Formulas for Computing
Capacitance and Inductance

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Explicit formulas are given for the computation of (1) the capacitance between conductors having a great variety of geometrical configurations, (2) the inductance, both self- and mutual, of circuits of various shapes, and (3) the electrodynamic forces acting between coils when carrying current. Formulas for skin effect and proximity effect in concentric cables and parallel wires are included. The formulas for the simpler configurations are given in terms of the elementary functions, whereas more complex shapes involve the use of Legendre polynomials, Legendre functions, and elliptic functions. One section is devoted to a discussion of the relation between the Legendre and the elliptic functions.

Introduction

This collection of formulas contains some that are commonly used in electrical work and some that have been specially developed for precision work at this Bureau. This is no attempt at completeness, for there is now available (since 1948) a revised third edition of the earlier compilation of formulas for inductance by Rosa and Grover [1]. This may be consulted for references to original memoirs and also for discussion of the most suitable formula for a given configuration or relative dimensions. Reference may also be made to Dr. Grover's [2] additions to these formulas in 1918 and to his book "Inductance calculations working formulas and tables."

Formulas for capacitance may be found in the second edition, 1924, of a work by J. H. Dellinger, L. E. Whittemore, and R. S. Ould [3]. This contains formulas for inductance and a few for capacitance. It is possible that the aggregate of researches on capacitance up to this time might amount to a collection of capacitance formulas as comprehensive as that of Rosa and Grover for inductance.

The formulas given here contain, in addition to elementary functions, the Legendre polynomials $P_n$, the Legendre functions $Q_{n-\frac{1}{2}}$ and $P_{n-\frac{1}{2}}$ and elliptic functions. It is shown in section 4 how the latter two may be found by use of tables of the two complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$.

These, together with the incomplete integrals $F(\phi, \mathcal{K})$ and $E(\phi, \mathcal{K})$, and $sn\, u$, $cn\, u$, $dn\, u$, etc., may be readily found from the 1947 Smithsonian elliptic functions tables by G. W. and R. M. Spenceley [4]. One point of superiority of this work over that of R. L. Hippisley [5] is that it proceeds by increments of $1^\circ$ in the modular angle instead of $5^\circ$.

Another purely mathematical table of elliptic functions and theta functions that has been found very useful is table 1, (1922) by H. Nagaoka and S. Sakurai [6]. The same authors in (1927) [7, table 2] produced a volume more directly applicable to the calculation.

1 Figures in brackets indicate the literature reference at the end of the paper.
of the force between coils and their self and mutual inductance. For the latter, Nagaoka [8] has also published three formulas that make use of the remarkable convergence rate of the series defining the theta functions in powers of the Jacobian parameter $q$.

Short tables of theta functions are given by Jahnke and Emde [9]. Also short tables of $F(\phi, k)$ and $E(\phi, k)$ were given by B. O. Peirce [10]. The recent work of W. Magnus and F. Oberhettinger [11] is very useful.

These volumes, especially the work of the Spenceleys put the computation of formulas with elliptic functions in quite a different light. Such formulas are not more difficult than those with sines, cosines, and logarithms.

In section 5 are placed a few notes on methods of deriving some of the formulas given here that are not generally available, or perhaps are unpublished. Where space permits, it has been attempted to summarize the entire electric field, on which capacitance is based, or the entire magnetic field underlying the inductance constants. Such a scheme seems desirable on a larger scale than is possible here. Each formula for capacitance requires the evaluation of the electric potential or field at every point of space. For each inductance $L$ or $M$, one must find the vector potential or magnetic field everywhere. The constants $C$, $L$, or $M$ represent a small byproduct, since they are derived from the fields by direct processes. A summary of the more important electric and magnetic fields that have been evaluated to date would probably fit present requirements better than further tabulation of capacitance and inductance.

1. Capacitance

The formulas for capacitance given in this paper are expressed in the centimeter-gram-second electrostatic system of units (unrationalized). If lengths in centimeters are substituted for the corresponding symbols in a formula, the resulting value of $C$ will be the capacitance in cgs electrostatic units. This value should be multiplied by $10/9$ (more precisely $10/c = 1.11277$) to obtain the capacitance in micromicrofarads. The formulas assume a dielectric constant of unity (in the cgs-esu system). If the space between electrodes is filled with a dielectric of permittivity $\varepsilon_r$ relative to empty space, the value of capacitance as computed from the formula should be multiplied by $\varepsilon_r$.

Alternatively, when expressed in the rationalized meter-kilogram-second-ampere (Giorgi) system of units each formula for capacitance would have an additional factor of $4\pi$ (by reason of the rationalization) and also a factor of $10^7/4\pi c^2$ (by reason of the conventionally chosen permittivity of free space). The net result is that with the dimensions expressed in meters, and after multiplying by the combined factor $1.11277 \times 10^{-10}$, the resulting value of $C$ is in farads.

The formulas for inductance and electromagnetic force given in this paper are expressed in the centimeter-gram-second electromagnetic system of units (unrationalized). In using the formulas, lengths should be expressed in centimeters, currents in abamperes (i.e., units of 10 amperes), and the permeability of space should be taken as unity. If this is done, the inductances as computed are in units of $10^{-9}$ henry, forces are in dynes, and torques in dyne-centimeters.

Alternatively in the rationalized meter-kilogram-second-ampere system the formulas should be multiplied by $1/4\pi$ (by reason of rationalization) and by $4\pi \cdot 10^{-7}$ (by reason of
the conventionally chosen permeability of free space). Then if dimensions are expressed in meters, and currents in amperes, the inductances as computed will be in henries, the forces in newtons, and the torques in newton-meters.

The first six figures illustrate two cases. In the axially symmetric cases the figures represent plane sections through the axis of symmetry. In the cylindrical case they are plane sections perpendicular to the endless generators. In this case the formulas give the capacitance \( C/l \) per unit length perpendicular to the plane of the figure.

### 1.1. Parallel Plates With Guard Planes

The separation \( c \) between the parallel plates should be small compared to the radius \( a_1 \) of the disk. Also the radius \( A \) of the plates should be large compared to \( a_2 \), so the field is practically uniform at some place between the edge of the disk and the outer edge of the plates \( (A>5a_2) \).

#### a. Coplanar Guard and Electrode [13]

![Diagram](image)

\[
\bar{a} = \frac{1}{2}(a_1 + a_2), \quad \text{and} \quad \frac{(a_2 - a_1)}{c} \text{ is small.}
\]

**Axial sym:**

\[
C = \frac{\bar{a}^2}{4c} \cdot \frac{\pi \bar{a}}{2} \left( \frac{a_2 - a_1}{2\pi c} \right)^2 \coth \frac{\pi \bar{a}}{c}.
\] (1.1)

**Cylindrical:**

\[
\frac{C}{l} = \frac{\bar{a}}{2\pi c} \cdot \frac{1}{2} \left( \frac{a_2 - a_1}{2\pi c} \right)^2 \coth \frac{\pi \bar{a}}{c}.
\] (1.2)

This capacitance is between the plane at potential \( V_0 \) and the electrode, including its plane face and its sides.
b. Electrode at Bottom of Hole in Guard [13]

(Capacitance Between the Plane at \( V_0 \) and the Face of Electrode at Bottom of Hole)

As in the preceding case, \( a/A \) must be small \( \left( a/A \lesssim \frac{1}{5} \right) \). Also the hole is not very shallow, and the clearance between the electrode and its guard is ignored.

\[
1 < \beta = \frac{d}{a} \ll \infty, \text{ and } 0 < \gamma = \frac{c}{a} \ll \frac{1}{5}.
\]

\[
\begin{align*}
C &= \frac{a}{\gamma} \sum_{s=1}^{\infty} \frac{\sinh a_s \gamma}{a_s^2 \sinh a_s (\beta + \gamma)}, \quad (1.3)
\end{align*}
\]

where \( a_1 = 2.4048, \ a_2 = 5.5201, \ a_3 = 8.6537, \ a_4 = 11.7915, \) and \( J_0(a_2) = 0 \). The first three terms are sufficient, with the conditions given above, for an accuracy of 1 in 200. \( J_0 \) is Bessel's function.

**Axial sym:**

\[
\frac{C}{l} = \frac{4q}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{2n-1} \sin (2n-1) \phi}{(2n-1)[1-(q^2)^{2n-1}]}, \quad (1.4)
\]

where

\[
\begin{align*}
q &= e^{-\left(\frac{\pi \beta}{2}\right)} \left[1 + 8 \cos^4 \theta \cos^{-4} \left(\frac{\pi \beta}{2}\right)\right] \\
\phi &= \theta - 8 \sin \theta \cos \theta e^{-4} \left(\frac{\pi \beta}{2}\right) \\
\theta &= \tan^{-1} \left(\frac{1}{\gamma}\right) = \tan^{-1} \left(\frac{a}{c}\right) \left( \theta \text{ in radians} \right).
\end{align*}
\]

In case the hole is very shallow \( (d/c \text{ small}) \), a better formula than (1.4) for the cylindrical case is

\[
\frac{C}{l} = \frac{a}{2\pi c - \frac{1}{2\pi^2}} \left\{ \frac{d}{c} \log \frac{1}{q_1} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \left[ \left(\frac{q_1}{r}\right)^n - (q_1 r)^n \right]}{n[1+q_1^2]^n} \right\}, \quad (1.4')
\]
where \( r \) and \( q_1 \) may be computed by

\[
\log r = \frac{\pi a}{c} - \frac{\epsilon}{1 + \epsilon \coth \frac{\pi a}{c}} \log \left( 1 - \frac{\epsilon}{2} \coth \left( \frac{\pi a}{c} \right) \right)
\]

\[
\log q_1 = \log \left( \frac{1 - \frac{\epsilon}{2} \coth \left( \frac{\pi a}{c} \right)}{2(1 - r^2)} \right).
\]

where \( \epsilon = d/c \).

Equation (1.4), like (1.4'), is exact with slot of any depth. Both ignore clearance between electrode and its guard. To take account of this (to first order) let \( 2a_1 \) denote the width of face of electrode; \( 2a_2 \), the width of slot; and \( d \), its depth. Then if \( a = (a_1 + a_2)/2 \),

\[
C/l = \frac{\overline{a}}{2\pi c} + \frac{1}{2\pi^2} \left[ \left( \frac{a_2 - a_1}{c} \right) \tan^{-1} \left( \frac{d}{a_2 - a_1} \right) - \frac{d}{c} \log \sqrt{d^2 + (a_2 - a_1)^2} \right] \tag{1.4''}
\]

neglecting terms of order

\[
[d^2 + (a_2 - a_1)^2] \log [d^2 + (a_2 - a_1)^2].
\]

1.2. Spheres or Cylinders

a. Concentric Case

![Figure 3](image)

Spheres:

\[
C = \frac{a_1 a_2}{a_2 - a_1} \tag{1.5}
\]

The capacity of one sphere alone \((a_2 \to \infty)\) is \( C = a_1 \).

Cylinders:

\[
\frac{C}{I} = \frac{1}{2 \log \frac{a_2}{a_1}} \tag{1.6}
\]
Equations (1.5) and (1.6) are limiting cases of (1.11) and (1.12).

The potential between the spheres is

$$V(r) = V_1 \frac{1 - a_2/r}{1 - a_2/a_1}$$

that between the cylinders is

$$V(r) = V_1 \frac{\log a_2/r}{\log a_2/a_1}.$$  

b. Plane With Sphere or Cylinder

(This is a limiting case of equations 1.14 and 1.15.)

When $h \to \infty$, $\gamma \to \log (4h^2/a^2)$ and (1.8) gives $C=a$, but (1.9) gives $C/l=0$, as it should, since the logarithmic potential becomes infinite at spatial infinity for any finite charge except zero.
c. Eccentric Spheres or Cylinders (Internal Case $0 < b < a_2 - a_1$)

\[ \overline{O_1O_2} = b = \text{the distance between centers (always positive)} \]

\[ 2bc = \sqrt{[(a_2 + a_1)^2 - b^2][(a_2 - a_1)^2 - b^2]} \quad \text{(positive)} \]

\[ \beta_1 = \log \frac{a_2^2 - a_1^2 - b^2 + 2bc}{2a_1 b} \quad \text{(positive)} \]

\[ \beta_2 = \log \frac{a_2^2 - a_1^2 + b^2 + 2bc}{2a_2 b} \quad \text{(positive)} \]

Spheres:

\[ C = 2c \sum_{n=0}^{\infty} \frac{e^{-(2n+1)\beta_1}}{1 - e^{-(2n+1)(\beta_1 - \beta_2)}} \]

Cylinders:

\[ \frac{C}{l} = \frac{1}{2(\beta_1 - \beta_2)^{-1}} \left[ 2 \log \frac{a_1^2 + a_2^2 - b^2 + 2bc}{2a_1 a_2} \right] \]
d. Eccentric Spheres or Cylinders (External Case, $b > a_1 + a_2$)

![Diagram of eccentric spheres or cylinders](image)

Figure 6

$\beta = \beta_2 > 0$  
$\beta = -\beta_1 < 0$

\[ b = 0_10_2 \]

\[ 2bc = V \left[ b^2 - (a_1 + a_2)^2 \right] \left[ b^2 - (a_1 - a_2)^2 \right] \quad \text{(positive)} \]

\[ \beta_1 = \log \left( \frac{b^2 + a_1^2 - a_2^2 + 2bc}{2a_1b} \right) \quad \text{(positive)} \]

\[ \beta_2 = \log \left( \frac{b^2 - a_1^2 + a_2^2 + 2bc}{2a_2b} \right) \quad \text{(positive)} \]

\[ \gamma = 2(\beta_1 + \beta_2) = 2 \log \left( \frac{b^2 - a_1^2 - a_2^2 + 2bc}{2a_1a_2} \right) \quad \text{(positive)} \]

Spheres:

\[ C = 2c \sum_{n=0}^{\infty} \frac{e^{-\left(n+\frac{1}{2}\right)\gamma}}{1-e^{-\left(n+\frac{1}{2}\right)\gamma}} = c \sum_{s=1}^{\infty} \frac{1}{s \sinh \frac{s\gamma}{2}} \quad (1.14) \]

Cylinders or parallel wires:

\[ C/l = \frac{1}{\gamma} \frac{1}{2(\beta_1 + \beta_2)} \quad (1.15) \]

Placing $b = a_1 + h$ and $a_1 \to \infty$, eq (1.14) and (1.15) go into eq (1.8 and 1.9), respectively.

1.3. Spheroids

a. A Thin Circular Disk of Radius $a$

\[ C = \frac{2a}{n} \quad (1.16) \]
This is a limiting case $b=0$, of the following formula.

### b. Oblate Spheroid

Major axis $2a$, minor axis $2b$:

$$
C = \frac{\sqrt{a^2 - b^2}}{\sin^{-1} \left( \frac{\sqrt{a^2 - b^2}}{a} \right)}.
$$

### c. Prolate Spheroid

Major axis $2a$, minor axis $2b$:

$$
C = \frac{\sqrt{a^2 - b^2}}{\log \left( \frac{a + \sqrt{a^2 - b^2}}{b} \right)}.
$$

### 1.4. Toroidal Surface

![Figure 7](image)

$a =$ radius of generating circle

\[ A = \overline{OC} > a \]

\[ \cosh \beta_1 = \frac{2}{k^2} - 1 = \frac{A}{a} \]

\[ k^2 = \frac{2a}{A + a} \text{ so } 0 < k < 1 \]

$$
C = \frac{4A^2 - a^2}{\pi} \sum_{n=0}^{\infty} \epsilon_n \frac{Q_{\frac{n}{2}}(\cosh \beta_1)}{P_{\frac{n-1}{2}}(\cosh \beta_1)} (\epsilon_0 = \frac{1}{2}, \epsilon_n = 1 \text{ if } n \neq 0).
$$

$P$ and $Q$ are the two Legendre functions with the same argument $A/a$. A method of finding these functions from tables of elliptic functions is given in section 4.
1.5. Conductor Bounded by Two Intersecting Spheres [14]
(Alone in space)

\[ a = \text{radius of arc on the right, semiaperture} = \theta \]

\[ a_1 = \frac{a \sin \theta}{\sin(\psi - \theta)} = \text{radius of arc on the left} \]

\[ \psi = \text{angle at which the arcs intersect.} \]

All figures may be obtained with the restrictions

\[ 0 < \theta < \pi \] and \[ \theta < \psi < 2\pi. \]

The capacitance for general \( \psi \) and \( \theta \) is \( C_\psi(\theta) \), where

\[
C(\theta) = \frac{a \sin \theta}{\psi} \left\{ \frac{\pi}{\sin \frac{\psi}{\theta}} - 8 \sum_{n=1}^{\infty} \sin^2 \frac{n\pi \theta}{\psi} \left[ \psi \left( \frac{n+1}{2} \right) - \psi \left( \frac{n\pi + 1}{2} \right) + \log \frac{\pi}{\psi} \right] \right\},
\]

where \( \psi \) denotes the psi-function, \( \Gamma'/\Gamma \), whose values may be taken from the tables of H. T. Davis, "Tables of the higher mathematical functions" (Principia Press, Bloomington, Ind., 1933).

The series (1.20) converges like \( \Sigma 1/\pi^2 \). For a much more rapid series converging like \( \Sigma n^{-14} \), see reference [21], where the cases are considered that have finite terms for capacitance(\( \psi/\pi \) rational).
The simplest of these is the limiting case \( \omega = 2\pi \), where the conductor is a thin shell with any aperture \( 2\theta \).

For \( \theta = \pi/2 \) this gives the capacitance of a hemispherical bowl

\[
C_{2\pi}(\pi/2) = a \left( 1 + \frac{1}{2} \frac{1}{\pi} \right) = 0.8183a.
\]

Orthogonal Spheres: (External, \( \omega = \pi/2 \))

\[
C(\theta) = a \left( 1 + \tan \theta \sin \theta \right) = a + a_1 - \frac{aa_1}{\sqrt{a^2 + a_1^2}},
\]

since \( a_1 = a \tan \theta \).
Orthogonal Spheres: \((\text{Internal, } \omega = 3\pi/2)\)

\[
C(\theta) = \frac{a}{\sqrt{3}} \left[ \sqrt{3} \cdot \frac{4}{3} + \frac{1}{2 \sin \frac{\theta}{3} \left( \frac{\theta}{3} + \frac{\pi}{3} \right)} + \frac{1}{2 \cos \frac{\theta}{3} \left( \cos \frac{\theta}{3} + \frac{\pi}{3} \right)} \right],
\]  

(1.23)

where \(a = a \tan \theta\) in this case also.

**Hemisphere: \((\omega = 3\pi/2 \text{ and } \theta = \pi/2)\)**

\[C_{3\pi/2}(\pi/2) = 2a \left( 1 - \frac{1}{\sqrt{3}} \right) = 0.8453a.\]  

(1.24)

More generally, for \(\omega = \pi/m\), where \(m > 1\),

\[
C(\theta) = a + a \sin \theta \sum_{t=1}^{\alpha-1} \left[ \frac{1}{\sin \left( \frac{t\pi}{m} + \theta \right)} - \frac{1}{\sin \frac{t\pi}{m}} \right],
\]  

(1.25)

and for \(\omega = 2\pi/m\), where \(m > 2\),

\[
C(\theta) = a - \frac{a}{\pi} (\theta - \sin \theta) + a \sin \theta \sum_{t=1}^{\alpha-1} \left[ \frac{1 - \left( \frac{2t + \theta}{m} + \pi \right)}{\sin \left( \frac{2t\pi}{m} + \theta \right)} - \frac{1 - \left( \frac{2t}{m} \right)}{\sin \left( \frac{2\pi t}{m} \right)} \right],
\]  

(1.26)

\[C_{\pi}(\theta) = a \text{ (complete sphere)}.\]
There are finite sums for capacitance when
\[ \omega = \left( \frac{2n-1}{2m} \right) \pi, \text{ where } 1 < n < 2m, \]
and
\[ \omega = \frac{2n}{2m-1} \pi, \text{ where } 1 < n < 2m-1. \]

2. Inductance and Electromagnetic Force

2.1. General Formulation

If the unit of length is the centimeter and the permeability of the conductors and
the surrounding media are unity, the formulas below give inductances in cgs electromag¬
etic units, that is in \(10^{-9}\) henry. If the electric currents \(I_1\) and \(I_2\) are in cgs elec¬
tromagnetic units, one of which is 10 amperes, the electromagnetic force is in dynes and
torque in dyne-centimeters.

The vector \(B\) of magnetic induction is the curl of a vector potential \(A\). If a wire of
appreciable cross section, and in the form of a closed circuit, carries a unit current
whose volume-density is the vector \(i_1\), the integral over the volume of the wire

\[ \iiint (i_1 \cdot A) dv_1 \]

is a scalar quantity which is called either the self-inductance of this current distribu-
tion, or its mutual inductance with the field, according as \(A\) is produced by this distribu-
tion alone, or entirely by currents other than itself.

If the magnetic permeability \(\mu\) is 1 everywhere, the vector potential \(A\) at any point
\(P_1\) due to a unit current in a wire No. 2, whose volume density of current at \(P_2\) is the
vector \(i_2\), is

\[ A = \iiint \frac{i_2 dv_2}{R} \]

integrated over the volume of wire No. 2, where \(R\) is the scalar distance from the fixed
point \(P_1\) to the point of integration in the volume-element \(dv_2\).

The mutual inductance \(M\) between the two current distributions is the repeated volume
integral

\[ M = \iiint dv_1 \iiint (i_1 \cdot i_2) \frac{dv_2}{R}. \]
When the wires shrink to mathematical, closed curves this becomes Neumann's double line integral

\[ N = \int ds_1 \int \cos(ds_1, ds_2) \frac{ds_2}{R}, \]

taken completely around both curves.

The self-inductance of a unit current distribution in a wire is

\[ L = \int \int \int dv \int \int \int \frac{(\mathbf{i} \cdot \mathbf{i}')}{R} dv'. \]

In the case of long straight wires, the formulas below apply for the uniform current distribution.

In the case of wires in the form of circular turns, or in form of helices, the few very accurate formulas given below apply to the "natural" current distribution (current density inversely proportional to the distance from the axis of symmetry).

The distinction between uniform and natural distribution is only of interest for precision measurements.

In the formulas for \( L \) and \( M \) to be given for parts of a closed circuit (such as \( L \) for a long straight wire alone, or \( M \) for two parallel ones), these expressions must be understood to represent only such contributions to the multiple integrals for \( L \) and \( M \) as may be written without specifying the nature of the return circuit whose contribution is, of course, to be evaluated by the same type of integral.

The majority of formulas for \( L \) and \( M \) that are given below fall into one or other of two classes, in each of which the above volume integrals are reduced to surface integrals over a plane section of the conductor.

**a. Axially Symmetric Configurations**

The first class is that of axially symmetric conductors for which the surface integrals are taken over a cross section in a plane through the axis of symmetry, say the \( x \)-axis, where \( (x, \rho, \phi) \) are cylindrical coordinates. The only component of the current density vector is \( \mathbf{i} = i(x, \rho) \), independent of longitude \( \phi \). The only component of the vector potential \( \mathbf{A} \) is \( A_\phi = A(x, \rho) \). The cylindrical components of the magnetic field are derived from \( \mathbf{A} \) by \( \mathbf{B} = \mu \mathbf{A} = \mu \text{curl} \mathbf{A} \), so

\[ \mu \mathbf{B}_x = \frac{1}{\rho} \frac{\partial}{\partial \rho} [\rho A(x, \rho)] \]

\[ \mu \mathbf{B}_\rho = -D_x A(x, \rho) \text{ and } \mathbf{B}_\phi = 0. \]

With \( \mu = 1 \) everywhere

\[ (A) \quad L \text{ or } M = 2\pi \int \int \rho' i(x', \rho') A(x', \rho') dx' d\rho', \]
where
\[ \iint t(x, \rho) \, dx \, d\rho = 1 \] (unit current).

The surface integrals are taken over an axial section in the \((x, \rho)\) half-plane. The symbol \(R\) has been used to denote the distance (in space) between two points \(P(x, \rho, \phi)\) and \(P'(x', \rho', \phi')\). We may designate by \(R\) this distance when the points are in the same axial plane \((\phi=\phi')\)

\[
R^2 = (x-x')^2 + (\rho-\rho')^2 \\
R^2 = (x-x')^2 + \rho^2 + \rho'^2 - 2\rho \rho' \cos(\phi-\phi') = 2\rho \rho' \left[ 1 + \frac{R^2}{2\rho \rho'} \cos(\phi-\phi') \right].
\]

There is the known Fourier series

\[
(B) \quad \frac{1}{R} = \frac{2}{\pi \sqrt{\rho \rho'}} \sum_{n=0}^{\infty} \epsilon_n Q_{n-1/2} \left( 1 + \frac{R^2}{2\rho \rho'} \right) \cos n(\phi-\phi'),
\]

where \(\epsilon_0 = 1/2\) and \(\epsilon_n = 1\) of \(n>0\), and \(Q_{n-1/2}\) is a Legendre function of the second kind with parameter \(n-1/2\). Its reduction to elliptic integrals is given in section 4.

Equation (B) is equivalent to

\[
(B') \quad \int_{-\pi}^{\pi} \frac{\cos n(\phi-\phi')}{R} \, d\phi' = \frac{2}{\pi \sqrt{\rho \rho'}} Q_{n-1/2} \left( 1 + \frac{R^2}{2\rho \rho'} \right).
\]

Since the only cylindrical component of current density is \(i_\phi = i(x, \rho)\) independent of \(\phi\), the only component of vector potential will be \(A_\phi = A(x, \rho)\) independent of \(\phi\). Hence, it would be sufficient to evaluate \(A\) for \(\phi=0\).

However, the volume integral defining \(A\) is the vector equation

\[
A = \iiint \frac{1(x', \rho') \, dy'}{R} = \iiint \rho' \, dx' \, dy' \, dz' \, \phi' = \int_{-\pi}^{\pi} \frac{i_\phi(x', \rho') \, d\phi'}{R}.
\]

This integral is the sum of many vectors that are not parallel, so that we may use rectangular coordinates, \(y = \rho \cos \phi\) and \(z = \rho \sin \phi\), and write

\[
i_y = i_\phi(x', \rho') \sin \phi' \quad \text{and} \quad i_z = i_\phi(x', \rho') \cos \phi',
\]
\[ A_y(x, \rho) = -\int \int \int_{-\pi}^{\pi} \rho' t(x', \rho') d\rho' dx' \int \cos \phi' d\phi' \]
\[ A_z(x, \rho) = \int \int \int_{-\pi}^{\pi} \rho' t(x', \rho') d\rho' dx' \int \sin \phi' d\phi' \]

Hence

\[ A_\phi = A(x, \rho) = -A_y(x, \rho) \sin \phi + A_z(x, \rho) \cos \phi = \int \int \rho' t(x', \rho') d\rho' dx' \int \cos (\phi - \phi') d\phi' \]

Consequently, by eq (B') with \( n=1 \),

\[ A_\phi = A(x, \rho) = \frac{2}{\sqrt{\rho}} \int \int S \sqrt{\rho'} t(x', \rho') Q_{\frac{1}{2}} \left( 1 + \frac{(x-x')^2 + (\rho-\rho')^2}{2\rho\rho'} \right) dx' d\rho', \]

so that \( A \) satisfies

\[ (C') \quad \left( D_x^2 + D_y^2 + \frac{1}{\rho} D_\rho - \frac{1}{\rho^2} \right) A = \begin{cases} -4\pi t(x, \rho) \text{ in } S \\ 0 \text{ outside } S \end{cases} \]

The integral in (C) is taken over any plane axial section \( S \) of the conductor, which may be of any shape. Its self-inductance \( L \) is therefore

\[ L = 2\pi \int \int S t(x, \rho) \cdot \rho A(x, \rho) dxd\rho \]

\[ = 4\pi \int \int S \rho^{1/2} t(x, \rho) dxd\rho \int \rho^{1/2} t(x', \rho') Q_{\frac{1}{2}} \left( 1 + \frac{(x-x')^2 + (\rho-\rho')^2}{2\rho\rho'} \right) dx' d\rho', \]

where

\[ \int \int S t(x, \rho) dxd\rho = 1. \]

Also from (C), the mutual inductance \( M \) between two coaxial wires with any shapes or size of axial sections \( S_1 \) and \( S_2 \) is

\[ M = 4\pi \int \int \rho_1^{1/2} t_1(x_1, \rho_1) dx_1 d\rho_1 \int \rho_2^{1/2} t_2(x_2, \rho_2) Q_{\frac{1}{2}} \left( 1 + \frac{(x_1-x_2)^2 + (\rho_1-\rho_2)^2}{2\rho_1\rho_2} \right) dx_2 d\rho_2, \]
where
\[ \int \int i_1(x_1, \rho_1) \, dx_1 \, d\rho_1 = \int \int i_2(x_2, \rho_2) \, dx_2 \, d\rho_2 = 1. \]

Letting both sections shrink to points gives
\[ (F) \]
\[ M = 4\pi \sqrt{a_1 \cdot a_2 \cdot q_{1/2}} \left( 1 + \frac{(x_1-x_2)^2 + (a_1-a_2)^2}{2a_1a_2} \right) \]
as the mutual inductance of two coaxial circular current filaments of radii \( a_1 \) and \( a_2 \), in the planes \( x_1 \) and \( x_2 \) (eq (2.1), page 1). See section 4 for the evaluation of the functions \( Q_{\pi/2} \) in terms of elliptic integrals.

If the section \( S \) shrinks to a point, eq (C) gives the vector potential \( A(x, \rho) \) at any point \( P(x, \rho) \) in space, that is produced by unit circular current of radius \( a \), in the plane \( x = 0 \), and coaxial with the \( x \)-axis

\[ A(x, \rho) = 2 \sqrt{a} Q_{1/2} \left( 1 + \frac{x^2 + (\rho-a)^2}{2a\rho} \right) = 2 \sqrt{a} \left[ \frac{2}{k} \left( \frac{K-E}{k^2} \right) \right], \]

where
\[ k^2 = \frac{4a\rho}{x^2 + (\rho+a)^2}; \]

\( k \) is the modulus of the complete elliptic integrals \( K \) and \( E \).

The cylindrical components of the magnetic field are given by \( H_x(x, \rho) = 1/\rho D_\rho (\rho A) \) and \( H_\rho(x, \rho) = -D_x A \), so

\[ H_x(x, \rho) = \frac{1}{\sqrt{x^2 + (\rho+a)^2}} \left[ K - E + \frac{2a(a-\rho)}{x^2 + (a-\rho)^2} \right] \]
\[ H_\rho(x, \rho) = \frac{4ax}{[x^2 + (\rho+a)^2]^{3/2}} \left[ \frac{E}{1-k^2} - 2 \left( \frac{K-E}{k^2} \right) \right], \]

which apply for any point \((x, \rho)\).

b. Cylindrical Configurations

The other class of formulas may be described (with rectangular coordinates \( x, y, z \)) as applying to a unit current with uniform current density \( i_x = i_y = 0, i_z = i(x, y) \) (independent of \( z \)). The total current +1 flows in the first conductor with cross section \( S \), its density being \( i_1 = 1/S \). The return current density in the second is \( i_2 = -1/S_2 \). The self-inductance "per unit length of the line" is denoted by \( L/l \), which means the self-induct-
ance of two cylinders having the same two end planes, these planes being separated by one unit of length.

\[ L/l = \frac{1}{S_1} \iint AdS_1 - \frac{1}{S_2} \iint AdS_2, \]

where \( A \) is the potential of both distributions. When \( \mu = 1 \) everywhere, the value of \( A \) at any point \( xy \) in the plane is (where \( A \) denotes \( A_x \), the only component of \( A \))

\[ A(x, y) = -\frac{2}{S_1} \iint \log R dS_1 + \frac{2}{S_2} \iint \log R dS_2, \]

where \( R \) is the distance from a point of integration \( P_1 \) in \( dS_1 \), (or from \( P_2 \) in \( dS_2 \)) to the general point \( P(x, y) \) in the same \( xy \) plane. This gives

\[ L/l = -\frac{2}{S_1} \iint dS_1 \iint \log R dS_1 - \frac{2}{S_2} \iint dS_2 \iint \log R dS_2 + \frac{4}{S_1 S_2} \iint dS_1 \iint \log R dS_2. \]

The first two integrals are frequently designated by \( L_1 \) and \( L_2 \), respectively, the third by

\[-2M_12, \]

but the separate integrals only have a meaning with reference to this equation of a "closed", or return, circuit. With this understanding, the self-inductance \( L/l \) per unit length of the line is written

\[ (G) \quad L/l = -2M_{12}/l + L_{1/l} + L_{2/l} = 2 \left[ 2 \log D_{12} - \log D_{11} - \log D_{22} \right], \]

where

\[ \log D_{12} = \frac{1}{S_1 S_2} \iint dS_1 \iint dS_2 \log R dS_2 \]

and similarly for \( D_{22} \).

This \( D_{12} \) defined by the repeated surface integral over the two coplanar areas \( S_1 \) and \( S_2 \) is called the g.m.d., or geometric-mean-distance, of the area \( S_1 \) from \( S_2 \), and \( D_{11} \) the geometric-mean-distance of the area \( S_1 \) from itself. The definition is consistent with the extension to the g.m.d. of one curve from another or of the g.m.d. of the line from itself.

In the case of a return circuit of two parallel wires whose circular sections have radii \( a_1 \) and \( a_2 \), it is easily found that

\[ (I) \quad \log D_{11} = \log a_1 - \frac{1}{4}, \quad \log D_{22} = \log a_2 - \frac{1}{4}, \]

\[ \text{18} \]
and

\[
\log D_{12} = \log b, \text{ where } b \text{ is the distance between centers.}
\]

The general formula (G) for the inductance \(L/\ell\) per unit length of the line,

\[
L/\ell = 2[2 \log D_{12} - \log D_{11} - \log D_{22}] \text{ gives eq (2.44).}
\]

For a tubular conductor whose cross section is an annular area with inner radius \(a_1\) and outer \(A_1\), it is found without difficulty that

(K)

\[
\log D_{11} = \log A_1 - \frac{a_1^2}{2(A_1^2 - a_1^2)} \left[ \frac{2a_1^2}{A_1^2 - a_1^2} \log \left( \frac{A_1}{a_1} \right) - 1 \right] - \frac{1}{4}
\]

When \(a_1 \to A_1\), the tube becomes infinitely thin and this approaches the finite limit

\[
\log D_{11} = \log A_1, \text{ so that } A_1 \text{ is the g.m.d. of the perimeter of the circle from itself.}
\]

When the current in tube No. 1 returns in a larger tube, No. 2, coaxial with it, whose annular section has inner radius \(a_2\) and outer \(A_2\), it is relatively simple to find that the g.m.d. between the two annule is given for \(0 < a_1 < A_1 < a_2 < A_2\) by

(L)

\[
\log D_{12} = \frac{1}{A_2^2 - a_2^2} \left[ A_2^2 - a_2^2 \log a_2 - a_2^2 \log a_2 \right] - \frac{1}{2}.
\]

Formula (G), \(L/\ell = 2[2 \log D_{12} - \log D_{11} - \log D_{22}]\) in this case leads to eq (2.46) by (K) and (L).

In checking for numerical errors, it may be noticed that the formula (G) for the self-inductance per unit length of a line (a return circuit) will be dimensionless. This is not true of the constituents \(\log D_{11}\) and \(\log D_{12}\), etc., as they do not involve logarithms of the ratio of two lengths.

In getting g.m.d., use may be made of any formal analogies to the logarithmic potentials of electrostatic distributions, for the same first integral occurs in both problems.

The logarithmic potential \(V\) of an endless cylinder of any cross section \(S\) with unit density per unit length perpendicular to the plane of \(S\) is

\[
V(x, r) = -2 \int \int_S \log R \, dx \, dy\).
\]

For example, when the section \(S\) is a circular area, or an annulus, \(V\) is the same at outside points as if the charge were all concentrated at the center. At points inside, a simple law prevails. This may be used to check the g.m.d. given above for circles and annuli.

When the total charge \(Q\) per unit length in finite space is zero, the logarithmic potential (like \(L/\ell\) for a return circuit) is dimensionless as to length. Hence the capacitance per unit length of endless cylinders, as in eq (1.1) to (1.15), involves logarithms of the ratio of two lengths. But the potential of a circular cylinder with charge \(Q\) has the value \(2Q \log r\) at outside points. This is not dimensionless.
A final example [15] may be quoted, the g.m.d. of a rectangular area from itself.
If the width of section is $w$ and breadth $b$,

\[ -4 \log D_{11} = -2 \log (w^2 + b^2) \left( \frac{w^2}{3b^2} \log \left( \frac{w^2 + b^2}{w^2} \right) + \frac{b^2}{3w^2} \log \left( \frac{b^2 + w^2}{b^2} \right) \right) \]

\[ -\frac{8}{3} \left( \frac{w}{b} \tan^{-1} \frac{b}{w} \tan^{-1} \frac{w}{b} \right) + \frac{25}{3} \]

\[ = \frac{1}{3\delta^2} F(\delta) - \frac{4\pi}{3} \delta - 4 \log w + \frac{25}{3}, \]

where $\delta = b/w$, and

\[ F(x) = -2x^4 \log x + (1-6x^2+x^4) \log (1+x^2) - 8x(1-x^2) \tan^{-1} x. \]

Since the expression for $\log D_{11}$ is symmetric in $b$ and $w$, this suggests the identity in $x$ that is easily verified.

\[ F(x) = x^4 F\left( \frac{1}{x} \right) - 12x^2 \log x - 4\pi x(1-x^2), \]

so that, from a power series for $F(x)$ in powers of $x$ valid, when $x<1$, we get by this, the series for $x>1$. These equations, (M), (N), (O), are used in deriving eq (2.47).

If two or more long parallel, cylindrical conductors all carry current in the same direction with the same uniform current density, they are effectively one conductor of cross section $S$. If the sections are the coplanar, nonoverlapping areas $S_1, S_2, S_3$, etc., then $S = S_1 + S_2 + S_3$... For example, with three such sections the g.m.d. of the compound area $S$ from itself is given by

\[ (S_1 + S_2 + S_3)^2 \log D_{11} = \int \int dS \int \log RdS \]

\[ = \int dS_1 \int \int \log RdS_1 + \int dS_2 \int \int \log RdS_2 + \int dS_3 \int \int \log RdS_3 \]

\[ + 2 \int dS_1 \int \int \log RdS_2 + 2 \int dS_2 \int \int \log RdS_3 + 2 \int dS_3 \int \int \log RdS_1 \]

\[ = \log(D_{11}) \cdot (D_{22}^2) \cdot (D_{33}^2) \cdot (D_{12}^{2S_1S_2}) \cdot (D_{23}^{2S_2S_3}) \cdot (D_{31}^{2S_3S_1}). \]

The generalization of this is not difficult, when each conductor carries a uniform but different current density. Weighting factors are introduced.
2.2. Circular Filaments and Circular Turns of Wire

a. Coaxial Circular Filaments [16]

Their mutual inductance $M$ is

$$M = 4\pi \sqrt{a_1 a_2} \left[ \frac{2(K-E)}{k} - kK \right] = 4\pi \sqrt{a_1 a_2} Q_y \left( \frac{2}{k^2} - 1 \right),$$

(2.1)

where the modulus $k$ of the complete elliptic integrals is given by

$$k^2 = \frac{4a_1 a_2}{x^2 + (a_1 + a_2)^2},$$

(2.2)

Their force of attraction is $X = -I_1 I_2 \frac{dM}{dx}$ or

$$X = \frac{I_1 I_2 x k}{\sqrt{a_1 a_2}} \left[ \frac{2 + k^2}{1 - k^2} - 2K \right].$$

(2.3)
b. Circles Whose Axes Intersect \[17\]

\[\mu_1 = \cos \alpha_1 \quad (0 \leq \alpha_1 \leq \pi)\]
\[\mu_2 = \cos \alpha_2 \quad (0 \leq \alpha_2 \leq \pi)\]
\[\mu = \cos \theta \quad (0 \leq \theta \leq \pi)\]

**Figure 14**

\[
N = 4\pi^2 (1-\mu_1^2)(1-\mu_2^2) \sum_{n=1}^{\infty} \frac{r_2^{n+1}}{r_1^n} \frac{P_n'(\mu_1) P_n' (\mu_2) P_n(\mu)}{n(n+1)} \quad \text{if} \quad r_2 < r_1
\]
\[
= 4\pi^2 (1-\mu_1^2)(1-\mu_2^2) \sum_{n=1}^{\infty} \frac{r_1^{n+1}}{r_2^n} \frac{P_n'(\mu_1) P_n' (\mu_2) P_n(\mu)}{n(n+1)} \quad \text{if} \quad r_2 > r_1
\]

whenever the values of \(a_2/a_1\). \(P_n(\mu)\) is the Legendre polynomial and \(P_n'(\mu) = (d/d\mu)P_n(\mu)\).

In the special case where the center \(C_2\) of circle 2 coincides with 0, \(r_2\) becomes \(a_2\) and \(a_2\) becomes \(\pi/2\), and (2.4) reduces to

\[
N = 2\pi^{3/2} a_2 (1-\mu_2^2) \sum_{n=1}^{\infty} (-1)^n \left(\frac{a_2}{r_1}\right)^{2n+1} \frac{\Gamma(n+\frac{3}{2})}{(n+1)!} P_{2n+1}'(\mu_1) P_{2n+1}(\mu).
\]

The torque \(T\) acting on either circle, tending to reduce the angle \(\theta\) between their axes is

\[
T = -\frac{\partial N}{\partial \theta} \sin \theta \frac{\partial N}{\partial \mu} = 2\pi^{3/2} a_2 \sin \theta (1-\mu_2^2) \sum_{n=1}^{\infty} (-1)^n \left(\frac{a_2}{r_1}\right)^{2n+1} \frac{\Gamma(n+\frac{3}{2})}{(n+1)!} P_{2n+1}'(\mu_1) P_{2n+1}(\mu).
\]

These converge for all values of \(r_1\), if \(a_2 < a_1\). \(N\) is positive when the currents circulate in the same sense around their axes \(OC_1\) and \(OC_2\).
c. Two Concentric Circles (Not Coaxial)  
(Special case of eq (2.4')

\[ a_1 = \overline{OA_1}, \quad \mu = \cos \theta \]
\[ a_2 = \overline{OA_2}, \quad a_2 < a_1 \]

\[ M = \frac{4\pi a_2^2}{a_1} \sum_{n=0}^{\infty} \left( \frac{a_2}{a_1} \right)^{2n} \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n!(n+1)!} P_{2n+1}(\mu). \]  

\[ T = \frac{4\pi I_1 I_2 a_2^2}{a_1} \sin \theta \sum_{n=0}^{\infty} \left( \frac{a_2}{a_1} \right)^{2n} \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n!(n+1)!} P_{2n+1}^{'}(\mu). \]  

Axes Parallel \((\theta=0, \mu=1), \quad T=0\)

\[ M = \frac{4\pi a_2^2}{a_1} \sum_{n=0}^{\infty} \left( \frac{a_2}{a_1} \right)^{2n} \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n!(n+1)!}. \]  

Axes Perpendicular \((\theta=\pi/2, \mu=0) \quad M=0\)

\[ T = \frac{8 I_1 I_2 \sqrt{\pi a_2^2}}{a_1} \sum_{n=0}^{\infty} \left( \frac{a_2}{a_1} \right)^{2n} \frac{\Gamma(n+\frac{1}{2})}{(n+1)!} \left( \frac{\Gamma(n+\frac{3}{2})}{n!} \right)^2. \]
d. Two parallel circles \[18\]

Case where \(0 < r < a_1 - a_2\).

\[
K = \frac{4\pi^{3/2}a_2^2}{a_1} \sum_{n=0}^{\infty} (-1)^n \left(\frac{r}{a_1}\right)^{2n} \frac{\Gamma(n+\frac{3}{2})}{n!} F_n P_{2n}(\mu),
\]

where \(F_n = F(n+\frac{1}{2}, n+\frac{3}{2}, 2; a_2^2/a_1^2)\) (hypergeometric series).

If the circles are coplanar, \((\theta = \pi/2, \mu = 0)\) and

\[
P_{2n}(\phi) = (-1)^n \frac{\Gamma(n+\frac{3}{2})}{n!}.
\]

Case where \(r > a_1 + a_2\)

\[
K = \frac{4\pi^{3/2}a_2^2}{a_1} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{a_1}{r}\right)^{2n+1} \frac{\Gamma(n+\frac{3}{2})}{(n-1)!} F_n P_{2n}(\mu),
\]

where \(F_n = F(-n, 1-n, 2; a_2^2/a_1^2)\).
For equal radii \( a_1 = a_2 = a \) and \( r > 2a \)

\[
M = 2\pi a \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{2a}{r} \right)^{2n+1} \frac{\Gamma(n+\frac{3}{2})\Gamma(n+\frac{1}{2})}{(n-1)!(n+1)!} P_{2n}(\mu). 
\] (2.12)

For two equal circles, coplanar and external, this becomes

\[
M = -2\pi a \sum_{n=1}^{\infty} \left( \frac{2a}{r} \right)^{2n+1} \frac{\Gamma^3(n+\frac{1}{2})}{(n-1)!n!(n+1)!} 
\] (2.13)

(The plus sign would apply when the currents circulate in opposite senses with respect to the normal to their plane.) The torque is zero. The force of repulsium along their line of centers is \( F = +I_1 I_2 \frac{\partial M}{\partial r} \), or

\[
F = +2\pi I_1 I_2 \sum_{n=1}^{\infty} \left( \frac{2a}{r} \right)^{2n+2} \frac{\Gamma(n+\frac{3}{2})\Gamma^2(n+\frac{1}{2})}{(n-1)!n!(n+1)!} 
\] (2.14)
Current density of the unit current is

\[ i = \frac{1}{\pi a^2 f} \left( \frac{\rho}{A} \right)^b \quad \text{(any } b) \],

where

\[ f = F \left( \frac{-b}{2}, \frac{1-b}{2}, 2; \frac{a^2}{A^2} \right) = 1 + \frac{b(b-1)}{8} \frac{a^2}{A^2} \text{ to 2d order} \]

\[ L = 4\pi a \cdot \left[ 1 + (2b+1) \frac{a^2}{8A^2} \right] \log \frac{8A}{a} - \frac{7}{4} \frac{(b-1)(b-2/3)}{16} \frac{a^2}{A^2} + \text{zero} \left( \frac{a^3}{A^3} \log \frac{A}{a} \right). \tag{2.15} \]

For uniform current distribution \( b = 0 \).
For "natural" current distribution \( b = -1 \).
For \( b = -\frac{5}{2} \), the magnetic field outside the wire is exactly the same as if the unit current were concentrated in a circular filament of radius \( \sqrt{A^2 - a^2} \).
f. Self-Inductance of a Circular Turn of Wire Near a Magnetic Medium

\[ L = L_{\text{air}} + \frac{\mu - 1}{\mu + 1} 4\pi \mu Q\frac{\gamma_i}{h} \left( \frac{2}{K^2} - 1 \right) \]

\[ = L_{\text{air}} + \frac{\mu - 1}{\mu + 1} 4\pi A \left[ \frac{2(X-E)}{k} - kX \right]. \quad (2.16) \]

(modulus \( k \) where \( k^2 = \frac{4\mu a}{4\pi a^2 + (A+a)^2} \).)

\( L_{\text{air}} \) may be computed by preceding case, (2.15) on the basis \( \mu = 1 \) everywhere.

In correcting the self- or mutual inductance of coils for the effect of thin lead-in wires, the diameter of the wire is important by affecting its self-inductance, but wires may be treated as linear conductors in estimating their mutual inductance.

g. Self-Inductance of a Wire

\[ L = 2 \left[ l \log \left( \frac{\sqrt{l^2+a^2}+a}{a} \right) - \sqrt{l^2+a^2} + \frac{l}{4} + a \right], \quad (2.17) \]

where \( l \) is its length and \( a \) its radius.
h. Mutual Inductance of Two Parallel Wires Having the Same End-Planes

\[ M = 2 \left( l \log \left( \frac{V \sqrt{l^2 + D^2 + l}}{D} \right) - V \sqrt{l^2 + D^2 + D} \right) \]  

(2.18)

where \( l \) is their length, and \( D \) the distance between centers. If the currents are in opposite directions, the sign of \( M \) is reversed.

i. Mutual Inductance of Two Parallel Wires Not Co-terminous

\[ c = \frac{C_1 + C_2}{2} \text{ where } C_1 \text{ and } C_2 \text{ are centers of the wires} \]

\[ M = \omega \left( c + \frac{l_2 - l_1}{2} \right) + \omega \left( c - \frac{l_2 + l_1}{2} \right) - \omega \left( c + \frac{l_2 + l_1}{2} \right) - \omega \left( c - \frac{l_2 - l_1}{2} \right), \]  

(2.19)

where

\[ w(x) = |x| \log \left( \frac{\sqrt{x^2 + D^2} + |x|}{D} \right) - \sqrt{x^2 + D^2}, \]

so that \( w(x) \) is an even function of \( x \).

This holds for collinear wires \((D=0)\) if they do not overlap. The sign of \( M \) is reversed if the currents have opposite directions.
j. Mutual Inductance of Two Equal Rectangles Lying in Parallel Planes

(One is the perpendicular projection of the other.) The distance between their planes is $d$, the length and breadth of each is $a$ and $b$, respectively.

Neumann's formula is

$$\text{\#} \over 4 = a \log \left[ \frac{(a+\sqrt{a^2+d^2})}{(a+\sqrt{a^2+b^2+d^2})} \right] + b \log \left[ \frac{(b+\sqrt{b^2+d^2})}{(b+\sqrt{a^2+b^2+d^2})} \right] + 2 \left[ \sqrt{a^2+b^2+d^2} - \sqrt{a^2+d^2} - \sqrt{b^2+d^2} + d \right]. \quad (2.20)$$

k. Self-Inductance of a Rectangle

![Figure 22](image)

$a =$ radius of wire, $l =$ length of rectangle, $b =$ breadth

$$L = 4 \left\{ (b+l) \log \left[ \frac{\sqrt{4(b+l)^2 + a^2} + a}{a} \right] - b \cdot \log \left[ \frac{\sqrt{b^2 + l^2} + b}{l} \right] - l \cdot \log \left[ \frac{\sqrt{b^2 + l^2} + l}{b} \right] + 2 \sqrt{b^2 + l^2 + a} \cdot \frac{3}{2} (b+l) - \frac{1}{2} \sqrt{4(b+l)^2 + a^2} \right\}. \quad (2.21)$$
2.3. Concentric Solenoids (Current Sheets) [20]

\[ N = 4\pi \left( \frac{N_1}{2b_1} \right) N_2 \pi a_2^2 \mu_1 \left\{ \frac{2(1-\mu_1^2)}{\mu_1 \mu_2} \sum_{s=1}^{\infty} \left( \frac{r_2}{r_1} \right)^{2s} P_{2s+1}(\mu) P'_{2s}(\mu_1) P'_{2s+2}(\mu_2) \right\} \] (2.22)

The series is relatively small compared to \( \mu \) when \( a_1/2b_1 \) is small. \( P_s(\mu) \) is Legendre polynomial and \( P'_s(\mu) = \frac{d[P_s(\mu)]}{d\mu} \).

If the coils carry currents of strength \( I_1 \) and \( I_2 \) (in cgs electromagnetic units of current, the torque on either coil, tending to decrease \( \theta \), is \( T = -I_1 I_2 \partial \mathbf{H}/\partial \theta = I_1 I_2 \sin \theta \partial \mathbf{H}/\partial \mu \), or

\[ T = 4\pi \left( \frac{N_1 I_1}{2b_1} \right) N_2 I_2 \pi a_2^2 \mu_1 \sin \theta \left\{ -2 \frac{(1-\mu_1^2)}{\mu_1 \mu_2} \sum_{s=1}^{\infty} \left( \frac{r_2}{r_1} \right)^{2s} P_{2s+1}(\mu) P'_{2s}(\mu_1) P'_{2s+2}(\mu_2) \right\} \] (2.23)

Case a. Axes perpendicular (\( \theta = \pi/2, \mu = 0 \))

\( N = 0 \)

\[ T = 4\pi \left( \frac{N_1 I_1}{2b_1} \right) N_2 I_2 \pi a_2^2 \mu_1 \left\{ \frac{1-\mu_1^2}{\mu_1 \mu_2} \sum_{s=1}^{\infty} (-1)^s \frac{(r_2)^{2s} \Gamma(s+\frac{1}{2}) P'_{2s}(\mu_1) P'_{2s+2}(\mu_2)}{(r_1)^{2s} \Gamma(s+\frac{1}{2}) (2s+2)(2s+3)} \right\} \] (2.24)
Case b. Axes parallel ($\theta = 0, \mu = 1$)

\[ T = 0 \]

\[ N = 4\pi \frac{N_1 N_2 \pi a_1^2 a_2^2}{2 \mu_1} \left\{ 1 - \frac{2(1 - \mu_1^2)}{\mu_1 \mu_2} \sum_{s=1}^{\infty} \left( \frac{r_2}{r_1} \right)^{2s} \frac{P'_{2s}(\mu_1)P'_{2s+2}(\mu_2)}{2s(2s+1)(2s+2)(2s+3)} \right\} \quad (2.25) \]

This may be computed in finite terms. (See coaxial coils, eq (2.30).)

### 2.4. Self-Inductance of a Cylindrical Current Sheet [21]

![Figure 24](image)

The sheet consists of \( N \) complete circular turns of thin tape without insulating space between them. Their diameter is \( D \); the total length of the cylinder is \( l \).

\[ L_s = \frac{4\pi N^2}{3} \sqrt{l^2 + D^2} \left[ K + E + \frac{D^2}{l^2} (E - k) \right] \],  \quad (2.26) \]

where \( k = D/\sqrt{l^2 + D^2} \) the modulus of the complete elliptic integral \( K \) and \( E \). The complementary modulus is \( k^1 = l/\sqrt{l^2 + D^2} \).
2.5. Self-Inductance of a Helical Wire

The length \( l \) of the equivalent sheet, is the distance from the center of the wire at start of first turn to center at end of last turn.

\[
L = L_s + l \left[ \log \left( \frac{1 + k'}{1 - k'} \right) + k' \log 4 \right] + \pi D \left\{ 2 N \left[ \frac{1}{4} \log \left( \frac{N \pi d}{l} \right) \right] + \frac{1}{3} \log \left( \frac{N \pi D}{l} \right) - \frac{4}{\pi^2} \left( \frac{E}{k} - 1 \right) \left[ 1 + \frac{1}{2} \left( \frac{N \pi d}{2 l} \right)^2 \right] \right. \\
- \frac{2}{3} \left[ \frac{K - E}{k} - \frac{k E}{2} \right] \frac{k'}{2k} \left( 1 - \frac{k'}{k} \sin^{-1} k \right) \left\} \right. \tag{2.27}
\]

This takes account of the relatively small axial component of current. \( L_s \) is given by (2.26), moduli \( k, k' \), as in (2.26).

2.6. Bifilar Mutual Inductor

Primary and secondary are helical wires identical in form, the turns of one midway between those of the other. Two cases are \( M_0 \) and \( M_n \). \( M_0 \) is their mutual inductance when the second helix is displaced axially from the first by one-half the pitch. When the second is displaced 180° in azimuth from the first, but with its extremities in the same end-plane as the first, their mutual inductance is designated by \( M_n \). The principal part of either is \( L_s \) given by (2.26), the self-inductance of the current sheet equivalent to primary or secondary. The moduli \( k \) and \( k' \) are the same as in (2.26).

\[
M_0 = L_s + l \left[ \log \left( \frac{1 + k'}{1 - k'} \right) + k' \log 4 \right] - \pi D \left\{ N \log 4 + \frac{1}{6} \log \left( \frac{4N D}{l} \right) + \frac{4}{\pi^2} \left( \frac{E}{k} - 1 \right) \left[ 1 + \frac{1}{2} \left( \frac{N \pi d}{2 l} \right)^2 \right] \right. \\
- \frac{1}{3} \left( \frac{K - E}{k} - \frac{k E}{2} \right) \right. \frac{k'}{2k} \left( 1 - \frac{k'}{k} \sin^{-1} k \right) \left\} \right. \tag{2.28}
\]
\[ M_n = L_s + l \left[ \log \frac{1 + k'}{1 - k'} + k' \log 4 \right] \]

\[-\pi D \left\{ N \log 4 + \frac{1}{6} \log \left( \frac{4N D}{l} \right) + \frac{4}{\pi^2} \left( \frac{E}{k} - 1 \right) \left[ 1 + 1/2 \left( \frac{N \pi d}{2l} \right)^2 \right] \right. \]

\[ + \frac{2}{3} \left[ \frac{K - E}{k} - \frac{k K}{2} \right] + \frac{1}{2k} \left[ 1 - \frac{1}{2k} \log \left( \frac{1 + k}{1 - k} \right) \right] \} \quad (2.29) \]
2.7. Coaxial Current Sheets [24]

The force of attraction (in dynes) is

\[ F = -\frac{2\pi N_1 N_2}{l_1 l_2} \left( \begin{array}{cc} c + \frac{l_2 + l_1}{2} & + \omega \left( \frac{l_2 - l_1}{2} \right) \\ -w \left( \frac{l_2 - l_1}{2} \right) & -w \left( \frac{l_2 + l_1}{2} \right) \end{array} \right). \] (2.30)

The total number of turns \( N_1 \) and \( N_2 \) are given by

\[ \frac{N_1 N_2}{l_1 l_2} = \frac{2\pi}{2\pi} \left( \begin{array}{cc} w \left( c + \frac{l_2 + l_1}{2} \right) + w \left( c + \frac{l_2 - l_1}{2} \right) \\ w \left( c + \frac{l_2 - l_1}{2} \right) - w \left( c + \frac{l_2 + l_1}{2} \right) \end{array} \right). \] (2.31)

where

\[ w(x) = x w'(x) + \frac{8(a_1 a_2)^2}{3k} \left[ K \left( \frac{2}{k^2} \right) - E \left( \frac{2}{k^2} \right) \right] \] (2.32)

This is an even function of \( x \). Its derivatives \( w'(x) \) is an odd function of \( x \), vanishing with \( x \), and given by

\[ w'(x) = \frac{2x}{k} \left[ K-E \right] \pm \left[ \frac{a_1 a_2}{a_1 + a_2} \right] \left[ KE(\theta, k^2) - E(\theta, k^2) \right] \] (2.33)

the + sign is for \( x \) positive, - , for \( x \) negative.

The complete elliptic integrals \( K \) and \( E \) have modulus \( k \), where

\[ k^2 = \frac{4a_1 a_2}{x^2 + (a_1 + a_2)^2}. \] (2.34)

The incomplete integrals \( F(\theta, k^2) \) and \( E(\theta, k^2) \) have the complementary modulus \( k' = \sqrt{1-k^2} \). Their amplitude \( \theta \) is computed by

\[ \sin \theta = \sqrt{1 + \left( \frac{x}{a_1 + a_2} \right)^2}, \text{ where } 0 < \theta < \frac{\pi}{2}. \]
The bracket with factor $2a_1^2 - a_2^2$ vanishes when $x=0$. If the coils are also concentric ($c=0$), the force vanishes, and $M$ becomes

$$M = \frac{4\pi N_1 N_2}{l_1 l_2} \left[ \omega \left( \frac{l_2 + l_1}{2} \right) - \omega \left( \frac{l_2 - l_1}{2} \right) \right]$$

(2.36)

Another special case is that in which the second sheet is replaced by a single turn of radius $a_2$ coaxial with the $x$-axis in the plane $x=c$. The mutual inductance between the circle and sheet 1 is

$$M = \frac{2\pi N_1}{l_1} \left\{ \omega \left( \frac{l_1}{2} \right) - \omega \left( \frac{-l_1}{2} \right) \right\}$$

(2.37)

$$M = \frac{2\pi N_1 I_1 I_2}{l_1} \left\{ \omega \left( \frac{-l_1}{2} \right) - \omega \left( \frac{l_1}{2} \right) \right\} = \frac{N_1 I_1}{l_1} N_2 (M_1 - M_2),$$

(2.38)

where

$$\omega(x) = 4\sqrt{\frac{a_1 a_2}{r}} \left[ \frac{R-E}{k} - \frac{kR}{2} \right],$$

(2.39)

so that the circle has mutual inductance $M_1$ with the nearest circular turn of the sheet, and $M_2$ with the farthest.

### 2.8. Toroidal Current Sheets

Current in tape winding of $N$ turns circulates around the core in planes through its axis of symmetry. Permeability of core is $\mu$.

**Case a. Core of Circular Section**

![Figure 27](image)

$$L = \frac{4\pi a^2 \mu N^2}{A + \sqrt{A^2 - a^2}}$$

(2.40)
2.9. Endless Return-Circuits
(Self-Inductance)

a. Concentric Cable (Special Case of 2.46)

\[ L/l = \frac{\mu_1}{2} + 2 \log \frac{a_2}{a_1} + \frac{\mu_2 a_2^2}{a_2^2 - a_1^2} \left[ \frac{2a_2^2}{a_2^2 - a_1^2} \log \left( \frac{a_2}{a_1} \right) - 1 \right] - \frac{\mu_2}{2} \]  
\[ (2.42) \]

\[ L/l = \frac{\mu_2}{2} + 2 \log \left( \frac{a_2}{a_1} \right) \text{ if } a_2 = a_1 \]  
\[ (2.43) \]

\[ L/l \sim 2 \log a_2/a_1 \text{ for high frequency (see (3.1))}. \]  

\( \mu_1 \) and \( \mu_2 \) are magnetic permeabilities. The current goes one way in the central wire and returns in the outer shell. The self-inductance of the line, per unit length, is, for low-frequency or direct current.
b. Two Parallel Wires (Nonmagnetic)

The self-inductance per unit length of the line when the unit current is in opposite directions in the two wires, and \( \mu = 1 \) everywhere, is given by the exact formula

\[
\frac{L}{l} = 1 + 2 \log \frac{b^2}{a_1 a_2}.
\]  

(2.44)

See (2.45) and (3.3). This is derived as in section 2, eq (I) and (J).
The positive constants $\beta_1$, $\beta_2$, and $\gamma$ are defined in eq (1.13). This $\gamma$ is the reciprocal of capacity (1.15). When $\mu_1 = \mu_2 = 1$, eq (2.45) reduces to (2.44). (See also (3.3) for high frequency in this circuit.)
d. Two Coaxial Tubes
\((\mu=1)\) (See section 2, equations \((K)\) and \((L)\)

\[
L/\ell = 2 \log \frac{a_2}{A_1} + \frac{A_2^2}{A_2^2 - a_2^2} \left[ \frac{2A_2^2}{A_2^2 - a_2^2} \log \left( \frac{A_2}{a_2} \right) - 1 \right] + \frac{a_1^2}{A_1^2 - a_1^2} \left[ \frac{2a_1^2}{A_1^2 - a_1^2} \log \left( \frac{A_1}{a_1} \right) - 1 \right].
\]

(2.46)

When \(a_1 \to 0\), the inner tube becomes a solid wire and this reduces to (2.42) with \(\mu_1 = \mu_2 = 1\).

When \(a_2 \to A_2\), \(L/\ell = \frac{1}{2} + \log \frac{A_2}{A_1} + \frac{a_1^2}{A_1^2 - a_1^2} \left[ \frac{2a_1^2}{A_1^2 - a_1^2} \log \left( \frac{A_1}{a_1} \right) - 1 \right].\)

When \(a_1 \to A_1\), \(L/\ell = -\frac{1}{2} + \log \frac{a_2}{A_1} + \frac{A_2^2}{A_2^2 - a_2^2} \left[ \frac{2A_2^2}{A_2^2 - a_2^2} \log \left( \frac{A_2}{a_2} \right) - 1 \right].\)

When \(a_1 \to A_1\) and \(a_2 \to A_2\), \(L/\ell = \log \frac{A_2}{A_1} \).

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Since $D_{11}=D_{22}$ (given by eq (M)), the formula (6) for self-inductance $L/l$ per unit length of line is

$$L/l = \frac{1}{3\delta} \left[ F(\gamma+\delta) + F(\delta) - \frac{1}{2} F(\gamma+2\delta) - \frac{1}{2} F(\gamma) \right] + 4\pi(\gamma+2\delta/3),$$

(2.47)

where as in equation (N)

$$F(x) = -2x^4 \log x + (1-6x^2+x^4)\log(1+x^2) - 8x(1-x^2)\tan^{-1}x.$$  

(2.48)

This satisfies the identical relation in $x$ (eq (O)).

$$F(x) = x^4 F\left(\frac{1}{x}\right) - 12x^2 \log x - 4\pi x(1-x^2).$$

(2.49)

When $x<1$, we find

$$F(x) = 2x^4 \log\frac{1}{x} - 7x^2 + \frac{25x^4}{6} + 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{2n(2n-1)(2n-2)(2n-3)(2n-4)}. $$

(2.50)

This is obtained by use of the series

$$\log(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{n} \quad \text{and} \quad x\tan^{-1}x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{2n-1}. $$

Equation (2.50) might be preferable to (2.48) for computation with thin flat, strips as in the figure, where $\gamma$, or $\delta$, or both, are small, that is $\delta$, or $b$, or both, are small compared to the width $\omega$.

In the other extreme ($\delta$ or $b$ or both large compared to $\omega$), the expansion of $F(x)$ is required for $x>1$. This is obtained from (2.50) by use of the identity (2.49) which gives for $x>1$

$$F(x) = 2(1-6x^2)\log x - 7x^2 + \frac{25}{6} - 4\pi x(1-x^2) + 48x^4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{-2n}}{2n(2n-1)(2n-2)(2n-3)(2n-4)}. $$

(2.51)
The derivation of (2.47) depends in part upon eq (M), which is

\[-4 \log D_{11} = \frac{1}{3 \delta^2} F(\delta) - \frac{4\pi}{3} - 4 \log \omega + \frac{25}{3}. \tag{2.52}\]

With the same function \(F(x)\), it is found that Gray's [26] formula for \(4 \log D_{12}\) may be written

\[4 \log D_{12} = \frac{1}{3 \delta^2} [F(\gamma + \delta) - \frac{1}{2} F(\gamma + 2 \delta) - \frac{1}{2} F(\gamma)] + 4\pi (\gamma + \delta) + 4 \log \omega - \frac{25}{3}, \tag{2.53}\]

which, with (2.52), gives (2.47).

The formula of Gray, corrected by Rosa [27] is

\[4w^2b^2 \log D_{12} = -\frac{25}{3} w^2 b^2 + \left[ (\delta + 2b)^2 \left( \frac{w^2 - (\delta + 2b)^2}{6} - \frac{w^4}{6} \right) \log [w^2 + (\delta + 2b)^2] \right]
\]

\[-2 \left[ (\delta + b)^2 \left( \frac{w^2 - (\delta + b)^2}{6} - \frac{w^4}{6} \right) \log [w^2 + (\delta + b)^2] + \left[ (\delta + b)^2 \left( \frac{w^2 - (\delta + b)^2}{6} - \frac{w^4}{6} \right) \log [w^2 + \delta^2] \right] \]

minus the same terms with \(w = 0,\)

\[\frac{4w}{3} \left\{ (\delta + 2b)^3 \tan^{-1} \frac{w}{\delta + 2b} + w^2 (\delta + 2b) \tan^{-1} \frac{\delta + 2b}{w} \right\}
\]

\[-2 \left\{ (\delta + b)^3 \tan^{-1} \frac{w}{\delta + b} + w^2 (\delta + b) \tan^{-1} \frac{\delta + b}{w} \right\} + g^3 \tan^{-1} \frac{w}{\delta} + w^2 g \tan^{-1} \frac{\delta}{w}. \tag{2.54}\]

By use of \(\log xy = \log x + \log y,\) and \(\tan^{-1} x = \pi/2 - \tan^{-1} 1/x,\) this formula may be put in the form (2.53).
3. Frequency Effects [28]

3.1. Skin Effect in Concentric Cable

The resistivities $\rho_1$ and $\rho_2$ are in electromagnetic cgs units, one of which is equal to $(10^{-9})$ ohm-cm (for copper $1/\rho \approx 0.0006$ and for iron $1/\rho \approx 0.0001$).

For frequency $f$, let $R_{f/1}$ and $L_{f/1}$ denote resistance and self-inductance of the line per centimeter length, when the current flows one way in the central wire and returns through the outer shell. (See eq (2.42) and (2.46) for low frequency.) For very high frequency

\[
R_{f/1} \approx \left[ \frac{\sqrt{\mu_1 \rho_1}}{a_1} + \frac{\sqrt{\mu_2 \rho_2}}{a_2} \right] \sqrt{f}
\]

(3.1)

\[
L_{f/1} \approx 2 \log \frac{a_2}{a_1} + \frac{1}{2\pi} \left[ \frac{\sqrt{\mu_1 \rho_1}}{a_1} + \frac{\sqrt{\mu_2 \rho_2}}{a_2} \right] \frac{1}{\sqrt{f}}
\]

(3.2)

For any frequency $f$, the resistance and inductance may be computed by use of certain tabulated functions, which are the real and imaginary parts of Bessel's (and Hankel's) function of the first kind, these having parameters 0 and 1 and argument $x\sqrt{f}$, where $x$ is a positive real.

The resistance $R_{f/1}$ and inductance $L_{f/1}$ are obtained by equating real and imaginary components in the complex equation

\[
L_{f/1} + \frac{R_{f/1}}{2\pi f} = 2 \log \frac{a_2}{a_1} + \frac{2\mu_2}{\rho_2} \frac{2\mu_1}{\rho_1} \frac{J_0(x_1\sqrt{f})}{x_1 \sqrt{f} J_1(x_1\sqrt{f})}
\]

(3.2)
where

\[ x_1 = 2\pi a_1 \sqrt{\frac{2\mu_1}{\rho_1}} f, \quad x_2 = 2\pi a_2 \sqrt{\frac{2\mu_2}{\rho_2}} f \] and

\[ x_3 = 2\pi a_3 \sqrt{\frac{2\mu_2}{\rho_2}} f \]

\[ g = \frac{J_0(x_2 V\sqrt{I}) \left[ \sqrt{V} H_1^{(1)}(x_3 V\sqrt{I}) \right] - \sqrt{V} J_1(x_3 V\sqrt{I}) H_0^{(1)}(x_2 V\sqrt{I})}{\left[ \sqrt{V} J_1(x_2 V\sqrt{I}) \right] \left[ \sqrt{V} H_1^{(1)}(x_3 V\sqrt{I}) \right] - \sqrt{V} J_1(x_3 V\sqrt{I}) H_0^{(1)}(x_2 V\sqrt{I})}. \]

where

\[ J_0(x V\sqrt{I}) = u_0(x) + i u_0(x) \]
\[ J_1(x V\sqrt{I}) = u_1(x) + i u_1(x) \]
\[ H_0^{(1)}(x V\sqrt{I}) = u_0(x) + i v_0(x) \]
\[ H_1^{(1)}(x V\sqrt{I}) = u_1(x) + i v_1(x) \]

The eight real functions \( u_n, v_n, U_n, V_n, (n=0,1) \) are tabulated in Jahnke-Emde's "Tables of functions," pages 246-258 (fourth edition, 1945) for values of \( x \) from 0 to 5.99. For larger values the asymptotic expansions may be used and lead to the high-frequency formulas given above.
3.2. Proximity Effect in Parallel Wires [29]

Current goes one way in one of the wires and returns in the other. See (2.44) and (2.45) for low frequency.

The resistance $R/l$ and self-inductance $L/l$ per unit length of the line (of both wires) for high frequency $f$, are given by

$$\begin{align*}
R/l & \sim \left[ \left(1 + \frac{a_1}{a_2} e^{-\gamma/2 + e^{-\gamma}} \right) \frac{\sqrt{\mu_1 \rho_1}}{a_1} + \left(1 + \frac{a_2}{a_1} e^{-\gamma/2 + e^{-\gamma}} \right) \frac{\sqrt{\mu_2 \rho_2}}{a_2} \right] \frac{Vf}{1 - e^{-\gamma}} \\
L/l & \sim \gamma + \frac{R/l}{2\pi f}
\end{align*}$$

(3.3)

where $\gamma$ is the reciprocal of the capacitance per unit length, so that by (1.15)

$$\frac{1}{C/l} = \gamma = 2 \log \left[ \frac{b^2 - a_1^2 - a_2^2 + \sqrt{b^2 - (a_1 + a_2)^2} \left[ b^2 - (a_1 - a_2)^2 \right]}{2a_1 a_2} \right]$$

$$= 2 \log \left[ \frac{2a_1 a_2}{b^2 - a_1^2 - a_2^2 - \sqrt{b^2 - (a_1 + a_2)^2} \left[ b^2 - (a_1 - a_2)^2 \right]} \right]$$

For equal wires of the same material, these become

$$\begin{align*}
R/l & \sim \frac{2b}{a} \sqrt{\frac{\mu \rho f}{b^2 - 4a^2}} \\
L/l & \sim 4 \log \left( \frac{b + \sqrt{b^2 - a^2}}{2a} \right) \frac{R/l}{2\pi f}
\end{align*}$$

(3.4)
3.3. Single Wire Parallel to the Earth

![Diagram of a single wire parallel to the Earth](image)

Figure 36

4. Legendre Functions That Occur in the Formulas

The Legendre polynomials $P_n(x)$ and their derivatives $P'_n(x)$, (where $n$ is a positive integer or zero), occur in formulas (2.4) to (2.12) and in (2.22) to (2.25). These satisfy the recurrence relations

\[ (2n+1)xP_n(x) = nP_{n-1}(x) + (n+1)P_{n+1}(x), \quad (4.1) \]
\[ (2n+1)(1-x^2)P'_n(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)]. \quad (4.2) \]

They are even or odd functions of $x$, according as $n$ is an even or odd integer

\[ P_n(x) = \sum_{s=0}^{n} (-1)^s \left( \frac{1-x}{2} \right)^s \frac{(s+n)!}{s! s! (n-s)!} \quad (4.3) \]
\[ P'_n(x) = \frac{1}{2} \sum_{s=0}^{n-1} (-1)^s \left( \frac{1-x}{2} \right)^s \frac{(s+n+1)!}{s! (s+1)! (n-1-s)!} \quad (4.4) \]

or in powers of $x$,

\[ P_{2n}(x) = (-1)^n \sum_{s=0}^{n} \frac{(-1)^s x^{2s} \Gamma(s+\frac{1}{2}+n)}{s! (n-s)! \Gamma(s+\frac{1}{2})} \quad (4.5) \]
\[ P_{2n+1}(x) = (-1)^n x \sum_{s=0}^{n} \frac{(-1)^s x^{2s} \Gamma(s+\frac{3}{2}+n)}{s! (n-s)! \Gamma(s+\frac{3}{2})} \quad (4.6) \]
The last two equations give

\[ P_0(x) = 1 \]
\[ P_1(x) = x \]
\[ P_2(x) = \frac{1}{2} (3x^2 - 1) \]
\[ P_3(x) = \frac{1}{2} (5x^3 - 3x) \]
\[ P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3) \]
\[ P_5(x) = \frac{1}{8} (63x^5 - 70x^3 + 15x) \]
\[ P_6(x) = \frac{1}{16} (231x^6 - 315x^4 + 105x^2 - 5) \]
\[ P_7(x) = \frac{1}{16} (429x^7 - 693x^5 + 315x^3 - 35x) \]
\[ P_8(x) = \frac{1}{128} (6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35) \]

(See references [5] and [11].)

The Legendre functions \( Q_n \) and \( P_n \) occur in the capacitance formula (1.19). The function \( Q_n \) appears in the general inductance formula (B) of section 2 and is the origin of the elliptic functions in (2.1) (2.16) (2.26) to (2.29). These are infinite series that occur frequently for real argument greater than 1, sometimes written \( \cosh \beta \), where \( \beta \) is a positive real quantity.

\[
Q_{\nu - \frac{1}{2}}(\cosh \beta) = \frac{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)} e^{-(\nu + \frac{1}{2})\beta} F\left(\frac{1}{2}, \nu + \frac{1}{2}, \nu + 1; e^{-2\beta}\right)
= e^{-(\nu + \frac{1}{2})\beta} \sum_{s=0}^{\infty} e^{-2s\beta} \frac{\Gamma(s + \frac{1}{2}) \Gamma(s + \nu + \frac{1}{2})}{s! \Gamma(s + \nu + 1)}
\]  

(4.7)

\[
P_{\nu - \frac{1}{2}}(\cosh \beta) = P_{\nu - \frac{1}{2}}(\cosh \beta) = F\left(\frac{1}{2}, \frac{1}{2} + \nu, 1; 1 - e^{-2\beta}\right)
= \frac{e^{-(\nu + \frac{1}{2})\beta}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \sum_{s=0}^{\infty} \frac{(1 - e^{-2\beta})s \Gamma(s + \frac{1}{2}) \Gamma(s + \nu + \frac{1}{2})}{s! \cdot s!}
\]  

(4.8)

\[
sinh^{2}\beta [Q_{\nu - \frac{1}{2}}(\cosh \beta)P'_{\nu - \frac{1}{2}}(\cosh \beta) - Q'_{\nu - \frac{1}{2}}(\cosh \beta)P_{\nu - \frac{1}{2}}(\cosh \beta)] = 1,
\]  

(4.9)

where \( P'(z) \) denotes \( dP(z)/dz \), etc.
The hypergeometric functions in (4.7) (4.8) may be transformed, leading in the following equivalent expressions;

\[ Q_{\nu, \frac{1}{2}} (\cosh \beta) = \sqrt{\pi} \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu + 1)} \left(\frac{1}{2} \operatorname{sech} \frac{\beta}{2}\right)^{2\nu + 1} P(\nu + 1/2, \nu + 1/2, 2\nu + 1; \operatorname{sech}^2 \frac{\beta}{2}) \quad (4.10) \]

\[ P_{\nu, \frac{1}{2}} (\cosh \beta) = \left(\operatorname{sech} \frac{\beta}{2}\right)^{1-2\nu} P\left(1/2 - \nu, 1/2 - \nu, 1; \tanh^2 \frac{\beta}{2}\right) \]
\[ = \left(\operatorname{sech} \frac{\beta}{2}\right)^{1+2\nu} P\left(1/2 + \nu, 1/2 + \nu, 1; \tanh^2 \frac{\beta}{2}\right) \quad (4.11) \]

The identity

\[ P(a, \beta, \gamma; z) = (1-z)^{\gamma-a-\beta} P(\gamma-a, \gamma-\beta, \gamma; z) \quad (4.12) \]

shows that \( P_{\nu, \frac{1}{2}} \) is an even function of \( \nu \).

The recurrence relations (4.1) and (4.2) become

\[ 2\nu \cosh \beta P_{\nu, \frac{1}{2}} (\cosh \beta) = (\nu + 1/2) P_{\nu + 1/2} (\cosh \beta) + (\nu - 1/2) P_{\nu - 1/2} (\cosh \beta) \quad (4.13) \]

\[ 2\nu \sinh^2 \beta P_{\nu, \frac{1}{2}} (\cosh \beta) = (\nu^2 - 1/4) \left[ P_{\nu + 1/2} (\cosh \beta) - P_{\nu - 1/2} (\cosh \beta) \right]. \quad (4.14) \]

The same formulas are satisfied by \( Q_{\nu, \frac{1}{2}} \).

In all these expressions \( \nu \) may be replaced by any integer \( n \). The functions \( P_{\nu, \frac{1}{2}} \) are even functions of \( \nu \), but the functions \( Q_{\nu, \frac{1}{2}} \) are not, except when \( \nu = n \). In that case eq (4.7) gives \( Q_{-n, \frac{1}{2}} (z) = Q_{n, \frac{1}{2}} (z) \), where \( n \) is any integer.

For the formulas given above \( \nu \) is an integer \( n \), so that it is not necessary to compute these functions by the series (4.7) or (4.8) in view of the many excellent tables of elliptic functions. If we find the two functions for \( n = 0 \) and \( n = 1 \), namely, \( Q_{0, \frac{1}{2}} \) and \( Q_{\frac{1}{2}} \), any other \( Q_{n, \frac{1}{2}} \) may be computed by (4.13).

Similarly, if \( P_{\frac{1}{2}} \) and \( P_{\frac{1}{2}} \) are known, the recurrence relation (4.13) gives the \( P_{n, \frac{1}{2}} \) for \( n > 1 \).

The complete elliptic integrals \( K(k) \) and \( E(k) \) with modulus \( k \), where \( 0 < k < 1 \), are given by

\[ K = \frac{\pi}{2} P\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} \quad (4.15) \]

\[ E = \frac{\pi}{2} P\left(-\frac{1}{2}, \frac{1}{2}, 1; k^2\right) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \theta} d\theta. \quad (4.16) \]

The same functions with complementary modulus \( k' = \sqrt{1-k^2} \) are denoted by \( K' \) and \( E' \), respectively. (These \( E' \), \( K' \), etc. are not derivatives.)
Legendre's relation between the four is

\[ K E' + K' E - K K' = \frac{\pi}{2} \]  \hspace{1cm} (4.17)

Let \[
\begin{align*}
\cosh \beta &= \frac{2}{K^2} - 1, \\
\sinh \beta &= \frac{2 K'}{K^2}, \\
\end{align*}
\]

so that

\[
\begin{align*}
k^2 &= \text{sech}^2 \beta / 2 \\
k'^2 &= \tanh^2 \beta / 2 \\
e^{-\beta} &= \frac{1}{1 + K'}.
\end{align*}
\] \hspace{1cm} (4.18)

The equations to be derived are

\[ Q_{-\frac{1}{2}}(\cosh \beta) = kK. \] \hspace{1cm} (4.19)

\[ Q_{\frac{1}{2}}(\cosh \beta) = 2 \left( \frac{K - E}{K} \right) - kK. \] \hspace{1cm} (4.20)

\[
\frac{\pi}{2} P_{-\frac{1}{2}}(\cosh \beta) = kK'.
\] \hspace{1cm} (4.21)

\[
\frac{\pi}{2} P_{\frac{1}{2}}(\cosh \beta) = \frac{2}{K'} E' - kK'.
\] \hspace{1cm} (4.22)

Placing \( \nu = 0 \) in eq (4.10) gives by reference to (4.15) and (4.18)

\[ Q_{-\frac{1}{2}}(\cosh \beta) = Q_{-\frac{1}{2}} \left( \frac{2}{K^2} - 1 \right) - \frac{\pi}{2} kP(\frac{1}{2}, \frac{1}{2}, 1, k^2) = kK, \]

which proves eq (4.19).

Similarly, taking \( \nu = 0 \) in (4.11) gives

\[ \frac{\pi}{2} P_{-\frac{1}{2}}(\cosh \beta) = kP(\frac{1}{2}, \frac{1}{2}, 1, k'^2) = kK', \]

which proves (4.21).

The proof of the remaining two equations, (4.20) and (4.22), is not so simple. For this we may place the notation of (4.18) in eq (B) and (B') of section 2, so that

\[
\begin{align*}
\cosh \beta &= 1 + \frac{R^2}{2 \rho \rho'} = 1 + \frac{(x-x')^2 + (\rho - \rho')^2}{2 \rho \rho'} = \frac{2}{K^2} - 1, \\
k^2 &= \frac{4 \rho \rho'}{(x-x')^2 + (\rho + \rho')^2}.
\end{align*}
\] \hspace{1cm} (4.23)
Equations (B) and (B') become (with $\alpha = \phi - \phi'$)

\[
\frac{1}{\sqrt{2(cosh \beta - cos \alpha)}} = \frac{k}{2\sqrt{1-k^2 cos^2 \alpha/2}} = \sum_{n=0}^{\infty} \epsilon_n Q_{n,1/2}(cosh \beta) cos n\alpha
\]  

(4.24)

\[
Q_{n,1/2}(cosh \beta) = Q_{n,1/2} \left( \frac{2}{k^2} - 1 \right) = \frac{1}{2} \int_0^{\pi/2} \frac{cos n\alpha d\alpha}{\sqrt{2(cosh \beta - cos \alpha)}} = (-1)^n k \int_0^{\pi/2} \frac{cos 2n\theta d\theta}{\sqrt{1-k^2 sin^2 \theta}}.
\]  

(4.25)

Taking $n=1$ in (4.25) gives

\[
Q_{1/2} \left( \frac{2}{k^2} - 1 \right) = k \int_0^{\pi/2} \frac{(2 sin^2 \theta - 1)}{\sqrt{1-k^2 sin^2 \theta}} d\theta = \frac{2}{k} \int_0^{\pi/2} \frac{1- (1- k^2 sin^2 \theta)}{\sqrt{1-k^2 sin^2 \theta}} d\theta - k \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 sin^2 \theta}} = 2 \left( \frac{k-E}{k} \right) - k \xi,
\]

which proves (4.20).

For the remaining eq (4.22) take $\nu=1$ in (4.11). This gives

\[
\frac{\pi}{2} P_{1/2}(cosh \beta) = \frac{\pi}{2} k^3 F \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} (1/2, k', 2) = \frac{\pi}{2} p(-1/2, -1/2, 1; k', 2) \text{ by } (4.12).
\]

By writing out the series for $E'$ and $K'$ it is readily found that

\[
2E' - k^2 K' = \frac{\pi}{2} p(-1/2, -1/2, 1; k', 2).
\]  

(4.26)

Hence $(\pi/2) P_{1/2}(cosh \beta) = 2E'/k - kK'$, which is eq (4.22) to be proved. Hence the function $Q_{n,1/2}(cosh \beta)$ and $P_{n,1/2}(cosh \beta)$ may be evaluated by use of any of the tables referred to in section 6 that give the complete elliptic integrals $K$ and $E$ as functions of the modulus $k$. This would apply to eq (1.19).

In case of the mutual inductance $M$ between two coaxial circles, the formula (2.2) gives

\[
M = 4\pi Q_{1/2}(2/k^2 - 1),
\]

and this is tabulated against $k^2$ in table 2 of Nagoaoka and Sakurai [7].

It is found that the functions

\[
Q = Q_{n,1/2} \left( 1 + \frac{(x-x')^2 + (\rho-\rho')^2}{2\rho\rho'} \right) \text{ and } P_{n,1/2} \left( 1 + \frac{(x-x')^2 + (\rho-\rho')^2}{2\rho\rho'} \right)
\]

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satisfy the partial differential equation

$$\left( D_x^2 + D_{\rho}^2 + \frac{\nabla^2 - n^2}{\rho^2} \right) Q = 0$$

(4.27)

in the cylindrical coordinates \((x, \rho)\), and also in \((x', \rho')\). The canonical expansions in various systems of coordinates of \(Q_{n-\Delta} \) with this argument are obtained in reference [12].

From (4.27) it is found that if

$$U_n(x, \rho) = \frac{2}{\sqrt{\rho}} \int_s \int_{\rho'} f(x', \rho') Q_{n-\Delta} \left( 1 + \frac{(x-x')^2 + (\rho-\rho')^2}{2\rho\rho'} \right) dS',$$

(4.28)

then

$$\left( D_x^2 + D_{\rho}^2 + \frac{\nabla^2 - n^2}{\rho^2} \right) \left( \rho^{\Delta} U_n \right) = 0, \text{ where } (x, \rho) \text{ is outside } S$$

(4.29)

$$= 4\pi \rho^{\frac{\Delta}{2}} f(x, \rho), \text{ where } (x, \rho) \text{ is inside } S$$

which may be written

$$\left( D_x^2 + D_{\rho}^2 + \frac{\nabla^2 - n^2}{\rho^2} \right) U_n = 0 \text{ when } (x, \rho) \text{ is outside } S$$

(4.30)

$$= -4\pi f(x, \rho) \text{ when } (x, \rho) \text{ is inside } S$$

For the case \(n=1, U_1 = A_\phi = \text{ the } \phi\text{-component of vector potential of a current distribution whose } \phi\text{-component of current density is } i_\phi = f(x, \rho). \text{ This is eq (C') of section 2.}

For the case \(n=0, U_0(x, \rho) \text{ is the axially symmetric potential } V, \text{ of a ring distribution of charge whose density is } f(x, \rho) \text{ in the ring of section } S. \text{ Hence}

$$V(x, \rho) = \frac{2}{\sqrt{\rho}} \int_s \int_{\rho'} f(x', \rho') Q_{n-\Delta} \left( 1 + \frac{(x-x')^2 + (\rho-\rho')^2}{2\rho\rho'} \right) dx' d\rho'$$

(4.31)

$$\nabla^2 V = (D_x^2 + D_{\rho}^2 + \frac{1}{\rho^2} D_\rho) V = 0 \text{ outside } S$$

(4.32)

$$= -4\pi f(x, \rho) \text{ inside } S$$

Hence the potential at \((x, \rho)\) due to a circular line charge \(\mathcal{M}\) in the plane \(x'\) with radius \(\rho'\) and coaxial with the \(x\)-axis is

$$V(x, \rho) = \frac{\mathcal{M}}{\pi\sqrt{\rho\rho'}} Q_{n-\Delta} \left( 1 + \frac{(x-x')^2 + (\rho-\rho')^2}{2\rho\rho'} \right)$$

(4.33)
5. Derivation of Some Formulas

5.1. Eccentric Spheres and Cylinders (Internal)

Equations (1.11) and (1.12) are derived by use of biaxial coordinates $\alpha$ and $\beta$, defined by the transformation

$$x + iy = ic \cot \left( \frac{\alpha + i\beta}{2} \right) \text{ where } c > 0$$

or

$$x = \frac{c \sinh \beta}{\cosh \beta \cos \alpha} \quad (5.1)$$
$$y = \frac{c \sin \alpha}{\cosh \beta \cos \alpha} \quad (5.2)$$

$$r = \sqrt{x^2 + y^2} = c \sqrt{\frac{\cosh \beta \cos \alpha}{\cosh \beta \cos \alpha}} \quad (5.3)$$

$$\sqrt{dx^2 + dy^2} = \frac{c \sqrt{d\alpha^2 + d\beta^2}}{\cosh \beta \cos \alpha} \quad (5.4)$$

The family of circles, $\beta =$ constant, has the equation

$$(x-c \coth \beta)^2 + y^2 = \left( \frac{c}{\sinh \beta} \right)^2, \text{ or } \coth \beta = \frac{x^2 + y^2 + c^2}{2cx}. \quad (5.5)$$

The orthogonal family of circular arcs, $\alpha = $ constant, is

$$x^2 + (y-c \cot \alpha)^2 = \left( \frac{c}{\sin \alpha} \right)^2, \text{ or } \cot \alpha = \frac{x^2 + y^2 - c^2}{2cy}. \quad (5.6)$$

The two-dimensional potential satisfies

$$\left( D_x^2 + D_y^2 \right) V = \left( \frac{\cosh \beta \cos \alpha}{c} \right)^2 (D_{\alpha}^2 + D_{\beta}^2) V = 0. \quad (5.7)$$

For the axially symmetric potential, Laplace's equation with cylindrical coordinates 
$$\left( D_x^2 + D_\rho^2 + \frac{1}{\rho} D_\rho \right) V = 0$$
becomes

$$\left( D_\alpha^2 + D_\beta^2 + \frac{1}{\sin^2 \alpha} \right) (\rho^{\frac{3}{2}} V) = 0. \quad (5.8)$$
The correspondence of the \((x, y)\) half-plane \((y > 0)\) and the \((\alpha, \beta)\) strip \((0 < \alpha < \pi)\), \((-\infty < \beta < \infty)\) is shown by the lettering in figure 37.

The three constants \(c\), \(\beta_1\), and \(\beta_2\) are determined by eq (5.5) in terms of the given radii \(a_1 = c/\sinh \beta_1\), \(a_2 = c/\sinh \beta_2\), and the distance between centers \(b = c (\coth \beta_2 - \coth \beta_1)\). The solution of these three equations for the case of internal circles as in figure 5 is given in (1.10). The inner circle \(\beta_1\) of figure 5 is the dotted semicircle of figure 37.

In the case of cylinders the two-dimensional potential between these cylinders is

\[
\varphi(\beta) = \left[ \frac{\beta_1 - \beta}{\beta_1 - \beta_2} \right] \varphi_2 \text{ for } \beta_2 < \beta < \beta_1.
\] (5.9)

The positive charge per unit length on cylinder 2 is \(Q_2 > 0\); the negative charge on 1 is \(Q_1\), where

\[
-Q_1 = Q_2 = \frac{-1}{4\pi} \int_{\pi} (\frac{\partial \varphi}{\partial \beta}) d\alpha = \frac{1}{2(\beta_1 - \beta_2)}.
\]
so that

\[
\frac{Q_2}{v_2} = C \frac{c}{m} = \frac{1}{2(\beta_1 - \beta_2)},
\]

which is eq (1.12).

To derive (1.11) for eccentric spheres, one within the other, we find the axially symmetric potential between the spheres, satisfying Laplace's equation in the form (5.8) for \( \beta_2 < \beta < \beta_1 \)

\[
V(\alpha, \beta) = v_2 \sqrt{2(cosh \beta - cos \alpha)} \sum_{n=0}^{\infty} \frac{e^{-(n+\frac{1}{2})\beta}}{\sinh(n+\frac{1}{2})(\beta_1 - \beta_2)} \sinh(n+\frac{1}{2})(\beta_1 - \beta_2) P_n(\mu),
\]

(5.10)

where \( \mu = cos \alpha \) and \( P_n(\mu) \) is the Legendre polynomial. This potential vanishes on the inner sphere \( \beta = \beta_1 \). To show that it has the constant value \( v_2 \) on the outer sphere where \( \beta = \beta_2 \), the normal series

\[
f(\mu) = \sum_{n=0}^{\infty} (n+1/2) P_n(\mu) \int_{-1}^{1} f(\mu') P_n(\mu') d\mu' \text{ for } -1 < \mu < 1
\]

may be used. Since \( \mu = cos \alpha \), we find for \( 0 < \beta, \)

\[
\int_{-1}^{1} \frac{P_n(\mu) d\mu}{\sqrt{2(cosh \beta - cos \alpha)}} = \frac{e^{-(n+\frac{1}{2})\beta}}{(n+\frac{1}{2})}
\]

(5.11)

which gives the normal series

\[
\frac{1}{\sqrt{2(cosh \beta - cos \alpha)}} = \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\beta} P_n(\mu) \text{ for } -1 < \mu < 1.
\]

(5.12)

(Equation (4.24) is the Fourier series for this same function.)

Taking \( \beta = \beta_2 \) in (5.12) shows that \( V(\alpha, \beta_2) = v_2 \). There is a positive charge \( Q_2 \) on sphere No. 2 and a negative charge \( Q_1 \) on No. 1, where (since \( y \) is now replaced by the cylindrical coordinate \( \rho \)

\[
Q_1 = Q_2 = -\frac{1}{4\pi} \int_0^\pi 2\pi R_1 (\frac{\partial V}{\partial \beta}) d\alpha = -\frac{c}{2} \int_0^\pi \frac{\sin \alpha}{(\cosh \beta_1 - \cos \alpha)} (\frac{\partial V}{\partial \beta_1}) d\alpha = -\frac{c}{2} \int_{-1}^{1} (\frac{\partial V}{\partial \beta_1}) (\cosh \beta_1 - \mu)
\]

This gives by use of (5.10)

\[
Q_2 = v_2 c \sum_{n=0}^{\infty} \frac{e^{-(n+\frac{1}{2})\beta_2}}{\sinh(n+\frac{1}{2})(\beta_1 - \beta_2)} (n+\frac{1}{2}) \int_{-1}^{1} \frac{P_n(\mu) d\mu}{\sqrt{2(cosh \beta_1 - \mu)}},
\]

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or by (5.11) with $\beta = \beta_1$

$$\frac{Q_2}{V_2} = c \sum_{n=0}^{\infty} \frac{e^{-(n+\frac{1}{2})\beta_2}}{\sinh(n+\frac{1}{2})(\beta_2 - \beta_1)} = 2c \sum_{n=0}^{\infty} \frac{e^{-(2n+1)\beta_1}}{1 - e^{(2n+1)(\beta_2 - \beta_1)}},$$

where $0 < \beta_2 < \beta_1$, which proves eq (1.11).

When the circles become coaxial $b \to 0$, but $c \to \infty$, and $2bc - a_2^2 - a_1^2$. Hence eq (1.11) reduces to the coaxial case (1.5) and eq (1.12) reduces to (1.6).

5.2. Eccentric Spheres and Cylinders (External)

Equations (1.14) and (1.15)

In this case the circles are external. The circle No. 1 on the left is $\beta = \beta_1 < 0$, and the derivation is made with $\beta_1$ negative. At the end we then replace $\beta_1$ by $-\beta_1$, so that in figure 37 circle No. 1 is $\beta = -\beta_1$, where $\beta_1 > 0$. This is done to keep the three constants $c$, $\beta_1$, and $\beta_2$ all positive, as stated in the three equations (1.13), which have been determined by use of (5.5).

Hence with $\beta_1$ negative, the potential between the cylinders is

$$V(\beta) = \left[ \frac{\beta - \beta_1}{\beta_2 - \beta_1} \right] V_2 \text{ for } 0 > \beta_1 < \beta < \beta_2 < 0. \quad (5.13)$$

As before,

$$-Q_1 = Q_2 = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left( \frac{\partial V}{\partial \beta_1} \right) d\alpha = \frac{1}{2(\beta_2 - \beta_1)} \int \frac{\partial V}{\partial \beta_1} d\alpha,$$

so

$$\frac{Q_2}{V_2} = \frac{1}{2(\beta_2 - \beta_1)},$$

which becomes (1.15) on replacing $-\beta_1$ by $\beta_1$.

For the case of spheres

$$V(\alpha, \beta) = V_2 \sqrt{2(\cosh \beta - \cos \alpha)} \sum_{n=0}^{\infty} \frac{e^{-(n+\frac{1}{2})\beta_2} \sinh(n+\frac{1}{2})(\beta - \beta_1)}{\sinh(n+\frac{1}{2})(\beta_2 - \beta_1)} P_n(\mu). \quad (5.14)$$

We now find

$$-Q_1 = Q_2 = \frac{c}{2} \int_{-1}^{1} \left( \frac{\partial V}{\partial \beta_1} \right) \frac{d\mu}{(\cosh \beta_1 - \mu)},$$

or

$$\frac{Q_2}{V_2} = c \sum_{n=0}^{\infty} \frac{e^{-(n+\frac{1}{2})\beta_2}}{\sinh(n+\frac{1}{2})(\beta_2 - \beta_1)} \left( n + \frac{1}{2} \right) \int_{-1}^{1} P_n(\mu) d\mu.$$

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Since $\beta_1$ is here negative, we must write eq (5.11)

$$
(n+1/2) \int_{-1/2}^{1/2} \frac{P_n(\mu)}{\sqrt{2(cosh(\beta_1-\mu))}} e^{-(\mu+1/2)|\beta_1|} = e^{+(\mu+1/2)|\beta_1|} \quad \text{for } \beta_1 < 0,
$$

so that

$$
C = c \sum_{n=0}^{\infty} \frac{e^{-(\mu+1/2)(\beta_1-\beta_1)}}{\sinh((n+1/2)(\beta_2-\beta_1))} = 2c \sum_{n=0}^{\infty} \frac{e^{-(\mu+1/2)\gamma}}{1-e^{-(\mu+1/2)\gamma}},
$$

where $\gamma = 2(\beta_2-\beta_1)$ when $\beta_1 < 0$ and $\beta_2 > 0$.

On reversing the sign of $\beta_1$, this gives eq (1.14). For the limiting cases $\beta_1 \to 0$, in which the sphere or cylinder on the left of figure 6 or figure 37 has an infinite radius, we may place $b=a_1+h$ and $a_2=a_1$. When $a=\infty$, $c=\sqrt{h^2-a^2}$, $\beta_2=\log(h+a^2-a^2)/a$, so eq (1.14) and (1.15) become (1.8) and (1.9), respectively. The potential between cylinder and plane $\beta=0$ (fig. 4)
is

$$
V(\beta) = \frac{\beta}{\beta_2} V_2 \quad \text{for} \quad 0 < \beta < \beta_2 = \log \frac{h+\sqrt{h^2-a^2}}{a}.
$$

Between the sphere and plane the potential is eq (5.14) with $\beta_1 = 0$.

5.3. Derivation of Equations (1.17) and (1.16) for Oblate Spheroid and Circular Disk

With oblate spheroidal coordinates $(a, \beta)$ the $(x, \rho)$ half-plane is represented on the $(a, \beta)$ strip $0 < a < \pi$, $0 < \beta < c$ by

$$
x + i \rho = c \sin(a + i \beta) \quad \text{where} \quad c > 0
$$
or

$$
x = -c \cos a \sinh \beta \quad \text{and} \quad \rho = c \sin a \cosh \beta
$$
so

$$
r = \sqrt{x^2 + \rho^2} = c \sqrt{\sinh^2 \beta + \sin^2 a}
$$
or

$$
\frac{x^2}{c^2 \sinh^2 \beta} + \frac{\rho^2}{c^2 \cosh^2 \beta} = 1 \quad \text{(confocal ellipses)}
$$
and

$$
\frac{-x^2}{c^2 \cos^2 a} + \frac{\rho^2}{c^2 \sin^2 a} = 1 \quad \text{(confocal hyperbolas)}.
$$
The equation for the axially symmetric potential

\[
\left( \frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial \beta^2} + \frac{1}{4 \sin^2 a} - \frac{1}{4 \cosh^2 \beta} \right) (\rho^2) = 0
\]

has solutions \( V = Q_n (i \sinh \beta) P_n (\cos a) \). The oblate spheroid \( \beta = \beta_1 \) has semiaxes \( a \) and \( b < a \), where

\[
c = \sqrt{a^2 - b^2}, \quad \sinh \beta_1 = b/c \quad \text{and} \quad \cosh \beta_1 = a/c
\]

\[
i Q_0 (i \sinh \beta) = \frac{i}{2} \log \frac{i \sinh \beta + 1}{i \sinh \beta - 1} = \sin^{-1} \left( \text{sech} \ \beta \right)
\]

so

\[
i Q_0 (i \sinh \beta_1) = \sin^{-1} \left( \frac{c}{a} \right)
\]

The potential outside the conducting spheroid \( \beta_1 \) at potential \( V_1 \) with charge \( \kappa_1 \) is (for \( \beta_1 < \beta < \infty \))

\[
V(\beta) = V_1 \frac{i Q_0 (i \sinh \beta)}{i Q_0 (i \sinh \beta_1)} = V_1 \frac{\sin^{-1} \left( \text{sech} \ \beta \right)}{\sin^{-1} \left( \frac{c}{a} \right)}.
\]

and \( \kappa_1 = \lim_{\beta \to \infty} \frac{r V(\beta)}{c} = \lim_{\beta \to \infty} \frac{c \sinh \beta V(\beta)}{\sin c \ \alpha} = \frac{c V_1}{\sin c \ \alpha} \), which is eq (1.17).

### 5.4. Derivation of Equation (1.18) for Prolate Spheroid

With prolate spheroidal coordinates the \((x, \rho)\) half-plane is represented on the \((a, \beta)\) strip \((0 < a < \pi), \ (0 < \beta < \infty)\) by

\[
x + i \rho = c \cos (a + i \beta), \ \text{where} \ c > 0, \ \text{or} \n
\[
x = c \cos a \ \cos \beta \ \text{and} \ \rho = c \sin a \ \sinh \beta, \ \text{so} \n
\[
r = \sqrt{x^2 + \rho^2} = c \sqrt{\sinh^2 \beta + \cos^2 a}.
\]

Hence

\[
\frac{x^2}{c^2 \cosh^2 \beta} + \frac{\rho^2}{c^2 \sinh^2 \beta} = 1 \ (\text{ellipses})
\]

\[
\frac{x^2}{c^2 \cos^2 a} - \frac{\rho^2}{c^2 \sin^2 a} = 1 \ (\text{hyperbolas}).
\]

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The equation for the axially symmetric potential

\[
\left( \frac{D_\alpha^2 + D_\beta^2}{4 \sin^2 \alpha + 4 \sinh^2 \beta} \right) (\rho^{\frac{3}{2}} V) = 0
\]

has solutions \( V = Q_n (\cosh \beta) P_n (\cos \alpha) \).

For the prolate spheroid \( \beta = \beta_1 \) with semiaxes \( a \) and \( b < a \)

\[
c = \sqrt{a^2 - b^2} \quad \text{and} \quad \cosh \beta_1 = a / c,
\]

\[
Q_0 (\cosh \beta) = \frac{1}{2} \log \frac{\cosh \beta + 1}{\cosh \beta - 1} = \log \coth \frac{\beta}{2} = \log \left( \frac{1 + e^{-\beta}}{1 - e^{-\beta}} \right),
\]

\[
Q_0 (\cosh \beta_1) = \log (a + c) / b.
\]

The potential outside the conducting spheroid \( \beta_1 \) at potential \( V_1 \) with charge \( N_1 \) is

\( (\text{for } \beta_1 < \beta < \infty) \)

\[
V(\beta) = V_1 \frac{Q_0 (\cosh \beta)}{Q_0 (\cosh \beta_1)} = V_1 \frac{1}{2} \log \cosh \beta / b = V_1 \frac{1}{\log (a + c) / b}, \tag{5.16}
\]

and

\[
N_1 = \lim_{r \to \infty} [r V(\beta)] = \lim_{\beta \to \infty} [c \sqrt{\sinh^2 \beta + \cos^2 a} V(\beta)]
\]

\[
= \frac{c V_1}{\log (a + c) / b'},
\]

which is eq (1.18).

5.5. Derivation of Equation (1.19) for a Toroid

The strip \((-\pi < \alpha < \pi), \,(0 < \beta < \infty)\) of the toroidal (or "ring") coordinates, represents the \((x, \rho)\) half-plane, if this is cut from zero to \( c \) along the \( \rho \)-axis. The equation \( x + i \rho = -c \cot (a + i \beta) / 2 \) gives

\[
x = \frac{-c \sin \alpha}{\cosh \beta - \cos \alpha} \quad \text{and} \quad \rho = \frac{c \sin \beta}{\cosh \beta - \cos \alpha}
\]

\[
r = \sqrt{x^2 + \rho^2} = c \sqrt{\frac{\cosh \beta + \cos \alpha}{\cosh \beta - \cos \alpha}}
\]

\[
\sqrt{dx^2 + dy^2} = \frac{c \sqrt{da^2 + d\beta^2}}{\cosh \beta - \cos \alpha}. \tag{5.18}
\]
The family of circles \( \beta = \text{constant} \), each member of which generates a toroidal surface by rotation around the \( x \)-axis, belongs to the equation

\[
X^2 + (\rho - \cot \alpha \coth \beta)^2 = \frac{c^2}{\sinh^2 \beta}.
\] (5.19)

The equation of the family of circular arcs, orthogonal to these circles, is

\[
(x + c \cot \alpha)^2 + \rho^2 = \frac{c^2}{\sin^2 \alpha}.
\] (5.20)

The equation for the axially symmetric potential

\[
\left( \frac{D^2 + \beta^2}{4 \sinh^2 \beta} \right) (\rho^{\beta^2}) = 0
\] (5.21)

has solutions of the form

\[
V = \sqrt{2(\cosh \beta - \cos \alpha)}(A \cos n \alpha + B \sin n \alpha)(CP_{n-y}(\cosh \beta) + DQ_{n-y}(\cosh \beta)).
\]

The third of eq (5.17) shows that spatial infinity, \( r=\infty \) corresponds to the point \( \alpha=\beta=0 \). The first two of these equations show that \( \beta=+\infty \) corresponds to \( x=0 \) and \( \rho=c=\) the radius of the focal circle.

If the generating circle of figure 7 has the equation \( \beta = \beta_1 \), it is evident from (5.19) that

\[
c = \sqrt{A^2 - a^2} \text{ and } \cosh \beta_1 = \frac{a}{c}.
\] (5.22)

If the toroidal surface has a constant potential \( V_1 \) and charge \( \Xi_1 \), the Newtonian potential at outside points where \( 0<\beta<\beta_1 \) is

\[
V(\alpha, \beta) = \frac{2V_1}{\pi} \sqrt{2(\cosh \beta - \cos \alpha)} \sum_{n=0}^{\infty} \epsilon_n Q_{n-y}(\cosh \beta_1) P_{n-y}(\cosh \beta) \cos n\alpha,
\] (5.23)

where \( \epsilon_0 = \frac{1}{2}, \epsilon_n = 1 \) for \( n \neq 0 \).

This vanishes at \( r=\infty \) (i.e., when \( \alpha=\beta=0 \)). On the surface \( \beta_1 \) it becomes

\[
V(\alpha, \beta_1) = V_1 \sqrt{2(\cosh \beta_1 - \cos \alpha)} \frac{2}{\pi} \sum_{n=0}^{\infty} \epsilon_n Q_{n-y}(\cosh \beta_1) \cos n\alpha
\]

\[
= V_1 \text{=constant by eq (4.24)}.
\]
By a fundamental property of Newtonian potentials, the charge \( \mathcal{N}_1 \) on the torus is

\[
\mathcal{N}_1 = \lim_{r \to \infty} \frac{\mathcal{V}}{r} = \lim_{a \to 0} c \sqrt{\frac{\cosh \beta + \cos \alpha}{\cosh \beta - \cos \alpha}} \mathcal{V}(\alpha, \beta)
\]

\[
= \frac{4cV}{\pi} \lim_{a \to 0} \sum_{n=0}^{\infty} \epsilon_n \frac{Q_{n-\frac{1}{2}}(\cosh \beta_1)}{P_{n-\frac{1}{2}}(\cosh \beta_1)} P_{n-\frac{1}{2}}(\cosh \beta) \cos n\alpha.
\]

Since \( P_{n-\frac{1}{2}}(1) = 1 \), this gives

\[
\frac{\mathcal{N}_1}{V_1} = \frac{4c}{\pi} \sum_{n=0}^{\infty} \epsilon_n \frac{Q_{n-\frac{1}{2}}(\cosh \beta_1)}{P_{n-\frac{1}{2}}(\cosh \beta_1)},
\]

which is eq (1.19), since \( c = \sqrt{A^2 - a^2} \) and \( \cosh \beta_1 = A/a \). The evaluation of these functions by elliptic integrals is discussed in section 4.

### 5.6. Self-Inductance of a Single Turn of Wire

**Equation (2.15)**

With cylindrical coordinates \((x, \rho)\) the vector potential \( A_{\phi} = A(x, \rho) \) at any point \((x, \rho)\) in space is by eq (C) of section 2.

\[
\rho^\frac{3}{2} A(x, \rho) = 2 \int \rho' \frac{1}{2} i(x', \rho') Q_{\frac{3}{2}} \left( 1 + \frac{D^2}{2\rho'} \right) dS'
\]

where \( D^2 = (x-x')^2 + (\rho-\rho')^2 \), and the integration is taken with respect to \((x', \rho')\) over the upper circle of radius \( a \) in figure 18.

Also by eq (D) of section 2 the self-inductance is given by

\[
L = 2\pi \int \rho^\frac{3}{2} i(\rho) \cdot \rho^\frac{3}{2} A(x, \rho) dS
\]

integrated with respect to \((x, \rho)\) over the same circular section. Since \((x, \rho)\) and \((x', \rho')\) are both points in this circle, we may use the expansion

\[
Q_{n-\frac{1}{2}} \left( 1 + \frac{D^2}{2\rho'} \right) = \frac{(-1)^{n+1}}{2\pi} \sum_{s=0}^{\infty} \left( \frac{-D^2}{4\rho'} \right)^{s} \frac{\Gamma(s+\frac{1}{2}+n)\Gamma(s+\frac{1}{2}+n)}{s!} \left[ \log \frac{D^2}{4\rho'} + \psi(s+\frac{1}{2}+n) + \psi(s+\frac{1}{2}+n) - 2\psi(s+1) \right].
\]

This is valid if \( D^2/4\rho' < 1 \), which will be true for all positions of the points \( P(x, \rho) \) and \( P'(x', \rho') \) both within the circle, provided that \( a < A/2 \), which will be true here,
since it is assumed that $a/A$ is so small that terms smaller than $a^2/8A^2 \log 8A/a$ may be neglected in comparison with 1.

Hence for $n=1$, eq (5.26) gives, to this approximation,

\[-2Q\frac{\lambda}{2}(\frac{D^2}{2\rho\rho'}) = \left[1 + \frac{3}{4}(\frac{D^2}{4\rho\rho'})\right] \log \frac{D^2}{4\rho\rho'} + 4(1 - \log 2) + \frac{1}{2}(1 - 6 \log 2) \frac{D^2}{4\rho\rho'}, \quad (5.27)\]

Let $y = \rho - A$ and $y' = \rho' - A$, so that $x$ and $y$ are rectangular coordinates with origin at the center of the circle. Then, to the second order in $a/A$,

\[-2Q\frac{\lambda}{2}(1 + \frac{D^2}{2\rho\rho'}) = \left[4 + 2 \log \frac{D}{8A} - \frac{y + y'}{A} + \frac{y^2 + y'^2}{2A^2} + \frac{D^2}{8A^2} \left(1 + 3 \log \frac{D}{8A}\right)\right]. \quad (5.28)\]

For the assumed current density

\[i = \frac{1}{\pi a^2 F}(\frac{\rho}{A})^b, \quad (5.29)\]

the total current is 1, which gives

\[F = F\left(\frac{b}{2}, \frac{1 - b}{2}, 2; \frac{a^2}{A^2}\right) = 1 + b(b - 1)\frac{a^2}{8A^2} + \ldots. \quad (5.30)\]

Then

\[\rho^\frac{1}{2} i(\rho) = \frac{A^\frac{1}{2}}{\pi a^2 F}(1 + \frac{y}{A})^b + \frac{C_1^2}{A^2}, = \frac{A^\frac{1}{2}}{\pi a^2 F} \left[1 + C_1^2 + C_2^2 \frac{y^2}{2A^2}\right], \quad (5.31)\]

where

\[C_1 = b + \frac{1}{2} \text{ and } C_2 = \frac{1}{2} \left(b^2 - \frac{1}{4}\right). \quad (5.32)\]

This gives

\[2\rho^\frac{1}{2} i(\rho')Q\frac{\lambda}{2}(1 + \frac{D^2}{2\rho\rho'}) = \frac{A^\frac{1}{2}}{\pi a^2 F} \left\{[1 + C_1^2 + C_2^2 \frac{y^2}{2A^2} - 4 + 2 \log \frac{D}{8A}]\right\} + \frac{y + y'}{A} + \frac{y^2 + y'^2}{2A^2} + \frac{D^2}{8A^2} \left(1 + 3 \log \frac{D}{8A}\right). \quad (5.33)\]

To use this expression in the integral (5.24) it is better to use polar coordinates, placing $x = r \cos \theta$ and $y = r \sin \theta$, so that

\[D^2 = r^2 - 2rr' \cos(\theta - \theta') + r'^2, \quad (5.34)\]

and
\[ \log D = \log r - \sum_{n=1}^{\infty} \left( \frac{r'}{r} \right)^n \cos n(\theta - \theta') \quad \text{if } r' \leq r \]

\[ = \log r' - \sum_{n=1}^{\infty} \left( \frac{r}{r'} \right)^n \cos n(\theta - \theta') \quad \text{if } r' > r. \quad (5.35) \]

The result of integrating eq (5.24) is

\[ \rho^{\frac{1}{2}} A(x, \rho) = -\frac{A^{\frac{1}{2}}}{F} \left[ f_0(r) + f_1(r, \theta) + f_2(r, \theta) \right] \]

where

\[ f_0(r) = 3 - 2 \left( 1 + \frac{a^2 C_2}{a^2} \right) \log \left( \frac{8A}{a} \right) + \frac{r^2}{a^2} \]

\[ f_1(r, \theta) = \left[ 1 + \frac{C_1 r^2}{2a^2} \right] \frac{r \sin \theta}{A} \]

\[ f_2(r, \theta) = \frac{a^2}{8A^2} \left[ \frac{9}{8} - 2C_1 + 7C_2 - \frac{3}{2} \log \frac{8A}{a} \right] \]

\[ + \frac{r^2}{8A^2} \left[ \frac{5}{2} C_2 - 3 \log \frac{8A}{a} + (4 - 2C_2) \sin^2 \theta \right] + \frac{r^4}{24a^2 A^2} \left[ \frac{9}{8} + C_2 + 2C_2 \sin^2 \theta \right] \]

(5.39)

With this, the integral (5.25) gives

\[ L = \frac{4\pi A}{F^2} \left\{ \left[ 1 + \left( 4C_2 + \frac{3}{2} \right) \frac{a^2}{8A^2} \right] \log \left( \frac{8A}{a} \right) - \frac{7}{4} \frac{a^2}{8A^2} \left[ 2C_1 \left( 1 + \frac{C_1}{3} \right) - \frac{22}{3} C_2 - \frac{7}{4} \right] \right\} \]

(5.40)

Finally, multiplying by the factor,

\[ \frac{1}{F^2} = 1 - 2b(b-1) \frac{a^2}{8A^2} = 1 - C_0 \frac{a^2}{8A^2}, \]

(5.41)

where \( C_0 = 2b(b-1) \), gives

\[ L = 4\pi A \left\{ \left[ 1 + \left( 4C_2 - C_0 + \frac{3}{2} \right) \frac{a^2}{8A^2} \right] \log \left( \frac{8A}{a} \right) - \frac{7}{4} \frac{a^2}{8A^2} \left[ \frac{7}{4} C_0 - 1 \right] + 2C_1 \left( 1 + \frac{C_1}{3} \right) - \frac{22}{3} C_2 \right\}. \]

(5.42)

On substituting the expressions given above for \( C_0, C_1, \) and \( C_2 \), it is found that

\[ L = 4\pi A \left\{ \left[ 1 + \left( 2b + 1 \right) \frac{a^2}{8A^2} \right] \log \left( \frac{8A}{a} \right) - \frac{7}{4} \frac{b(b-1)(b-2/3)}{16} \left( \frac{a}{A} \right)^2 \right\}, \]

(5.43)

which is eq (2.15).
Exact expressions for the magnetic field and inductance of any toroid with this current distribution may be found as normal series of ring functions, using the toroidal coordinates of reference [20]. It is thus found that for $b=-5/2$ the external magnetic field is the same as if the total current were concentrated in the focal circle. This is true for the more general case $I_\phi=C\rho^{-5/2}f(\beta)$.

To get an expression for the potential $A(x,\rho)$ when the point $P(x,\rho)$ is outside the circle of figure 18, the approximation (5.27) based on (5.26) cannot be used unless the distance of $P$ from the center $C$ is small compared to $A$. When this distance is of the order of magnitude of $A$ or greater, while $P(x',\rho')$ remains in the circle, it is sufficient to use Taylor's series, with $(x,\rho)$ fixed and the variables $x'/A$ and $y'/A=(\rho'-A)/A$ small. For brevity, let

$$g=1+\frac{(x-x')^2+(\rho-\rho')^2}{2\rho\rho'}=\frac{(x-x')^2+\rho^2+\rho'^2}{2\rho\rho'}$$

so $g=g_0$ when $x'=y'=0$. Then, to the second order

$$Q_{1/2}(g) = Q_{1/2}(g_0) + (x'Q_x + y'Q_y) + \frac{1}{2} (x'^2 Q_{xx} + y'^2 Q_{yy} + 2 x' y' Q_{xy}),$$

(5.44)

where $Q_{xx}$ is the value of $\frac{D_x^2}{A^2}Q_{1/2}(g)$ when $g=g_0(x'=y'=0)$ and similarly, $Q_{xy}$ and $Q_{yy}$ = $\frac{D_y^2}{A^2}Q_{1/2} = \frac{D_y^2}{A^2}Q_{1/2}$.

From eq (4.27), with $n=1$ and variables $x',\rho'$, we find an exact expression when $x'=0$ and $\rho'=A$.

$$Q_{xx}+Q_{yy} = \frac{3}{4A^2}Q_{1/2}(g_0).$$

(5.45)

By use of (5.44) with the current in (5.31) in eq (5.24), it is found that

$$A(x,\rho) = 2\sqrt{\frac{A}{\rho}} \left\{ \left[ 1 + \left( b + \frac{1}{2} \right) \frac{a^2}{8A^2} \right] Q_{1/2}(g_0) + \frac{a^2}{8A^2} \left( b + \frac{1}{2} \right) 2A Q_{y} \right\},$$

(5.46)

where

$$2A Q_y = -2 \left( \frac{g_0 - A}{\rho} \right) Q_{1/2}(g_0) = - \left( \frac{g_0 - A}{g_0 - 1} \right) \left[ g_0 Q_{1/2}(g_0) - Q_{1/2}(g_0) \right],$$

(5.47)

and

$$g_0 = 1 + \frac{x^2 + (\rho-A)^2}{2A\rho}.$$  

(5.48)
Equation (5.46) is valid when \( q_0 - 1 \) is not small. Hence there remains a gap, not here considered, between the ranges of validity of the two equations (5.46) and (5.36), which could only be bridged by an equation more complicated than either. Applications of (5.46) that would require the retention of the second-order terms are exceedingly rare. It is generally sufficient to consider the total current concentrated in a filament with trace at center of the circular section of the wire.

5.7. Derivation of Equation (2.16) for Self-Inductance of a Single Turn of Wire Near a Magnetic Medium

Referring to figure 19 let \( A_a(x, \rho) \) denote the value at any point \( P(x, \rho) \) in space due to any axially symmetric distribution of currents when \( \mu = 1 \) everywhere. These currents are all to the left of the boundary plane \( x = x_0 \).

Similarly, let \( A_2(x, \rho) \) denote the potential at any point to the left of \( x = x_0 \) that would be produced (with \( \mu = 1 \) everywhere) by a fictitious distribution of currents that is the image of the existing distribution by reflection in the plane \( x = x_0 \).

Then the potential \( A(x, \rho) \) due to the actual currents in the presence of the magnetic material with \( \mu \neq 1 \), where \( x_0 < x \), is in the air, where \( -\infty < x < x_0 \),

\[
A(x, \rho) = A_a(x, \rho) + \frac{\mu - 1}{\mu + 1} A_2(x, \rho),
\]

and in the material, where \( x_0 < x < +\infty \),

\[
A(x, \rho) = \frac{2\mu}{\mu + 1} A_a(x, \rho).
\]

By this definition of \( A_a(x, \rho) \) and \( A_2(x, \rho) \) it is evident that at the plane \( x = x_0 \), \( A_a = A_2 \), and \( D_x A_a = -D_x A_2 \) identically in \( \rho \).

Consequently, \( A \) is continuous, which makes \( B_x \) continuous. Also the continuity of \( H_\rho \) is assured by that of \( D_x A/\mu \).

The inductance of the turn of wire near the material as figure 19 is by (5.49)

\[
L = L_{air} + \frac{\mu - 1}{\mu + 1} \int \int i(x, \rho) \rho A(x, \rho) dS
\]

integrated over a circular section of the wire. This integration could be effected for the form of current in eq (5.31) by use of eq (5.46), assuming that \( 2x_0 \) is not small compared to \( A \). Formula (2.16) assumes that the fictitious current producing \( A_2(x, \rho) \) is a filament of radius \( a \) coaxial with the x-axis in the plane \( x = 2x_0 \).

For this approximation, we place in (5.51)

\[
\rho A(x, \rho) = 2A \frac{1}{2} \left( 1 + \frac{4x_0^2 + (A - a)^2}{2Aa} \right) = 2A \frac{1}{2} \left( \frac{2}{k^2} - 1 \right) = 2A \left[ \frac{2(\kappa - E)}{\kappa} - k \right] ,
\]

where

\[
k^2 = \frac{4Aa}{4x_0^2 + (A + a)^2}.
\]

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Since \( \int \int t dS = 1 \), this gives

\[
L = L_{\text{air}} + \frac{\mu - 1}{\mu + 1} 4\pi A \left[ \frac{2(1-R)}{R} - k K \right], \tag{5.54}
\]

where \( L_{\text{air}} \) is given by 5.43).

### 5.8. Derivation of Equations (2.40) and (2.41) for the Self-Inductance of Toroidal Current Sheets (Tape Winding)

With ideal tape windings the current circulates as indicated by the arrow in figure 27. There is no external field and the internal field of the unit current is \( H_0 = 2N/\rho \), where \( N \) is the number of turns, and \( \rho \) is the distance of a point from the axis of revolution. The inductance \( L \) is equal to twice the integral defining total electrokinetic energy \( T \).

\[
T = \frac{1}{8\pi} \iint \mu H^2 d\nu
\]

integrated over all space. Hence

\[
L = \frac{\mu}{4\pi} \iint H^2 d\nu = 2\mu N^2 \iint \frac{dS}{\rho}
\]

integrated over the axial section. For circular and rectangular axial sections shown in figure 27 and 28, this results in eq (2.40) and (2.41), respectively.

### 5.9. Derivation of Equation (2.45) for Self-Inductance per Unit Length of Two Parallel Wires of Magnetic Material

Referring to figure 31, the current +1 flows upward perpendicular to paper with uniform current density \( i_1 = \frac{1}{\pi a_1^2} \) in cylinder No. 1. The current density in cylinder No. 2 is \( i_2 = -\frac{1}{\pi a_2^2} \).

The only components of current density and of vector potential are the \( z \)-components where the \( z \)-axis is upward perpendicular to the paper. The general field equations \( B = \mu H = \text{curl} A \) and \( \text{curl} H = 4\pi I \) give

\[
B_x = D_y A, \quad B_y = -D_x A, \quad B_z = 0,
\]

where \( A(x, y) = A_z \). Hence

\[
(D_x^2 + D_y^2) A = \frac{4\mu_1}{a_1^2} \text{ in cylinder 1}
\]

\[
= \frac{4\mu_2}{a_2^2} \text{ in cylinder 2}
\]

\[
= 0 \text{ in the air between them.} \tag{5.55}
\]
The boundary conditions at the surface of each wire are:

\[ A \text{ is continuous (continuity of normal component of } B), \]

\[ \frac{1}{\mu} \frac{\partial A}{\partial n} \text{ is continuous (continuity of tangential } H). \]

With plane polar coordinates \((r_1, \theta_1)\) with origin at center \(O_1\) of wire No. 1.

\[
(D_x^2 + D_y^2) A = \frac{1}{r_1} D_{r_1} \left( r_1 D_{\theta_1} A \right) + \frac{1}{r_1^2} D_{\theta_1}^2 A.
\]

Similarly, with polar coordinates \((r_2, \theta_2)\) with center at \(O_2\)

\[
(D_x^2 + D_y^2) A = \frac{1}{r_2} D_{r_2} \left( r_2 D_{\theta_2} A \right) + \frac{1}{r_2^2} D_{\theta_2}^2 A.
\]

Hence let

\[
A = U - \mu_1 \left( \frac{r_1^2}{a_1^2} \right) \text{ in wire No. 1}
\]

\[
= U + \mu_2 \left( \frac{r_2^2}{a_2^2} \right) \text{ in wire No. 2}
\]

\[
= U \text{ in the air}
\]

Then \((D_x^2 + D_y^2) U = 0\) everywhere.

At \(r_1 = a_1\),

\[
U_0 = U_i - \mu_1 \text{ and } D_{r_1} U_0 = \frac{1}{\mu_1} D_{r_1} U_i - \frac{2}{a_1}.
\]

At \(r_2 = a_2\)

\[
U_0 = U_i + \mu_2 \text{ and } D_{r_2} U_0 = \frac{1}{\mu_2} D_{r_2} U_i + \frac{2}{a_2},
\]

where \(U_0\) means outside, \(U_i\) inside the wire.
The self-inductance of the line per centimeter length

\[ L/cm = \frac{1}{\pi a^2} \int \int A dS_1 - \frac{1}{\pi a^2} \int \int A dS_2 \]

\[ = -\frac{1}{2}(\mu_1 + \mu_2) + \frac{1}{\pi a^2} \int \int A dS_1 - \frac{1}{\pi a^2} \int \int A dS_2. \]  

(5.62)

The biaxial coordinates \( a, \beta \), see reference \(^{[16]}\), are suitable for constructing the harmonic function \( U(a, \beta) \) that satisfies the four boundary conditions in (5.56) and (5.57). We follow the procedure adopted in deriving the potential \( V \) in eq (5.14), that is, we take as the equation of circle No. 1 of figure 37 the equation \( \beta = \beta_1 \), where \( \beta_1 < 0 \). In the end result we change the sign of \( \beta_1 \) to make all the constants \( c, \beta_1, \) and \( \beta_2 \) positive, as given in the three eq (1.13).

By (5.4) the surface element \( dS \) for integrating over a circular area bound by the circle \( \beta = \text{constant} \) is

\[ dS = \frac{c^2 d\alpha d\beta}{(\cosh \beta - \cos \alpha)^2}, \]

where \( d\alpha, d\beta > 0 \). Now \( \sinh \beta_1 = -c/a_1 \) and \( \sinh \beta_2 = c/a_2 \). Hence after an expression for \( U(a, \beta) \) is found, eq (5.62) becomes

\[ L/cm = -\left(\frac{\mu_1 + \mu_2}{2}\right) + \frac{2 \sinh^2 \beta_1}{\pi} \int_0^{\beta_1 < 0} d\beta \int_0^\pi U(a, \beta) \frac{d\alpha}{(\cosh \beta - \cos \alpha)^2} \]

\[ - \frac{2 \sinh^2 \beta_2}{\pi} \int_{\beta_2 > 0}^{\infty} d\beta \int_0^\pi U(a, \beta) \frac{d\alpha}{(\cosh \beta - \cos \alpha)^2} \].  

(5.63)

\( U(a, \beta) \) will be found as a series in \( \cos n\alpha \), so the following integrals will be required.

\[ \frac{1}{\pi} \int_0^{\pi} \frac{\cos n\alpha d\alpha}{(\cosh x - \cos \alpha)^2} = \frac{e^{-nx}(n + \coth x)}{\sinh^2 x} = 4e^{-nx} \sum_{s=1}^{\infty} s(s+n)e^{-2(s+n)x} \]

if \( 0 < x \).

From this we find, when \( 0 < \beta \),

\[ 2 \sinh^2 \beta \int_{\beta}^{\infty} \frac{e^{-2nx}(n + \coth x)}{\sinh^2 x} dx = e^{-2n\beta}. \]

(5.65)

The function \( U(a, \beta) \) that satisfies the four boundary conditions in (5.60) and (5.61) is
In wire No. 1, where \(-\infty < \beta < \beta_1 < 0\):

\[ U(a, \beta) = \mu_1 - C_0 - 2\beta_1 + \sum_{n=1}^{\infty} A_n e^{n(\beta - \beta_1)} \cos n\alpha. \]  

(5.66)

In wire No. 2, where \(0 < \beta_2 < \beta < \infty\):

\[ U(a, \beta) = -\mu_2 - C_0 - 2\beta_2 + \sum_{n=1}^{\infty} B_n e^{-n(\beta - \beta_2)} \cos n\alpha. \]

(5.67)

In the air between them, where \(\beta_1 < \beta < \beta_2\):

\[ U(a, \beta) = -C_0 - 2\beta_1 + \sum_{n=1}^{\infty} \left[ \frac{A_n \sinh n(\beta_2 - \beta) + B_n \sinh n(\beta - \beta_1)}{\sinh n(\beta_2 - \beta_1)} \right] \cos n\alpha, \]

(5.68)

where

\[ C_0 = \sum_{n=1}^{\infty} \frac{A_n \sinh n\beta_2 - B_n \sinh n\beta_1}{\sinh n(\beta_2 - \beta_1)}. \]

(5.69)

This makes \(U = A\) vanish at spatial infinity (\(a = \beta = 0\)).

The boundary conditions require

\[ A_n e^{n\beta_1} = \frac{2(1+\epsilon_1)}{n(1-\epsilon_1 \epsilon_2 e^{-n\gamma})} \left[ (1+\epsilon_2 e^{-n\gamma}) e^{2n\beta_1} - (1+\epsilon_2) e^{-n\gamma} \right] \]

(5.70)

\[ B_n e^{-n\beta_1} = \frac{-2(1+\epsilon_2)}{n(1-\epsilon_1 \epsilon_2 e^{-n\gamma})} \left[ (1+\epsilon_1 e^{-n\gamma}) e^{-2n\beta_1} - (1+e^{-n\gamma}) \right], \]

(5.71)

where

\[ \epsilon_1 = \frac{\mu_1 - 1}{\mu_1 + 1}, \quad \epsilon_2 = \frac{\mu_2 - 1}{\mu_2 + 1}, \quad \text{and} \quad \gamma = 2(\beta_2 - \beta_1). \]

(5.72)

Performing the integrations in (5.63) by use of (5.68) and (5.69) gives

\[
\frac{L}{cm} = \pm \frac{\mu_1 + \mu_2}{2} + 2(\beta_2 - \beta_1) + \sum_{n=1}^{\infty} \left( A_n e^{n\beta_1} - B_n e^{-n\beta_2} \right) + \sum_{n=1}^{\infty} \frac{1}{n(1-\epsilon_1 \epsilon_2 e^{-n\gamma})} \left[ (1+\epsilon_1)(1+\epsilon_2 e^{-n\gamma}) e^{2n\beta_1} \right. \\
+ (1+\epsilon_1)(1+\epsilon_2 e^{-n\gamma}) e^{-2n\beta_2} \left. - 2(1+\epsilon_1)(1+\epsilon_2) e^{-n\gamma} \right]
\]

(5.73)
To obtain positive constants for computing we next reverse the sign of \( \beta_1 \), so that 
\[
\gamma = 2(\beta_1 + \beta_2),
\]
as in eq (1.13), where \( \beta_1, \beta_2 \) and \( c \) are all positive. After this change we find that when \( \mu_1 = \mu_2 = 1 \) the formula reduces to the known correct expression, say \( L_0 \), that is given in (2.44), where

\[
L_0 - 1 = 2 \log \frac{b^2}{a_1 a_2} = 2(\beta_1 + \beta_2) + 2 \sum_{n=1}^{\infty} \frac{1}{n} (e^{-2n\beta_1} + e^{-2n\beta_2} - e^{-\gamma}).
\]

Subtracting this from the expression for \( L \) (with positive \( \beta_1 \)) gives the eq (2.45).

6. References


[13] C. Snow, A standard of small capacitance, J. Research NBS 42 (March 1949) RP1970. The two-dimensional case is based on the transformation with theta-function, p. 297, eq (37). The preceding cases (1.1), (1.2) are based upon a more general transformation with theta-functions in which the clearance is not neglected as it is in figure 2. Experimental methods of evaluating edge corrections are described by A. H. Scott and H. L. Curtis, J. Research NBS 22, 747 (1939) RP1217.

[14] C. Snow, Potential problems and capacitance for a conductor bounded by two intersecting spheres, J. Research NBS 43, 377 (Oct. 1949) RP2032. The potential field is found in finite terms when \( \omega = \pi \nu/m \), where \( m \) is any positive integer, but \( \nu \) is either 1, 2, 3, or 4, and the case \( \nu = 3 \) and 4 involve elliptic funtions. The capacitance is also found for a conductor consisting of two unequal spheres in external contact.


[16] H. L. Curtis and C. M. Sparks, Formulas, tables and curves for computing the mutual inductance of two coaxial circles, BS Sci. Pap. 19, 541-576 (1923-24); also see, sec. 2, eq (F) and sec. IV.


[22] C. Snow, BS Sci. Pap. 537, 21, 431 (1926),

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