities in question. This theorem presumes a normal law for the distribution of each of the above quantities (7). This normal law is known to hold, under very general conditions, for limits obtained from the addition of a large number of independent random vectors; in our case the latter may be realized by the two-dimensional vectors formed by the real and imaginary parts of those contributions to (5) that result from narrow frequency intervals. The applicability of the central limit theorem then is justified by assuming independent phases of $F(\omega)$ in the various ω intervals. We further point to the analogy existing between the quasi-monochromatic signal $h(t)$ under consideration and that describing white noise which has passed through a filter with a frequency characteristic given by the energy spectrum $F(\omega)$. In view of this analogy, Booker, Ratcliffe, and Shinn [1] [1] could deduce fading properties, such as discussed in this article, from Rice's [2] theory for white noise; the latter theory also uses the central limit theorem.

All fading properties to be discussed can be derived from the joint probability density $f(x_1, x_2, x_3, x_4)$ defined such that $f\,dx_1\,dx_2\,dx_3\,dx_4$ represents the probability of finding x_1, x_2, x_3, x_4 in prescribed infinitesimal intervals dx_1, dx_2, dx_3, dx_4. The central limit theorem, assumed to hold for the above reasons, then states that f should be proportional to an exponential the exponent of which is a bilinear form in the four mentioned variables.

The coefficients of the bilinear form determining f can be connected with the so-called second-order moments or covariance elements $\langle x_1x_2\rangle$, $\langle x_1x_3\rangle$, etc. This connection will be indicated in the next section for an arbitrary number of variables, instead of the four variables associated with our fading problem.

4. Evaluation of the Coefficients in the Central-Limit Theorem in Terms of the Covariance Elements

Let $x_1, x_2, \ldots, x_n$ be n real random variables with zero mean values and a joint probability density satisfying the central-limit theorem. The probability $f\,dx_1\,dx_2\ldots dx_n$ of having $x_1, \ldots, x_n$ in specified infinitesimal intervals then depends, under very general conditions, on a function of the form

$$f=C\exp\left\{-\sum_{i=1}^{n}\sum_{k=1}^{n}A_{ik}x_ix_k\right\}. \tag{8}$$

It is well known that both C and the coefficients A_{ik} can be expressed in terms of the second-order moments

$$\Lambda_{ik}=\langle x_ix_k\rangle=\int dx_1\int dx_2\ldots\int dx_n\cdot f(x_1,\ldots,x_n)x_ix_k. \tag{9}$$

[1] Figures in brackets indicate the literature references at the end of this Circular.

These averages, the existence of which is to be assumed in view of the applicability of the central-limit theorem, constitute the elements of the so-called covariance matrix Λ. The evaluation of (9), together with that of the normalization condition

$$\int dx_1 \int dx_2 \ldots \int dx_n \cdot f(x_1, \ldots, x_n) = 1,$$

leads to relations from which C and all coefficients A_{ik} can be derived. The final explicit formula [3] for the probability density (8) then proves to read

$$f = \frac{\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}(\Lambda)_{ik}^{-1}x_i x_k\right\}}{(2\pi)^{n/2}|\Lambda|^{1/2}}, \tag{10}$$

in which $(\Lambda)_{ik}^{-1}$ represents the ik element of the reciprocal Λ^{-1} of the covariance matrix, and $|\Lambda|$ the determinant of the original matrix Λ.

5. Covariance Matrix of the Variables (7) for a Random Signal Expressed in Terms of the Energy Spectrum $W(\omega)$

The matrix elements referring to the signal (2), such as $\langle x_1x_2\rangle$ and $\langle x_1y_1\rangle$, can be connected with the Fourier transform $F(\omega)$ of (3). It is sufficient to investigate the quantity $\langle x_1(x_2+iy_2)\rangle$. In fact, its real and imaginary part yield at once $\langle x_1x_2\rangle$ and $\langle x_1y_2\rangle$, while the other relevant elements can be obtained by interchanging the roles of the subscripts 1 and 2, or by letting x_2, y_2 approach x_1, y_1.

The evaluation of $\langle x_1(x_2+iy_2)\rangle$ will be based on the assumption that our ensemble associated with the various possible functions for the phase of $F(\omega)$ should be ergodic. This leads to the following representation of the mentioned quantity as the limit of a time average over a large interval $2T$

$$\langle x_1(x_2+iy_2)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt\, x(t)\{x(t+\tau)+iy(t+\tau)\}.$$

The limit exists since our definitions of $x(t)$ and $y(t)$ according to (5) involve a well defined asymptotically linear increase of the integral itself with T if $F(\omega)$ has been given. As a matter of fact, the integral proves (see eq 12) to depend linearly on the product of two factors $F(\omega)$ each of which increases porportional to $T^{\frac{1}{2}}$ if we assume a well defined variance $\langle h^2\rangle = H^2$, say. The latter proportionality follows from Parseval's theorem to be deduced from (3) which reads in the case under consideration

$$\int_{-T}^{T} h^2(t)\, dt = 2\int_{-\infty}^{\infty} d\omega F(\omega)\int_{-\infty}^{\infty} d\omega' F(\omega')\,\frac{\sin T(\omega+\omega')}{\omega+\omega'};$$

for large values of T this tends to

$$\int_{-T}^{T} h^2(t)\, dt = 2\pi\int_{-\infty}^{\infty} d\omega F(\omega)F^*(\omega).$$

The left-hand side can be approximated by $2T\,H^2$ which involves the mentioned proportionality of $F(\omega)$ with $T^{\frac{1}{2}}$. In view of these remarks we arrive at the following approximation for large T

$$\langle x_1(x_2+iy_2)\rangle = \frac{1}{2T}\int_{-T}^{T} dt\, x(t)\{x(t+\tau)+iy(t+\tau)\}. \tag{11}$$

However, we emphasize that the explicit dependence on T does drop out in all *ratios* of averages of the type (11), while our results will be expressed in terms of such very ratios. The T values occurring in the numerator and denominator of the latter should then be identical before passing to the limit for $T\to\infty$.

We can substitute half the sum of (5) and of the corresponding conjugated complex quantity for the first factor of the integrand of (11), whereas the second factor is obtained by replacing t in (5) by $t+\tau$. Hence

$$<x_1(x_2+iy_2)>=\frac{1}{T}\int_{-T}^{T}dt\left[\int_0^\infty d\omega F^*(\omega)e^{-i(\omega-\omega_0)t}+\int_0^\infty d\omega F(\omega)e^{i(\omega-\omega_0)t}\right]\int_0^\infty d\omega' F^*(\omega')e^{-i(\omega'-\omega_0)(t+\tau)},$$

or, after changing the orders of integration (which should not affect the result)

$$<x_1(x_2+iy_2)>=\frac{1}{T}\int_0^\infty d\omega F^*(\omega)\int_0^\infty d\omega' F^*(\omega')e^{-i(\omega'-\omega_0)\tau}\int_{-T}^{T}dt e^{-i(\omega+\omega'-2\omega_0)t}$$

$$+\frac{1}{T}\int_0^\infty d\omega F(\omega)\int_0^\infty d\omega' F^*(\omega')e^{-i(\omega'-\omega_0)\tau}\int_{-T}^{T}dt e^{i(\omega-\omega')t}.$$

In view of the large value of T we replace the integration limits $\pm T$ by $\pm\infty$ while applying the relation

$$\int_{-\infty}^{\infty}dt e^{i\alpha t}=2\pi\delta(\alpha).$$

We thus obtain

$$<x_1(x_2+iy_2)>=\frac{2\pi}{T}\int_0^\infty d\omega F^*(\omega)\int_0^\infty d\omega' F^*(\omega')e^{-i(\omega'-\omega_0)\tau}\delta(\omega'+\omega-2\omega_0)$$

$$+\frac{2\pi}{T}\int_0^\infty d\omega F(\omega)\int_0^\infty d\omega' F^*(\omega')e^{-i(\omega'-\omega_0)\tau}\delta(\omega-\omega').$$

The singular point of the delta function in the first term, that is $\omega'=2\omega_0-\omega$, is situated inside the integration interval $0<\omega<\infty$ only if $\omega<2\omega_0$. Therefore, the first integral in this term reduces to the interval $0<\omega<2\omega_0$. No such reduction of the integration interval occurs for the second term. Hence the integrations with respect to ω' yield

$$<x_1(x_2+iy_2)>\sim\frac{2\pi}{T}\int_0^{2\omega_0}d\omega F^*(\omega)\ F^*(2\omega_0-\omega)\,e^{-i(\omega_0-\omega)\tau}+\frac{2\pi}{T}\int_0^\infty d\omega F(\omega)F^*(\omega)e^{-i(\omega-\omega_0)\tau}. \tag{12}$$

According to (4) we can substitute $W(\omega)$ for the nonexponential part of the integrand of the second term. This term is completely independent of the phases of $F(\omega)$. As to the first term, our assumption of phases of $F(\omega)$ that are random for the most part, involves a very low correlation between the phases of $F^*(\omega)$ and $F^*(2\omega_0-\omega)$, provided the arguments ω and $2\omega_0-\omega$ are not almost equal. The latter only occurs at ω near ω_0 and we therefore expect a value of the first term which is lower according as the randomness of the phases of $F(\omega)$ becomes more complete. A perfect randomness would exclude, according to (5), any discrimination between the statistical properties of $x(t)$ and those of $y(t)$. The impossibility of such discrimination applies to our previous concept of a random fluctuating component for which, therefore, we assume the vanishing of the first term of (12). As a matter of fact, our later treatment (in section 18) of quasi-random fluctuating components, which also accounts for this first term, will show how the in general existing anisotropy with respect to the properties of $x(t)$ and $y(t)$ only disappears for a vanishing value of the first term of (12).

Restricting ourselves provisionally to random fluctuations, we put

$$<x_1(x_2+iy_2)>=\frac{2\pi}{T}\int_0^\infty d\omega W(\omega)e^{-i(\omega-\omega_0)\tau}. \tag{13}$$

Obviously, an interchange of the indices 1 and 2 should correspond to replacing τ by $-\tau$. This substitution changes (13) into its conjugated complex value. Hence

$$<x_2(x_1+iy_1)>=\{<x_1(x_2+iy_2)>\}^*,$$

so as to have

$$<x_2y_1>=-<x_1y_2>.$$

Moreover, taking $x_2 \equiv x_1$ and $y_2 \equiv y_1$ has to correspond to $\tau=0$. We then infer from the real and imaginary part of (13), $W(\omega)$ being real

$$
\begin{aligned}
&<x_1^2>=<x_2^2>=\frac{2\pi}{T}\int_0^\infty d\omega W(\omega), \\
&<x_1 y_1>=0.
\end{aligned}
$$

Both moments t_1 and t_2 being arbitrary, we also have $<x_2 y_2>=0$. The symmetry relations here derived are understandable. The slow change of $x(t)$ and $y(t)$ over a quarter cycle involves the following relation when replacing t in (2) by $t+\pi/(2\omega_0)$:

$$
h\left(t+\frac{\pi}{2\omega_0}\right)=-x\left(t+\frac{\pi}{2\omega_0}\right)\cdot \sin(\omega_0 t)+y\left(t+\frac{\pi}{2\omega_0}\right)\cos(\omega_0 t)\sim -x(t)\sin(\omega_0 t)+y(t)\cos(\omega_0 t).
$$

Comparing this representation with the original (2), we infer that the statistical properties should be invariant (in our case of statistical isotropy) with respect to the combined substitutions $x \to y$ and $y \to -x$. Therefore, e. g., $<xy>=<y\,(-x)>$; this implies the property $<xy>=0$, that is, the statistical independence of the value of $h(t)$ at two moments separated by a quarter cycle. Moreover, the invariance with respect to the substitution $x \to y$ involves the other relation $<x_1\, x_2>=<y_1\, y_2>$ for the random fluctuating component. All these symmetry relations could also be derived, assuming the existence of ensemble averages such as $<x_1\, x_2>$, from the condition that the resulting average $<h(t)h(t+\tau)>$ should be independent of the time (condition of stationariness).

6. Explicit Form of the Central-Limit Theorem for the Variables x_1, x_2, y_1, y_2 in the Case of a Random Signal

For the variables in question the elements of the covariance matrix are given by

$$
\Lambda_{ik}=<x_i x_k> \qquad (i=1,2,3,4;\; k=1,2,3,4),
$$

in which x_3 and x_4 are to be identified with y_1 and y_2 respectively. Hence all matrix elements are represented in the following scheme:

$$
\Lambda=\begin{Vmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & \Lambda_{42} & \Lambda_{43} & \Lambda_{44} \end{Vmatrix} = \begin{Vmatrix} <x_1^2> & <x_1x_2> & <x_1y_1> & <x_1y_2> \\ <x_2x_1> & <x_2^2> & <x_2y_1> & <x_2y_2> \\ <y_1x_1> & <y_1x_2> & <y_1^2> & <y_1y_2> \\ <y_2x_1> & <y_2x_2> & <y_2y_1> & <y_2^2> \end{Vmatrix}. \tag{14}
$$

In this section we further consider a random signal (the corresponding evaluation for a quasi-random signal is given in section 19). According to the symmetry relations of the previous section, the elements of (14) either prove to be zero or can be reduced to one of the three following quantities:

(a) $<x_1^2>=<x_2^2>=<y_1^2>=<y_2^2>$, to be marked $<x^2>$; (b) $<x_1x_2>=<y_1y_2>$; or (c) $<x_1y_2>$.

We thus obtain:

$$
\Lambda=\begin{Vmatrix} <x^2> & <x_1x_2> & 0 & <x_1y_2> \\ <x_1x_2> & <x^2> & -<x_1y_2> & 0 \\ 0 & -<x_1y_2> & <x^2> & <x_1x_2> \\ <x_1y_2> & 0 & <x_1x_2> & <x^2> \end{Vmatrix}. \tag{15}
$$

The corresponding reciprocal matrix reads as follows:

$$
\Lambda^{-1}=\frac{\begin{Vmatrix} <x^2> & -<x_1x_2> & 0 & -<x_1y_2> \\ -<x_1x_2> & <x^2> & <x_1y_2> & 0 \\ 0 & <x_1y_2> & <x^2> & -<x_1x_2> \\ -<x_1y_2> & 0 & -<x_1x_2> & <x^2> \end{Vmatrix}}{(<x^2>)^2-(<x_1x_2>)^2-(<x_1y_2>)^2}.
$$

This can be verified by determining the $i\,j$ element of $\Lambda\Lambda^{-1}$ from a multiplication of the terms of the ith row of Λ by the corresponding terms of the jth column of Λ^{-1}; the addition of the four products thus formed for each ij set then yields (while taking into account also the denominator of Λ^{-1}) zero for $i \neq j$ and 1 for $i=j$. Hence we arrive at $\Lambda\Lambda^{-1}=1$, as should be.

All relevant matrices prove to be symmetric in the case under consideration, so as to have, e. g., $(\Lambda^{-1})_{ik}=(\Lambda^{-1})_{ki}$. The bilinear form in (10) thus amounts to

$$\sum_{i=1}^{4}\sum_{k=1}^{4}(\Lambda^{-1})_{ik}x_ix_k=(<x^2>\cdot x_1^2-<x_1x_2>\cdot x_1x_2+0-<x_1y_2>\cdot x_1y_2-<x_1x_2>\cdot x_2x_1$$

$$+<x^2>\cdot x_2^2+<x_1y_2>\cdot x_1y_1+0+0+<x_1y_2>\cdot y_1x_2+<x^2>\cdot y_1^2-<x_1x_2>\cdot y_1y_2$$

$$-<x_1y_2>\cdot y_2x_1+0-<x_1x_2>\cdot y_2y_1+<x^2>\cdot y_2^2)[(<x^2>)^2-(<x_1x_2>)^2-(<x_1y_2>)^2]^{-1}$$

$$=\frac{<x^2>\cdot(x_1^2+y_1^2+x_2^2+y_2^2)-2<x_1x_2>\cdot(x_1x_2+y_1y_2)-2<x_1y_2>\cdot(x_1y_2-x_2y_1)}{(<x^2>)^2-(<x_1x_2>)^2-(<x_1y_2>)^2}.$$

Moreover, the evaluation of the determinant of (15) yields

$$|\Lambda|=\{(<x^2>)^2-(<x_1x_2>)^2-(<x_1y_2>)^2\}^2.$$

The final form of (10) then results, in the case under consideration, in the following expression (taking into account that $n=4$):

$$f=\frac{\exp\left\{\dfrac{-\dfrac{<x^2>}{2}\cdot(x_1{}^2+y_1{}^2+x_2{}^2+y_2{}^2)+<x_1x_2>\cdot(x_1x_2+y_1y_2)+<x_1y_2>\cdot(x_1y_2-x_2y_1)}{(<x^2>)^2-(<x_1x_2>)^2-(<x_1y_2>)^2}\right\}}{4\pi^2\{(<x^2>)^2-(<x_1x_2>)^2-(<x_1y_2>)^2\}}. \tag{16}$$

We observe the dependence of the exponent only on expressions which are invariant with respect to a "rotation" of the rectangular axes x and y that are connected with the definition of x_1, y_1 and of x_2, y_2. In fact, introducing (for both times t_1 and t_2) polar coordinates according to $x+iy=r\,e^{i\Phi}$, the quantities occurring in (16) prove to be given by

$$x_1{}^2+y_1{}^2+x_2{}^2+y_2{}^2=r_1{}^2+r_2{}^2,$$
$$x_1x_2+y_1y_2=r_1r_2\cos(\varphi_2-\varphi_1),$$
$$x_1y_2-x_2y_1=r_1r_2\sin(\varphi_2-\varphi_1);$$

each of these expressions is independent of the special position of the axes.

7. Probability Density $f(x_1, x_2, y_1, y_2)$ for a Random Signal Expressed in Terms of the Energy Spectrum

According to (13) all averages $<\quad>$ occurring in (16) depend only on the form of the energy spectrum $W(\omega)$, the dependence on T being apparent (compare section 5). Moreover, the explicit occurrence of T is eliminated by introducing, apart from the simplest average $<x^2>$, only the ratios $<x_1x_k>/<x^2>$, in which T should be identical in the numerator and denominator. In fact, these ratios then follow from a division of (13) by its value for $\tau=0$; the latter value amounts to $<(x^2{}_1+ix_1y_1)>=<x^2>+0$. Hence

$$\frac{<x_1(x_2+iy_2)>}{<x^2>}=\frac{\int_0^\infty d\omega\,W(\omega)e^{-i(\omega-\omega_0)\tau}}{\int_0^\infty d\omega\,W(\omega)}=ze^{i\Psi},\text{ say.} \tag{17}$$

Obviously, the real quantities z and ψ are completely determined by the energy spectrum $W(\omega)$, ω_0 and τ being given. The real and imaginary part of (17) yield

$$<x_1x_2>=<x^2>z\cos\psi,$$
$$<x_1y_2>=<x^2>z\sin\psi,$$

which involves the further relation

$$(<x^2>)^2-(<x_1x_2>)^2-(<x_1y_2>)^2=(<x^2>)^2.(1-z^2).$$

The substitution of the three latter relations into (16) leads to the following alternative form for the joint probability density of x_1, y_1, x_2, and y_2 for a random signal:

$$f=\frac{\exp\left\{-\frac{(x_1^2+y_1^2+x_2^2+y_2^2)-2z\cos\psi(x_1x_2+y_1y_2)-2z\sin\psi(x_1y_2-x_2y_1)}{2<x^2>\cdot(1-z^2)}\right\}}{4\pi^2(<x^2>)^2.(1-z^2)}. \tag{18}$$

We observe the connection of z and ψ with correlation properties, as expressed by the dependence of z and ψ on the time interval $\tau=t_2-t_1$. On the other hand, the third parameter $<x^2>$ is directly associated with the variance (that is, the square of the rms value) of the quasi-monochromatic signal (2). In fact, we derive from (2), by considering the smallness of the change of $x(t)$ and $y(t)$ over one cycle of $2\pi/\omega_0$ time units, the following variance:

$$<h^2>=<x^2(t).\cos^2(\omega_0t)>+<y^2(t).\sin^2(\omega_0t)>+2<x(t)y(t)\cos(\omega_0t)\sin(\omega_0t)>$$
$$\sim<x^2>\cdot\tfrac{1}{2}+<y^2>\cdot\tfrac{1}{2}+2<xy>\cdot 0=\tfrac{1}{2}<x^2>+\tfrac{1}{2}<x^2>,$$

or

$$H^2=<h^2>\sim<x^2>\cdot \tag{19}$$

8. Probability Density for a Random Signal Imposed on a Constant Signal

The constant signal can be described by $h_0(t)=R_0\cos(\omega_0t)$, R_0 being independent of time. The superposition of the random component $h(t)$ given by (2), considered so far, leads to a final signal represented by

$$H(t)=R_0\cos(\omega_0t)+h(t)=\{R_0+x(t)\}\cos(\omega_0t)+y(t)\sin(\omega_0t). \tag{20}$$

The phase has been fixed so that $t=0$ does correspond to a maximum of the background signal.

We introduce new polar coordinates according to

$$R_0+x(t)=R(t)\cos\Phi(t);\qquad y(t)=R(t)\sin\Phi(t). \tag{21}$$

The representation (20) then transforms into

$$\begin{aligned}H(t)&=R(t)\{\cos\Phi(t)\cos(\omega_0t)+\sin\Phi(t)\sin(\omega_0t)\}\\&=R(t)\cos\{\omega_0t-\Phi(t)\}.\end{aligned} \tag{22}$$

In other words, $R(t)$ and $\Phi(t)$ are the amplitude and phase of the envelope of the complete signal $H(t)$.

This amplitude and phase being measurable quantities, we are interested in the probability distribution that is obtained from (18) by passing to the values of the amplitude and phase at the two times in question, that is for $t=t_1$, and $t=t_2$. The four new variables to be considered therefore are represented by

$$R_1=R(t_1);\qquad R_2=R(t_2);\qquad \Phi_1=\Phi(t_1);\qquad \Phi_2=\Phi(t_2).$$

According to (21) the complete transformation formulas read as follows

$$\left.\begin{aligned}x_1&=R_1\cos\phi_1-R_0; & x_2&=R_2\cos\phi_2-R_0;\\ y_1&=R_1\sin\phi_1; & y_2&=R_2\sin\phi_2.\end{aligned}\right\} \tag{23}$$

The probability of finding the original variables x_1, x_2, y_1, y_2 within specified infinitesimal intervals becomes as follows when passing to the new variables, while taking into account the theory of Jacobi's functional determinant:

$$fdx_1dx_2dy_1dy_2=f\times\begin{vmatrix} \frac{\partial x_1}{\partial R_1} & \frac{\partial x_1}{\partial R_2} & \frac{\partial x_1}{\partial \Phi_1} & \frac{\partial x_1}{\partial \Phi_2} \\ \frac{\partial x_2}{\partial R_1} & \frac{\partial x_2}{\partial R_2} & \frac{\partial x_2}{\partial \Phi_1} & \frac{\partial x_2}{\partial \Phi_2} \\ \frac{\partial y_1}{\partial R_1} & \frac{\partial y_1}{\partial R_2} & \frac{\partial y_1}{\partial \Phi_1} & \frac{\partial y_1}{\partial \Phi_2} \\ \frac{\partial y_2}{\partial R_1} & \frac{\partial y_2}{\partial R_2} & \frac{\partial y_2}{\partial \Phi_1} & \frac{\partial y_2}{\partial \Phi_2} \end{vmatrix} dR_1dR_2d\Phi_1d\Phi_2$$

$$=GdR_1dR_2d\Phi_1d\Phi_2,\ \text{say.}$$

In view of (23) the determinant becomes

$$\begin{vmatrix} \cos\Phi_1 & 0 & -R_1\sin\Phi_1 & 0 \\ 0 & \cos\Phi_2 & 0 & -R_2\sin\Phi_2 \\ \sin\Phi_1 & 0 & R_1\cos\Phi_1 & 0 \\ 0 & \sin\Phi_2 & 0 & R_2\cos\Phi_2 \end{vmatrix}=R_1R_2.$$

Hence, the joint probability with respect to the new variables is given by $G=R_1R_2f$. The former expression (18) for f next has to be transformed with the aid of the substitutions (23). The corresponding evaluation of the three relevant quantities occurring in (18) yield

$$x_1^2+y_1^2+x_2^2+y_2^2=2R_0^2+R_1^2+R_2^2-2R_0(R_1\cos\Phi_1+R_2\cos\Phi_2);$$
$$x_1x_2+y_1y_2=R_0^2-R_0(R_1\cos\Phi_1+R_2\cos\Phi_2)+R_1R_2\cos(\Phi_2-\Phi_1);$$
$$x_1y_2-x_2y_1=R_0(R_1\sin\Phi_1-R_2\sin\Phi_2)+R_1R_2\sin(\Phi_2-\Phi_1).$$

Some further elementary reduction of the linear combination of these quantities that occurs in the exponent of (18) finally leads to the following formula for the new probability density $G=R_1R_2f$ (replacing $<x^2>$ by H^2, in accordance with eq 19):

$$G=\frac{R_1R_2}{4\pi^2H^4(1-z^2)}\exp\left\{\begin{array}{c}-[2R_0^2+R_1^2+R_2^2-2R_0^2z\cos\psi-2R_0(R_1\cos\Phi_1+R_2\cos\Phi_2)\\ +2R_0R_1z\cos(\Phi_1+\psi)+2R_0R_2z\cos(\Phi_2-\psi)-2R_1R_2z\cos(\Phi_2-\Phi_1-\psi)]\\ [2H^2(1-z^2)]^{-1}.\end{array}\right\} \tag{24}$$

9. Expansions of $z(\tau)$ and $\psi(\tau)$ for Small τ

As an introduction to the limiting procedure to be applied in the next section, we investigate the behaviour of $z(\tau)$ and $\psi(\tau)$ for small τ. The quantity represented by (17) reduces to unity at $\tau=0$, so as to have $z(0)=1$, $\psi(0)=0$. Further, the following series results when expanding the exponential in (17):

$$z(\tau)e^{i\psi(\tau)}=1-i\tau\frac{\int_0^\infty d\omega W(\omega)(\omega-\omega_0)}{\int_0^\infty d\omega W(\omega)}-\frac{\tau^2}{2}\frac{\int_0^\infty d\omega W(\omega)(\omega-\omega_0)^2}{\int_0^\infty d\omega W(\omega)}+i\frac{\tau^3}{6}\frac{\int_0^\infty d\omega W(\omega)(\omega-\omega_0)^3}{\int_0^\infty d\omega W(\omega)}+\ .\ .\ .\ . \tag{25}$$

The successive quotients of two integrals represent the averages $\overline{\omega-\omega_0}$, $\overline{(\omega-\omega_0)^2}$, $\overline{(\omega-\omega_0)^3}$ etc., of $\omega-\omega_0$, $(\omega-\omega_0)^2$, $(\omega-\omega_0)^3$ etc., provided we define by $W(\omega)\, d\omega$ the probability that ω shall lie in the interval between ω and $\omega+d\omega$. This definition implies the following average of the frequency itself:

$$\bar{\omega}=\frac{\int_0^\infty d\omega W(\omega)\omega}{\int_0^\infty d\omega W(\omega)}. \tag{26}$$

We next assume $\bar{\omega}=\omega_0$. In other words, the random component $h(t)$, which may originate from scattering effects influencing the original background signal, has an average frequency (defined by the energy spectrum) equaling that of this latter signal. The second term of (25) then drops out and the relation in question can be put in the form;

$$z(\tau)e^{i\psi(\tau)}=1-\frac{\tau^2}{2}\overline{(\omega-\omega_0)^2}+\frac{i\tau^3}{6}\overline{(\omega-\omega_0)^3}+\ldots.$$

The real and imaginary parts start as follows:

$$\left.\begin{aligned} z(\tau)\cos\psi(\tau)&=1-\frac{\tau^2}{2}\overline{(\omega-\omega_0)^2}+\ldots, \\ z(\tau)\sin\psi(\tau)&=\frac{\tau^3}{6}\overline{(\omega-\omega_0)^3}+\ldots, \end{aligned}\right\} \tag{27}$$

whereas an addition of the squares of these latter expansions will lead to:

$$z^2(\tau)=1-\tau^2\overline{(\omega-\omega_0)^2}+\ldots. \tag{28}$$

We shall mark the frequency variance $\overline{(\omega-\omega_0)^2}$ by σ_ω^2, σ_ω then being the rms deviation (relative to the energy spectrum) of the frequency from its average value $\bar{\omega}=\omega_0$. The relation (28) is then equivalent to

$$1-z^2(\tau)=\tau^2\sigma_\omega{}^2+\ldots. \tag{29}$$

It shows how the expression (24) for the probability density G of amplitudes and phases degenerates at $\tau=0$. At the same time, the fraction constituting the exponent of (24) proves to have a well defined limit for $\tau\rightarrow 0$, this limit being of great importance for the analysis in the next section.

10. Probability Density for the Amplitude, the Phase and Their Time Derivatives for a Random Signal Imposed on a Constant Signal

This probability density enables the derivation of the most elementary properties of fading rapidity, namely the average numbers of crossings of either the amplitude or the phase through a prescribed level. The probability in question follows from the probability (24) for the amplitude and phase at two different moments by letting their separation τ tend to zero. This limiting procedure can be worked out by substituting the two following Taylor expansions into (24):

$$\left.\begin{aligned} R_2=R(t_2)&=R(t_1)+\tau\dot{R}(t_1)+\frac{\tau^2}{2}\ddot{R}(t_1)+\ldots \\ \Phi_2=\Phi(t_2)&=\Phi(t_1)+\tau\dot{\Phi}(t_1)+\frac{\tau^2}{2}\ddot{\Phi}(t_1)+\ldots. \end{aligned}\right\} \tag{30}$$

Henceforth we shall omit the subscript 1, t_1 being the only time to be considered so far. The differentials dR_2 and $d\Phi_2$ can now be represented as follows provided we consider R_1, ϕ_1, and τ as fixed quantities,

$$\left.\begin{aligned} dR_2&=\tau d\dot{R}+\frac{\tau^2}{2}d\ddot{R}+\ldots, \\ d\Phi_2&=\tau d\dot{\Phi}+\frac{\tau^2}{2}d\ddot{\Phi}+\ldots. \end{aligned}\right\} \tag{31}$$

These values imply that in any double integration concerning, e. g., R_1 and R_2, the integration with respect to R_2 should be performed first.

In (24) the function G can be considered as depending explicitly on R_1, R_2, Φ_1, Φ_2, τ, the dependence on τ being contained in the occurrence of $z(\tau)$ and $\psi(\tau)$. The probability of finding R_1, R_2, Φ_1, Φ_2 within prescribed infinitesimal intervals thus leads to the following expression when substituting (30) and (31):

$$G(R_1,R_2,\Phi_1,\Phi_2,\tau)dR_1dR_2d\Phi_1d\Phi_2=G\left(R,R+\tau\dot{R}+\frac{\tau^2}{2}\ddot{R}\ldots,\Phi,\Phi+\tau\dot{\Phi}+\frac{\tau^2}{2}\ddot{\Phi}\ldots,\tau\right)$$

$$dR\left(\tau d\dot{R}+\frac{\tau^2}{2}d\ddot{R}+\ldots\right)d\Phi\left(\tau d\dot{\Phi}+\frac{\tau^2}{2}d\ddot{\Phi}+\ldots\right).$$

For small τ the right-hand side can be represented by

$$\tau^2 dR d\dot{R} d\Phi d\dot{\Phi} G\left(R,R+\tau\dot{R}+\frac{\tau^2}{2}\ddot{R}\ldots,\Phi,\Phi+\tau\dot{\Phi}+\frac{\tau^2}{2}\ddot{\Phi}\ldots,\tau\right),$$

so that the joint probability density G for R, $\dot{R}$, Φ, $\dot{\Phi}$ does result from the limit, if any,

$$G(R,\dot{R},\Phi,\dot{\Phi})=\lim_{\tau\to 0}\left\{\tau^2 G\left(R,R+\tau\dot{R}+\frac{\tau^2}{2}\ddot{R}\ldots,\Phi,\Phi+\tau\dot{\Phi}+\frac{\tau^2}{2}\ddot{\Phi}\ldots,\tau\right)\right\}. \tag{32}$$

The evaluation of (32) with the aid of (24) amounts to the following expression if the numerator in the exponent of (24) is represented by $E\ (R_1,R_2,\Phi_1,\Phi_2,\tau)$:

$$G(R,\dot{R},\Phi,\dot{\Phi})=\frac{R^2}{4\pi^2H^4}\lim_{\tau\to 0}\left[\frac{\tau^2}{1-z^2(\tau)}\cdot\exp\left\{-\frac{E(R_1,R_2,\Phi_1,\Phi_2,\tau)}{2H^2\{1-z^2(\tau)\}}\right\}\right]_{R_1\to R;\ \Phi_1\to\Phi;\ R_2\to R+\tau\dot{R}+\frac{\tau^2}{2}\ddot{R}\ldots;\ \Phi_2\to\Phi+\tau\dot{\Phi}+\frac{\tau^2}{2}\ddot{\Phi}\ldots} \tag{33}$$

In view of (29) we can apply the limit

$$\lim_{\tau\to 0}\frac{\tau^2}{1-z^2(\tau)}=\frac{1}{\sigma_\omega^2},$$

with the aid of which (33) can be transformed into:

$$G=\frac{R^2}{4\pi^2H^4\sigma_\omega^2}\exp\left[-\frac{1}{2H^2\sigma_\omega^2}\cdot\lim_{\tau\to 0}\left\{\frac{E(R_1,R_2,\Phi_1,\Phi_2,\tau)}{\tau^2}\right\}_{R_1\to R;\ \Phi_1\to\Phi;\ R_2\to R+\tau\dot{R}+\frac{\tau^2}{2}\ddot{R}\ldots;\ \Phi_2\to\Phi+\tau\dot{\Phi}+\frac{\tau^2}{2}\ddot{\Phi}\ldots}\right] \tag{34}$$

In the τ limit yet to be determined, we can omit all terms of E that are proportional to $z\sin\psi$; in fact, these terms being of the order of τ^3 [compare (27)], their contribution will be zero after the division by τ^2. An ordering of the remaining terms of E leads to the following representation of the τ limit in question:

$$\lim_{\tau\to 0}\left[\frac{\{1-z(\tau)\cos\psi(\tau)\}\{2R_0^2-2R_0(R_1\cos\Phi_1+R_2\cos\Phi_2)\}+R_1^2+R_2^2-2R_1R_2z(\tau)\cos\psi(\tau)\cos(\Phi_2-\Phi_1)}{\tau^2}\right]_{R_1\to R;\ \Phi_1\to\Phi;\ R_2\to R+\tau\dot{R}+\frac{\tau^2}{2}\ddot{R}\ldots;\ \Phi_2\to\Phi+\tau\dot{\Phi}+\frac{\tau^2}{2}\ddot{\Phi}\ldots} \tag{35}$$

According to (27) the ratio $(1-z\cos\psi)/\tau^2$ tends to $\frac{1}{2}\overline{(\omega-\omega_0)^2}=\sigma_\omega^2/2$, whereas the factor $2R_0^2-2R_0(R_1\cos\Phi_1+R_2\cos\Phi_2)$ approaches the limiting value $2R_0^2-4R_0R\cos\Phi$. Moreover, the substitutions indicated for Φ_1 and Φ_2 yield the following value for $\cos(\Phi_2-\Phi_1)$ up to second-order terms:

$$\cos(\Phi_2-\Phi_1)=\cos\left(\tau\dot{\Phi}+\frac{\tau^2}{2}\ddot{\Phi}+\ldots\right)=1-\frac{\tau^2}{2}\dot{\Phi}^2\ldots.$$

We perform the necessary substitutions in the remaining terms while applying, moreover, the relation $z \cos \psi = 1 - \frac{1}{2}\tau^2\sigma_\omega^2 \ldots$ [see (27)]. The limit (35) can then further be reduced to

$$\frac{\sigma_\omega^2}{2}(2R_0^2 - 4R_0R\cos\Phi)$$

$$+\lim_{\tau\to 0} \frac{R^2 + \left(R + \tau\dot{R} + \frac{\tau^2}{2}\ddot{R}\ldots\right)^2 - 2R\left(R + \tau\dot{R} + \frac{\tau^2}{2}\ddot{R}\ldots\right)\left(1 - \frac{\tau^2}{2}\sigma_\omega^2\ldots\right)\left(1 - \frac{\tau^2}{2}\dot{\Phi}^2\ldots\right)}{\tau^2}.$$

The elementary evaluation of the second term leads to the final expression

$$\sigma_\omega^2(R_0^2 - 2R_0R\cos\Phi) + \dot{R}^2 + R^2(\sigma_\omega^2 + \dot{\Phi}^2),$$

which thus constitutes the value of the limit occurring in (34). Hence, the joint probability density of the amplitude, phase, and the corresponding time derivatives reads explicitly

$$G(R,\dot{R},\Phi,\dot{\Phi}) = \frac{R^2}{4\pi^2 H^4\sigma_\omega^2}\exp\left\{-\frac{R_0^2 - 2R_0R\cos\Phi + R^2 + (\dot{R}^2 + R^2\dot{\Phi}^2)/\sigma_\omega^2}{2H^2}\right\}. \tag{36}$$

11. Distribution of the Amplitude and its Time Derivative for a Random Signal Imposed on a Constant Signal

The function $G(R,\dot{R})$ will be defined such that $G(R,\dot{R})dRd\dot{R}$ represents the probability of finding R and $\dot{R}$ in prescribed infinitesimal intervals. Obviously, $G(R,\dot{R})$ is obtained by integrating $G(R,\dot{R},\Phi,\dot{\Phi})$ over all possible values of Φ and $\dot{\Phi}$, that is, over the intervals $0 < \Phi < 2\pi$ and $-\infty < \dot{\Phi} < \infty$. Hence

$$G(R,\dot{R}) = \frac{R^2}{4\pi^2 H^4\sigma_\omega^2}\cdot\exp\left\{-\frac{R_0^2 + R^2 + \dot{R}^2/\sigma_\omega^2}{2H^2}\right\}\cdot\int_0^{2\pi} d\Phi\exp\left(\frac{R_0R\cos\Phi}{H^2}\right)\int_{-\infty}^{\infty} d\dot{\Phi}\exp\left(-\frac{R^2\dot{\Phi}^2}{2H^2\sigma_\omega^2}\right). \tag{37}$$

The first integral equals $2\pi\, I_0(R_0R/H^2)$, $I_0(x)$ being the zero-order Bessel function for imaginary argument. The second integral transforms by the substitution

$$u = \frac{R}{\sqrt{2}\sigma_\omega H}\dot{\Phi}$$

into a Poisson integral yielding the value $(2\pi)^{\frac{1}{2}} H\,\sigma_\omega/R$. Taking into account the values of both integrals, we obtain by working out (37),

$$G(R,\dot{R}) = \frac{R}{(2\pi)^{1/2}H^3\sigma_\omega}\cdot I_0\left(\frac{R_0R}{H^2}\right)\cdot\exp\left(-\frac{R_0^2 + R^2 + \dot{R}^2/\sigma_\omega^2}{2H^2}\right). \tag{38}$$

This probability function enables the determination of the rapidity of the amplitude fading (see the theory of section 13).

The distribution of the amplitude R is given by the probability $G(R)dR$ of finding an amplitude between R and $R+dR$. This new distribution is found by integrating (38) over all possible values of $\dot{R}$, that is by determining the following integral which is again of the Poisson type:

$$\int_{-\infty}^{\infty} d\dot{R}\exp\left(-\frac{\dot{R}^2}{2H^2\sigma_\omega^2}\right) = (2\pi)^{1/2}H\sigma_\omega.$$

The resulting distribution function reads

$$G(R) = \frac{R}{H^2}\cdot I_0\left(\frac{R_0R}{H^2}\right)\cdot\exp\left(-\frac{R_0^2 + R^2}{2H^2}\right). \tag{39}$$

This function should satisfy the normalization condition $\int_0^\infty dRG(R)=1$. The latter is easily verified with the aid of the following integral property of Bessel functions [4]:

$$\int_0^\infty dr\,rI_0(ar)e^{-br^2}=\frac{1}{2b}\,e^{a^2/(4b)}. \tag{40}$$

We consider the two limiting cases of:

(a) absence of the background signal ($R_0=0$). The distribution (39) simplifies to

$$G(R)=\frac{R}{H^2}e^{-R^2/(2H^2)}, \tag{41}$$

which is the well-known *Rayleigh distribution.* The latter is characterized by the variance $\langle R^2\rangle=2\langle h^2\rangle=2H^2$; this relation is in accordance with the respresentation (22) for the complete signal;

(b) predominance of the background signal ($R_0>>H^2/R$). A substitution of the asymptotic approximation for I_0 leads to the following approximation of (39):

$$G(R)\sim\frac{1}{(2\pi)^{1/2}H}\left(\frac{R}{R_0}\right)^{1/2}e^{-(R-R_0)^2/(2H^2)}. \tag{42}$$

12. Distribution of the Phase and its Time Derivative for a Random Signal Imposed on a Constant Signal

The function $G(\Phi,\dot{\Phi})$ defining the probability $G(\Phi,\dot{\Phi})\,d\Phi\,d\dot{\Phi}$ of finding Φ and $\dot{\Phi}$ in prescribed infinitesimal intervals is obtained by integrating $G(R,\dot{R},\Phi,\dot{\Phi})$ over all possible values of R and $\dot{R}$. The range of R contains all positive numbers, that of $\dot{R}$ also the negative numbers. Hence,

$$G(\Phi,\dot{\Phi})=\int_0^\infty dR\int_{-\infty}^\infty d\dot{R}G(R,\dot{R},\Phi,\dot{\Phi}),$$

or, in view of (36),

$$G(\Phi,\dot{\Phi})=\frac{\exp\{-R_0^2/(2H^2)\}}{4\pi^2H^4\sigma_\omega^2}\int_0^\infty dRR^2\exp\left\{\frac{2R_0R\cos\Phi-R^2-R^2\dot{\Phi}^2/\sigma_\omega^2}{2H^2}\right\}\int_{-\infty}^\infty d\dot{R}\exp\left\{-\frac{\dot{R}^2}{2H^2\sigma_\omega^2}\right\}.$$

The second integral, once again of the Poisson type, amounts to $(2\pi)^{1/2}\,\sigma_\omega H$. The substitution yields

$$G(\Phi,\dot{\Phi})=\frac{\exp\left\{-\frac{R_0^2}{2H^2}\right\}}{(2\pi)^{3/2}\sigma_\omega H^3}\int_0^\infty dRR^2\exp\left\{\frac{R_0R\cos\Phi}{H^2}-\frac{R^2(1+\dot{\Phi}^2/\sigma_\omega^2)}{2H^2}\right\}, \tag{43}$$

which expression might also be reduced to error functions. However, formula (43) can be applied straightforwardly for the derivation of the rapidity of phase fading (see section 15).

The distribution of the phase Φ itself, characterized by the probability $G(\Phi)\,d\Phi$ of finding Φ in a prescribed infinitesimal interval, is obtained by integrating (43) over all values of $\dot{\Phi}$. This integration amounts to the further Poisson integral,

$$\int_{-\infty}^\infty d\dot{\Phi}\exp\left\{-\frac{R^2\dot{\Phi}^2}{2H^2\sigma_\omega^2}\right\}=\frac{(2\pi)^{1/2}\sigma_\omega H}{R}.$$

The new probability density

$$G(\Phi)=\int_{-\infty}^\infty d\dot{\Phi}G(\Phi,\dot{\Phi})$$

then proves to be given, after a simple reduction, by the formula

$$G(\Phi)=\frac{\exp\left\{-\frac{R_0^2\sin^2\Phi}{2H^2}\right\}}{2\pi H^2}\int_0^\infty dRR\exp\left\{-\frac{(R-R_0\cos\Phi)^2}{2H^2}\right\}. \tag{44}$$

Here, too, the relevant normalization condition, that is $\int_0^{2\pi} G(\Phi)d\Phi=1$, is easily verified by applying, once again, the Bessel function property (40); in this case the I_0 function results from the integration of (44) over Φ, the ultimate exponent of (44) being proportional to $\cos\Phi$. Further, by splitting the nonexponential factor R of (44) into $(R-R_0\cos\Phi)+R_0\cos\Phi$ and by introducing the new integration variable $s=R-R_0\cos\Phi$, the formula (44) breaks up into the sum of an exponential and an error function, viz,

$$G(\Phi)=\frac{1}{2\pi}\exp\left\{-\frac{R_0^2}{2H^2}\right\}+\frac{R_0\cos\Phi}{2^{3/2}\pi^{1/2}H}\exp\left\{-\frac{R_0^2\sin^2\Phi}{2H^2}\right\}\cdot\operatorname{erfc}\left\{-\frac{R_0\cos\Phi}{2^{1/2}H}\right\}; \tag{45}$$

the error function erfc introduced here is defined in the conventional way by

$$\operatorname{erfc}(x)=\frac{2}{\pi^{1/2}}\int_x^\infty e^{-s^2}ds.$$

The phase density (45) is symmetric with respect to the phase $\Phi=0$ of the constant component $R_0\cos(\omega_0 t)$ [see (22)]. We therefore restrict ourselves when considering the corresponding integrated distribution, to the quantity

$$P(\Phi>\Phi_0)=\int_{\Phi_0}^{\pi}d\Phi\, G(\Phi). \qquad (0<\Phi_0<\pi).$$

It has been discussed numerically by Norton, Schultz, and Yarbrough [5] and also by Norton, Vogler, Mansfield, and Short [6].

We consider once more the two limiting cases mentioned at the end of the previous section:

(a) absence of the background signal. The substitution $R_0=0$ reduces (45) to the constant $1/(2\pi)$. In other words, the phases are homogeneously distributed for a pure random signal. As a matter of fact, this randomness of the phases constitutes the well known basis of the Rayleigh distribution (41) for the corresponding amplitudes;

(b) predominance of the background signal. The argument of erfc in (45) becomes large and negative for the most important phases, viz, acute angles not near $\pi/2$. We can then apply the corresponding approximation

$$\operatorname{erfc}(-a)=2-\operatorname{erfc}(a)\sim 2-\frac{1}{\sqrt{\pi}}\frac{e^{-a^2}}{a},$$

the substitution of which into (45) leads to

$$G(\Phi)\sim\frac{R_0\cos\Phi}{(2\pi)^{1/2}H}\cdot\exp\left\{-\frac{R_0^2\sin^2\Phi}{2H^2}\right\}-\frac{1}{2\pi}\exp\left\{-\frac{R_0^2}{2H^2}\right\}.$$

For the dominating small values of Φ, the second term can be neglected, whereas $\sin\Phi$ and $\cos\Phi$ may be replaced by Φ and 1, respectively. This leads to the final approximation

$$G(\Phi)\sim\frac{R_0}{(2\pi)^{1/2}H}\cdot\exp\left\{-\frac{R_0^2}{2H^2}\Phi^2\right\}, \tag{46}$$

which is simply Gaussian and satisfies the usual normalization condition for $-\infty<\Phi<\infty$.

13. Number of Crossings Through a Specific Level

In order to fix our ideas we consider the amplitude, though the following considerations may just as well be applied, mutatis mutandis, to the phase. We ask for the average number $N(a)$ of crossings per unit time interval of the amplitude through a specific level a. Any crossing through a, from below and within the time interval $t_0<t<(t_0+\Delta t)$, is characterized by the inequalities:

$$R(t_0)<a<R(t_0+\Delta t).$$

The following approximation holds for small Δt:

$$R(t_0) < a < R(t_0) + \Delta t \cdot \dot{R}(t_0),$$

from which we derive

$$a - \Delta t \cdot \dot{R}(t_0) < R(t_0) < a.$$

According to the probability density $G(R,\dot{R})$ (applied to the moment t_0) the over-all probability of crossings from below through a within Δt therefore is given by the following expression if we limit ourselves to $\dot{R}$ values within a specified infinitesimal interval:

$$d\dot{R} \int_{a-\Delta t \cdot \dot{R}(t_0)}^{a} dR\, G(R,\dot{R}) \sim d\dot{R} \cdot \Delta t \cdot \dot{R}(t_0) \cdot G(a,\dot{R}).$$

All crossings from below are connected with a positive value of $\dot{R}(t_0)$. Hence their total number during the time interval Δt is given by

$$\Delta t \int_0^\infty d\dot{R} \cdot \dot{R} \cdot G(a,\dot{R}).$$

Obviously, an equal number of crossings occurs on the average in the opposite direction, that is, from above. The total average number of any crossings (per unit time interval) through a therefore results after a multiplication of the last quantity by $2/\Delta t$. Hence we obtain

$$N(a) = 2 \int_0^\infty d\dot{R} \cdot \dot{R} \cdot G(a,\dot{R}). \tag{47}$$

This quantity is characteristic for the rapidity of the fading and was first derived by S. O. Rice [2]. Its maximum (as a function of a) can be compared with Ratcliffe's [7] definition of fading rate. The latter can be represented as follows,

$$\lim_{\tau \to 0} \frac{|R(t+\tau) - R(t)|}{\tau \cdot R(t)} = \frac{\langle |\dot{R}(t)| \rangle}{\langle R(t) \rangle}; \tag{48}$$

it may yield significantly differing results (see section 16).

14. Number of Crossings of the Amplitude Through a Specific Level for a Random Signal Imposed on a Constant Signal

An application of (47) to the joint probability density (38) for the amplitude and its time derivative yields the following quantity for the number in question:

$$N_A(a) = 2 \int_0^\infty d\dot{R}\dot{R} \frac{a}{(2\pi)^{1/2} H^3 \sigma_\omega} \cdot I_0\left(\frac{R_0 a}{H^2}\right) \cdot \exp\left(-\frac{R_0^2 + a^2 + \dot{R}^2/\sigma_\omega^2}{2H^2}\right).$$

The necessary integration reduces to

$$\int_0^\infty d\dot{R}\dot{R} \exp\left(-\frac{\dot{R}^2}{2H^2\sigma_\omega^2}\right) = H^2\sigma_\omega^2,$$

and yields the following well-known expression [2]

$$N_A(a) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sigma_\omega}{H} \cdot a \cdot I_0\left(\frac{R_0}{H^2} a\right) \cdot \exp\left(-\frac{R_0^2 + a^2}{2H^2}\right). \tag{49}$$

[2] This expression is equivalent to eq (4.8) on page 125 in reference 2.

This function proves to be proportional to the probability density (39) for the amplitude at $R=a$. We immediately verify the relation

$$N_A(a)=\left(\frac{2}{\pi}\right)^{1/2} H\sigma_\omega G(a). \tag{50}$$

Hence the number of crossings through a specific level increases in proportion to the probability of finding an amplitude near that level.

A substitution into (50) of the two limiting expressions (41) and (42) leads to the following approximations for the level-crossing numbers:

(a) Small background signal

$$N_A(a)\sim\left(\frac{2}{\pi}\right)^{1/2}\frac{\sigma_\omega}{H}\cdot a\cdot \exp\left(-\frac{a^2}{2H^2}\right); \tag{51}$$

(b) predominant background signal

$$N_A(a)\sim\frac{\sigma_\omega}{\pi}\left(\frac{a}{R_0}\right)^{1/2}\cdot \exp\left\{-\frac{(a-R_0)^2}{2H^2}\right\}. \tag{52}$$

15. Number of Crossings of the Phase Through a Specific Level for a Random Signal Imposed on a Constant Signal

A further application of (47) to the joint probability density (43) of the phase and its time derivative yields the following double integral for the average number of level crossings (per unit time interval) for the phase:

$$\begin{aligned} N_\Phi(\varphi)&=2\int_0^\infty d\dot{\Phi}\dot{\Phi}G(\varphi,\dot{\Phi})\\ &=2\int_0^\infty d\dot{\Phi}\dot{\Phi}\frac{\exp\{-R_0^2/(2H^2)\}}{(2\pi)^{3/2}\sigma_\omega H^3}\int_0^\infty dRR^2 \exp\left\{\frac{R_0R\cos\varphi}{H^2}-\frac{R^2(1+\dot{\Phi}^2/\sigma_\omega^2)}{2H^2}\right\}.\end{aligned}$$

We can invert the order of integration and next evaluate the $\dot{\Phi}$ integral according to

$$\int_0^\infty d\dot{\Phi}\dot{\Phi}\exp\left(-\frac{R^2}{2H^2\sigma_\omega^2}\dot{\Phi}^2\right)=\frac{H^2\sigma_\omega^2}{R^2}.$$

The remaining integral over R yields

$$N_\Phi(\varphi)=\frac{\sigma_\omega}{2^{1/2}\pi^{3/2}H}\cdot\exp\left(-\frac{R_0^2}{2H^2}\right)\cdot\int_0^\infty dR\exp\left\{-\frac{(R^2-2RR_0\cos\varphi)}{2H^2}\right\}. \tag{53}$$

This result reads as follows in terms of an error function

$$N_\Phi(\varphi)=\frac{\sigma_\omega}{2\pi}\cdot\exp\left(-\frac{R_0^2\sin^2\varphi}{2H^2}\right)\cdot\operatorname{erfc}\left\{-\frac{R_0\cos\varphi}{2^{1/2}H}\right\}. \tag{54}$$

Here again, there exists a simple connection with the corresponding probability density, that is the function (45) for the special value φ of the phase Φ. In fact, a comparison of (54) and (45) shows the validity of the following linear relation between $\cos\varphi N_\Phi(\varphi)$ and $G(\varphi)$:

$$\cos\varphi N_\Phi(\varphi)=\left(\frac{2}{\pi}\right)^{1/2}\frac{\sigma_\omega}{R_0}H\left\{G(\varphi)-\frac{1}{2\pi}\exp\left(-\frac{R_0^2}{2H^2}\right)\right\}. \tag{55}$$

The two previous limiting cases are investigated most conveniently with the aid of (54). We find:
(a) small background signal. The substitution $R_0=0$ yields the approximation

$$N_\Phi(\varphi)\sim\frac{\sigma_\omega}{2\pi}, \tag{56}$$

which is a number of level crossings independent of the level under consideration. Such a result had to be expected in view of the complete randomness of the phases for the random signal constituting the limiting case $R_0=0$;

(b) predominant background signal. The substitution $R_0=\infty$ in the error function of (54) here leads to the approximation

$$N_\Phi(\varphi)\sim\frac{\sigma_\omega}{\pi}\cdot\exp\left\{-\frac{R_0^2\sin^2\varphi}{2H^2}\right\} \tag{57}$$

for the most important domain in which $\cos\varphi>0$.

16. Comparison of the Amplitude Fading and the Phase Fading for a Random Signal Imposed on a Constant Signal

The rapidity of fading is characterized in a representative way by the number of crossings (per unit time interval) of the level that is traversed most frequently. The latter level, $\varphi_{\max}$ say, is very simple in the case of phase fading, the function (54) being maximum at $\varphi=0$; the phase value $\varphi_{\max}=0$ crossed most frequently thus coincides with that of the background signal. The substitution $\varphi=0$ in (54) thereupon yields the following value for the rapidity of phase fading as defined above:

$$N_\Phi(\varphi_{\max})=N_\Phi(0)=\frac{\sigma_\omega}{2\pi}\cdot\operatorname{erfc}\left\{-\frac{R_0}{2^{1/2}H}\right\}. \tag{58}$$

In the case of the amplitude, however, the determination of the level most crossed, $a_{\max}$ say, becomes much more complicated. In fact, the evaluation of this level with the aid of (49), by working out the relation $N'_A(a_{\max})=0$, leads to the transcendental equation

$$\frac{\frac{d}{d\lambda}\{(\lambda)_0I\lambda\}}{\lambda^2I_0(\lambda)}=\frac{H^2}{R_0^2} \tag{59}$$

for the quantity $\lambda=R_0a_{\max}/H^2$. Therefore, the number of crossings $N_A(a_{\max})$ through the amplitude level most frequently traversed can only be determined numerically. However, this number becomes very simple in the two limiting cases discussed previously; a comparison of the rapidities of the amplitude fading and the phase fading can then be made at once. In fact, we have:

(a) small background signal. According to (51) $a_{\max}$ is to be derived from the equation

$$\frac{d}{da}\left\{a\cdot\exp\left(-\frac{a^2}{2H^2}\right)\right\}=0,$$

yielding $a_{\max}=H$. Hence we find, again in view of (51),

$$N_A(a_{\max})=\left(\frac{2}{\pi e}\right)^{1/2}\sigma_\omega. \tag{60}$$

This quantity can be compared with the fading rate according to Ratcliffe's definition (48). The latter can be evaluated by determining the probability density of $\dot{R}$ from an integration of (38) with respect to R, while applying (40). The fading rate thus found, viz, $(2/\pi)\,\sigma_\omega$, differs from (60) merely by a factor of the order of unity.

A comparison of (60) with the corresponding quantity (56) for the phase yields

$$\frac{N_A(a_{\max})}{N_\Phi(\varphi_{\max})}=\frac{\sigma_\omega\cdot\{2/(\pi e)\}^{1/2}}{\sigma_\omega/(2\pi)},\ \text{or}$$
$$\frac{N_A(a_{\max})}{N_\Phi(\varphi_{\max})}=2\cdot\left(\frac{2\pi}{e}\right)^{1/2}=3\cdot04. \tag{61}$$

Hence, the rapidity of the amplitude fading proves to be about three times larger than that of the phase fading in the case of a pure random signal. This result is completely independent of the special form of the energy spectrum;

(b) predominant background signal. According to (52) the most frequently traversed amplitude level ($a_{\max}$) has to be determined, in this case, from the equation

$$\frac{d}{da}\left[a^{1/2}\exp\left\{-\frac{(a-R_0)^2}{2H^2}\right\}\right]=0.$$

In view of the assumption $R_0^2>>H^2$, the only root is given by $a_{\max}\sim R_0$. Hence we find, applying (52) once again,

$$N_A(a_{\max})\sim\frac{\sigma_\omega}{\pi}. \tag{62}$$

Ratcliffe's definition (48) here leads to a fading rate $(2/\pi)^{1/2}H\sigma_\omega/R_0$, which might have a very different value. On the other hand, we conclude from (57)

$$N_\Phi(\varphi_{\max})=N_\Phi(0)=\frac{\sigma_\omega}{\pi}. \tag{63}$$

We thus arrive at the very simple property that the rapidities of both amplitude fading and phase fading tend to one and the same quantity in this second limiting case.

17. An Extension of the Preceding Theory in View of a Rigorous Treatment of Scatter Propagation

The main physical assumption introduced so far concerned the hardly correlated phases of the Fourier transform $F(\omega)$ at different frequencies; the corresponding fluctuating component $h(t)$ was termed random. The assumption involved a zero value of the first term of (12); in its turn, this property led to the symmetry relations mentioned at the end of section 5. However, the latter only hold approximately if $h(t)$ represents a signal due to (tropospheric or ionospheric) scatter propagation. This can be shown as follows.

The so-called Born approximation (characterized physically by the neglect of the effects of multiple scattering) for the scattered field $E_{sc}(P)$ due to a primary field $E_{pr}(P)$ reads as follows if we assume (a) both fields to be proportional to exp $(-i\omega t)$, (b) the terms of the wave equation that depend explicitly on the gradient or on the time derivative of the refractive index n to be negligible, (c) the refractive index to be given by $n(P,t)=1+\delta n\ (P,t)$:

$$E_{sc}(P,t)=E_{pr}(P,t)\ \frac{k_0^2TP}{2\pi}\iiint_V d\tau_Q\ \frac{\delta_n(Q;t-QR/c)}{TQ.QP}\ e^{ik_0(TQP-TP)}. \tag{64}$$

In this formula T represents the position of the transmitter, P that of the receiver, Q an arbitrary point inside the scattering volume V, $d\tau_Q$ a volume element of the latter, while TP, TQ, QP, and $TQP=TQ+QP$ are the distances referring to the separations of these points; moreover

$$k_0=\omega/c=2\pi/\lambda.$$

The basic expression (64) can be derived as follows. The relevant approximative wave equation for the electric field reads

$$(\Delta+k_0^2n)E=0.$$

This can be put in the form

$$(\Delta+k_0^2)E=-k_0^2\delta nE_{pr}, \tag{65}$$

if the actual field is replaced by the primary field in the correction term proportional to δn, such in accordance with the Born approximation. The well known method for solving inhomogeneous wave equations leads to a solution of (65) that consists of the sum of the primary field E_{pr} (satisfying the homogeneous equation) and of the additional scattered field. The expression (64) is obtained for the latter from the retarded potential solution of (65) when taking into account the proportionality of $E_{\text{pr}}(Q,t)$ to $\exp(ik_0TQ)/TQ$.

The field corresponding to (64) for a time factor $\exp(i\omega t)$ is obtained by replacing ω by $-\omega$ or k_0 by $-k_0$. According to the principal of linear combination the real primary field

$$E_{\text{pr}}(P,t)=E_0(P)\cos(\omega t)=\frac{E_0(P)}{2}e^{i\omega t}+\frac{E_0(P)}{2}e^{-i\omega t}$$

then produces a scattered field given by

$$E_{\text{sc}}(P,t)=E_0(P)\frac{k_0^2TP}{2\pi}\iiint_V d\tau_Q\frac{\delta n(Q;t-QR/c)}{TQ\cdot QP}\cos\{k_0(TQP-TP)-\omega t\}.$$

This field is of the form (2) for $h(t)=E_{\text{sc}}(P,t)$ provided $x(t)$ and $y(t)$ are defined as the real and imaginary part of the relation

$$x(t)+iy(t)=E_0\frac{k_0^2TP}{2\pi}\iiint_V d\tau_Q\frac{\delta n(Q;t-QR/c)}{TQ\cdot QP}\cdot\exp\{ik_0(TQP-TP)\}. \tag{66}$$

However, relations such as $\langle x^2\rangle=\langle y^2\rangle$ then at best hold approximately, as will be discussed in the next section.

18. Basic Correlation Functions of the Extended Theory

We drop the assumption of hardly correlated phases for the Fourier transform $F(\omega)$ of $h(t)$ which led to the symmetry relations of section 5. The only hypotheses left for the fluctuating component $h(t)$, now termed a quasi-random signal, concern the applicability of the ergodic relations of the type (11) and of the central limit theorem. The latter concerns the joint distribution of $x(t_1)$, $y(t_1)$, $x(t_2)$, and $y(t)_2)$ which quantities constituted the values of $x(t)$ and $y(t)$ at two times separated by an interval $\tau=t_2-t_1$. Both terms of (12) are now to be accounted for. The substitution $\omega=\omega_0+\omega'$ transforms the first term of (12) into an integral over the interval $-\omega_0<\omega'<\omega_0$; the new integrand being an even function of ω', the ω' interval can further be reduced to $0<\omega'<\omega_0$. We thus obtain the following complex correlation function

$$\begin{aligned}\langle x_1(x_2+iy_2)\rangle&=\frac{2\pi}{T}\int_0^\infty d\omega F(\omega)F^*(\omega)e^{-i(\omega-\omega_0)\tau}+\frac{4\pi}{T}\int_0^{\omega_0}d\omega F^*(\omega_0+\omega)F^*(\omega_0-\omega)\cos(\omega\tau)\\&=a(\tau)+b(\tau),\ \text{say}.\end{aligned} \tag{67}$$

As a matter of fact, $a(\tau)$ and $b(\tau)$ thus defined are independent of T since $F^*(\omega)$ increases in proportion with $T^{1/2}$ (see section 5) for the large values of T under consideration.

A derivation completely analogous to that of (12) and (67) leads to the alternative complex correlation function

$$\langle y_1(x_2+iy_2)\rangle=\frac{2\pi i}{T}\int_0^\infty d\omega F(\omega)F^*(\omega)e^{-i(\omega-\omega_0)\tau}-\frac{4\pi i}{T}\int_0^{\omega_0}d\omega F^*(\omega_0+\omega)F^*(\omega_0-\omega)\cos(\omega\tau),$$

so as to have

$$\langle y_1(x_2+iy_2)\rangle=ia(\tau)-ib(\tau). \tag{68}$$

An analogous computation yields $\langle x+iy\rangle=(2\pi/T)F^*(\omega_0)$; hence $\langle x\rangle=\langle y\rangle=0$ still holds provided the phase of $F(\omega_0)$ may be considered as random.

The statistical properties to be derived hereafter depend on both $a(\tau)$ and $b(\tau)$. According to (67) and (68) these basic functions of τ can be expressed in terms of the two complex correlation functions considered here. We find

$$a(\tau)=\tfrac{1}{2}<(x_1-iy_1)(x_2+iy_2)>; \quad b(\tau)=\tfrac{1}{2}<(x_1+iy_1)(x_2+iy_2)>. \tag{69}$$

The special value $\tau=0$ corresponds to $x_1\equiv x_2$ and $y_1\equiv y_2$; it leads to the expressions

$$a(0)=\tfrac{1}{2}\{<x^2>+<y^2>\}; \quad b(0)=\tfrac{1}{2}\{<x^2>-<y^2>\}+i<xy>, \tag{70}$$

which clearly show the dependence of the anisotropy $<x^2>\neq<y^2>$ on the function $b(\tau)$. We further mention the connection of $a(\tau)$ with the original correlation function of $h(t)$ which proves to read

$$<h(t)h(t+\tau)>=\mathrm{Re}\{e^{-i\omega\tau}a(\tau)\}.$$

According to (69) the complete functions $a(\tau)$ and $b(\tau)$ can be deduced from averages concerning the quantities $x_1\mp iy_1$ and x_2+iy_2. In the scatter propagation theory these quantities follow from (66) by substituting t_1 and t_2 respectively for t. The products $(x_1\mp iy_1)$ (x_2+iy_2) can then be represented by six-fold integrals in which each pair of three integrations refers to either of two independent integration points Q and Q'. In these integrals the average procedure indicated by the symbol $<>$ affects only products of the type $\delta n(Q;t_1)$ $\delta n(Q';t_2)$, the average $<\delta n(Q;t_1)\ \delta n(Q';t_2)>$ of which constitutes a correlation function with respect to both space and time; its value depends on the physical model to be introduced. The evaluation of $a(\tau)$ and $b(\tau)$, outlined here, results in the expressions

$$a(\tau)=\frac{k_0^4E_0^2\cdot TP^2}{8\pi^2}\int_V\frac{d\tau_Q}{TQ\cdot QP}\int_V\frac{d\tau_{Q'}}{TQ'\cdot Q'P}\cdot<\delta n(Q;t_1-QP/c)\cdot\delta n(Q';t_2-Q'P/c)>\cdot e^{ik_0(TQ'P-TQP)}, \tag{71}$$

and

$$b(\tau)=\frac{k_0^4E_0^2\cdot TP^2}{8\pi^2}\int_V\frac{d\tau_Q}{TQ\cdot QP}\int_V\frac{d\tau_{Q'}}{TQ'\cdot Q'P}\cdot<\delta n(Q;t_1-QP/c)\cdot\delta n(Q';t_2-Q'P/c)>\cdot e^{ik_0(TQ'P+TQP-2TP)}. \tag{72}$$

The simplified theory of sections 1 through 16 amounts to taking $b(\tau)=0$. As a matter of fact, $b(\tau)$ will be much smaller than $a(\tau)$ in very general circumstances, as may be explained as follows. The correlation function occurring in (71) and (72) is maximum for $Q\equiv Q'$. Hence sets of neighbouring points Q and Q' will provide the main contribution to these two 6-fold integrals. For each such set the exponential phase in (71) is near the value zero corresponding to $Q=Q'$; in other words, the contributions from the sets in question are almost in phase. On the other hand, the corresponding exponential phase in (72) approaches for $Q'\to Q$ to the value $2k_0(TQP-TP)$ which may be widely different for sets $Q\sim Q'$ situated in various parts of the scattering volume V; the resulting interference will reduce (72) to a very low value, compared to that of (71), especially in the case of large distances TP and a small value of the scale of turbulence l.

At small distances, however, the interference effects in (72) are no longer as pronounced as those in (71), and $a(\tau)$ and $b(\tau)$ may become of equal order of magnitude. This is shown numerically in an investigation by Fannin [8] which is based on the Gaussian correlation function

$$<\delta n(Q;t_1)\delta n(Q';t_2)>=<\delta n^2>\cdot\ \exp\left(-\frac{u^2\tau^2}{l^2}-\rho^2\right),$$

in which

$$\rho^2=\frac{(x_{Q'}-x_Q-\tau v_x)^2+(y_{Q'}-y_Q-\tau v_y)^2+(z_{Q'}-z_Q-\tau v_z)^2}{l^2};$$

the parameter u occurring here can be interpreted as the rms value of a wind velocity with random direction, v_x, v_y, v_z as the components of the mean wind velocity $v=(v_x^2+v_y^2+v_z^2)^{1/2}$, l as a scale of turbulence (assumed as isotropic). The quantity $q=\lambda TP/(4\pi l^2)$ proves to be the decisive parameter, large values of which lead to limiting conditions characterized by $<x^2>=<y^2>$, that is, to the conditions of our simplified theory for $b(\tau)=0$. The asymmetry $\langle x^2\rangle \neq \langle y^2\rangle$ is shown numerically by Fannin[3]. The relation $\langle x^2\rangle = \langle y^2\rangle$ only holds at $q=\infty$ while the sum $\langle x^2\rangle + \langle y^2\rangle$ proves to be independent of q. Similar results have been derived by Rice [9] for the other correlation function

$$\langle \delta n(Q;t_1)\delta n(Q';t_2)\rangle = \langle \delta n^2\rangle \cdot \rho K_1(\rho).$$

19. Joint Probability Density of Amplitude, Phase and Their Time Derivatives for a Quasi-Random Signal Imposed on a Constant Signal

The simplifications inherent to the simplified theory ($b\equiv 0$) enabled the explicit derivation of the joint probability density (24) for the amplitudes R_1 and R_2 and the phases Φ_1 and Φ_2 at two times separated by a finite interval τ. The fading properties were obtained thereafter by passing (with the aid of a limiting procedure at $\tau=0$) to the joint probability density for the amplitude R, phase Φ and their time derivatives $\dot{R}$ and $\dot{\Phi}$ at one and the same moment. However, the former probability density becomes very complicated in the extended theory, but the much simpler expression for the latter density may then be arrived at as discussed in this section; another simpler derivation (using the central-limit theorem for the distribution of $x, y, \dot{x}, \dot{y}$) will be indicated in the next section. The relation (32) expressing the limiting procedure still holds, as well as the connection (see section 8) between the probability densities in polar and cartesian coordinates, viz,

$$G(R_1,R_2,\Phi_1,\Phi_2)=R_1R_2f(x_1,x_2,x_3,x_4).$$

A combination of both these connections yields

$$\begin{aligned} G(R,\dot{R},\Phi,\dot{\Phi}) &= \operatorname*{Lim}_{\tau\to 0}\,[\tau^2 G\{R(t),R(t+\tau),\phi(t),\phi(t+\tau),\tau\}] \\ &= R^2 \operatorname*{Lim}_{\tau\to 0}\,[\tau^2 f\{x(t),x(t+\tau),y(t),y(t+\tau)\}]. \end{aligned}$$

In view of (23) we further have,

$$\left.\begin{aligned} x_1 &= x(t)=R(t)\cos\Phi(t)-R_0, \\ x_2 &= x(t+\tau)=R(t+\tau)\cos\Phi(t+\tau)-R_0=R\cos\Phi-R_0+\tau(\dot{R}\cos\Phi-R\sin\Phi\dot{\Phi})+\ldots, \\ x_3 &= y(t)=R(t)\sin\Phi(t), \\ x_4 &= y(t+\tau)=R(t+\tau)\sin\Phi(t+\tau)=R\sin\Phi+\tau(\dot{R}\sin\Phi+R\cos\Phi\dot{\Phi})+\ldots, \end{aligned}\right\} \tag{73}$$

so as to arrive finally at

$$\begin{aligned} G(R,\dot{R},\Phi,\dot{\Phi}) = R^2 \operatorname*{Lim}_{\tau\to 0}\,\{\tau^2 f[&R\cos\Phi-R_0, R\cos\Phi-R_0+\tau(\dot{R}\cos\Phi-R\sin\Phi\dot{\Phi})+\ldots, \\ &R\sin\Phi, R\sin\Phi+\tau(\dot{R}\sin\Phi+R\cos\Phi\dot{\Phi})+\ldots]\}. \end{aligned} \tag{74}$$

The function f is given explicitly by (10) in which $(\Lambda)^{-1}_{ik}$ can be replaced by $M_{ik}/|\Lambda|$ if M_{ik} represents the cofactor of Λ that corresponds to the matrix element labeled ik. A substitution of (10) into (74) thereupon yields

$$G(R,\dot{R},\Phi,\dot{\Phi}) = \frac{R^2}{4\pi^2}\operatorname*{Lim}_{\tau\to 0}\frac{\tau^2}{\{\Lambda(\tau)\}^{1/2}}\cdot\exp\left\{-\frac{1}{2\Lambda(\tau)}\sum_{i=1}^{4}\sum_{k=1}^{4} M_{ik}(\tau)x_i(\tau)x_k(\tau)\right\}. \tag{75}$$

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[3] Figure 1 in both references [8]; the two ordinates of this diagram can be interpreted as representing the parameters

$$\langle y^2\rangle/(\langle y^2\rangle)_{TP=\infty} \text{ and } \langle x^2\rangle/(\langle x^2\rangle)_{TP=\infty}$$

as functions of q.

The functions $M_{ik}(\tau)$ are the cofactors of the matrix (14), the elements of which can be represented as follows in terms of $a(\tau)$ and $b(\tau)$, applying (69) and (70):

$$\Lambda=\begin{Vmatrix} a(0)+\mathrm{Re}\, b(0) & \mathrm{Re}\{a(\tau)+b(\tau)\} & \mathrm{Im}\, b(0) & \mathrm{Im}\{a(\tau)+b(\tau)\} \\ \mathrm{Re}\{a(\tau)+b(\tau)\} & a(0)+\mathrm{Re}\, b(0) & -\mathrm{Im}\{a(\tau)-b(\tau)\} & \mathrm{Im}\, b(0) \\ \mathrm{Im}\, b(0) & -\mathrm{Im}\{a(\tau)-b(\tau)\} & a(0)-\mathrm{Re}\, b(0) & \mathrm{Re}\{a(\tau)-b(\tau)\} \\ \mathrm{Im}\{a(\tau)+b(\tau)\} & \mathrm{Im}\, b(0) & \mathrm{Re}\{a(\tau)-b(\tau)\} & a(0)-\mathrm{Re}\, b(0) \end{Vmatrix}. \tag{76}$$

The limit (75) now proves to depend on the four lowest-order terms of the expansions of $a(\tau)$ and $b(\tau)$ with respect to τ. These expansions are simplified when assuming, once again, $\bar{\omega}=\omega_0$ (see section 9); we then have, applying the definition of $a(\tau)$ in (67),

$$a'(0)=-\frac{2\pi i}{T}\int_0^\infty d\omega F(\omega)F^*(\omega)(\omega-\omega_0)=-\frac{2\pi i}{T}\int_0^\infty d\omega W(\omega)(\omega-\omega_0)=-\frac{2\pi i}{T}(\bar{\omega}-\omega_0).$$

Hence our assumption involves $a'(0)=0$. Moreover, both Re $a(\tau)$ and $b(\tau)$ are recognized as even functions of τ, and Im $a(\tau)$ as an odd function of τ. According to these properties special τ powers do not occur in the expansions of $a(\tau)$ and $b(\tau)$. The remaining terms can be represented as follows, restricting ourselves to terms up to the fourth order:

$$\left.\begin{aligned} \mathrm{Re}\, a(\tau)&=a(0)+\frac{\tau^2}{2}a''(0)+\frac{\tau^4}{24}a''''(0)+\ldots, \\ \mathrm{Im}\, a(\tau)&=\frac{\tau^3}{6}\mathrm{Im}\, a'''(0)+\ldots=-i\frac{\tau^3}{6}a'''(0)+\ldots, \\ \mathrm{Re}\, b(\tau)&=\mathrm{Re}\, b(0)+\frac{\tau^2}{2}\mathrm{Re}\, b''(0)+\frac{\tau^4}{24}\mathrm{Re}\, b''''(0)+\ldots, \\ \mathrm{Im}\, b(\tau)&=\mathrm{Im}\, b(0)+\frac{\tau^2}{2}\mathrm{Im}\, b''(0)+\frac{\tau^4}{24}\mathrm{Im}\, b''''(0)+\ldots. \end{aligned}\right\} \tag{77}$$

The substitution of these expressions into the cofactors of (76) shows that the τ expansion of any M_{ik} does start as follows:

$$M_{ik}(\tau)=l_{ik}\tau^2+m_{ik}\tau^4+\ldots,$$

whereas the lowest order term of $\Lambda(\tau)$ proves to be proportional to τ^4. Hence (75) may be replaced by,

$$G(R,\dot{R},\Phi,\dot{\Phi})=\frac{R^2}{4\pi^2}\cdot\lim_{\tau\to0}\frac{\tau^2}{\{\Lambda(\tau)\}^{1/2}}\cdot\exp\left[-\frac{1}{2}\lim_{\tau\to0}\frac{\tau^4}{|\Lambda(\tau)|}\left\{\sum_{i=1}^4\sum_{k=1}^4 l_{ik}\lim_{\tau\to0}\frac{x_i(\tau)x_k(\tau)}{\tau^2}+\sum_{i=1}^4\sum_{k=1}^4 m_{ik}x_i(0)x_k(0)\right\}\right]. \tag{78}$$

All coefficients $l_{ik}=l_{ki}$ are to be determined individually, each of them being connected with a different product $x_i(\tau)x_k(\tau)$. The substitution of (77) into the various cofactors of (76), and a reduction of the dominating τ contributions of the latter with the aid of elementary properties of determinants, leads to values for l_{ik} that can be represented by the following matrix scheme:

$$\|l_{ik}\|=\{a^2(0)-|b(0)|^2\}.\begin{Vmatrix} -\mathrm{Re}\{a''(0)-b''(0)\} & \mathrm{Re}\{a''(0)-b''(0)\} & \mathrm{Im}\, b''(0) & -\mathrm{Im}\, b''(0) \\ \mathrm{Re}\{a''(0)-b''(0)\} & -\mathrm{Re}\{a''(0)-b''(0)\} & -\mathrm{Im}\, b''(0) & \mathrm{Im}\, b''(0) \\ \mathrm{Im}\, b''(0) & -\mathrm{Im}\, b''(0) & -\mathrm{Re}\{a''(0)+b''(0)\} & \mathrm{Re}\{a''(0)+b''(0)\} \\ -\mathrm{Im}\, b''(0) & \mathrm{Im}\, b''(0) & \mathrm{Re}\{a''(0)+b''(0)\} & -\mathrm{Re}\{a''(0)+b''(0)\} \end{Vmatrix}.$$

The resulting expression for the first bilinear form in (78) can be put in the form,

$$\sum_{i=1}^{4}\sum_{k=1}^{4} l_{ik}x_i(\tau)x_k(\tau)=\{a^2(0)-|b(0)|^2\}\cdot[-\mathrm{Re}\{a''(0)-b''(0)\}\cdot(x_1-x_2)^2$$
$$\mathrm{Re}\{a''(0)+b''(0)\}\cdot(x_3-x_4)^2+2\mathrm{Im}\ b''(0)\cdot(x_1-x_2)(x_3-x_4)]$$

The corresponding limit

$$\operatorname*{Lim}_{\tau\to 0}\sum\sum l_{ik}\frac{x_i(\tau)x_k(\tau)}{\tau^2}$$

is next obtained by applying the following relations resulting from (73):

$$\operatorname*{Lim}_{\tau\to 0}\frac{x_1-x_2}{\tau}=-\dot{R}\cos\Phi+R\sin\Phi\dot{\Phi};\qquad \operatorname*{Lim}_{\tau\to 0}\frac{x_3-x_4}{\tau}=-\dot{R}\sin\Phi-R\cos\Phi\dot{\Phi}.$$

We next consider the second bilinear form occurring in (78). The rather simple values of the quantities $x_1(0)=x_2(0)=R\cos\Phi-R_0$ and $x_3(0)=x_4(0)=R\sin\Phi$ involve the following reduction:

$$\sum_{i=1}^{4}\sum_{k=1}^{4} m_{ik}x_i(0)x_k(0)=(m_{11}+2m_{12}+m_{22})(R\cos\Phi-R_0)^2$$
$$+(m_{33}+2m_{34}+m_{44})R^2\sin^2\Phi+2(m_{13}+m_{14}+m_{23}+m_{24})(R\cos\Phi-R_0)R\sin\Phi.$$

The new coefficients occurring here can be determined by first adding the relevant cofactors M_{ik}, and by reducing their sum to a simplest form with the aid of elementary rules for determinants; the determinant then obtained for each of these coefficients proves to be of the order of τ^4 when applying (77). The coefficient of the dominating term (proportional to τ^4) thus obtained involves the expressions

$$m_{11}+2m_{12}+m_{22}=\{a(0)-\mathrm{Re}\ b(0)\}\cdot\{a''(0)^2-|b''(0)|^2\},$$
$$m_{33}+2m_{34}+m_{44}=\{a(0)+\mathrm{Re}\ b(0)\}\cdot\{a''(0)^2-|b''(0)|^2\},\text{ and}$$
$$m_{13}+m_{14}+m_{23}+m_{24}=-\mathrm{Im}\ b(0)\cdot\{a''(0)^2-|b''(0)|^2\}.$$

Finally, another reduction with the aid of elementary rules for determinants converts the complete determinant $|\Lambda(\tau)|$ into a form most suitable for a direct substitution of the expansions (77). In this form all elements referring to $\tau=0$ are replaced by differences between the values of the corresponding elements at an arbitrary τ and those at $\tau=0$. The proportionality of the lowest order approximation for $|\Lambda(\tau)|$ to τ^4 is then recognized at once; the evaluation of the corresponding determinant leads to the limit,

$$\operatorname*{Lim}_{\tau\to 0}\frac{|\Lambda(\tau)|}{\tau^4}=\{a^2(0)-|b(0)|^2\}\cdot\{a''^2(0)-|b''(0)|^2\}.$$

The final substitution of all these results into (78) yields the following explicit expression for the joint probability density under consideration:

$$G(R,\dot{R},\Phi,\dot{\Phi})=\frac{R^2}{4\pi^2\{a^2(0)-|b(0)|^2\}^{1/2}\cdot\{a''^2(0)-|b''(0)|^2\}^{1/2}}$$
$$\cdot\exp\left[-\frac{\{a(0)-\mathrm{Re}\ b(0)\}(R\cos\Phi-R_0)^2-2\ \mathrm{Im}\ b(0)\ (R\cos\Phi-R_0)R\sin\Phi+\{a(0)+\mathrm{Re}\ b(0)\}R^2\sin^2\Phi}{2\{a^2(0)-|b(0)|^2\}}\right]$$
$$\cdot\exp\left[\frac{\begin{array}{r}\mathrm{Re}\ \{a''(0)-b''(0)\}(\dot{R}\cos\Phi-R\sin\Phi\dot{\Phi})^2-2\ \mathrm{Im}\,b''(0)(\dot{R}\cos\Phi-R\sin\Phi\dot{\Phi})\\ (\dot{R}\sin\Phi+R\cos\Phi\dot{\Phi})+\mathrm{Re}\ \{a''(0)+b''(0)\}(\dot{R}\sin\Phi+R\cos\Phi\dot{\Phi})^2\end{array}}{2\{a''^2(0)-|b''(0)|^2\}}\right].\qquad(79)$$

20. Preceding Probability Density in Terms of the Orthogonal Parameters $x(t)$ and $y(t)$

The intricacy of (79) is partly due to the fact that the amplitude $R(t)$ and the phase $\Phi(t)$, though being the most directly observable quantities, do not represent the most convenient parameters for a description of the complete signal $H(t)$ of (20). In this respect $x(t)$ and $y(t)$ constitute simpler elements; they can be considered as those components of the quasi-random contribution $h(t)$ that are in phase and in quadrature respectively, with the steady signal $R_0 \cos(\omega_0 t)$. The relations between the amplitude, the phase and the latter orthogonal components are illustrated geometrically by figure 1.

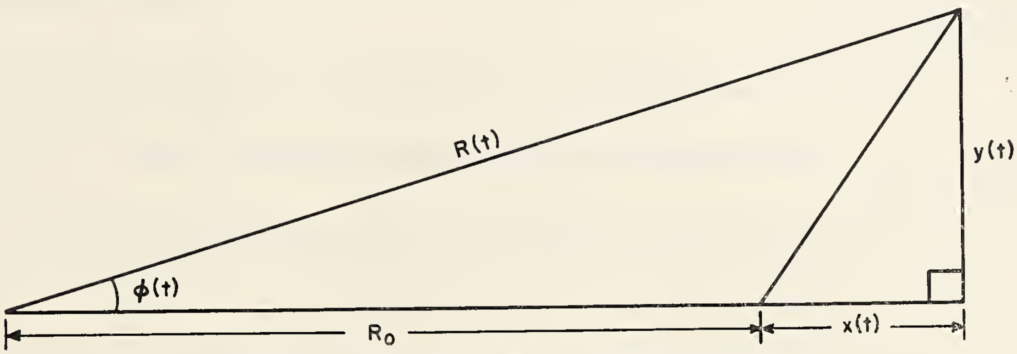

FIGURE 1. *Relation between the amplitude and phase.*

We next investigate the equivalent of (79) in terms of $x(t)$, $y(t)$, and their time derivatives $\dot{x}(t)$, $\dot{y}(t)$. The connection between these new variables and the former set R, Φ, $\dot{R}$, $\dot{\Phi}$ is given by (compare eq 21),

$$\left.\begin{aligned} x&=R\cos\Phi-R_0; & \dot{x}&=\dot{R}\cos\Phi-R\sin\Phi\,\dot{\Phi},\\ y&=R\sin\Phi; & \dot{y}&=\dot{R}\sin\Phi+R\cos\Phi\,\dot{\Phi}.\end{aligned}\right\}\tag{80}$$

The evaluation of the corresponding Jacobi determinant involves the relation,

$$dx\,dy\,d\dot{x}\,d\dot{y}=R^2\,dR\,d\Phi\,d\dot{R}\,d\dot{\Phi}.$$

Hence the former probability density $G(R, \dot{R}, \Phi, \dot{\Phi})$ and the new one $g\,(x, \dot{x}, y, y)$ are connected as follows:

$$R^2\,g(x,\dot{x},y,\dot{y})=G(R,\dot{R},\Phi,\dot{\Phi}).$$

Therefore, the function g is obtained by dividing (79) by R^2 while introducing the new variables x, $\dot{x}$, y, $\dot{y}$ in accordance with (80). The result reads

$$g(x,\dot{x},y,\dot{y})=\frac{1}{4\pi^2}\,\frac{\exp\left[-\dfrac{\{a(0)-\mathrm{Re}\,b(0)\}x^2-2\,\mathrm{Im}\,b(0)xy+\{a(0)+\mathrm{Re}\,b(0)\}y^2}{2\{a^2(0)-|b(0)|^2\}}\right]}{\{a^2(0)-|b(0)|^2\}^{1/2}}$$

$$\times\frac{\exp\left[-\dfrac{-\{a''(0)-\mathrm{Re}\,b''(0)\}\dot{x}^2+2\,\mathrm{Im}\,b''(0)\dot{x}\dot{y}-\{a''(0)+\mathrm{Re}\,b''(0)\}\dot{y}^2}{2\{a''(0)^2-|b''(0)|^2\}}\right]}{\{a''(0)^2-|b''(0)|^2\}^{1/2}}.\tag{81}$$

The derivation of this result and the equivalent expression (79) is essentially based on a heuristic method deducing, with the aid of a limiting procedure, the distribution of R, $\dot{R}$, Φ, $\dot{\Phi}$ from that of R_1, R_2, Φ_1, and Φ_2. The same result can also be arrived at in a simpler way as follows. We assume the applicability of the central-limit theorem when deriving the probability density of the orthogonal quantities x, $\dot{x}$, y, $\dot{y}$. We then have to consider the matrix

$$\Lambda_\tau = \begin{Vmatrix} \langle x^2\rangle & \langle xy\rangle & \langle x\dot{x}\rangle & \langle x\dot{y}\rangle \\ \langle yx\rangle & \langle y^2\rangle & \langle y\dot{x}\rangle & \langle y\dot{y}\rangle \\ \langle \dot{x}x\rangle & \langle \dot{x}y\rangle & \langle \dot{x}^2\rangle & \langle \dot{x}\dot{y}\rangle \\ \langle \dot{y}x\rangle & \langle \dot{y}y\rangle & \langle \dot{y}\dot{x}\rangle & \langle \dot{y}^2\rangle \end{Vmatrix}, \tag{82}$$

all elements of which can be connected with $a(\tau)$ and $b(\tau)$. In fact, we first derive by differentiating (69), remembering that $t_2=t_1+\tau$,

$$a'(\tau)=\tfrac{1}{2}\langle(x_1-iy_1)(\dot{x}_2+i\dot{y}_2)\rangle; \qquad b'(\tau)=\tfrac{1}{2}\langle(x_1+iy_1)(\dot{x}_2+i\dot{y}_2)\rangle. \tag{83}$$

We next replace τ by $-\tau$ which corresponds to interchanging the subscripts 1 and 2. Next, a further differentiation yields,

$$a''(-\tau)=-\tfrac{1}{2}\langle(\dot{x}_1+i\dot{y}_1)(\dot{x}_2-i\dot{y}_2)\rangle; \qquad b''(-\tau)=-\tfrac{1}{2}\langle(\dot{x}_1+i\dot{y}_1)(\dot{x}_2+i\dot{y}_2)\rangle. \tag{84}$$

The substitution $\tau=0$ in (83) and (84) leads to the other relations,

$$a'(0)=\tfrac{1}{2}\langle(x-iy)(\dot{x}+i\dot{y})\rangle; \qquad b'(0)=\tfrac{1}{2}\langle(x+iy)(\dot{x}+i\dot{y})\rangle, \tag{85}$$

$$a''=-\tfrac{1}{2}\langle(\dot{x}^2+\dot{y}^2)\rangle; \qquad b''(0)=-\tfrac{1}{2}\langle(\dot{x}+i\dot{y})^2\rangle. \tag{86}$$

On the other hand, we have $a'(0)=b'(0)=0$ [see (77)]. The real and imaginary parts of (85) therefore involve the properties $\langle x\dot{x}\rangle=\langle y\dot{y}\rangle=\langle x\dot{y}\rangle=\langle\dot{x}y\rangle=0$. The remaining elements of (82) follow from the real and imaginary parts of (70) and (86). The evaluation of (82) thus yields the final form

$$\Lambda_\tau = \begin{Vmatrix} a(0)+\operatorname{Re} b(0) & \operatorname{Im} b(0) & 0 & 0 \\ \operatorname{Im} b(0) & a(0)-\operatorname{Re} b(0) & 0 & 0 \\ 0 & 0 & -a''(0)-\operatorname{Re} b''(0) & -\operatorname{Im} b''(0) \\ 0 & 0 & -\operatorname{Im} b''(0) & -a''(0)+\operatorname{Re} b''(0) \end{Vmatrix}.$$

The evaluation of the corresponding reciprocal matrix Λ_τ^{-1} leads to (81) when applying the general formula (10).

The expression (81), which has thus been derived in two different ways, is of special interest for large background signals. The following expansions with respect to R_0^{-1} are then applicable:

$$R(t)=\{(R_0+x)^2+y^2\}^{1/2}=R_0+x+\frac{y^2}{2R_0}+\ldots,$$

$$\Phi(t)=\arctan\frac{y}{R_0+x}=\frac{y}{R_0}-\frac{xy}{R_0^2}+\frac{x^2y-(1/3)y^3}{R_0^3}+\ldots.$$

We infer that y/R_0 and x represent approximately the phase, and the deviation of the amplitude from its approximate average value R_0 for those signals. It turns out that the statistical properties of the limiting case in question can be derived much more conveniently from (81) than from (79).

21. Elimination of the Time Derivatives for a Quasi-Random Signal Imposed on a Constant Signal

We return to the probability density (79) expressed in terms of the amplitude and the phase. All distribution functions become much more complicated than in the simplified case in which $b(\tau)=0$ (random signal superimposed on a constant signal). We first derive the probability density $G(R,\Phi)$ defined such that $G(R,\Phi)dRd\Phi$ constitutes the joint probability of finding R and Φ in prescribed infinitesimal intervals. The function $G(R,\Phi)$ is arrived at by integrating (79) over both $\dot{R}$ and $\dot{\Phi}$. The integration over $\dot{\Phi}$ (from $-\infty$ to $+\infty$) is simplest; it only concerns the second exponential factor of (79), the exponent of which is a polynomial of second degree in $\dot{\Phi}$. Hence this integration over $\dot{\Phi}$ amounts, after a shift of the integration variable, to the evaluation of a Poisson integral. Obviously, the expression resulting from this first integration represents the three-dimensional joint probability density $G(R,\dot{R},\Phi)$ for $R,\dot{R}$, and Φ; we obtain

$$G(R,\dot{R},\Phi)=\frac{1}{(2\pi)^{3/2}\{a^2(0)-|b(0)|^2\}^{1/2}}\cdot\frac{R}{\{-a''(0)-\mathrm{Re}\,b''(0)\cdot\cos(2\Phi)-\mathrm{Im}\,b''(0)\cdot\sin(2\Phi)\}^{1/2}}$$
$$\cdot\exp\left[-\frac{\dot{R}^2}{2\{-a''(0)-\mathrm{Re}\,b''(0)\cdot\cos(2\Phi)-\mathrm{Im}\,b''(0)\cdot\sin(2\Phi)\}}\right]$$
$$\cdot\exp\left[-\frac{\{a(0)-\mathrm{Re}\,b(0)\}(R\cos\Phi-R_0)^2-2\,\mathrm{Im}\,b(0)\,(R\cos\Phi-R_0)R\sin\Phi+\{a(0)+\mathrm{Re}\,b(0)\}R^2\sin^2\Phi}{2\{a^2(0)-|b(0)|^2\}}\right]. \tag{87}$$

The further integration over $\dot{R}$ (also from $-\infty$ to $+\infty$) involves another Poisson integral. The two dimensional probability density $G(R,\Phi)$, then obtained, reads

$$G(R,\Phi)=\frac{\exp\left[-\frac{R_0^2\{a(0)-\mathrm{Re}\,b(0)\}}{2\{a^2(0)-|b(0)|^2\}}\right]}{2\pi\{a^2(0)-|b(0)|^2\}^{1/2}}R$$
$$\cdot\exp\left(-\frac{\{a(0)-\mathrm{Re}\,b(0)\cdot\cos(2\Phi)-\mathrm{Im}\,b(0)\cdot\sin(2\Phi)\}R^2-2[\{a(0)-\mathrm{Re}\,b(0)\}\cos\Phi-\mathrm{Im}\,b(0)\sin\Phi]R_0R}{2\{a^2(0)-|b(0)|^2\}}\right). \tag{88}$$

The individual distribution functions $G(R)$ and $G(\Phi)$ for the amplitude and phase can now be derived by integrating this expression with respect to Φ and R, respectively (see the next two sections).

We also mention the joint probability density $g(x,y)$ for x and y, which is obtained by eliminating $\dot{x}$ and $\dot{y}$ from (81). This elimination is performed by integrating the second exponential of (81) over $\dot{x}$ and $\dot{y}$ (from $-\infty$ to $+\infty$); it amounts to the evaluation of a two-dimensional Poisson integral of the type

$$\iint_{-\infty}^{\infty} d\xi d\eta\, e^{-(a\xi^2+2b\xi\eta+c\eta^2)}=\frac{\pi}{(ac-b^2)^{1/2}}. \tag{89}$$

The application of this formula to the second exponential of (81) yields the required probability density, viz,

$$g(x,y)=\frac{\exp\left[-\frac{\{a(0)-\mathrm{Re}\,b(0)\}x^2-2\,\mathrm{Im}\,b(0)xy+\{a(0)+\mathrm{Re}\,b(0)\}y^2}{2\{a^2(0)-|b(0)|^2\}}\right]}{2\pi\cdot\{a^2(0)-|b(0)|^2\}^{1/2}}. \tag{90}$$

The probability density of a quasi-random signal thus appears as the product of two Gaussian distributions for two properly chosen rectangular components. Further, the distribution given by (90) amounts to a Rayleigh distribution for the amplitude $(x^2+y^2)^{1/2}$ only if there exists complete symmetry with respect to x and y (see also the next section).

22. Distribution of the Amplitude for a Quasi-Random Signal Imposed on a Constant Signal

The relevant distribution function is obtained at once by integrating (88) over Φ, so as to have,

$$G(R)=\frac{\exp\left[-\frac{R_0^2\{a(0)-\mathrm{Re}\,b(0)\}}{2\{a^2(0)-|b(0)|^2\}}\right]}{2\pi\cdot\{a^2(0)-|b(0)|^2\}^{1/2}}\cdot R\exp\left[-\frac{a(0)}{2\{a^2(0)-|b(0)|^2\}}R^2\right]$$

$$\times\int_0^{2\pi}\exp\left(\frac{R^2\{\mathrm{Re}\,b(0)\cdot\cos(2\Phi)+\mathrm{Im}\,b(0)\cdot\sin(2\Phi)\}+2R_0R[\{a(0)-\mathrm{Re}\,b(0)\}\cos\Phi-\mathrm{Im}\,b(0)\sin\Phi]}{2\{a^2(0)-|b(0)|^2\}}\right)\cdot d\Phi. \tag{91}$$

The expression in front of the integral has the form of a Rayleigh distribution; unfortunately, the integral cannot be reduced to elementary functions in the most general case. However, simple results are obtained in the two well-known limiting cases, viz.

(a) absence of the background signal ($R_0=0$). The integral of (91) reduces to that for a Bessel function I_0 of imaginary argument. We obtain,

$$G(R)=\frac{R\cdot\exp\left[-\frac{a(0)}{2\{a^2(0)-|b(0)|^2\}}R^2\right]\cdot I_0\left[\frac{|b(0)|}{2\{a^2(0)-|b(0)|^2\}}R^2\right]}{\{a^2(0)-|b(0)|^2\}^{1/2}}. \tag{92}$$

In the simplified case $b(\tau)\equiv 0$ this reduces to,

$$G(R)=\frac{R\exp\left\{-\frac{R^2}{2a(0)}\right\}}{a(0)}. \tag{93}$$

The assumption $b(\tau)\equiv 0$ involves $\langle x^2\rangle=\langle y^2\rangle$ and $a(0)=\langle x^2\rangle$ (see eq 70); it shows the required equality of (93) with the former expression (41). A comparison of (92) and (93) demonstrates how the Rayleigh distribution of the simplified theory (fluctuating component random) is modified in the extended theory (fluctuating component quasi-random) by both a change of its parameter and by the addition of a correction factor given by a zero-order Bessel function; however, for this case, $b(\tau)$ usually is very small and the distribution is almost of the Rayleigh type.

(b) predominance of the background signal. Instead of considering the limiting behaviour of (91) for large R_0, we apply the approximation $x\sim R-R_0$ (see section 21) and derive the distribution for R from that of x. The latter is obtained by integrating (90) with respect to y (from $-\infty$ to $+\infty$); we find,

$$g(x)=\frac{\exp\left[-\frac{x^2}{2\{a(0)+\mathrm{Re}\,b(0)\}}\right]}{[2\pi\{a(0)+\mathrm{Re}\,b(0)\}]^{1/2}}. \tag{94}$$

We arrive at the corresponding distribution for R by replacing x by $R-R_0$ so as to obtain

$$G(R)=\frac{\exp\left[-\frac{(R-R_0)^2}{2\{a(0)+\mathrm{Re}\,b(0)\}}\right]}{[2\pi\{a(0)+\mathrm{Re}\,b(0)\}]^{1/2}}. \tag{95}$$

This function reduces for $b=0$ to (42) (neglecting the difference between R/R_0 and 1), as it should.

23. Distribution of the Phase for a Quasi-Random Signal Imposed on a Constant Signal

The function in question is obtained by integrating (88) over R from 0 to ∞. The integral in the resulting formula, viz,

$$G(\Phi)=\frac{\exp\left[-\frac{R_0^2\{a(0)-\mathrm{Re}\,b(0)\}}{2\{a^2(0)-|b(0)|^2\}}\right]}{2\pi\cdot\{a^2(0)-|b(0)|^2\}^{1/2}}$$

$$\cdot\int_0^\infty R\cdot\exp\left[-\frac{a(0)-\mathrm{Re}\,b(0)\cos(2\Phi)-\mathrm{Im}\,b(0)\sin(2\Phi)}{2\{a^2(0)-|b(0)|^2\}}R^2+\frac{\{a(0)-\mathrm{Re}\,b(0)\}\cos\Phi-\mathrm{Im}\,b(0)\sin\Phi}{a^2(0)-|b(0)|^2}R_0R\right]dR, \tag{96}$$

could be expressed in terms of an error function, if needed.

We again discuss the two limiting cases:

(a) absence of the background signal ($R_0=0$). The integral becomes elementary and leads to the following result,

$$G(\Phi)=\frac{\{a^2(0)-|b(0)|^2\}^{1/2}}{2\pi\{a(0)-\mathrm{Re}\,b(0)\cos(2\Phi)-\mathrm{Im}\,b(0)\sin(2\Phi)\}}. \tag{97}$$

The complete randomness of phases, expressed by the constant value of $G(\Phi)$ in the simplified theory, is removed if $b(0)\neq 0$. The most probable Φ value occurs at $\Phi=\frac{1}{2}\arg b(0)$;

(b) predominance of the background signal. In view of the approximation $\Phi\sim y/R_0$ the distribution of the phase can be derived from that of y. The latter is given by a formula similar to (94) in which $\mathrm{Re}\,b(0)$ has to be replaced by $-\mathrm{Re}\,b(0)$. When replacing y by $R_0\Phi$ in the formula for $g(y)$, we get the approximate phase distribution,

$$G(\Phi)=R_0\frac{\exp\left[-\frac{R_0^2}{2\{a(0)-\mathrm{Re}\,b(0)\}}\Phi^2\right]}{[2\pi\{a(0)-\mathrm{Re}\,b(0)\}]^{1/2}}. \tag{98}$$

24. Number of Crossings of the Amplitude Through a Specific Level for a Quasi-Random Signal Imposed on a Constant Signal

This number is obtained by an application of (47). The probability density $G(R,\dot{R})$, then to be known, follows from an integration of the other probability density $G(R,\dot{R},\Phi)$ of (87) over Φ, viz,

$$G(R,\dot{R})=\int_0^{2\pi}G(R,\dot{R},\Phi)\,d\Phi.$$

Hence the number in question is given by the double integral,

$$N_A(R)=2\int_0^\infty d\dot{R}\,\dot{R}\int_0^{2\pi}d\Phi\,G(R,\dot{R},\Phi). \tag{99}$$

If the order of integration is inverted, the integral over $\dot{R}$ becomes elementary. The remaining integral leads to the expression,

$$N_A(R)=\frac{R\cdot\exp\left[-\frac{a(0)R^2+\{a(0)-\mathrm{Re}\,b(0)\}R_0^2}{2\{a^2(0)-|b(0)|^2\}}\right]}{2^{1/2}\pi^{3/2}\{a^2(0)-|b(0)|^2\}^{1/2}}\cdot\int_0^{2\pi}\{-a''(0)-\mathrm{Re}\,b''(0)\cdot\cos(2\Phi)-\mathrm{Im}\,b''(0)\cdot\sin(2\Phi)\}^{1/2}$$

$$\cdot\exp\left(\frac{R^2\{\mathrm{Re}\,b(0)\cdot\cos(2\Phi)+\mathrm{Im}\,b(0)\cdot\sin(2\Phi)\}+2R_0R[\{a(0)-\mathrm{Re}\,b(0)\}\cos\Phi-\mathrm{Im}\,b(0)\sin\Phi]}{2\{a^2(0)-|b(0)|^2\}}\right)d\Phi. \tag{100}$$

Discussion of the two limiting cases,

(a) absence of the background signal ($R_0=0$). Even in this simplest case the integral in (100) does not reduce to an elementary function. However, neglecting $b''(0)$ with respect to $a''(0)$, we get the approximation,

$$N_A(R)\sim\left(\frac{2}{\pi}\right)^{1/2}\left\{\frac{-a''(0)}{a^2(0)-|b(0)|^2}\right\}^{1/2}\cdot R\cdot\exp\left[-\frac{a(0)}{2\{a^2(0)-|b(0)|^2\}}R^2\right]\cdot I_0\left[\frac{|b(0)|}{2\{a^2(_0)-|b(0)|^2\}}R^2\right]; \qquad (101)$$

(b) predominance of the background signal. According to the approximation $x\sim R-R_0$, the number of crossings of the amplitude through R can be reduced to that of x through $R-R_0$. From (47) we deduce,

$$N_x(R-R_0)=2\int_0^\infty d\dot{x}\dot{x}g(R-R_0,\dot{x}), \qquad (102)$$

in which the joint probability density $g(x,\dot{x})$ for x and $\dot{x}$ results from an integration of $g(x,\dot{x},y,\dot{y})$, represented by (81), over both y and $\dot{y}$. The result reads,

$$g(x,\dot{x})=\frac{\exp\left[-\frac{x^2}{2\{a(0)+\mathrm{Re}\,b(0)\}}+\frac{\dot{x}^2}{2\{a''(0)+\mathrm{Re}\,b''(0)\}}\right]}{2\pi\cdot\{a(0)+\mathrm{Re}\,b(0)\}^{1/2}\cdot\{-a''(0)-\mathrm{Re}\,b''(0)\}^{1/2}}. \qquad (103)$$

When substituting this function into (102), the evaluation of N_x becomes elementary. The final formula reads,

$$N_A(R)=\frac{\{-a''(0)-\mathrm{Re}\,b''(0)\}^{1/2}}{\pi\cdot\{a(0)+\mathrm{Re}\,b(0)\}^{1/2}}\cdot\exp\left[-\frac{(R-R_0)^2}{2\{a(0)+\mathrm{Re}\,b(0)\}}\right]. \qquad (104)$$

25. Number of Crossings of the Phase Through a Specific Level for a Quasi-Random Signal Imposed on a Constant Signal

The application of (47) here requires the knowledge of the probability density $G(\Phi,\dot{\Phi})$ for Φ and $\dot{\Phi}$. It results from an integration of the expression (79) for $G(R,\dot{R},\Phi,\dot{\Phi})$ over both R (from 0 to ∞) and $\dot{R}$ (from $-\infty$ to ∞). The integration over $\dot{R}$, though tedious, is elementary. It leads to the intermediate probability density,

$$G(R,\Phi,\dot{\Phi})=\int_{-\infty}^{\infty}G(R,\dot{R},\Phi,\dot{\Phi})\,d\dot{R}$$

$$=\frac{\exp\left[-\frac{R_0^2\{a(0)-\mathrm{Re}\,b(0)\}}{2\{a^2(0)-|b(0)|^2\}}\right]}{(2\pi)^{3/2}\{a^2(0)-|b(0)|^2\}^{1/2}\{\lambda(\Phi)\}^{1/2}}\cdot R^2\cdot\exp\left[-\left\{\frac{\dot{\Phi}^2}{2\lambda(\Phi)}+\frac{a(0)-\mathrm{Re}\,b(0)\cdot\cos(2\Phi)-\mathrm{Im}\,b(0)\cdot\sin(2\Phi)}{2\{a^2(0)-|b(0)|^2\}}\right\}R^2\right]$$
$$\times\exp\left[\frac{\{a(0)-\mathrm{Re}\,b(0)\}\cos\Phi-\mathrm{Im}\,b(0)\sin\Phi}{a^2(0)-|b(0)|^2}R_0R\right],$$

in which

$$\lambda(\Phi)\equiv-a''(0)+\mathrm{Re}\,b''(0)\cdot\cos(2\Phi)+\mathrm{Im}\,b''(0)\cdot\sin(2\Phi). \qquad (105)$$

The further integration over R yields the required probability density,

$$G(\Phi,\dot{\Phi})=\frac{\exp\left[-\frac{R_0^2\{a(0)-\mathrm{Re}\,b(0)\}}{2\{a^2(0)-|b(0)|^2\}}\right]}{(2\pi)^{3/2}\{a^2(0)-|b(0)|^2\}^{1/2}\{\lambda(\Phi)\}^{1/2}}$$
$$\times\int_0^\infty dR\cdot R^2\cdot\exp\left[-\frac{\dot{\Phi}^2R^2}{2\lambda(\Phi)}-\frac{a(0)-\mathrm{Re}\,b(0)\cos(2\Phi)-\mathrm{Im}\,b(0)\sin(2\Phi)}{2\{a^2(0)-|b(0)|^2\}}R^2\right.$$
$$\left.+\frac{\{a(0)-\mathrm{Re}\,b(0)\}\cos\Phi-\mathrm{Im}\,b(0)\sin\Phi}{a^2(0)-|b(0)|^2}R_0R\right], \qquad (106)$$

which might also be expressed in terms of an error function. The application of (47) according to

$$N_{\Phi}(\varphi)=2\int_0^\infty d\dot{\Phi}\,\dot{\Phi}\,G(\varphi,\dot{\Phi}) \tag{107}$$

yields the average number of crossings (per unit time) of the phase through a specific level φ. By substituting (106) into (107) we get a double integral which can be integrated at once with respect to $\dot{\Phi}$ after inverting the order of the integrations. The remaining integral over R can be transformed, applying a shift of the integration variable, into an error function. The final result reads,

$$N_{\phi}(\varphi)=\frac{\{\lambda(\varphi)\}^{1/2}}{\pi^{3/2}}\frac{\exp\left[-\frac{R_0^2\{a(0)-\mathrm{Re}\,b(0)\}}{2\{a^2(0)-|b(0)|^2\}}\right]}{\{a(0)-\mathrm{Re}\,b(0)\cos(2\Phi)-\mathrm{Im}\,b(0)\sin(2\Phi)\}^{1/2}}\cdot e^{u^2(\varphi)}\int_{-u(\varphi)}^{\infty}e^{-s^2}ds, \tag{108}$$

in which $\lambda(\varphi)$ is defined by (105) and u by

$$u(\varphi)=\frac{R_0[\{a(0)-\mathrm{Re}\,b(0)\}\cos\varphi-\mathrm{Im}\,b(0)\sin\varphi]}{2^{1/2}\{a^2(0)-|b(0)|^2\}^{1/2}\{a(0)-\mathrm{Re}\,b(0)\cos(2\varphi)-\mathrm{Im}\,b(0)\sin(2\varphi)\}^{1/2}}. \tag{109}$$

Discussion of the two limiting cases,

(a) absence of the background signal ($R_0=0$). In view of the value $u=0$, applying here, (108) reduces to

$$N_{\phi}(\varphi)=\frac{\{-a''(0)+\mathrm{Re}\,b''(0)\cos(2\varphi)+\mathrm{Im}\,b''(0)\sin(2\varphi)\}^{1/2}}{2\pi\cdot\{a(0)-\mathrm{Re}\,b(0)\cos(2\varphi)-\mathrm{Im}\,b(0)\sin(2\varphi)\}^{1/2}}; \tag{110}$$

(b) predominance of the background signal. The approximation $\Phi\sim y/R_0$ (see section 20) enables a reduction to the number of crossings of y through the level $R_0\varphi$. This number is given by

$$N_y(R_0\varphi)=2\int_0^\infty d\dot{y}\,\dot{y}\,g(R_0\varphi,\dot{y}). \tag{111}$$

The distribution function $g(y,\dot{y})$ can be derived along the same lines as the function $g(x,\dot{x})$ of (103). The formula for $g(y,\dot{y})$ is obtained by replacing x, $\dot{x}$, Re b (0), Re b'' (0) by y, $\dot{y}$, $-$Re b (0), $-$Re $b''(0)$ respectively, in $g(x,\dot{x})$. Substituting this formula in (111) for $y=R_0\varphi$, we find

$$N_{\Phi}(\varphi)=\frac{\{-a''(0)+\mathrm{Re}\,b''(0)\}^{1/2}}{\pi\cdot\{a(0)-\mathrm{Re}\,b(0)\}^{1/2}}\cdot\exp\left[-\frac{R_0^2}{2\{a(0)-\mathrm{Re}\,b(0)\}}\varphi^2\right]. \tag{112}$$

26. General Conclusions

All statistical properties derived depend on two complex correlation functions $a(\tau)$ and $b(\tau)$. The analysis has been based, apart from reasonable mathematical assumptions (e. g., integrations interchangeable in double integrals), on the applicability of the central limit theorem when deriving the distribution of orthogonal quantities connected with the fluctuating component, on the ergodic hypothesis in order to reduce ensemble averages to time averages, and on the symmetry of the energy spectrum as expressed by the relation $\overline{\omega}=\omega_0$ (see section 9). According to its definition (see eq 67), the function $a(\tau)$ is proportional to the Fourier transform of the energy spectrum $W(\omega)$ of the fluctuating part of the signal, whereas $b(\tau)$ is also connected with the influence of the phase in the Fourier synthesis of the fluctuating part. In the case of scatter propagation $a(\tau)$ and $b(\tau)$ are completely determined by the autocorrelation function (with respect to both space and time) for the refractive index (see eq 71 and 72). For a scattered signal superimposed on a constant background signal (being the ordinary signal in the case of line-of-sight propagation), $b(\tau)$ characterizes the asymmetries between the components $x(t)$ and $y(t)$ of the scattered signal that are in phase and in quadrature with the background signal. This is in particular obvious from the relation (see eq 70)

$$\langle x^2\rangle-\langle y^2\rangle=2\mathrm{Re}\,b(0).$$

The circumstances are simplest in the two following limiting cases:

(a) Absence of the background signal, occurring approximately for propagation over distances well beyond the transmitter's horizon. The amplitude of the scattered signal satisfies a Rayleigh distribution, and that of its phase a homogeneous distribution, provided $b(\tau)$ may be neglected altogether (see eq 93 and 97). Fortunately, the numerical evaluation of special models for the correlation coefficient of the refractive index shows that $b(\tau)$ approaches zero at the large distances connected with this limiting case. The simplified theory (sections 1 through 16) based on $b(\tau)\equiv 0$ then constitutes a fair approximation. A striking result of this theory concerns the ratio of the fading rates of the amplitude and of the phase; this ratio should be 3.04, independent of the special correlation function introduced for the refractive index (see eq 61);

(b) predominance of the background signal, as occurring for line of sight propagation at distances well within the transmitter's horizon. In this case the influence of the function $b(\tau)$ is largest; the distribution properties depend on the quantities (see eq 70):

$$\left.\begin{aligned} a(0)+\mathrm{Re}\, b(0)&=\langle x^2\rangle \\ a(0)-\mathrm{Re}\, b(0)&=\langle y^2\rangle, \end{aligned}\right\} \tag{113}$$

the fading rates also on the other quantities:

$$\left.\begin{aligned} -a''(0)-\mathrm{Re}\ b''(0)&=\langle \dot{x}^2\rangle, \\ -a''(0)+\mathrm{Re}\ b''(0)&=\langle \dot{y}^2\rangle. \end{aligned}\right\} \tag{114}$$

The two latter relations follow from (86). Applying (113), the distribution functions of the amplitude and of the phase, both of which are normal (see eq 95 and 98), can be put in the form.

$$G(R)=\frac{\exp\left\{-\dfrac{(R-R_0)^2}{2\langle x^2\rangle}\right\}}{(2\pi\langle x^2\rangle F)^{1/2}}, \qquad G(\varphi)=R_0\frac{\exp\left\{-\dfrac{R_0^2\varphi^2}{2\langle y^2\rangle}\right\}}{(2\pi\langle y^2\rangle)^{1/2}}.$$

Similarly, applying both (113) and (114), the average numbers of crossings per unit time through the levels most frequently passed ($R=R_0$ for the amplitude, $\varphi=0$ for the phase) can then be represented by (see eq 104 and 112)

$$N_{A,\max}=\frac{1}{\pi}\left(\frac{\langle \dot{x}^2\rangle}{\langle x^2\rangle}\right)^{1/2}, \qquad N_{\Phi,\max}=\frac{1}{\pi}\left(\frac{\langle \dot{y}^2\rangle}{\langle y^2\rangle}\right)^{1/2}.$$

Therefore, the fading rates of the amplitude and of the phase differ only insofar as asymmetries do exist with respect to the in-phase and the in-quadrature component of the scattered field (relative to the background field). Observations comparing these fading rates may therefore decide whether such asymmetries play an important role.

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27. References

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