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TECHNIQUES FOR COMPUTING REFRACTION OF RADIO WAVES IN THE TROPOSPHERE

BY
E.J.DUTTON AND G.D.THAYER

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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E. J. Dutton and G. D. Thayer

Eight methods of computing atmospheric refraction of radio rays are discussed with appropriate theoretical background. These methods are:

(1) The high-angle, or astronomical, refraction case
(2) The statistical method
(3) The low-angle, or terrestrial, refraction case (Schulkin's method)
(4) The four-thirds earth model
(5) The exponential model
(6) The initial gradient correction method
(7) The departures-from-normal method
(8) A graphical method (Weisbrod's and Anderson's method).

Sample computations are included for each of the above methods.

TECHNIQUES FOR COMPUTING REFRACTION OF RADIO WAVES IN THE TROPOSPHERE

E. J. Dutton and G. D. Thayer

1. INTRODUCTION

If a radio ray is propagated in free space, where there is no atmosphere, the path followed by the ray is a straight line. However, a ray that is propagated through the earth's atmosphere encounters variations in atmospheric refractive index along its trajectory that cause the ray-path to become curved. The geometry of this situation is shown in figure 1. Figure 1 defines the variables of interest. The total angular refraction of the ray-path between two points is designated by the Greek letter τ, and is commonly called the "bending" of the ray. The atmospheric radio refractive index, n, always has values slightly greater than unity near the earth's surface (e.g., 1.0003), and approaches unity with increasing height. Thus ray paths usually have a curvature that is concave downward, as shown in figure 1; for this reason downward bending is usually defined as being positive.

If it is assumed that the refractive index is a function only of height above the surface of a smooth, spherical earth (i.e., it is assumed that the refractive index structure is horizontally homogeneous), then the path of a radio ray will obey Snell's law for polar co-ordinates:

$$n_2 r_2 \cos \theta_2 = n_1 r_1 \cos \theta_1 ; \qquad (1)$$

the geometry and variables used with this equation are shown in figure 2. With this assumption τ may be obtained from the following integral:

$$\tau_{1,2} = - \int_{n_1, \theta_1}^{n_2, \theta_2} \cot \theta \, \frac{dn}{n} , \qquad (2)$$

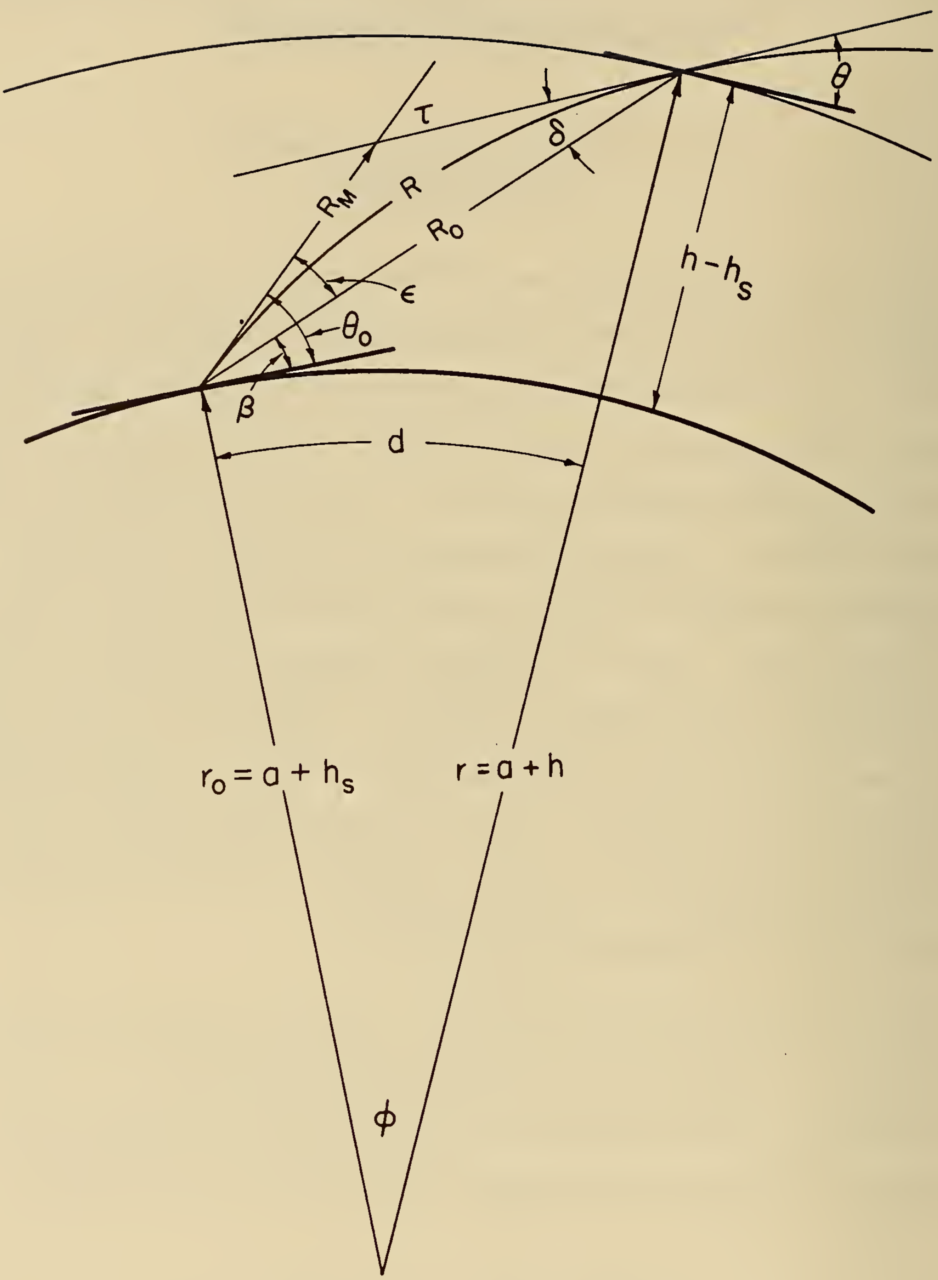

FIGURE 1. Geometry of the Refraction of Radio Waves

which can be derived as shown in Appendix I, and also by Smart [1931].

The elevation angle error, ϵ, is an important quantity to the radar engineer since it is a measure of the difference between the apparent elevation angle, θ_o, to a target, as indicated by radar, and the true elevation angle. Under the same assumption made previously ϵ is given as a function of τ, n, and θ by

$$\epsilon = \text{Arctan}\left\{\frac{\cos\tau - \sin\tau\,(\tan\theta) - \frac{n}{n_s}}{\frac{n}{n_s}\tan\theta_o - \sin\tau - \cos\tau\tan\theta}\right\}. \tag{3}$$

The apparent range to a target, as indicated by a radar, is defined as an integrated function of n along the ray path,

$$R_e = \int_0^R n\,dR = \int_0^h \frac{ndh}{\sin\theta}. \tag{4}$$

However, the maximum range error (R_e minus the true range) likely to be encountered is only about 200 meters, hence the evaluation of (4) is not of great importance unless one is dealing with an interferometer or phase-measuring system.

The preceding material should suffice to show the importance of radio ray bending in radar systems evaluation and allied types of radio propagation work. Unfortunately, the integral for τ (2) cannot be evaluated directly without a knowledge of the behavior of n as a function of height. Consequently, the approach of the many workers in this field has been along two distinct lines: the use of numerical integration techniques and approximation methods to evaluate τ without full knowledge of n as a function of height, and the construction of model n-atmospheres in order to evaluate average atmospheric refraction.

The following sections are devoted to a discussion of these methods.

2. LIMITATIONS TO RADIO RAY-TRACING

The user should keep in mind that the equations given in the preceding section are subject to the following restrictions of ray-tracing:

(1) The refractive index should not change appreciably in a wavelength.

(2) The fractional change in the spacing between neighboring rays (initially parallel) must be small in a wavelength.

Condition (1) will be violated if there is a discontinuity in the refractive index (which will not occur in nature), or if the gradient of refractive index, dn/dr, is very large, in which case condition (2) will also be violated. Condition (1) should be satisfied if

$$\frac{(dn/dh) \text{ per km}}{N} < 0.002\, f_{kc},$$

where refractivity, N, is defined as $N = (n-1) \times 10^6$ and f_{kc} is the carrier frequency in kilocycles [Bean and Thayer, 1959]. Condition (2) is a basic requirement resulting from Fermat's principle for geometrical optics. An atmospheric condition for which both conditions (1) and (2) are violated is known as "trapping" of a ray, and it can occur whenever a layer of refractive index exists with a vertical decrease of N greater than 157 N-units per kilometer. A layer of this type is called a "duct", and the mode of propagation through such a layer is similar to that of a waveguide [Booker and Walkinshaw, 1946]. Taking into account refractive index gradients, a cutoff frequency may be derived for waveguide-like propagation through a ducting layer [Kerr, 1951].

In addition to the above limitations, it should be remembered that the postulate of horizontal homogeneity, made in order to use equation (1), is not realized under actual atmospheric conditions; some degree of horizontal inhomogeneity is always present.

3. AN APPROXIMATION FOR HIGH INITIAL ELEVATION ANGLES

A method may be derived for determining ray-bending from a knowledge only of n at the end points of the ray path, if it is assumed that the initial elevation angle is large. Equation (2) in terms of refractivity, N, is equal to

$$\tau_{1,2} \cong -\int_{N_1, \theta_1}^{N_2, \theta_2} \cot\theta \, dN \cdot 10^{-6}, \tag{5}$$

assuming $n \cong 1$ in the denominator. Integration by parts yields:

$$\tau_{1,2} = -\int_{N_1, \theta_1}^{N_2, \theta_2} \cot\theta \, dN \cdot 10^{-6} = -\left[N \cot\theta \cdot 10^{-6}\right]_{N_1, \theta_1}^{N_2, \theta_2} - \int_{\theta_1, N_1}^{\theta_2, N_2} \frac{N}{\sin^2\theta} \, d\theta \cdot 10^{-6}. \tag{6}$$

Note that the fraction, $N/\sin^2\theta$, becomes smaller with increasing θ for values of θ close to 90°. If point 1 is taken at the surface, then $\theta_1 = \theta_o$ and $N_1 = N_s$. Then for $\theta_o = 10^\circ$, $N_2 = 0$ and $\theta_2 = \pi/2$, the last term of (6) amounts to only 3.5 percent of the entire equation, and for the same values of N_2 and θ_2 but with $\theta_o = 87$ mr ($\sim 5^\circ$) the second term of (6) is still relatively small ($\sim$ 10 percent). Thus it would seem reasonable to assume that for

$$\theta_o \geq 87 \text{ mr } (\sim 5^\circ),$$

the bending, $\tau_{1,2}$, between the surface and any point, r, is given sufficiently well by

$$\tau_{1,2} = -\left[N \cot\theta \times 10^{-6}\right]_{N_s, \theta_o}^{N_r, \theta_r}, \tag{7}$$

or $$\tau_{1,2} = N_s \cot\theta_o \times 10^{-6} - N_r \cot\theta_r \times 10^{-6}. \tag{8}$$

The term $-N_r \cot\theta_r \times 10^{-6}$ is practically constant and small with respect to the first term, for a given value of θ_o and r, in the range $\theta_o \geq 87$ mr. Thus $\tau_{1,2}$ is seen to be essentially a linear function of N_s in the range $\theta_o \geq 87$ mr. For bending through the entire atmosphere, (to a point where $N_r = 0$), and for $\theta_o > 87$ mr, (8) reduces to

$$\tau = N_s \cot\theta_o \times 10^{-6}. \tag{9}$$

For initial elevation angles less than about 5^o the errors inherent in this method exceed 10 percent (except near the surface) and rise quite rapidly with decreasing θ_o.

4. THE STATISTICAL METHOD

Another method for determining high-angle bending is the statistical linear regression technique developed by Bean, Thayer, and Cahoon [1959]. It has been found that for normal conditions and all heights the right-hand integral of (6) is approximately a linear function of N_s (θ_o, r constant) for $\theta_o > 17$ mr ($\sim 1^o$) and that the second term of (8) tends to be constant. Thus (6) reduces to a linear equation,

$$\tau_{1,2} = bN_s + a, \tag{10}$$

where b and a are constants (as in tables I - IX) and N_s is the surface refractivity.

The form of (10) is very attractive, since it implies two things: 1) $\tau_{1,2}$ may be predicted with some accuracy as a function only of N_s (surface height and θ_o constant), a parameter which may be observed from simple surface measurements of the common meteorological elements of temperature, pressure, and humidity.

2) The simple linear form of the equation indicates that, given a large number of observed $\tau_{1,2}$ versus N_s values for many values of h and θ_o, the expected (or best estimate) values of b and a can be obtained by the standard method of statistical linear regression. This is what was done to obtain values listed in tables I - IX.

Tables I - IX also show the values of the standard error of estimate, SE, to be expected in predicting the bending, and the correlation coefficients, r, for the data used in predicting the lines. Linear interpolation can be used between the heights given to obtain a particular height that is not listed in the tables. For more accurate results, plot the values of τ from the tables (for desired N_s) against height, and then plot the values of the standard error of estimate on the same graph. Then connect these points with a smooth curve. This will permit one to read the τ value and the SE value directly for a given height.

5. SCHULKIN'S METHOD

Schulkin has presented a relatively simple, numerical integration method of calculating bending for N-profiles obtained from ordinary significant-level radiosonde (or "RAOB") data [Schulkin, 1952]. The N-profile obtained from the RAOB data consists of a series of values of N for different heights; one then assigns to N(h) a linear variation with height in between the tabulated profile points, so that the resulting N versus height profile is that of a series of interconnected linear segments. Under this assumption, (2) is integrable over each separate linear N-segment of the profile (after dropping the n term in the denominator, which can result in an error of no more than 0.04 percent in the result), yielding the following result:

$$\Delta\tau_{1,2}(\mathrm{rad}) \cong - \int_{n_1,\,\theta_1}^{n_2,\,\theta_2} \cot\theta \, dn \cong \frac{2(n_1 - n_2)}{\tan\theta_1 + \tan\theta_2} ,$$

or

$$\Delta\tau_{1,2}(\mathrm{mr}) \cong \frac{2(N_1 - N_2) \times 10^{-3}}{\tan\theta_1 + \tan\theta_2} . \tag{11}$$

For the conditions stated above, this result is accurate to within 0.04 percent or better of the true value of $\Delta\tau_{1,2}$, an accuracy that is usually better than necessary. Thus it is possible to simplify (11) further by substituting θ for $\tan\theta$; this introduces an additional error that is less than 1 per cent if θ is under 10° (~ 175 mr). Now (11) becomes

$$\Delta\tau_{1,2}(\mathrm{mr}) \cong \frac{2(N_1 - N_2)}{\underset{(\mathrm{mr})}{\theta_1} + \underset{(\mathrm{mr})}{\theta_2}} , \quad \theta_1 \text{ and } \theta_2 \text{ in mr.} \tag{12}$$

where θ may be determined from (33), Appendix I.

The bending for the whole profile can now be obtained by summing up the $\Delta\tau_{1,2}$ for each pair of profile levels:

$$\tau_n\,(\mathrm{mr}) \cong \sum_{k=0}^{n} \frac{2(N_k - N_{k+1})}{\underset{(\mathrm{mr})}{\theta_k} + \underset{(\mathrm{mr})}{\theta_{k+1}}} . \tag{13}$$

This is Schulkin's result. The degree of approximation of (13) is quite high, and thus most recent "improved" methods of calculating τ will reduce to Schulkin's result for the accuracy obtainable from RAOB or other similar data. Thus, provided that the N-profile is known, (13) is the <u>most useful</u> form for computing bending (for all practical purposes) that should concern the communications or radar engineer. Some other methods have been published which are actually the same as Schulkin's, but have some additional desirable features; e.g., the method of Anderson [1958] employs a graphical approach to avoid the extraction of square roots to obtain θ_k.

6. THE FOUR-THIRDS EARTH MODEL

Perhaps the earliest attempt to utilize a model of atmospheric refractive index for the solution of problems in microwave radio propagation dates back to 1933, when Schelling, Burrows, and Ferrell [1933] published their discovery that radio propagation through an atmosphere with a constant refractive index gradient of $-\frac{1}{4}a$, where "a" is the radius of the earth, was equivalent, for purposes of calculation, to radio propagation over an airless world of radius 4a/3. This was a great simplification, since it meant that for the calculation of radio field strengths, etc., the atmospheric refractive index could be ignored provided that 4a/3 was entered in the calculations instead of "a" wherever it appeared. This method quickly became known as the "four-thirds-earth," and has formed the backbone, until very recently, of radio refraction calculations since its introduction.

The 4/3 earth method, as originally proposed, suffers from two serious shortcomings, only one of which may be overcome by use of this kind of a model. They are as follows:

a) The gradient of refractive index near the earth's surface that is implied by the ratio 4/3 (~ -40N/km) is valid only for certain areas and at certain times, e.g., temperate areas in winter; the gradient implied is less than average for temperate climates in summer, always much below average for tropical climates, and greater than average for arctic climates.

b) The gradient of refractive index implied by the 4/3 earth model is nearly constant, decreasing with height at uniform rate, and thus the values of refractive index implied quickly reach unrealistically low values; free space value (N = 0) is attained at about eight-kilometer-height.

The first of these drawbacks may be avoided by a simple modification of the original 4/3 earth theory. All that is required is to pick a value of the "effective earth's radius factor", e.g., 4/3, which is consistent with the meteorological data that are available for the area under consideration. Hence, a location that has a normal gradient of refractivity near the surface, of -100 N-units/kilometer, would have an associated effective earth's radius factor of 11/4, and the effective earth's radius for this location would be 11a/4, or about 17,500 km.

The shortcoming of the 4/3 earth model listed under "b)" above is an objection to the effective earth's radius theory in general, and hence cannot be avoided by a change in the size of the effective earth's radius factor (except by making the factor a function of height).

With the above considerations the following recommendation is made: when dealing with problems involving ground-to-ground communications systems or other types of low-altitude radio propagation problems where the ray paths involved do not exceed one, or at most two, kilometers above the earth's surface, the effective earth's radius method should be used to solve the associated refraction problems. The user should refer to the tables in Appendix II, where effective earth's radius factors are tabulated along with other refractivity variables. Table A-1 may be entered with N_s and table A-2 may be entered with $\Delta N(N_s)$ subtracted from the N value at one kilometer above the surface. In both these tables

Probable Errors When using Effective Earth's Radius Model for a Ray with $\theta_o = 0$; using $h = d^2/2\,ka$.

True Height of Ray for an Exponential Profile:	Calculated Height and Percent Errors	
	For Normal Conditions $dN/dh \cong -50/km$	For Superrefraction: $dN/dh \cong -100/km$
1.000 km	0.987 km, 1.3% error	~ 0.95 km, 5% error
2.000 km	1.950 km, 2.5% error	~ 1.8 km, 10% error

linear interpolation will suffice for any practical problem. The variables listed in these tables are for the exponential model of N(h) that is covered in the following subsection.

When the effective earth's radius treatment is used, height is calculated as a function of distance, for a ray with $\theta_o = 0$, with the equation $h = d^2/2ka$, where d is the distance, k is the effective earth's radius factor, and a is the true radius of the earth (~6373 km). The table above will serve as a guide to the errors likely to be incurred when using this equation, assuming as a true atmosphere an exponential N(h) profile as given in the following subsection.

7. THE EXPONENTIAL MODEL

An exponentially decreasing refractive index in the troposphere has been recognized for some time [Bauer, Mason, and Wilson, 1958; Anderson, 1958]. Recently Bean and Thayer [1959] introduced an exponential model for N(h) based on an analysis of observed profiles from many climatic areas (mostly in the U.S.). With this model, the value of N as a function of height is given by the equation

$$N(h) = N_s \exp\{-c_e h\}, \tag{14}$$

where c_e is a function only of N_s, and is thus a constant for any given profile, and h is the height above the surface. The quantity c_e can be related to N_s and ΔN by

$$c_e = \ln \left\{ \frac{N_s}{N_s + \Delta N} \right\}, \tag{15}$$

so that a relationship between N_s and ΔN would fix each exponential profile of form (14) as a function of the single variable, N_s. A brief description of the development of such a relationship is given in the following paragraphs.

If it is assumed that N(h) is indeed an exponential function of height, then the gradient of N(h) would also be an exponential function of height. The most extensive amount of data with which to evaluate the coefficients in the exponential is that of ΔN (the value of N at one kilometer minus the surface value, N_s) which has received wide application in radio propagation problems. Thus one would expect

$$\frac{\Delta N}{\Delta h} = k_1 \exp\{-k_2 h\} \tag{16}$$

to take the form

$$\Delta N = k_1 \exp\{-k_2\},$$

for the special case of $\Delta h = h = 1$ kilometer. Examination of available ΔN data reveals that k_2 is dependent upon N_s, i.e., the higher the surface value of N the greater the expected drop in N over one kilometer. Further examination indicates that

$$k_2 \doteq k_3 N_s,$$

and the resultant equation,

$$\Delta N = k_1 \exp\{-k_3 N_s\}, \tag{17}$$

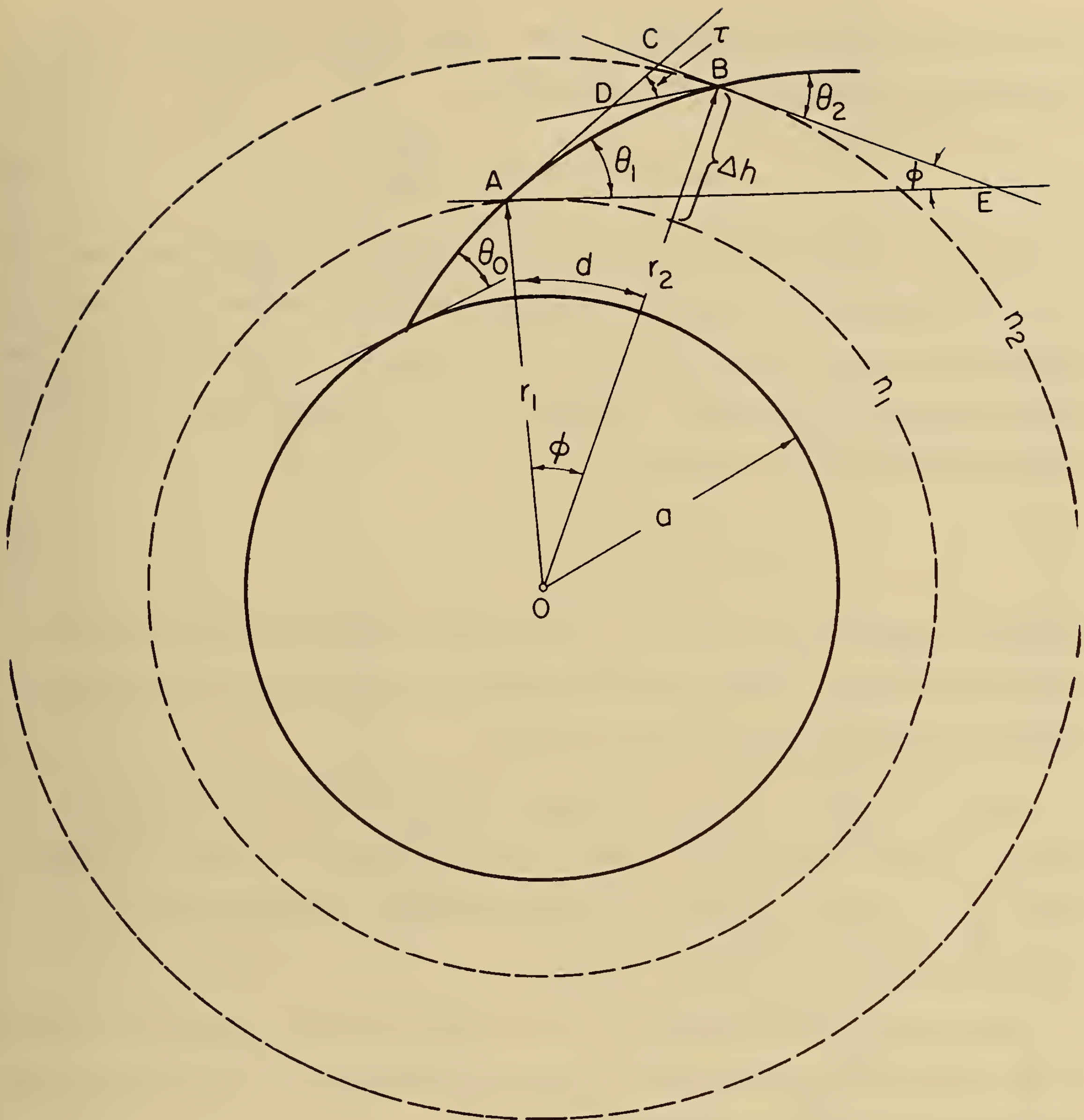

FIGURE 2. Bending Geometry on a Spherical Earth with Concentric Layers

may be solved by least squares. The least squares determination is facilitated by converting (17) to the form

$$\ln|\Delta N| = -k_3 N_s + \ln k_1, \tag{18}$$

that is, expressing the natural logarithm of ΔN as a linear function of N_s. The values of k_1 and k_3 are established from 888 sets of 8-year means of ΔN and N_s from 45 U.S. weather stations. The results of this study are shown graphically in figure 3, and the least squares exponential fit of $\overline{\Delta N}$ and $\overline{N}_s$ is given by

$$\Delta N = -7.32 \exp\{0.005577 N_s\}. \tag{19}$$

With this equation the C.R.P.L. Exponential Reference Atmosphere [Bean and Thayer, 1959a] is determined; the profiles are completely defined by equations (14), (15), and (19).

Ray tracings have been computed for this model covering more than the normal range of N_s, and the results found in Tables X through XVII may be used to predict τ for any normal combination of N_s, θ_o, and height [Bean and Thayer, 1959a].

The exponential atmosphere is considered to be an adequate solution to the bending problem for any θ_o larger than about 10 milliradians and all heights above one kilometer.

8. THE INITIAL GRADIENT CORRECTION METHOD

The importance of the initial gradient in radio propagation, where the initial elevation angle of a ray path is near zero, has long been recognized. For example, if $dN/dh = -1/a$ (the reciprocal of the earth's radius), then the equation for τ is indeterminate, an expression of the fact that the ray path remains at a constant height above the earth's surface. This is called

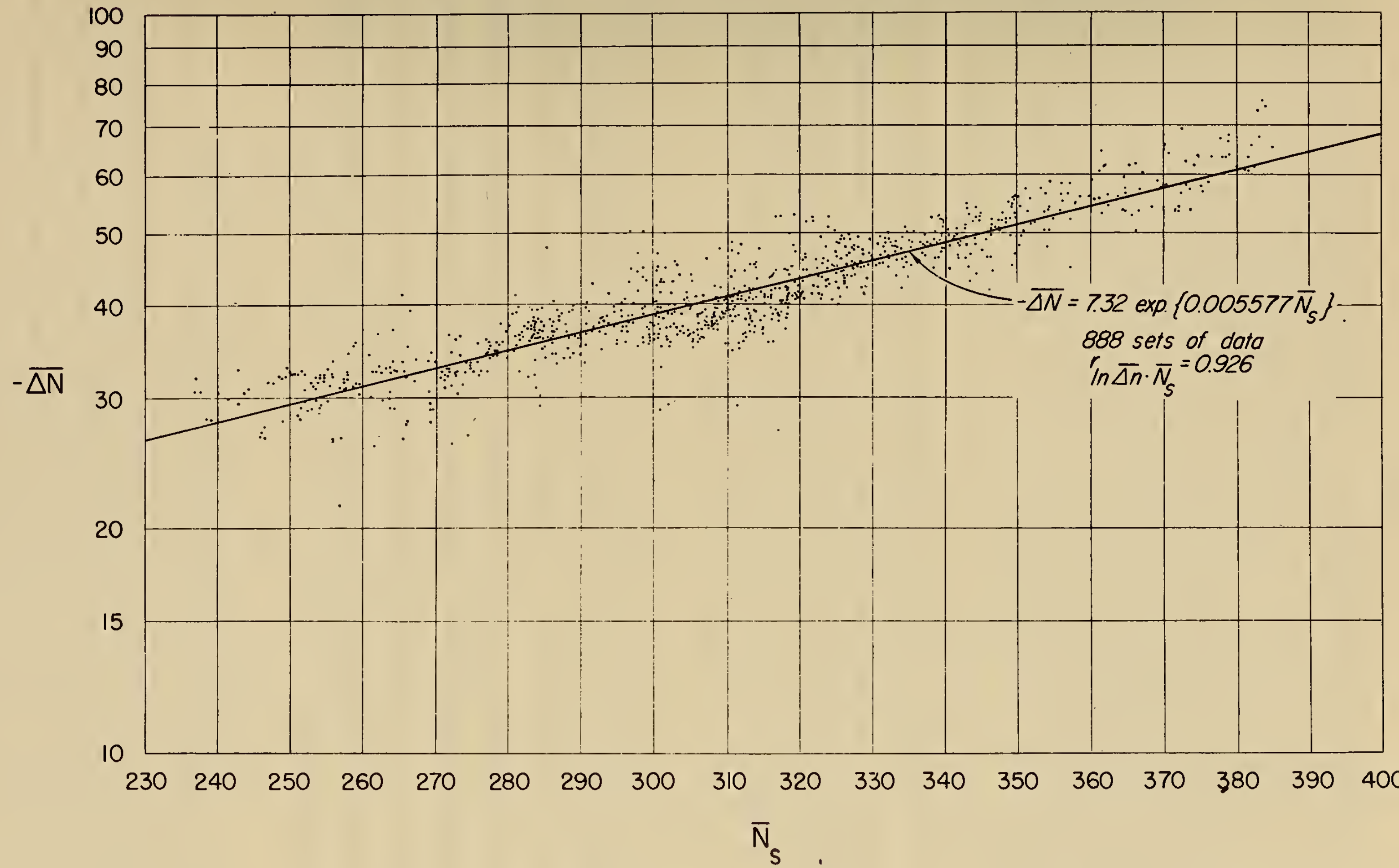

FIGURE 3. Regression of ln $\overline{\Delta N}$ upon $\overline{N}_s$ for 0300 and 1500 GMT

ducting, or trapping of the radio ray. The effect of anomalous initial N-gradients on ray propagation at elevation angles near zero, and for gradients less than ducting, ($|dN/dh| < 157$ N units/km, or $dN/dh > -157$ N units/km), may also be quite large. A method has been developed for correcting the predicted refraction (from the exponential reference atmosphere) to account for anomalous initial N-gradients, assuming that the actual value of the initial gradient is known [Bean and Thayer, 1959b]. The result is

$$\tau_h = \tau_h (N_s, \theta_o) + [\tau_{100}(N_s*, \theta_o) - \tau_{100}(N_s, \theta_o)], \qquad (20)$$

where $\tau_h (N_s) = \tau$ at height h, for the exponential reference atmosphere corresponding to N_s, and N_s* is the N_s for the exponential reference atmosphere that has the same initial gradient as the observed initial gradient; τ_{100} is τ at a height of 100 meters.

This procedure has the effect of correcting the predicted bending by assuming that the observed initial gradient exists throughout a surface layer 100 meters thick, calculating the bending at the top of the 100-meter-thick layer, and then assuming that the atmosphere behaves according to the exponential reference profile corresponding to the observed value of N_s for all heights above 100 meters. This approach has proved quite successful in predicting τ for initial elevation angles under 10 milliradians, and will, of course, predict trapping when it occurs.

9. THE DEPARTURES-FROM-NORMAL METHOD

A method of calculating bending by the use of the exponential model of N(h) together with an observed N(h) profile is given by Bean and Dutton [1960]. This method is primarily intended to point out the difference between actual ray-bending and the average bending that is predicted by the exponential N(h) profile and is a powerful method of identifying air mass refraction effects.

The exponential model described in subsection 7, can be expected to represent average refractivity profile characteristics at any given location, but it cannot be expected to depict accurately any single refractivity profile selected at random, even though it may occasionally do so. In order to study the differences between individual observed N(h) profiles and the mean profiles predicted by the exponential model, a variable called the A-unit has been developed; it is defined simply as the sum of the observed N at any height, h, and the drop in N from the surface value, N_s, to the height, h, which is predicted by the exponential profile for the given value of N_s.

Thus

$$A(N_s, h) = N(h) + N_s (1-\exp\{-c_e h\}). \tag{21}$$

Thus (21) adds to N(h) the average decrease of N with height, so that if a particular profile should happen, by coincidence, to be the same as the corresponding exponential profile, the value of $A(N_s, h)$ for this profile would be equal to N_s for all heights. The above analysis shows that the difference between $A(N_s, h)$ from N_s, $\delta A(N_s h)$, is a measure of the departure of N(h) from the normal, exponential profile:

$$\delta A(N_s, h) = A(N_s, h) - N_s = N(h) - N_s \exp\{-c_e h\}. \tag{22}$$

It seems logical that the application of the A unit to bending would indicate the departures of bending from normal, in some way, just as it indicates departures of refractivity, N, from normal. This is indeed the case as can be seen in figure 4, where for an N_s = 313.0 exponential atmosphere, A(313.0, h) is plotted on one set of graphs for various typical air masses, and the corresponding bending departures from normal are shown in the

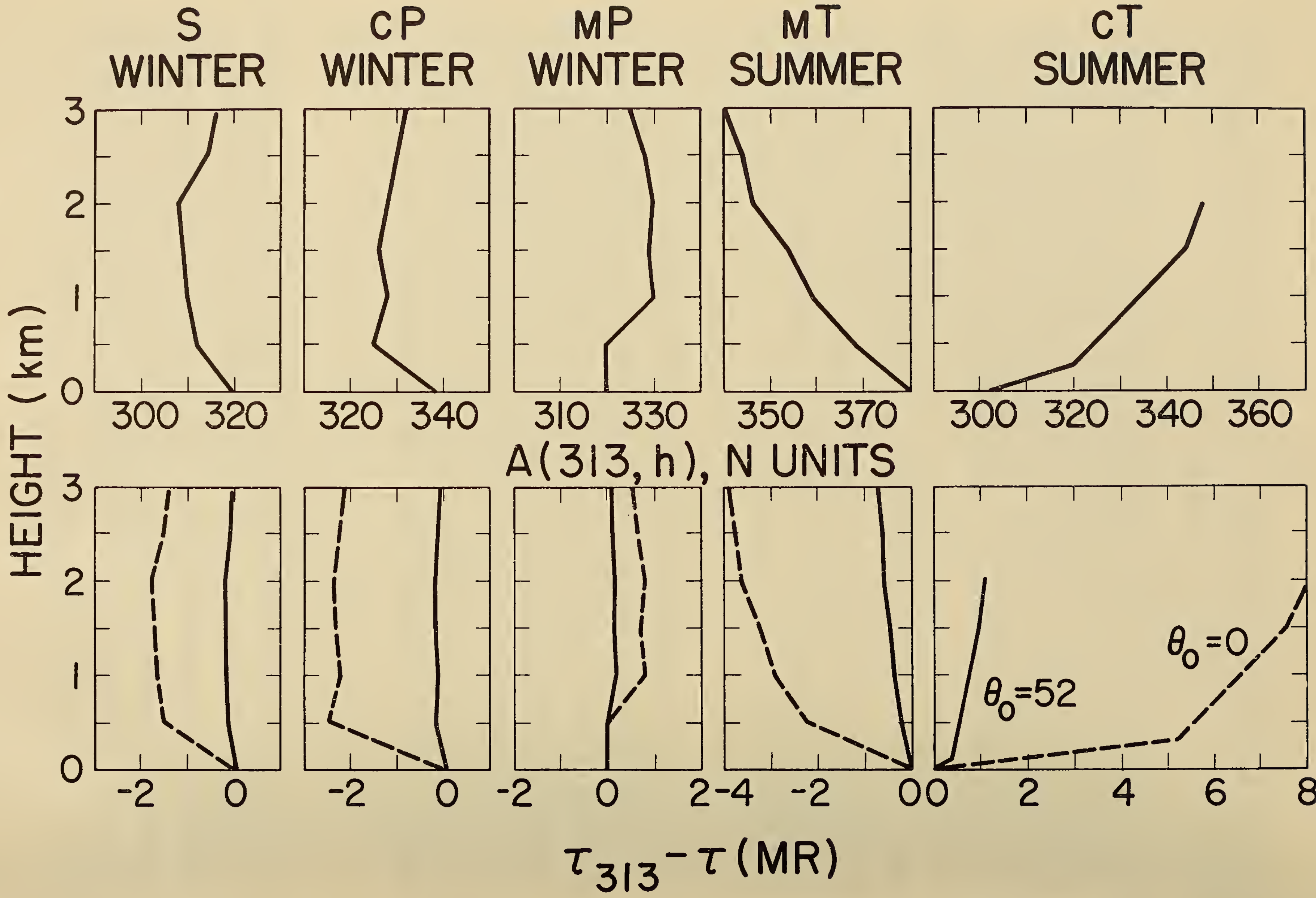

FIGURE 4. A-Unit Profiles for Typical Air Masses and Refraction Deviation from Normal.

second set of graphs corresponding to the same air masses. Obviously, the bending departures between layers are highly analogous to the A unit variation. It can be seen from figure 4 that the similarity exists, although it is less, for higher initial elevation angles. The similiarity also decreases with increasing height, owing to the fact that the bending departures from normal are an integrated effect and at low initial elevation angles, are more sensitive to N-variations at the lower heights. This causes an apparent damping of the bending departures from normal at greater heights. However, the A-unit variation is not similarly influenced; hence a loss of similarity arises at large heights above the earth's surface.

If (21) is differentiated and substituted into (2) the following equation results:

$$\underset{(\text{rad})}{\tau_{0,h}} \cong \underset{(\text{rad})}{\tau_{N_s}(h)} + \sum_{k=0}^{k_h} - \frac{2}{\underset{(\text{rad})}{\theta_k} + \underset{(\text{rad})}{\theta_{k+1}}} \left[\Delta A(N_s)\right]_{N_k}^{N_{k+1}} \times 10^{-6}, \qquad (23)$$

where

$$\Delta A(N_s) = \Delta N(h) + \Delta\left[N_s\left\{1-\exp(-c_e h)\right\}\right] = \Delta N(h) + N_s c_e \exp(-c_e h)\Delta h$$

$\tau_{N_s}(h)$ is the value of τ tabulated for various atmospheres in tables X - XVIII, θ_k and θ_{k+1} are in milliradians and must be <u>from the N_s exponential atmosphere used</u>. $\Delta A(N_s)$ is obtained from subtraction of the A value at layer level, k, from the value of A at layer, k+1. The A value may be obtained by adding any given N(h) value, obtained from RAOB or other similar data, to a value of $N_s[1-\exp\{-ch\}]$ for the same height which may be obtained from figure 5. Since $\tau_{N_s}(h)$ has been calculated only for a few of the exponential atmospheres, these being the N_s = 200.0, 252.9, 289.0, 313.0, 344.5, 377.2, 404.9, and 450.0 atmospheres, one of these

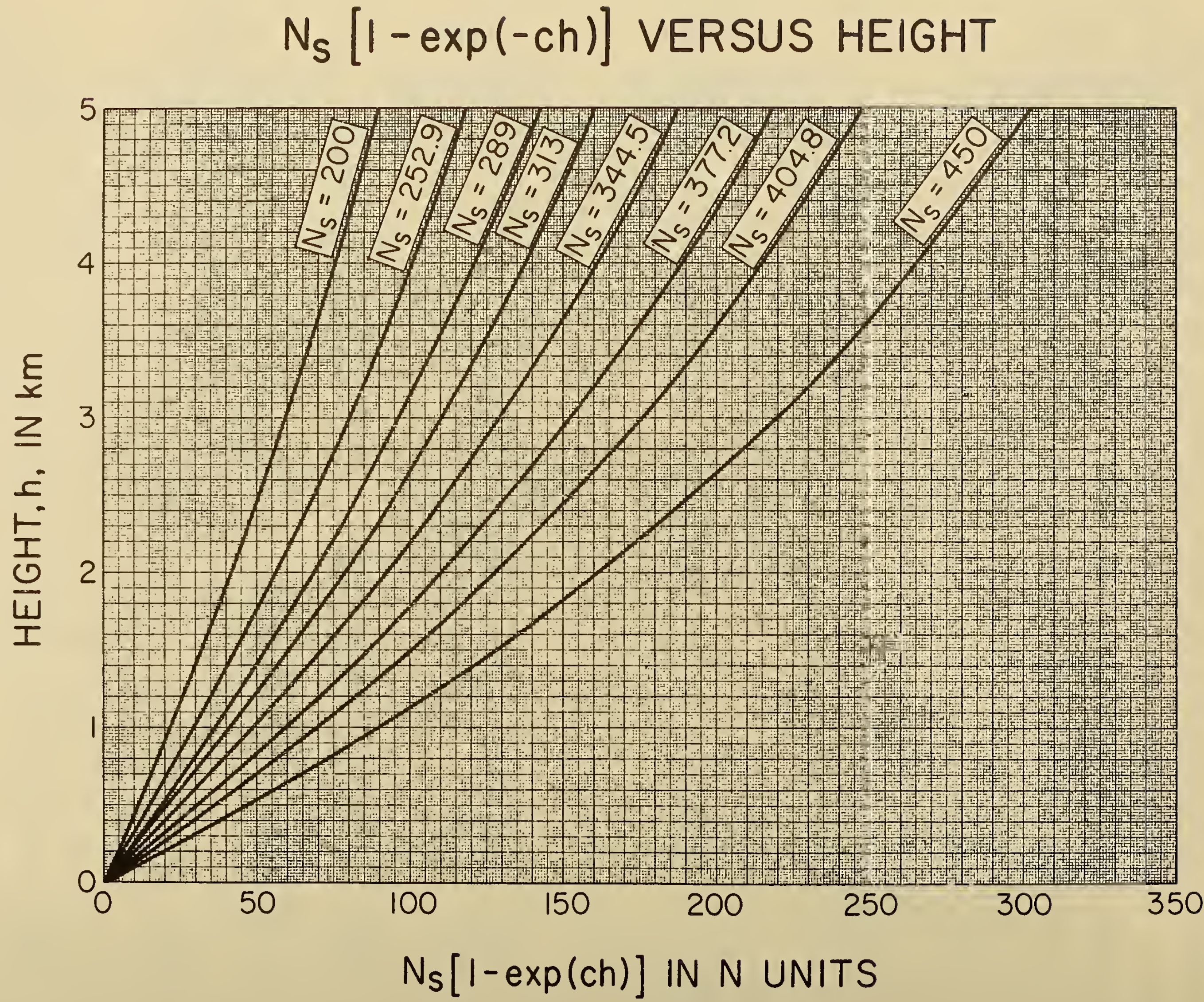
N_S [1 - exp(-ch)] VERSUS HEIGHT
HEIGHT, h, IN km
N_S[1-exp(ch)] IN N UNITS
0
1
2
3
4
5
0
50
100
150
200
250
300
350
N_S = 200
N_S = 252.9
N_S = 289
N_S = 313
N_S = 344.5
N_S = 377.2
N_S = 404.8
N_S = 450

FIGURE 5.

atmospheres must be used in the calculation of bending by the departures method. The selection of the particular atmosphere to be used is based on the value of the gradient of N, dN/dh, between the surface of the earth and the first layer considered. In table XVIII are shown the ranges of the gradient for the choice of a particular exponential atmosphere.

10. A GRAPHICAL METHOD

Weisbrod and Anderson [1959] present a handy graphical method for computing refraction in the troposphere. Rewriting and enlarging (11), one obtains

$$\tau(mr) = \sum_{k=0}^{n} \frac{N_{k+1} - N_k}{500\,(\tan\theta_k + \tan\theta_{k+1})}, \qquad (24)$$

where τ will be the total bending through n layers. Terms for the denominator can be determined from figure 6. Equation (24) is essentially Schulkin's result with only the approximation, $\tan\theta_k \cong \theta_k$, for small angles omitted.

The procedure in using figure 6 is as follows: Enter on the left margin at the appropriate $N_s - N(h)$. Proceed horizontally to the proper height, h, interpolating between curves if necessary. Use the solid height curves when $N_s - N(h)$ is positive and the dashed curves when $N_s - N(h)$ is negative. Then proceed vertically to the assumed θ_o and read 500 tan θ along the right margin.

11. SAMPLE CALCULATIONS

The following problem will serve to illustrate the application of the various methods of calculating bending.

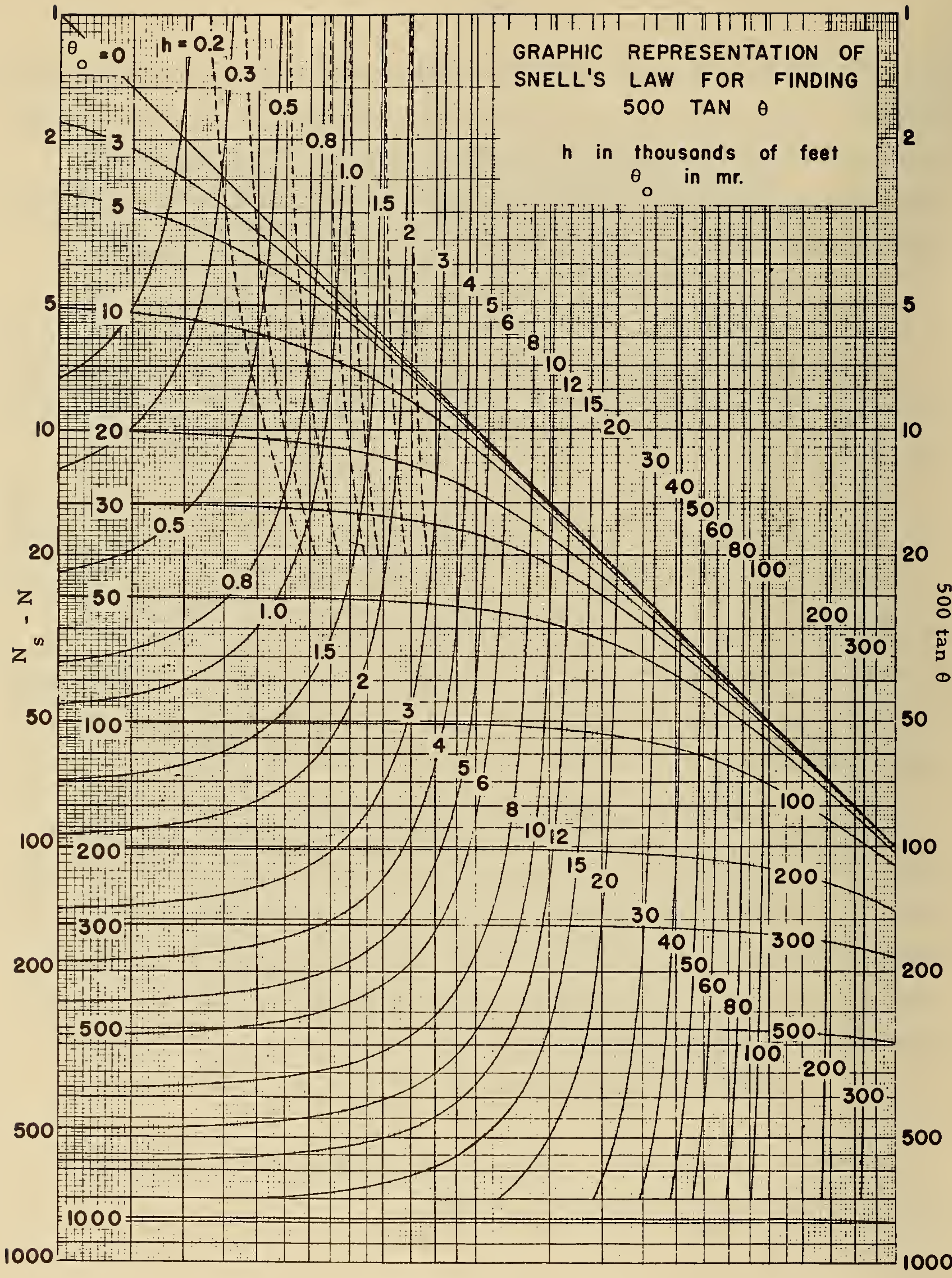
GRAPHIC REPRESENTATION OF SNELL'S LAW FOR FINDING 500 TAN θ
h in thousands of feet
θo in mr.
θo = 0
h = 0.2
0.3
0.5
0.8
1.0
1.5
2
3
4
5
6
8
10
12
15
20
30
40
50
60
80
100
200
300
500
1000
Ns - N
500 tan θ

FIGURE 6.

A particular daily set of RAOB readings from Truk in the Caroline Islands yields the following data:

height above the surface (km)	N value (N units)
0.000	400.0 = N_s
0.340	365.0
0.950	333.5
3.060	237.0
4.340	196.5
5.090	173.0
5.300	172.0
5.940	155.0
6.250	152.0
7.180	134.0
7.617	125.5
9.660	98.0
10.870	85.0

What is the total bending up to the 3.270 km level at initial elevation angles of 0, 10 mr, 52.4 mr (3^{o}) and 261.8 mr (15^{o}) by (a) Schulkin's approach, (b) the exponential model, (c) the initial gradient method, (d) the departures from normal method, (e) the use of regression lines, and (f) the graphical method of Weisbrod and Anderson? Since the gradient between the ground and the first layer is

$$\frac{\Delta N}{\Delta h} = \frac{365.0 - 400.0}{0.340} = -102.9 \text{ N units/ km},$$

and this is a decrease of N per km that is less than the -157 N units/ km required for ducting, no surface duct is present. However, should a surface duct have been present, it would have been necessary to calculate the angle of penetration,

$$\theta_p = \sqrt{2[N_s - N_h - 156.9\,(\Delta h)(\text{in km})]},$$

to find the smallest initial elevation angle that yields a non-trapped ray. Any initial elevation angle less than θ_p <u>cannot</u> be used in bending calculations.

(a) Schulkin's approach of (13) yields the results shown in table XIX for 0 mr, table XX for 10 mr, table XXI for 52.4 mr and table XXII for 261.8 mr, where θ_{k+1} is determined from (33) in Appendix I using $r_k = a + h_k$, and a is the radius of the earth,

$$a = 6370 \text{ km}.$$

It should be remembered that $\theta_k = 0$, 10, 52.4, or 261.8 mr only for the first-level calculation, and that thereafter θ_k is equal to the θ_{k+1} computed for the preceding layer, e.g., for the second layer of table XIX ($\theta_o = 0$ mr), $\theta_k = 6.15$ mr, which is the θ_{k+1} calculated for the first layer.

(b) The exponential model solution may be found by using tables X through XVII. Interpolation will usually be necessary for N_s, θ_o, and height; this interpolation may be done linearly. In practice, one of these three variables will often be close enough to a tabulated value that interpolation will not be necessary, thus reducing from 7 to 3 the number of interpolations necessary. Since in the problem for $N_s = 404.9$, $h = 10.0$ km and $\theta_o = 10$ mr

$$\tau_{0,\, 10.0\, (10\,\text{mr})} = 15.084 \text{ mr},$$

and at h = 20.00 km, $\theta_o = 10$ mr.

$$\tau_{0,\, 20.00\, (10\,\text{mr})} = 15.946 \text{ mr},$$

and thus by linear interpolation for h = 10.870 km, $\theta_o = 10$ mr.

$$\tau_{0,\, 10.870,\, (10\,\text{mr})} = 15.084 + (15.946 - 15.084)\ \frac{10.870 - 10.00}{20.00 - 10.00}$$

$$= 15.159 \text{ mr}.$$

Similarly for $N_s = 377.2$ in the exponential tables,

$$\tau_{0,\ 10.870,\ (10\ mr)} = 13.120\ mr.$$

Again using linear interpolation, but now between the $N_s = 377.2$ and $N_s = 344.5$ atmospheres, the desired value of τ at 3.270 km for $N_s = 360.0$ and $\theta_o = 10$ mr is obtained.

Thus

$$\tau_{0,\ 10.870\ (10\ mr)} = 13.120 + (15.159 - 13.120)\ \frac{400.0 - 377.2}{404.9 - 377.2}$$

$$= 14.798\ mr.$$

For the $\theta_o = 0, 52.4$, and 261.8 mr cases, by similar calculations, using linear interpolation:

$$\tau_{0,\ 10.870,\ (0\ mr)} = 21.386\ mr,$$

$$\tau_{0,\ 10.870,\ (52.4\ mr)} = 5.816\ mr,$$

$$\tau_{0,\ 10.870,\ (261.8\ mr)} = 1.270\ mr.$$

(c) The initial gradient correction method may be used if one determines the N_s* which corresponds to the observed initial gradient and then applies (20). The initial N gradient is $-102.9\ \frac{\text{N units}}{\text{km}}$, which, as can be seen from table XVIII, corresponds to the N_s 450.0 exponential atmosphere. Therefore, using the exponential tables of Bean and Thayer [1959] and (20) to determine the bending for the $\theta_o = 0$ mr case,one finds by linear interpolation

$$\tau_{10,000\ (0)} = \tau_{10,000}(400.0, 0\ mr) + [\tau_{100}(450.0, 0mr) - \tau_{100}(400.0, 0\ mr)]$$

$$= 21.309\ mr + [5.908 - 3.657]\ mr = 23.560\ mr.$$

The $\tau_{100}(400.0, 0 \text{ mr})$ is determined by linear interpolation between the 404.9 and 377.2 atmospheres. At h = 20.0 km as given in the tables

$$\tau_{20,000\,(0\,mr)} = \tau_{20,000}(400.0, 0\,mr) + [\tau_{100}(450.0, 0\,mr) - \tau_{100}(400.0, 0\,mr)]$$

$$= 22.191 + [5.908 - 3.657]\ mr$$

$$= 24.442\ mr.$$

Hence by linear interpolation

$$\tau_{10.870\,(0\,mr)} = 23.560 + [24.442 - 23.560] \cdot \frac{10,870 - 10,000}{20,000 - 10,000}$$

$$= 23.637\ mr.$$

The bendings for θ_o = 10, 52.4, and 261.8 mr are as given below:

$$\theta_o = 10;\ \tau_{10,870\,(10)} = 15.053\ mr$$

$$\theta_o = 52.4;\ \tau_{10,870\,(52.4)} = 5.864\ mr$$

$$\theta_o = 261.8;\ \tau_{10,870\,(261.8)} = 1.280\ mr.$$

(d) To use the departures-from-normal method of determining bending it is first necessary to know the atmosphere which must be used for the calculation. In the problem,

$$-\frac{dN}{dh}\bigg|_{initial} = 102.9\ N\ units/km,$$

which is within the range of the N_s = 450.0 exponential atmosphere, as can be seen from table XVIII. Thus one will use table XVII to determine the θ's and the τ's in the N_s = 450.0 exponential atmosphere, or one can use the exponential atmosphere tables of Bean and Thayer and (33) in Appendix I.

For an $N_s = 450.0$ atmosphere:

$$\tau_{N_s(0\ mr)}(10.870) = 30.776 \text{ mr},$$

$$\tau_{N_s(10\ mr)}(10.870) = 19.414 \text{ mr},$$

$$\tau_{N_s(52.4\ mr)}(10.870) = 7.024 \text{ mr},$$

$$\tau_{N_s(261.8\ mr)}(10.870) = 1.506 \text{ mr}.$$

Equation (33) of Appendix I should be used for the θ interpolation in preference to linear interpolation, although, if no tables or other facilities are present at the engineering site for easy acquisition of square roots, linear interpolation will suffice. Proceeding in table XVII with (33) of Appendix I for the first layer at h = 0.340 km and the $\theta_o = 0$ mr case:

$$\theta = \sqrt{\theta_o^2 + \frac{2(r_1 - r_o)}{r_o} \times 10^6 - 2(N_s - N_1)}$$

= 6.388 mr. The remaining θ's for the various layers are shown in table XXIII. To determine the value of A at the bottom and top of the layer, one makes use of (21) or figure 5. First, however, one must determine the value of c in (21) to be used. Usually interpolation will be necessary in table XVIII, but in the $N_s = 450.0$ case it is not possible, and thus the straight $N_s = 450.0$ exponential atmosphere values are used. From (21)

$$A(N_s, h) = N(h) + N_s [1 - \exp(-ch)],$$

and for the layer running from h = 0 to h = 0.340 km, figure 5 yields

$$377.2 [1 - \exp(-c0)] = 0.0,$$

and $\quad 450.0[1-\exp(-c \times 0.340)] = 32.8,$

and therefore, $\quad A(450.0, 0) = 400.0 + 0 = 400.0,$

and $\quad A(450.0, 0.340) = 365.0 + 32.8 = 397.8,$

whence

$$\Delta A = A(450.0, 0.340) - A(450.0, 0) = 397.8 - 400.0 = -2.2 \text{ N units.}$$

Therefore, the departure term of (23):

$$\frac{-2}{\theta_k + \theta_{k+1}} \Big[\Delta A(N_s) \Big]_{N_k}^{N_{k+1}} ,$$

becomes

$$- \frac{2}{0 + 6.388} \Big[-2.2 \Big]_{400.0}^{365.0} = + 0.689 \text{ mr.}$$

The remaining calculations are tabulated in table XXIII for the $\theta_o = 0$ mr case, in table XXIV for the $\theta_o = 10$ mr case, in table XXV for the $\theta_o = 52.4$ mr case, and in table XXVI for the $\theta_o = 261.8$ mr case. The sum of the departures for the 0 mr case is

$$\sum_{k=0}^{k_n} \frac{-2}{\theta_k + \theta_{k+1}} \Big[\Delta A \, (N_s) \Big]_{N_k}^{N_{k+1}} = -5.335 \text{ mr.}$$

(e) Determination of the bending is required in part (e) of the problem by using regression lines. By (10), using table VII and VIII, it is found for the $\theta_o = 0$ mr case that at 10.0 km (from table VII)

$$\tau_{0,\,10.0} = (0.1149)(400.0) - 18.5627 \pm 7.5227$$

$$= 27.3973 \pm 7.5227 \text{ mr},$$

and at 20.0 km (from table VIII)

$$\tau_{0,\,20.0} = (0.1165)(400.0) - 17.9573 \pm 7.5131$$

$$= 28.6427 \pm 7.5131 \text{ mr}.$$

Thus, by linear interpolation

$$\tau_{1,2} = \tau_{0,\,10.87} = 27.3973 + (28.6427 - 27.3973)\,\frac{10.87 - 10.00}{20.00 - 10.00} \pm 7.5227$$

$$\mp (7.5227 - 7.5131)\,\frac{10.87 - 10.00}{20.00 - 10.00}\,,$$

$$\tau_{1,2} = 27.5056 \pm 7.5218 \text{ mr}.$$

Similarly for the remaining θ's,

$$\tau_{1,2(10\text{ mr})} = 13.9548 \pm 0.9701 \text{ mr},$$

$$\tau_{1,2(53.4\text{ mr})} = 5.2186 \pm 0.0817 \text{ mr},$$

$$\tau_{1,2(261.8\text{ mr})} = 1.2695 \pm 0.0158 \text{ mr}.$$

(f) Determination of the bending by means of the graphical method of Weisbrod and Anderson yields, from figure 6, for 500 tan θ for the first layer:

h(m)	At θ_o = 0 mr	At θ_o = 10 mr	At θ_o = 52.4 mr	At θ_o = 261.8 mr
0.000	0.0	5.0	26.2	134.0
0.340	3.0	5.8	27.0	134.0

which yields for the bending in the first layer.

At θ_o = 0 mr	At θ_o = 10 mr	At θ_o = 52 mr	At θ_o = 261.8 mr
11.67 mr	3.24 mr	0.66 mr	0.13 mr

Similarly, the bending for the entire profile may be obtained, and shown to be

At θ_o = 0 mr	At θ_o = 10 mr	At θ_o = 52.4 mr	At θ_o = 261.8 mr
24.42	14.00 mr	5.32 mr	1.18 mr

The answers to the several parts of the problem are summarized in the table which follows on page 25. Bending values for the assumed profile, from a method which exponentially interpolated layers between given layers and then integrated between resulting layers, assuming <u>only</u> a linear decrease of refractivity between interpolated layers, are included for the sake of comparison. The computations were performed on a digital computer.

The reason that the answers to part (e) vary so radically from the remaining answers for the θ_o = 0 mr case and not so much for the θ_o = 261.8 mr case is the fact that the accuracy of the regression line method increases with increasing initial elevation angle, θ_o. It must be remembered that the statistical regression technique, like the exponential model, is an adequate solution to the bending problem for all θ_o's <u>larger</u> than about 10 mr, and all heights above one kilometer.

The reason that the answers in part (f) and part (a) agree more closely than with any other of the answers is because (24) is, as mentioned before, Schulkin's result with only the approximation, $\tan \theta_k \cong \theta_k$ for small angles, omitted. For this individual profile the bending obtained from an exponential atmosphere does not give particularly accurate bendings; however, for 22 five-year mean refractivity profiles, figure 8 shows that exponential bending predicts accurately within 1 percent of the average bending for these five-year means. Figure 7 shows the r.m.s. error in

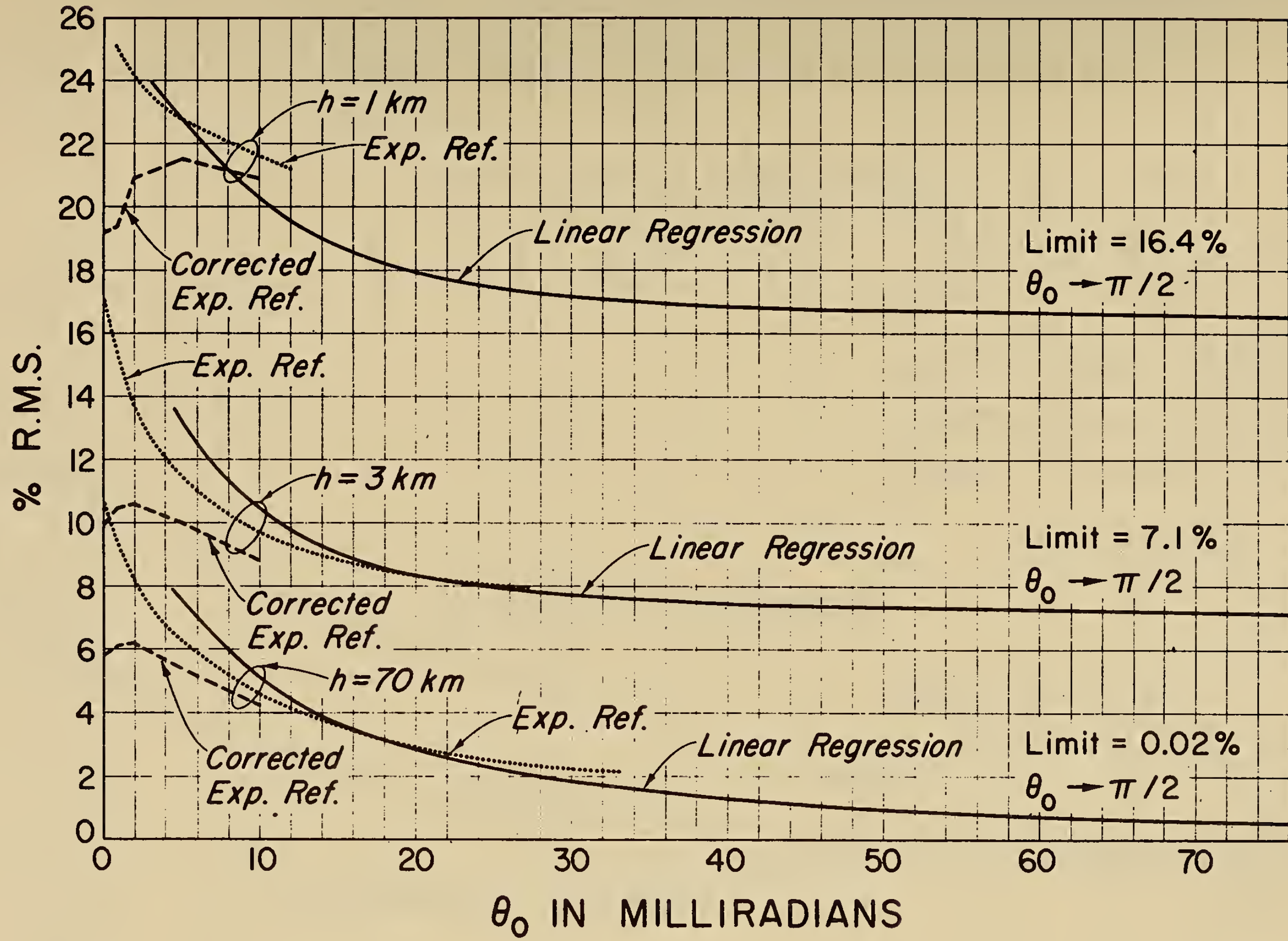

FIGURE 7. Percent R M.S. Error of Predicting Refraction by Three Methods.

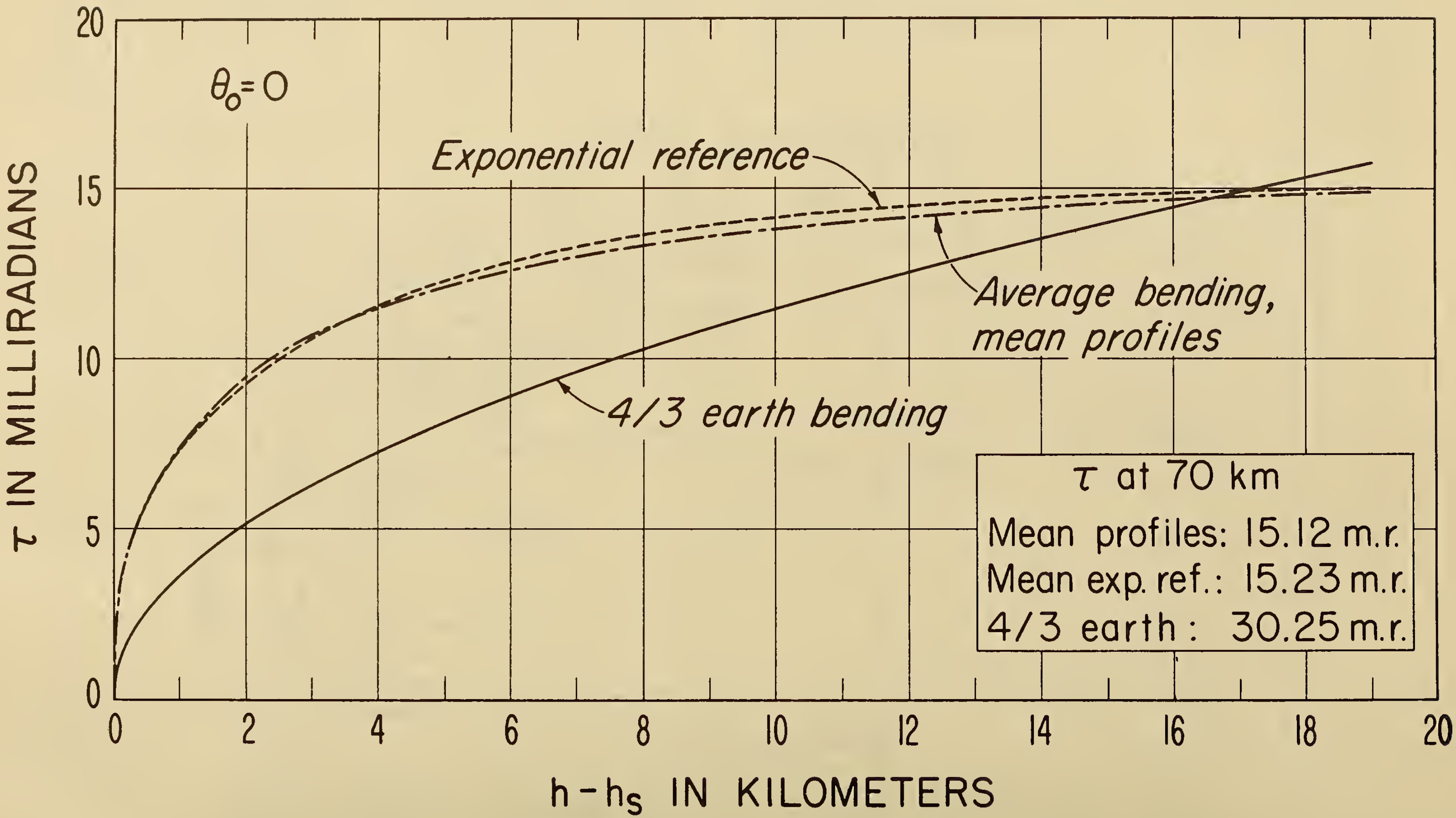

FIGURE 8. Comparison of Mean Refraction with Model Atmosphere Refraction

Summary Table of Refraction Results for the Sample Computation

Problem Part	Method used	Bending in mr at $\theta_o = 0$ mr	Bending in mr at $\theta_o = 10$ mr	Bending in mr at $\theta_o = 52.4$ mr	Bending in mr at $\theta_o = 261.8$ mr
a.	Schulkin's Method	24.248	14.008	5.341	1.196
b.	Exponential Model	21.386	14.798	5.816	1.270
c.	Initial Gradient Correction Method	23.637	15.053	5.864	1.280
d.	Departures from Normal Method	25.441	14.858	5.350	1.143
e.	Statistical Regression Method	27.506 ±7.522	13.955 ±0.9701	5.2186 ±0.0817	1.2695 ±0.0158
f.	Graphical Method	24.42	14.00	5.32	1.168
Comparison (exponential layer interpolation) bending		24.171	14.104	5.343	1.168

predicting bending at various heights as a per cent of mean bending (not including super-refraction).

In summary, it is recommended that the communications engineer either use the statistical regression technique or the exponential tables of Bean and Thayer [1959] without interpolation (i.e., pick the values of height, N_s, and θ_o that are closest to the given parameters) for a quick and facile bending result, keeping in mind the restrictions on these methods. However, as mentioned before, use of Schulkin's method is recommended if accuracy is the primary incentive.

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APPENDIX I

The approximate relation between θ_1 and θ_2 is derived here. This relation holds for small increments of height and small θ's. The relationship was used in making all sample computations in preceding sections. Since for small θ's

$$\cos\theta_1 \cong 1 - \frac{\theta_1^2}{2} \text{ and } \cos\theta_2 \cong 1 - \frac{\theta_2^2}{2}, \tag{25}$$

and knowing

$$r_2 = r_1 + \Delta h \text{ (figure 2)}$$

then substituting in (1) yields

$$n_2(r_1 + \Delta h)\left(1 - \frac{\theta_2^2}{2}\right) \cong n_1 r_1 \left(1 - \frac{\theta_1^2}{2}\right), \tag{26}$$

or

$$n_2 r_1 + n_2 \Delta h - n_2 r_1 \frac{\theta_2^2}{2} - n_2 \Delta h \frac{\theta_2^2}{2} \cong n_1 r_1 - n_1 r_1 \frac{\theta_1^2}{2}. \tag{27}$$

Dividing by r_1

$$n_2 + \frac{n_2 \Delta h}{r_1} - \frac{n_2 \theta_2^2}{2} - n_2 \frac{\Delta h}{r_1} \frac{\theta_2^2}{2} \cong n_1 - n_1 \frac{\theta_1^2}{2}. \tag{28}$$

Since the term $-n_2 \frac{\Delta h}{r_1} \frac{\theta_2^2}{2}$ is small with respect to the other terms of (28) it may be neglected, and thus:

$$n_2 + \frac{n_2 \Delta h}{r_1} - n_2 \frac{\theta_2^2}{2} \cong n_1 - n_1 \frac{\theta_1^2}{2} \tag{29}$$

or

$$-n_2 \frac{\theta_2^2}{2} \cong -n_1 \frac{\theta_1^2}{2} - \frac{n_2 \Delta h}{r_1} + (n_1 - n_2). \tag{30}$$

If one now divides both sides of equation (30) by n_2 and assumes $\frac{n_1-n_2}{n_2} \cong n_1-n_2$ and $\frac{n_1}{n_2} \cong 1$ (30) may be arranged to yield

$$\theta_2 \cong \sqrt{\theta_1^2 + \frac{2\Delta h}{r_1} - 2(n_1-n_2)} \quad . \tag{31}$$

Writing (31) in terms of N units,

$$\theta_2\,(mr) \cong \sqrt{\theta_1^2 + \frac{2\,\Delta h}{r_1} \times 10^6 - 2(N_1-N_2)} \tag{32}$$

if θ_1 is in milliradians.

Generalizing (32) for the kth and the (k+1)th layers,

$$\theta_{k+1}(mr) \cong \sqrt{\theta_k^2(mr) + \frac{2(r_{k+1}-r_k)}{r_k} \times 10^6 - 2(N_k-N_{k+1})} \quad \cdot \tag{33}$$

Also from the geometry shown in figure 2, a useful relationship for $\tau_{1,2}$ can be obtained. Tangent lines drawn at A and B will be respectively perpendicular to r_1 and r_2, since r_1 and r_2 describe spheres of refractive indices n_1 and n_2 concentric with o. Therefore,

$$\text{angle AEC} = \text{angle AOB} = \phi \ ;$$

also, in triangle AEC

$$\text{angle ACE} = 180^{\circ} - \text{angle CAE} - \text{angle AEC} = 180^{\circ} - \theta_1 \ \phi \, . \tag{34}$$

But from triangle DCB

$$\text{angle ACE} = \text{angle DCB} = 180^{\circ} - \tau_{1,2} - \theta_2 \tag{35}$$

Since angle DBC and θ_2 are vertical angles, (34) and (35) are equal. Thus

$$180^\circ - \tau_{1,2} - \theta_2 = 180^\circ - \theta_1 - \phi,$$

or

$$\tau_{1,2} = \phi + (\theta_1 - \theta_2). \qquad (36)$$

Now since ϕ in radians $= d/a$, where d is distance along the earth's surface:

$$\tau_{1,2} = \frac{d}{a} + (\theta_1 - \theta_2), \qquad (37)$$

or the bending of a ray between any two layers is given in terms of the distance, d, along the earth's surface from the transmitter (or receiver), the earth's radius, a, and the elevation angles θ_1 and θ_o (in radians) at the beginning and end of the layer.

If one considers figure 9, Snell's law in polar coordinates, and the refraction formula (equation (2)) may be obtained from the more familiar form of Snell's law. *

Assume that the earth is spherical and that the atmosphere is arranged in spherical layers. In figure 9 let C be the center of the earth, O the observer and COZ the direction of his zenith. Let n and n+dn be the indices of refraction in two adjacent thin layers M and M'. Let LP be the section of a ray in M' which finally reaches the observer at O. At P it is refracted along PQ. Similarly, it is refracted at the surfaces between successive layers and the final infinitesimal element of its path is TO.

Draw the radii CP and CQ. Let angle PQF = θ, and angle LPS = $\theta + d\theta$, and angle QPF = ψ. Then, since the radius CP is perpendicular at

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* The authors feel that the derivation in the previous edition was not, in essence, a strong enough derivation of Snell's law in polar co-ordinates. The authors also feel that more detail is contained in the derivation than in some other derivations, yet it remains mathematically simple. By employing the calculus of variations, Snell's law in polar-co-ordinates can be obtained from Fermat's principle.

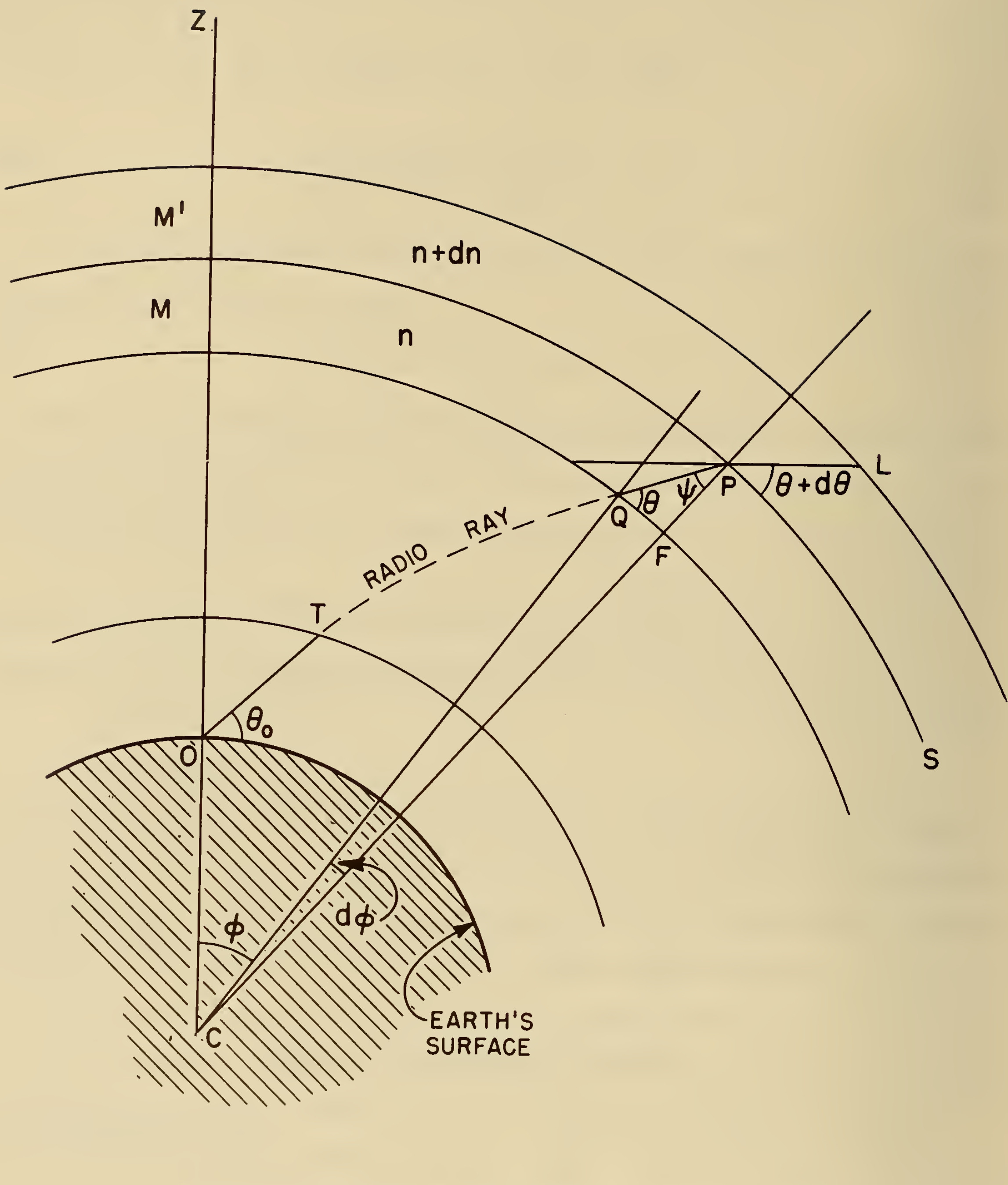

FIGURE 9. Differential Geometry of Radio Ray Refraction

P to the bounding surface between layers M and M', by Snell's law we have

$$(n+dn) \sin [90^{o} - (\theta+d\theta)] = \sin \psi. \qquad (38)$$

Now from the triangle CQP*, in which CQ = r and CP = r+dr, and angle CQP = $90^{o} + \theta$. we have, from the law of sines,

$$r \sin(90^{o} + \theta) = r \sin (90^{o} - \theta) = (r + dr) \sin \psi. \qquad (39)$$

Eliminating $\sin \psi$ from equations (38) and (39), we then have

$$(n + dn)(r + dr) \sin [90^{o} -(\theta + d\theta)] = nr\sin (90^{o} - \theta),$$

or

$$(n + dn)(r + dr) \cos (\theta + d\theta) = nr\cos \theta. \qquad (40)$$

Multiplication of the (n + dn) and (r + dr) terms, ignoring differential products, yields

$$(nr + ndr + rdn) \cos (\theta + d\theta) = nr\cos \theta,$$

or

$$(nr + ndr + rdn) [\cos \theta \cos (d\theta) - \sin \theta \sin (d\theta)] = nr\cos \theta. \qquad (41)$$

Since $\cos (d\theta) \cong 1$, and $\sin (d\theta) \cong d\theta$, another multiplication, again ignoring products of differentials, yields:

$$ndr \cos \theta + rdn \cos \theta - nr \sin \theta \, d\theta = 0,$$

or, dividing all terms by $nr\cos \theta$,

$$\frac{dr}{r} + \frac{dn}{n} - \tan \theta \, d\theta = 0. \qquad (42)$$

Now if (42) is integrated between any two thin layers of refractive indices n_1 and n_2, whose radial distances from the earth's center are r_1 and r_2,

* The assumption involved in this triangle is that the path of the ray in M' is a straight line, which of course, can only be true in an isotropic medium. Hence, it can only be true for an infinitesimal layer in the troposphere. Thus only a differential form of Snell's law (eq. 40) in polar coordinated can be obtained by the use of the geometry of figure 9, not the finite form (eq. 43) which has the same appearance.

and the initial elevation angles of a radio ray entering the layers are θ_1 and θ_2:

$$\int_{r_1}^{r_2} \frac{dr}{r} + \int_{n_1}^{n_2} \frac{dn}{n} - \int_{\theta_1}^{\theta_2} \tan \theta d\theta = \ell n \frac{r_2}{r_1} + \ell n \frac{n_2}{n_1} + \ell n \frac{\cos \theta_2}{\cos \theta_1} = 0$$

or, taking antilogs of both sides,

$$\frac{r_2 n_2 \cos \theta_2}{r_1 n_1 \cos \theta_1} = 1,$$

whence

$$n_1 r_1 \cos \theta_1 = n_2 r_2 \cos \theta_2, \tag{43}$$

which is Snell's law for polar co-ordinates, given by (1).

In figure 9, it can be seen that

$$\tan \theta \cong \frac{PF}{QF} = \frac{dr}{rd\phi}, \tag{44}$$

where ϕ is the angle at the earth's center between r and COZ. Substituting (44) in (42)

$$\tan \theta d\phi + \frac{dn}{n} - \tan \theta d\theta = 0,$$

or

$$(d\phi - d\theta) \tan \theta = - \frac{dn}{n} . \tag{45}$$

Since, by considering equation (37) for infinitesimal angles.

$$d\tau = d\phi - d\theta,$$

or, in (45)

$$d\tau \tan \theta = - \frac{dn}{n} ,$$

$$d\tau = - \cot \theta \frac{dn}{n} . \tag{46}$$

Integration of (46) yields (2).

APPENDIX II

TABLES OF REFRACTION VARIABLES FOR THE EXPONENTIAL REFERENCE ATMOSPHERE

The following table of estimated maximum errors should serve as a guide to the accuracy of the tables.

Errors in elevation angle, θ:

$\theta_o \leq 4$ milliradians	± 0.00005 mr.	nearly independent of N_s.
4 mr. $< \theta_o <$ 100 mr.	± 0.000005 mr.	
$\theta_o \geq$ 100 mr.	± 0.00004 mr.	

Errors in τ, ϵ (in milliradians):

$N_s =$	450	404.8	377.2	344.5	313	252.9	200
$\theta_o = 0$ ±	0.001	0.00065	0.0005	0.0004	0.0003	0.0002	0.0002
$\theta_o = 1^\circ$	0.0003	0.00015	0.0001	0.00008	0.00006	0.00005	0.00005
$\theta_o = 3^\circ$	0.00004	0.000025	0.00002	0.000017	0.000015	0.000013	0.000012

Errors in R_o, R, R_e, or Δh (in meters):

$N_s =$	450	404.8	377.2	344.5	313	252.9	200
$\theta_o = 0$ ±	5.0	2.7	1.8	1.2	0.8	0.65	0.6
$\theta_o \geq 1^\circ$	0.4	0.3	0.25	0.2	0.17	0.15	0.14

Assume that error in ΔR or ΔR_e is ± 0.5% or ± 0.1 meters, whichever is larger.

Table A-1

N_s	$-\Delta N$	c_e	$-dN_o$	k
200	22.3317700	0.118399435	23.6798870	1.17769275
210	23.6125966	0.119280212	25.0488444	1.18991401
220	24.9668845	0.120458179	26.5007993	1.20315637
230	26.3988468	0.121916361	28.0407631	1.21752719
240	27.9129385	0.123642065	29.6740955	1.23314913
250	29.5138701	0.125626129	31.4065323	1.25016295
260	31.2066224	0.127862319	33.2442030	1.26873080
270	32.9964614	0.130346887	35.1936594	1.28904048
280	34.8889558	0.133078254	37.2619112	1.31131073
290	36.8899932	0.136056720	39.4564487	1.33579768
300	39.0057990	0.139284287	41.7852861	1.36280330
310	41.2429556	0.142764507	44.2569972	1.39268608
320	43.6084233	0.146502381	46.8807620	1.42587494
330	46.1095611	0.150504269	49.6664087	1.46288731
340	48.7541501	0.154777865	52.6244741	1.50435338
350	51.5504184	0.159332141	55.7662495	1.55104840
360	54.5070651	0.164177379	59.1038565	1.60393724
370	57.6332884	0.169325150	62.6503054	1.66423593
380	60.9388149	0.174788368	66.4195799	1.73349938
390	64.4339281	0.180581312	70.4267116	1.81374807
400	68.1295015	0.186719722	74.6878887	1.90765687
410	72.0370324	0.193220834	79.2205420	2.01884302
420	76.1686780	0.200103517	84.0434770	2.15232187
430	80.5372922	0.207388355	89.1769927	2.31525447
440	85.1564647	0.215097782	94.6430240	2.51823286
450	90.0405683	0.223256247	100.4653113	2.77761532

Table A-2

$-\Delta N$	N_s	c_e	$-dN_o$	k
20	180.226277	.117626108	21.1993155	1.15617524
22	197.316142	.118216356	23.3259953	1.17457412
24	212.917967	.119594076	25.4637276	1.19366808
26	227.270255	.121491305	27.6113599	1.21348565
28	240.558398	.123746115	29.7681671	1.23406110
30	252.929362	.126255291	31.9336703	1.25543336
32	264.501627	.128950180	34.1075325	1.27764560
34	275.372099	.131783550	36.2895127	1.30074523
36	285.621054	.134721962	38.4794288	1.32478398
38	295.315731	.137741207	40.6771452	1.34981825
40	304.513148	.140823306	42.8825481	1.37590934
42	313.261483	.143955014	45.0955611	1.40312414
44	321.602888	.147125889	47.3161107	1.43153539
46	329.573439	.150328075	49.5441405	1.46122250
48	337.204713	.153555418	51.7796106	1.49227226
50	344.524418	.156803056	54.0224815	1.52477960
52	351.557000	.160067149	56.2727266	1.55884863
54	358.324138	.163344614	58.5303179	1.59459364
56	364.845143	.166633002	60.7952415	1.63214058
58	371.137293	.169930326	63.0674811	1.67162830
60	377.216108	.173234984	65.3470266	1.71321044
62	383.095581	.176545680	67.6338699	1.75705732
64	388.788373	.179861358	69.9280046	1.80335830

Table A-2
(Continued)

$-\Delta N$	N_s	c_e	$-dN_o$	k
66	394.305974	.183181171	72.2294300	1.85232456
68	399.658845	.186504431	74.5381454	1.90419225
70	404.856538	.189830583	76.8541525	1.95922635
72	409.907798	.193159183	79.1774555	2.01772514
74	414.820650	.196489873	81.5080567	2.08002556
76	419.602477	.199822385	83.8459677	2.14650999
78	424.260086	.203156494	86.1911914	2.21761358
80	428.799768	.206492043	88.5437400	2.29383429
82	433.227348	.209828917	90.9036251	2.37574437
84	437.548229	.213167031	93.2708570	2.46400458
86	441.767432	.216506335	95.6454475	2.55938222
88	445.889634	.219846812	98.0274147	2.66277367
90	449.919193	.223188453	100.4167688	2.77523207
92	453.860184	.226531281	102.8135290	2.89800399
94	457.716416	.229875327	105.2177108	3.03257531
96	461.491458	.233220637	107.6293319	3.18073184
98	465.188659	.236567271	110.0484114	3.34463902
100	468.811163	.239915290	112.4749663	3.52694820

Table A-3

k	N_s	$-\Delta N$	c_e	$-dN_o$
1.0	0.0	0.0	0.0	0.0
1.2	217.689023	24.6471681	0.120160519	26.1576259
1.3	275.037959	33.9367000	0.131692114	36.2203304
1.4	312.297111	41.7747176	0.143600133	44.8459068
1.5	339.003316	48.4839018	0.154339490	52.3215987
1.6	359.298283	54.2941700	0.163827653	58.8629945
1.7	375.341242	59.3759008	0.172203063	64.6349115
1.8	388.391792	63.8586055	0.179626805	69.7655765
1.9	399.243407	67.8426334	0.186242834	74.3562234
2.0	408.424907	71.4070090	0.192172034	78.4878451
2.1	416.304322	74.6148487	0.197514185	82.2260091
2.2	423.146728	77.5171828	0.202351472	85.6243634
2.3	429.148472	80.1557288	0.206751820	88.7272277
2.4	434.458411	82.5649192	0.210771674	91.5715264
2.5	439.191718	84.7734613	0.214458304	94.1883108
2.6	443.438906	86.8054237	0.217851443	96.6038056
2.7	447.272272	88.6811886	0.220984823	98.8403838
2.8	450.750273	90.4181120	0.223887193	100.9172131
∞	523.299600	135.5109159	0.299693586	156.829534
4/3	289.036274	36.6922523	0.135758874	39.2392391

Tables of Coefficients, Standard Errors of Estimate, and Correlation Coefficients for use in the Statistical Method

Table I, $h - h_s = 0.1$ km

θ_o	r	b	a	S.E.
0.0	0.2665	0.0479	-8.7011	6.7277
1.0	0.2785	0.0257	-4.1217	3.4363
2.0	0.2881	0.0162	-2.3732	2.0960
5.0	0.3048	0.0073	-0.9181	0.8792
10.0	0.1915	0.0053	-0.6085	1.0551
20.0	0.2070	0.0025	-0.2639	0.4555
52.4	0.2100	0.0009	-0.0973	0.1688
100.0	0.2105	0.0005	-0.0507	0.0879
200.0	0.2105	0.0002	-0.0250	0.0435
400.0	0.2107	0.0001	-0.0120	0.0208
900.0	0.2108	0.00004	-0.0040	0.0070

Table II, $h - h_s = 0.2$ km

θ_o	r	b	a	S.E.
0.0	0.2849	0.05801	-10.4261	7.5726
1.0	0.2979	0.0348	-5.6431	4.3330
2.0	0.3104	0.0239	-3.5707	2.8357
5.0	0.3415	0.0117	-1.5287	1.2512
10.0	0.2306	0.0073	-0.7436	1.1990
20.0	0.2550	0.0035	-0.3184	0.5122
52.4	0.2604	0.0013	-0.1162	0.1890
100.0	0.2610	0.0007	-0.0603	0.0983
200.0	0.2613	0.0003	-0.0299	0.0486
400.0	0.2613	0.0002	-0.0143	0.0233
900.0	0.2604	0.00005	-0.0047	0.0078

Table III, $h - h_s = 0.5$ km

θ_o	r	b	a	S.E.
0.0	0.3615	0.0769	-14.6443	7.6170
1.0	0.3997	0.0510	-9.0567	4.4954
2.0	0.4369	0.0384	-6.5408	3.0395
5.0	0.5205	0.0228	-3.6605	1.4376
10.0	0.3933	0.0140	-1.9055	1.2733
20.0	0.4563	0.0071	-0.8926	0.5365
52.4	0.4731	0.0027	-0.3308	0.1966
100.0	0.4753	0.0014	-0.1721	0.1022
200.0	0.4760	0.0007	-0.0851	0.0505
400.0	0.4761	0.0003	-0.0408	0.0242
900.0	0.4764	0.0001	-0.0137	0.0081

Table IV, $h - h_s = 1.0$ km

θ_o	r	b	a	S.E.
0.0	0.3936	0.0840	-15.1802	7.6151
1.0	0.4620	0.0607	-10.3739	4.5217
2.0	0.5238	0.04918	-8.2066	3.1040
5.0	0.6348	0.0337	-5.4816	1.5931
10.0	0.5718	0.0224	-3.2378	1.2574
20.0	0.6598	0.0124	-1.6959	0.5531
52.4	0.6823	0.0049	-0.6495	0.2071
100.0	0.6851	0.0026	-0.3388	0.1080
200.0	0.6859	0.0013	-0.1676	0.0534
400.0	0.6860	0.0006	-0.0803	0.0256
900.0	0.6864	0.0002	-0.0270	0.0086

Table V, $h - h_s = 2.0$ km

θ_o	r	b	a	S.E.
0.0	0.4524	0.0985	-17.7584	7.5391
1.0	0.5490	0.0752	-12.9451	4.4420
2.0	0.6316	0.0636	-10.7566	3.0277
5.0	0.7707	0.0475	-7.8969	1.5234
10.0	0.7634	0.0345	-5.3712	1.1421
20.0	0.8515	0.02111	-3.1571	0.5086
52.4	0.8668	0.0089	-1.2770	0.2003
100.0	0.8679	0.0047	-0.6705	0.1057
200.0	0.8681	0.0023	-0.3323	0.0524
400.0	0.8682	0.0011	-0.1593	0.0252
900.0	0.8684	0.0004	-0.0535	0.0084

Table VI, $h - h_s - 5.0$ km

θ_o	r	b	a	S.E.
0.0	0.4962	.0.1115	-19.1704	7.5676
1.0	0.6101	0.0881	-14.3543	4.4401
2.0	0.7030	0.0764	-12.1589	3.0001
5.0	0.8504	0.0601	-9.2514	1.4422
10.0	0.8674	0.0464	-6.6445	1.0420
20.0	0.9484	0.0308	-4.0706	0.4028
52.4	0.9674	0.0139	-1.6236	0.1426
100.0	0.9695	0.0075	-0.8348	0.0739
200.0	0.9701	0.0037	-0.4098	0.0365
400.0	0.9702	0.0018	-0.1960	0.0175
900.0	0.9703	0.0006	-0.0658	0.0059

Table VII, $h - h_s = 10.0$ km

θ_o	r	b	a	S.E.
0.0	0.5099	0.1149	-18.5627	7.5227
1.0	0.6290	0.0915	-13.7469	4.3895
2.0	0.7250	0.0799	-11.5514	2.9443
5.0	0.8734	0.0635	-8.6434	1.3733
10.0	0.8950	0.0498	-6.0729	0.9713
20.0	0.9723	0.0338	-3.5012	0.3179
52.4	0.9907	0.0157	-1.1441	0.0844
100.0	0.9927	0.0085	-0.5084	0.0406
200.0	0.9931	0.0043	-0.2310	0.0197
400.0	0.9932	0.0020	-0.1078	0.0094
900.0	0.9932	0.0007	-0.0359	0.0032

Table VIII, $h - h_s = 20.0$ km

θ_o	r	b	a	S.E.
0.0	0.5155	0.1165	-17.9573	7.5131
1.0	0.6367	0.0931	-13.1413	4.3763
2.0	0.7336	0.0814	-10.9463	2.9281
5.0	0.8815	0.0651	-8.0397	1.3521
10.0	0.9028	0.0514	-5.4747	0.9573
20.0	0.9785	0.0353	-2.9228	0.2909
52.4	0.9968	0.0169	-0.6738	0.0535
100.0	0.9984	0.0093	-0.1802	0.0203
200.0	0.9986	0.0047	-0.0467	0.0096
400.0	0.9986	0.0023	-0.0161	0.0046
900.0	0.9986	0.0008	-0.0048	0.0016

Table IX, $h - h_s = 70.0$ km

θ_o	r	b	a	S.E.
0.0	0.5174	0.1170	-17.9071	7.5113
1.0	0.6391	0.0936	-13.0912	4.3738
2.0	0.7361	0.0820	-10.8960	2.9251
5.0	0.8837	0.0656	-7.9895	1.3481
10.0	0.9051	0.0519	-5.4209	0.9539
20.0	0.9797	0.0358	-2.8696	0.2862
52.4	0.9979	0.0173	-0.6246	0.0445
100.0	0.9997	0.0096	-0.1402	0.0095
200.0	1.0000	0.0048	-0.0212	0.0013
400.0	1.0000	0.0024	-0.0027	0.0002
900.0	1.0000	0.0008	-0.0002	0.0001

TABLE X

Refraction Variables in the N(h) = 200 exp {-0.1184h} Atmosphere

	θ_o = 0		θ_o = 1		θ_o = 10		θ_o = 30		θ_o = 52.4		θ_o = 261.7	
Ht. (km)	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ
0.01	1.63199	2.90015	1.91400	1.62424	10.132	0.0235	30.044	0.0079	52.385	0.0045	261.804	0.0009
0.02	2.30816	4.09935	2.51547	2.69096	10.263	0.0467	30.089	0.0157	52.411	0.0090	261.809	0.0018
0.05	3.65007	6.47399	3.78457	4.93673	10.645	0.1143	30.221	0.0392	52.487	0.0225	261.824	0.0044
0.10	5.16332	9.13630	5.25926	7.53121	11.254	0.2215	30.441	0.0779	52.614	0.0448	261.849	0.0088
0.20	7.30576	12.8710	7.37388	11.2173	12.384	0.4183	30.877	0.1537	52.867	0.0889	261.899	0.0175
0.50	11.5688	20.1014	11.6120	18.4052	15.292	0.9111	32.153	0.3699	53.622	0.2167	262.049	0.0429
1.00	16.4006	27.8619	16.4311	26.1453	19.209	1.5398	34.189	0.6965	54.866	0.4163	262.301	0.0832
2.00	23.2996	37.8826	23.3210	36.1529	25.355	2.4319	37.983	1.2464	57.306	0.7697	262.810	0.1571
5.00	37.2658	53.6331	37.2792	51.8941	38.584	3.9221	47.836	2.3512	64.258	1.5470	264.378	0.3319
10.00	53.4356	64.7135	53.4450	62.9715	54.362	5.0024	61.274	3.2671	74.795	2.2620	267.074	0.5136
20.00	76.7811	71.3961	76.7876	69.6533	77.428	5.6623	82.423	3.8699	92.906	2.7753	272.578	0.6674
70.00	146.154	73.4185	146.157	71.6756	146.494	5.8633	149.180	4.0617	155.187	2.9502	299.014	0.7329

TABLE XI

Refraction Variables in the N(h) =252.9 exp{-0.1262h} Atmosphere

	$\theta_o = 0$		$\theta_o = 1$		$\theta_o = 10$		$\theta_o = 30$		$\theta_o = 52.4$		$\theta_o = 261.7$	
Ht. (km)	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ
0.01	1.581	0.4036	1.871	0.2223	10.124	0.0317	30.042	0.0106	52.384	0.0061	261.804	0.0012
0.02	2.236	0.5706	2.449	0.3698	10.247	0.0630	30.083	0.0212	52.408	0.0122	261.809	0.0024
0.05	3.536	0.9010	3.675	0.6812	10.607	0.1544	30.208	0.0528	52.479	0.0303	261.823	0.0059
0.10	5.003	1.2713	5.102	1.0415	11.182	0.2996	30.414	0.1050	52.598	0.0604	261.846	0.0118
0.20	7.081	1.7904	7.151	1.5535	12.253	0.5670	30.824	0.2073	52.836	0.1198	261.893	0.0235
0.50	11.222	2.7934	11.266	2.5501	15.031	1.2392	32.029	0.4988	53.548	0.2919	262.034	0.0577
1.00	15.929	3.8654	15.960	3.6191	18.807	2.0987	33.965	0.9390	54.727	0.5599	262.272	0.1118
2.00	22.683	5.2392	22.705	4.9910	24.789	3.3145	37.608	1.6781	57.058	1.0326	262.758	0.2102
5.00	36.484	7.3588	36.498	7.1093	37.829	5.3144	47.230	3.1461	63.808	2.0593	264.272	0.4398
10.00	52.641	8.7979	52.650	8.5480	53.581	6.7162	60.582	4.3283	74.230	2.9774	266.920	0.6714
20.00	76.120	9.6161	76.126	9.3662	76.773	7.5239	81.808	5.0644	92.361	3.6024	272.397	0.8574
70.00	145.812	9.8376	145.816	9.5876	146.152	7.7439	148.845	5.2742	154.865	3.7934	298.850	0.9283

TABLE XII

Refraction Variables in the $N(h) = 289 \exp\{-0.1357h\}$ Atmosphere

	$\theta_o = 0$		$\theta_o = 1$		$\theta_o = 10$		$\theta_o = 30$		$\theta_o = 52.4$		$\theta_o = 261.7$	
Ht. (km)	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ
0.01	1.534	0.5123	1.831	0.2776	10.117	0.0391	30.039	0.0131	52.382	0.0075	261.804	0.0015
0.02	2.170	0.7234	2.389	0.4628	10.233	0.0775	30.078	0.0261	52.405	0.0149	261.808	0.0029
0.05	3.432	1.1417	3.575	0.8557	10.573	0.1901	30.196	0.0650	52.472	0.0373	261.821	0.0073
0.10	4.857	1.6101	4.959	1.3107	11.117	0.3691	30.390	0.1290	52.584	0.0742	261.843	0.0145
0.20	6.876	2.2662	6.949	1.9572	12.136	0.6998	30.778	0.2547	52.809	0.1471	261.888	0.0289
0.50	10.908	3.5305	10.953	3.2131	14.798	1.5338	31.921	0.6126	53.483	0.3581	262.021	0.0707
1.00	15.506	4.8753	15.538	4.5539	18.450	2.6011	33.769	1.1521	54.606	0.6856	262.248	0.1368
2.00	22.138	6.5831	22.161	6.2592	24.291	4.1039	37.282	2.0542	56.844	1.2600	262.712	0.2560
5.00	35.825	9.1609	35.839	8.8354	37.194	6.5300	46.723	3.8189	63.434	2.4876	264.183	0.5292
10.00	52.013	10.8393	52.022	10.5133	52.964	8.1634	60.037	5.1900	73.786	3.5477	266.800	0.7950
20.00	75.641	11.7296	75.647	11.4034	76.298	9.0419	81.362	5.9891	91.966	4.2246	272.267	0.9952
70.00	145.584	11.9412	145.587	11.6151	145.925	9.2522	148.621	6.1895	154.651	4.4067	298.742	1.0624

TABLE XIII

Refraction Variables in the N(h) = 313.0 exp{-0.1438h} Atmosphere

	$\theta_o = 0$		$\theta_o = 1$		$\theta_o = 10$		$\theta_o = 30$		$\theta_o = 52.4$		$\theta_o = 261.7$	
Ht. (km)	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ
0.01	1.496	0.6014	1.800	0.3214	10.111	0.0447	30.037	0.0150	52.381	0.0086	261.804	0.0017
0.02	2.116	0.8500	2.341	0.5384	10.221	0.0889	30.075	0.0299	52.403	0.0171	261.808	0.0034
0.05	3.347	1.3419	3.494	0.9990	10.545	0.2183	30.186	0.0745	52.467	0.0427	261.820	0.0084
0.10	4.737	1.8927	4.842	1.5332	11.065	0.4244	30.372	0.1480	52.574	0.0851	261.841	0.0167
0.20	6.709	2.6635	6.783	2.2922	12.042	0.8058	30.741	0.2921	52.788	0.1687	261.883	0.0331
0.50	10.652	4.1457	10.699	3.7641	14.610	1.7702	31.834	0.7025	53.431	0.4102	262.011	0.0810
1.00	15.163	5.7167	15.196	5.3302	18.164	3.0060	33.613	1.3202	54.509	0.7844	262.228	0.1563
2.00	21.703	7.6946	21.726	7.3050	23.896	4.7380	37.025	2.3491	56.676	1.4372	262.677	0.2915
5.00	35.316	10.6243	35.330	10.2328	36.703	7.4894	46.334	4.3356	63.147	2.8133	264.116	0.5966
10.00	51.547	12.4649	51.557	12.0728	52.507	9.2793	59.635	5.8325	73.459	3.9667	266.712	0.8844
20.00	75.302	13.3858	75.309	12.9936	75.962	10.1878	81.047	6.6578	91.688	4.6644	272.175	1.0899
70.00	145.418	13.5824	145.421	13.1903	145.759	10.3833	148.459	6.8439	154.494	4.8332	298.662	1.1519

TABLE XIV

Refraction Variables in the $N(h) = 344.5 \exp\{-0.1568h\}$ Atmosphere

	$\theta_o = 0$		$\theta_o = 1$		$\theta_o = 10$		$\theta_o = 30$		$\theta_o = 52.4$		$\theta_o = 261.7$	
Ht. (km)	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ
0.01	1.435	0.7521	1.749	0.3926	10.102	0.0537	30.034	0.0180	52.380	0.0103	261.803	0.0020
0.02	2.030	1.0632	2.263	0.6612	10.204	0.1068	30.069	0.0359	52.399	0.0206	261.807	0.0040
0.05	3.211	1.6782	3.363	1.2341	10.503	0.2624	30.171	0.0894	52.458	0.0513	261.819	0.0100
0.10	4.546	2.3662	4.655	1.8997	10.985	0.5107	30.342	0.1775	52.557	0.1021	261.838	0.0200
0.20	6.442	3.3278	6.519	2.8453	11.895	0.9720	30.684	0.3504	52.754	0.2021	261.877	0.0397
0.50	10.245	5.1702	10.294	4.6739	14.316	2.1433	31.701	0.8420	53.352	0.4910	261.995	0.0969
1.00	14.623	7.1074	14.657	6.6046	17.715	3.6454	33.373	1.5801	54.362	0.9365	262.198	0.1864
2.00	21.026	9.5163	21.050	9.0096	23.282	5.7374	36.633	2.8020	56.420	1.7076	262.623	0.3455
5.00	34.553	12.9757	34.567	12.4665	35.970	8.9749	45.755	5.1119	62.725	3.2972	264.018	0.6960
10.00	50.886	15.0271	50.896	14.5172	51.858	10.9677	59.064	6.7691	72.997	4.5676	266.587	1.0107
20.00	74.849	15.9624	74.856	15.4524	75.513	11.8902	80.627	7.6053	91.317	5.2725	272.053	1.2171
70.00	145.205	16.1315	145.208	15.6215	145.546	12.0581	148.250	7.7650	154.294	5.4173	298.561	1.2698

TABLE XV

Refraction Variables in the $N(h) = 377.2 \exp\{-0.1732 h\}$ Atmosphere

	$\theta_o = 0$		$\theta_o = 1$		$\theta_o = 10$		$\theta_o = 30$		$\theta_o = 52.4$		$\theta_o = 261.7$	
Ht. (km)	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ
0.01	1.354	0.9641	1.683	0.4865	10.091	0.0650	30.031	0.0217	52.377	0.0125	261.803	0.0024
0.02	1.915	1.3628	2.161	0.8255	10.182	0.1292	30.061	0.0434	52.395	0.0249	261.806	0.0049
0.05	3.031	2.1505	3.192	1.5535	10.449	0.3181	30.153	0.1081	52.447	0.0620	261.817	0.0121
0.10	4.293	3.0309	4.408	2.4019	10.883	0.6203	30.306	0.2147	52.535	0.1234	261.834	0.0242
0.20	6.089	4.2591	6.171	3.6073	11.708	1.1840	30.612	0.4236	52.712	0.2442	261.869	0.0479
0.50	9.712	6.6004	9.764	5.9289	13.940	2.6239	31.533	1.0172	53.252	0.5921	261.975	0.1167
1.00	13.922	9.0376	13.958	8.3570	17.141	4.4741	33.072	1.9056	54.178	1.1258	262.161	0.2237
2.00	20.166	12.0142	20.191	11.3283	22.509	7.0292	36.146	3.3646	56.106	2.0402	262.557	0.4115
5.00	33.634	16.1172	33.649	15.4282	35.089	10.8508	45.066	6.0491	62.224	3.8727	263.902	0.8127
10.00	50.138	18.3758	50.148	17.6861	51.125	13.0418	58.422	7.8583	72.478	5.2508	266.449	1.1514
20.00	74.370	19.2920	74.376	18.6021	75.038	13.9450	80.182	8.6751	90.925	5.9372	271.924	1.3509
70.00	144.981	19.4269	144.984	18.7370	145.323	14.0790	148.031	8.8024	154.084	6.0524	298.455	1.3926

TABLE XVI

Refraction Variables in the N(h) = 404.9 exp{-0.1898h} Atmosphere

	$\theta_o = 0$		$\theta_o = 1$		$\theta_o = 10$		$\theta_o = 30$		$\theta_o = 52.4$		$\theta_o = 261.7$	
Ht. (km)	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ
0.01	1.268	1.2071	1.615	0.5853	10.080	0.0762	30.027	0.0255	52.375	0.0146	261.802	0.0029
0.02	1.794	1.7061	2.054	1.0018	10.160	0.1517	30.054	0.0509	52.391	0.0292	261.805	0.0057
0.05	2.840	2.6917	3.011	1.9035	10.396	0.3738	30.134	0.1268	52.437	0.0727	261.814	0.0142
0.10	4.026	3.7918	4.148	2.9585	10.780	0.7305	30.269	0.2518	52.514	0.1446	261.830	0.0283
0.20	5.718	5.3234	5.805	4.4578	11.519	1.3986	30.540	0.4966	52.671	0.2860	261.860	0.0561
0.50	9.155	8.2262	9.209	7.3331	13.558	3.1163	31.365	1.1915	53.153	0.6922	261.956	0.1364
1.00	13.199	11.2147	13.237	10.3092	16.559	5.3297	32.774	2.2284	53.996	1.3123	262.124	0.2603
2.00	19.298	14.7946	19.324	13.8820	21.735	8.3616	35.669	3.9184	55.800	2.3641	262.493	0.4754
5.00	32.757	19.5234	32.772	18.6069	34.249	12.7431	44.415	6.9465	61.754	4.4145	263.794	0.9214
10.00	49.469	21.9397	49.479	21.0224	50.469	15.0836	57.849	8.8659	72.017	5.8678	266.327	1.2758
20.00	73.963	22.8146	73.969	21.8972	74.634	15.9459	79.805	9.6438	90.593	6.5197	271.816	1.4641
70.00	144.792	22.9208	144.795	22.0033	145.134	16.0513	147.846	9.7439	153.906	6.6100	298.365	1.4966

TABLE XVII

Refraction Variables in the N(h) = 450 exp{-0.2232 h} Atmosphere

	$\theta_o = 0$		$\theta_o = 1$		$\theta_o = 10$		$\theta_o = 30$		$\theta_o = 52.4$		$\theta_o = 261.7$	
Ht. (km)	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ	θ	τ
0.01	1.064	1.8854	1.460	0.8154	10.056	0.1000	30.019	0.0334	52.371	0.0191	261.802	0.0037
0.02	1.506	2.6642	1.808	1.4279	10.113	0.1993	30.038	0.0667	52.382	0.0382	261.804	0.0075
0.05	2.389	4.2000	2.589	2.7880	10.281	0.4924	30.095	0.1661	52.414	0.0952	261.810	0.0186
0.10	3.394	5.9079	3.538	4.3997	10.560	0.9662	30.191	0.3299	52.470	0.1893	261.821	0.0371
0.20	4.845	8.2725	4.948	6.6952	11.112	1.8628	30.389	0.6504	52.583	0.3740	261.843	0.0733
0.50	7.867	12.6813	7.930	11.0468	12.723	4.2047	31.014	1.5581	52.947	0.9017	261.915	0.1773
1.00	11.567	17.0868	11.610	15.4280	15.290	7.2492	32.152	2.9040	53.621	1.6987	262.049	0.3357
2.00	17.422	22.1065	17.451	20.4350	20.088	11.3482	34.691	5.0619	55.180	3.0219	262.365	0.6040
5.00	31.037	28.1060	31.053	26.4286	32.607	16.8348	43.162	8.7161	60.860	5.4548	263.591	1.1262
10.00	48.287	30.7108	48.297	29.0324	49.310	19.3503	56.841	10.7493	71.211	6.9761	266.115	1.4919
20.00	73.296	31.4553	73.303	29.7768	73.974	20.0835	79.188	11.4081	90.050	7.5252	271.639	1.6487
70.00	144.482	31.5161	144.485	29.8376	144.825	20.1439	147.542	11.4653	153.615	7.5768	298.218	1.6670

TABLE XVIII

Initial N Gradients, ΔN_ϵ, in the C.R.P.L.
Exponential Reference Atmosphere*

Range of ΔN_ϵ (N units/km)					N(h)
		ΔN_ϵ	$\leq$	27.55	200 exp(-0.1184h)
27.55	<	ΔN_ϵ	$\leq$	35.33	252.9 exp(-0.1262h)
35.33	<	ΔN_ϵ	$\leq$	47.13	289 exp(-0.1257h)
42.13	<	ΔN_ϵ	$\leq$	49.52	313 exp(-0.1428h)
49.52	<	ΔN_ϵ	$\leq$	59.68	344.5 exp(-0.1568h)
59.68	<	ΔN_ϵ	$\leq$	71.10	377.2 exp(-0.1732h)
71.10	<	ΔN_ϵ	$\leq$	88.65	404.9 exp(-0.1898h)
88.65	<	ΔN_ϵ	$\leq$		450 exp(-0.2232h)

* Note height, h, is in kilometers.

TABLE XIX

Schulkin's Method Sample Computation for $\theta_o = 0$ mr

k	h_k (km)	N_k (N units)	$a + h_k$ (km)	$2(h_{k+1} - h_k)$ (km)	$2(N_k - N_{k+1})$ (N units)	$\frac{2(h_{k+1}-h_k)\times 10^6}{a + h_k}$	$(\theta_{k+1})^2$ $(mr)^2$	θ_{k+1} (mr)	$\frac{\theta_k + \theta_{k+1}}{2}$ (mr)	$\tau_{k,k+1}$ (mr)
0	0	400.0	6370.	0.680	70	106.75	36.75	6.062	3.031	11.547
1	0.340	365.0	6370.340	1.220	63	191.51	165.26	12.855	9.458	3.331
2	0.950	333.5	6370.950	4.220	193	662.38	634.64	25.192	19.024	5.073
3	3.060	237.0	6373.060	2.560	81	401.69	955.33	30.908	28.050	1.444
4	4.340	196.5	6374.340	1.500	47	235.32	1143.65	33.818	32.363	0.726
5	5.090	173.0	6375.090	0.420	2	65.88	1207.53	34.750	34.284	0.029
6	5.300	172.0	6375.300	1.280	34	200.77	1374.30	37.072	35.911	0.473
7	5.940	155.0	6375.940	0.620	6	97.24	1465.54	38.282	37.677	0.080
8	6.250	152.0	6376.250	1.860	36	291.71	1721.25	41.488	39.885	0.451
9	7.180	134.0	6377.180	0.874	17	137.05	1841.30	42.910	42.199	0.242
10	7.617	125.5	6377.617	4.086	55	640.68	2426.98	49.264	46.087	0.597
11	9.660	98.0	6379.660	2.420	26	379.33	2780.31	52.729	50.996	0.255
12	10.870	85.0	6380.870							

$$\sum \tau_{k,k+1} = 24.248 \text{ mr.}$$

TABLE XX

Schulkin's Method Sample Computation for $\theta_o = 10$ mr.

k	h_k (km)	N_k (N units)	$a + h_k$ (km)	$2(h_{k+1} - h_k)$ (km)	$2(N_k - N_{k+1})$ (N units)	$\frac{2(h_{k+1}-h_k)\times 10^6}{a+h_k}$	$(\theta_{k+1})^2$ $(mr)^2$	θ_{k+1} (mr)	$\frac{\theta_k + \theta_{k+1}}{2}$ (mr)	$\tau_{k,k+1}$ (mr)
0	0	400.0	6370.	0.680	70	106.75	136.75	11.694	10.847	3.227
1	0.340	365.0	6370.340	1.220	63	191.51	265.26	16.287	13.990	2.252
2	0.950	333.5	6370.950	4.220	193	662.38	734.64	27.104	21.696	4.448
3	3.060	237.0	6373.060	2.560	81	401.69	1055.33	32.486	29.795	1.359
4	4.340	196.5	6374.340	1.500	47	235.32	1243.65	35.265	33.876	0.694
5	5.090	173.0	6375.090	0.420	2	65.88	1307.53	36.160	35.712	0.028
6	5.300	172.0	6375.300	1.280	34	200.77	1474.30	38.397	37.278	0.456
7	5.940	155.0	6375.940	0.620	6	97.24	1565.54	39.567	38.982	0.077
8	6.250	152.0	6376.250	1.860	36	291.71	1821.25	42.676	41.122	0.438
9	7.180	134.0	6377.180	0.874	17	137.05	1941.30	44.060	43.368	0.196
10	7.617	125.5	6377.617	4.086	55	640.68	2526.98	50.269	47.164	0.583
11	9.660	98.0	6379.660	2.420	26	379.33	2880.31	53.669	51.969	0.250
12	10.870	85.0	6380.870							

$$\sum \tau_{k,k+1} = 14.008 \text{ mr.}$$

TABLE XXI

Schulkin's Method Sample Computation for $\theta_o = 52.4$ mr (3^o)

k	h_k (km)	N_k (N units)	$a + h_k$ (km)	$2(h_{k+1} - h_k)$ (km)	$2(N_k - N_{k+1})$ (N units)	$\frac{2(h_{k+1}-h_k)\times 10^6}{a + h_k}$	$(\theta_{k+1})^2$ $(mr)^2$	θ_{k+1} (mr)	$\frac{\theta_k + \theta_{k+1}}{2}$ (mr)	$\tau_{k,k+1}$ (mr)
0	0	400.0	6370.	0.680	70	106.75	2782.51	52.750	52.575	0.666
1	0.340	365.0	6370.340	1.220	63	191.51	2911.02	53.954	53.352	0.590
2	0.950	333.5	6370.950	4.220	193	662.38	3380.40	58.141	56.048	1.722
3	3.060	237.0	6373.060	2.560	81	401.69	3701.09	60.837	59.489	0.681
4	4.340	196.5	6374.340	1.500	47	235.32	3889.41	62.365	61.601	0.381
5	5.090	173.0	6375.090	0.420	2	65.88	3953.29	62.875	62.620	0.016
6	5.300	172.0	6375.300	1.280	34	200.77	4120.06	64.188	63.532	0.268
7	5.940	155.0	6375.940	0.620	6	97.24	4211.30	64.895	64.542	0.046
8	6.250	152.0	6376.250	1.860	36	291.71	4467.01	66.836	65.866	0.273
9	7.180	134.0	6377.180	0.874	17	137.05	4587.06	67.728	67.282	0.126
10	7.617	125.5	6377.617	4.086	55	640.68	5172.74	71.922	69.825	0.394
11	9.660	98.0	6379.660	2.420	26	379.33	5526.07	74.338	73.130	0.178
12	10.870	85.0	6380.870							

$$\sum \tau_{k,k+1} = 5.341 \text{ mr.}$$

TABLE XXII

Schulkin's Method Sample Computation for $\theta_o = 261.8$ mr (15^o).

k	h_k (km)	N_k (N units)	$a + h_k$ (km)	$2(h_{k+1} - h_k)$ (km)	$2(N_k - N_{k+1})$ (N units)	$\frac{2(h_{k+1}-h_k)\times 10^6}{a + h_k}$	$(\theta_{k+1})^2$ (mr)^2	θ_{k+1} (mr)	$\frac{\theta_k + \theta_{k+1}}{2}$ (mr)	$\tau_{k,k+1}$ (mr)
0	0	400.0	6370.	0.680	70	106.75	68575.99	261.870	261.835	0.134
1	0.340	365.0	6370.340	1.220	63	191.51	68704.50	262.115	261.992	0.120
2	0.950	333.5	6370.950	4.220	193	662.38	69173.88	263.009	262.562	0.368
3	3.060	237.0	6373.060	2.560	81	401.69	69494.57	263.618	263.314	0.154
4	4.340	196.5	6374.340	1.500	47	235.32	69682.89	263.975	263.796	0.089
5	5.090	173.0	6375.090	0.420	2	65.88	69746.77	264.096	264.036	0.004
6	5.300	172.0	6375.300	1.280	34	200.77	69913.54	264.412	264.254	0.064
7	5.940	155.0	6375.940	0.620	6	97.24	70004.78	264.584	264.498	0.011
8	6.250	152.0	6376.250	1.860	36	291.71	70260.49	265.067	264.782	0.068
9	7.180	134.0	6377.180	0.874	17	137.05	70380.54	265.293	265.180	0.032
10	7.617	125.5	6377.617	4.086	55	640.68	70966.22	266.395	265.844	0.103
11	9.660	98.0	6379.660	2.420	26	379.33	71319.55	267.057	266.726	0.049
12	10.870	85.0	6380.870							

$$\sum \tau_{k,k+1} = 1.196 \text{ mr.}$$

TABLE XXIII

Departures Method Sample Computation for $\theta_o = 0$ mr.

h	N(h)	A(N, h)	ΔA	θ	$\frac{2}{\theta_k + \theta_{k+1}}$	departure term	$\tau(h)_{N_s}$	$\tau_{1,2}$
0	400.0	400.0		0				
0.340	365.0	397.8	- 2.2	6.388	0.3131	+0.689		
0.950	333.5	419.5	+21.7	11.230	0.1135	-2.463		
3.060	237.0	459.7	+40.0	22.683	0.0590	-2.359		
4.340	196.5	475.7	+16.0	28.195	0.0393	-0.629		
5.090	173.0	478.5	+ 2.8	31.383	0.0336	-0.094		
5.300	172.0	484.1	+ 5.6	32.191	0.0315	-0.176		
5.940	155.0	485.6	+ 1.5	34.654	0.0299	-0.045		
6.250	152.0	490.5	+ 4.9	35.811	0.0284	-0.139		
7.180	134.0	493.4	+ 2.9	39.123	0.0267	-0.077		
7.617	125.5	493.3	- 0.1	40.581	0.0251	+0.003		
9.660	98.0	495.9	+ 2.6	47.236	0.0228	-0.059		
10.870	85.0	495.2	- 0.7	50.645	0.0204	+0.014	30.776	25.441

$$\sum = -5.335$$

TABLE XXIV

Departures Method Sample Computation for $\theta_o = 10$ mr.

0	400.0	400.0		10.000				
0.340	365.0	397.8	- 2.2	11.873	0.0914	+0.201		
0.950	333.5	419.5	+21.7	15.038	0.0743	-1.613		
3.060	237.0	459.7	+40.0	24.789	0.0502	-2.008		
4.340	196.5	475.7	+16.0	29.947	0.0365	-0.585		
5.090	173.0	478.5	+ 2.8	32.938	0.0318	-0.089		
5.300	172.0	484.1	+ 5.6	33.712	0.0300	-0.168		
5.940	155.0	485.6	+ 1.5	36.069	0.0287	-0.043		
6.250	152.0	490.5	+ 4.9	37.181	0.0273	-0.134		
7.180	134.0	493.4	+ 2.9	40.383	0.0258	-0.075		
7.617	125.5	493.3	- 0.1	41.801	0.0243	+0.002		
9.660	98.0	495.9	+ 2.6	48.282	0.0222	-0.058		
10.870	85.0	495.2	- 0.7	51.630	0.0200	+0.014	19.414	14.858

$$\sum = -4.556$$

TABLE XXV

Departures Method Sample Computation for $\theta_o = 52.4$ mr (3^o)

h	N(h)	$A(N_s, h)$	ΔA	θ	$\frac{2}{\theta_k + \theta_{k+1}}$	departure term	$\tau(h)_{N_s}$	$\tau_{1,2}$
0	400.0	400.0		52.4				
0.340	365.0	397.8	- 2.2	52.750	0.0190	+0.042		
0.950	333.5	419.5	+21.7	53.550	0.0188	-0.408		
3.060	237.0	459.7	+40.0	57.061	0.0181	-0.724		
4.340	196.5	475.7	+16.0	59.567	0.0171	-0.274		
5.090	173.0	478.5	+ 2.8	61.045	0.0166	-0.046		
5.300	172.0	484.1	+ 5.6	61.477	0.0163	-0.091		
5.940	155.0	485.6	+ 1.5	62.792	0.0161	-0.024		
6.250	152.0	490.5	+ 4.9	63.433	0.0158	-0.078		
7.180	134.0	493.4	+ 2.9	65.368	0.0155	-0.045		
7.617	125.5	493.3	- 0.1	66.277	0.0152	+0.002		
9.660	98.0	495.9	+ 2.6	70.512	0.0146	-0.038		
10.870	85.0	495.2	- 0.7	72.920	0.0139	+0.010	7.024	5.350

$$\sum = -1.674$$

TABLE XXVI

Departures Method Sample Computation for $\theta_o = 261.8$ mr (15^o)

0	400.0	400.0		261.8				
0.340	365.0	397.8	- 2.2	261.880	0.0038	+0.008		
0.950	333.5	419.5	+21.7	262.035	0.0038	-0.083		
3.060	237.0	459.7	+40.0	262.758	0.0038	-0.152		
4.340	196.5	475.7	+16.0	263.308	0.0038	-0.061		
5.090	173.0	478.5	+ 2.8	263.633	0.0038	-0.011		
5.300	172.0	484.1	+ 5.6	263.733	0.0038	-0.021		
5.940	155.0	485.6	+ 1.5	264.034	0.0038	-0.006		
6.250	152.0	490.5	+ 4.9	264.184	0.0038	-0.019		
7.180	134.0	493.4	+ 2.9	264.647	0.0038	-0.011		
7.617	125.5	493.3	- 0.1	264.872	0.0038	+0.000		
9.660	98.0	495.9	+ 2.6	265.934	0.0038	-0.010		
10.870	85.0	495.2	- 0.7	266.592	0.0038	+0.003	1.506	1.143

$$\sum = -0.363$$

DEPARTMENT OF COMMERCE

UNITED STATES OF AMERICA