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Calibration of Time Response of Thermometers:

Concepts and Model Calculations

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t. Technical Note 959

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Calibration of Time Response of Thermometers:

Concepts and Model Calculations

Raymond D. Mountain and G. W. Mulholland

Some of the conceptual problems associated with the calibration of the time response of a temperature sensor are examined in this report. The discussion is in terms of a time response function which characterizes the way a given sensor responds to changes in the temperature of its surroundings. A series of model calculations of the response function for idealized sensors are used to investigate the general features of the response functions. Important features are the sensitivity of these functions to (i) material properties of the sensor, (ii) the type of thermal coupling of the sensor with the environment and (iii) the geometry of the sensor. These features must be considered in the design of procedures for calibrating the time response of thermometers.

Key Words: Calibration; response time; temperature; thermal response; thermometer; time response function.

I. INTRODUCTION

The concept of calibration of dynamical measurements is inherently more complex than is the concept of calibration of quasi-static (say, temperature or pressure) measurements. This increased complexity is due to the increased number of quantities whose characteristics must be quantitatively understood if a calibration is to be possible. Also, it is necessary to clarify the relationship between the quantity actually measured (e.g. electrical resistance) and the time variation of the property inferred from that measurement.

For example, equilibrium temperatures are often determined by measuring the electrical resistance of a platinum wire for which the temperature-resistance relation has been determined under conditions of thermodynamic equilibrium. Two such wires, of quite different diameter, geometrical arrangement and mounting will yield when calibrated sensibly identical determinations of equilibrium temperatures. This is not the case for the time response of these "thermometers" to changes in temperature. The response will depend upon the aforementioned factors and will be especially sensitive to the properties of the materials and to the thickness of the materials used in the mounting. Also it will depend upon the nature of the thermal contact of the "thermometer" with the medium undergoing thermal change involving the thermal properties of both the "thermometer" and the medium as well as the nature of the thermal gradients over the surface of the sensor-medium boundary. All of these must be quantitatively understood before a meaningful calibration of the sensor can be obtained. In addition, the sensitivity of the calibration to variations of these conditions must be known if the sensor is to be used in situations which do not exactly match those used in obtaining the calibration.

Some of the features involved in calibrating the time response of a temperature sensor are examined in this report. Model calculations have been performed to indicate how a sensor would respond to changes in the temperature of surrounding material. Some general features of these calculations have been extracted and form the basis for the discussion of what is involved in establishing a calibration of time response.

First let us specify what we mean by time response. Suppose that initially the sensor and surroundings are in thermal equilibrium at an arbitrarily defined T=0. Then, at an initial time, t=0, the surroundings undergo a change of temperature, $T^e(t)$. The time response of the sensor is known when the temperature of the sensor T(t) is known as a function of $T^e(t)$. For the situations discussed here, (those for which the temperature dependenc of the thermodynamic and transport properties can be ignored), we show in Sec. III-A that the relationship between T(t) and $T^e(t)$ takes the form

$$T(t) = T^e(t) - \int_0^t dy\, G(t-y) \frac{dT^e(y)}{dy} \quad . \tag{1}$$

If one knows G(t), one can then, using Eq. (1), predict the response of the sensor to the external temperature $T^e(t)$. In particular, the response function G(t) describes how the sensor responds to a step function change in external temperature applied at t=0. If $T^e(t)$ is a unit step change as shown in Fig. 1, then

$$T(t) = 1 - G(t) \tag{2}$$

as shown in Fig. 2, so G(t) has the form of Fig. 3.

It is important to keep in mind the distinction between the calibraton of the time response of a sensor under well characterized conditions and the application of this calibration to the mesurement of a time varying temperature. For the purposes of calibration, the external temperature $T^e(t)$ is known and controlled. This greatly simplifies the analysis since the effect the sensor has on the temperature of the medium surrounding it need not be considered.

The task of calibrating the time response of a temperature sensor is therefore one of determining G(t) under specificied conditions. To be useful it is important to know how sensitive this function is to changes in those conditions and how well G(t) can be determined. Model calculations of G(t) have been made to investigate its properties and to specifically explore the following questions:

1. How should G(t) be represented explicitly?

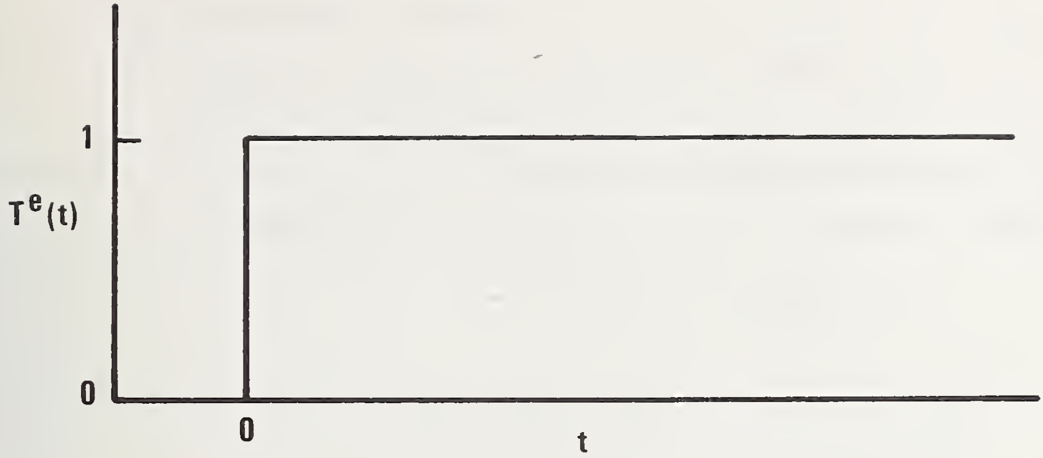

Fig. 1. A unit step change in external temperature.

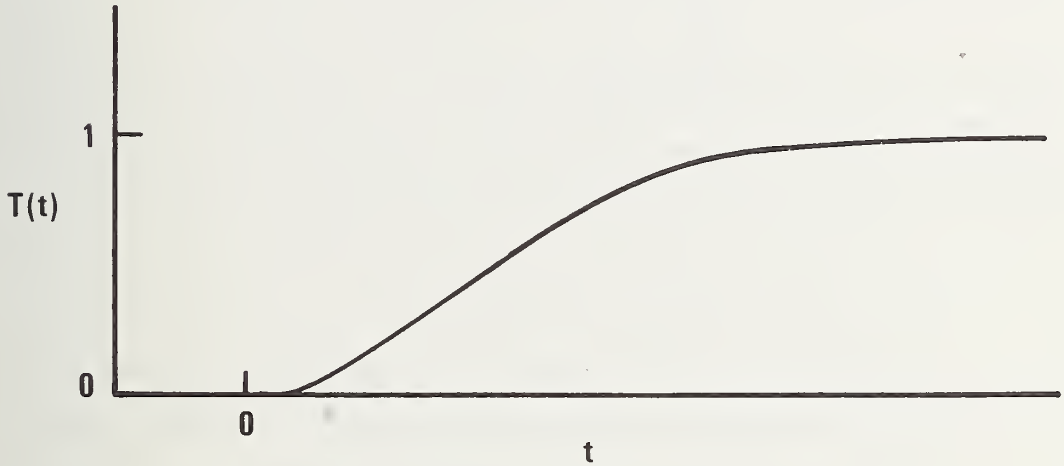

Fig. 2. The temperature response of a hypothetical thermometer.

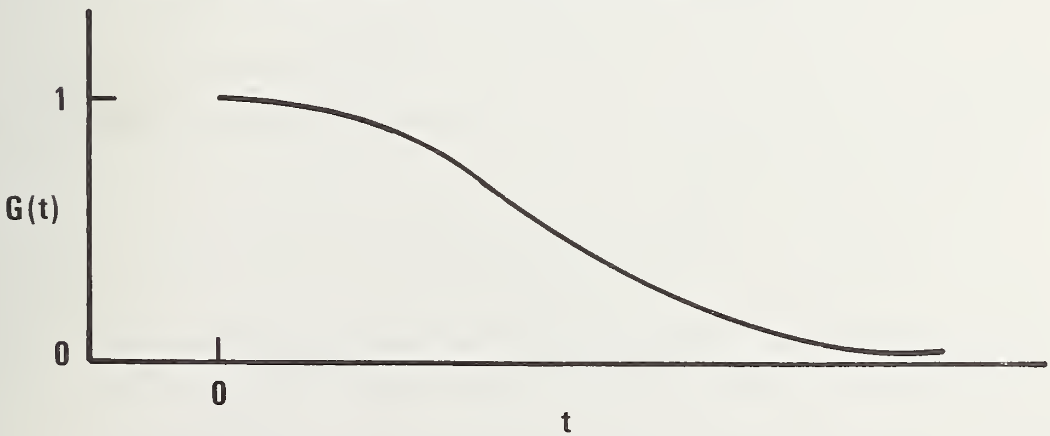

Fig. 3. The response function for the hypothetical thermometer of Fig. 2.

2. What sort of information is needed to determine $G(t)$ from experiment?

3. What are the limitations on the $T(t)$, $T^e(t)$, $G(t)$ relationship?

4. How sensitive is $G(t)$ to changes in material properties, in boundary conditions and in operating features?

The model calculations are of the following type. The temperature equation

$$\frac{\partial T(r,t)}{\partial t} = \chi \nabla^2 T(r,t) \tag{3}$$

is solved for plates, cylinders and spheres with boundary conditions of either

a) temperature continuity at the sensor surroundings interface, or

b) Newton's law of cooling which states that q, the heat flow at the surface, is proportional to the temperature discontinuity at the surface,

$$q = -\lambda \left.\frac{\partial T}{\partial r}\right|_{\text{surface}} = H\,[T(t) - T^e(t)]_{\text{surface}}. \tag{4}$$

In Eqs. 3 and 4, $\chi = \lambda/(\rho_o C_p)$ is the thermal diffusivity, ρ_o is the density, C_p the specific heat at constant pressure and λ is the thermal conductivity of the sensor. The heat transfer coefficient H is a property of the material surrounding the sensor. We use the initial condition $T(r,0) = 0$ for the temperature of the sensor. We have shown in IIIA that only the step function external temperature change case need be considered for the linear problem defined by Eq. (3) and boundary conditions (a) or (b). The symmetries of the plate, cylinder and sphere have been exploited so that only one spatial dimension is involved. This greatly simplifies the calculations and makes it possible to extract the essential features of the problem more readily. These calculations are not new and can be found in various texts on heat transport.[1-3] For the sake of completeness we do include some of the pertinent details of the calculations and show that the numerical evaluation of the infinite series expressions obtained for G(t) is straightforward and accurate. Time response functions have been usefully employed in some engineering studies[4].

Section II of this report contains a discussion of the factors involved in time response determinations. The details of the calculations on which this discussion is based are found in Section III. The third section contains a series of "appendices" to sections I and II and can be consulted by those interested in the details of the calculations.

II. DISCUSSION: GENERAL FEATURES OF THE TIME RESPONSE OF A TEMPERATURE SENSOR

The solutions to the temperature equation, Eq. (3), for a <u>unit</u> step function change at t=0 can be expressed in the form

$$T(r,t) = 1 - \sum_{j=1}^{\infty} a_j f_j(r) \exp[-\chi t \alpha_j^2/\ell^2] \tag{5}$$

where the set of coefficients $\{a_j\}$ are determined by the initial conditions, the set of coefficients $\{\alpha_j\}$ are obtained from solutions of a dispersion relation which reflects the boundary conditions, and the set of functions $\{f_j(r)\}$ reflect the geometry of the sensor. The coefficients $\{\alpha_j\}$ are ordered so that $\alpha_j < \alpha_{j+1}$. The quantities χ and ℓ are the thermal diffusivity and a characteristic length (e.g. the radius) of the sensor. The ratio ℓ^2/χ is the "natural" time unit for the sensor. When the sensor is a composite of two or more "layers", the form remains the same with the χ and ℓ being the properties of the active part only and with the coefficients a_j and α_j reflecting the complexity introduced by the layering of materials.

A comparison of Eqs. (2) and (5) shows that the response function can be represented as an infinite series of exponential functions with decreasing relaxation times $\tau_j = \ell^2/\chi\alpha_j^2$. Except for very short times, only a few terms in the series are significant and when

$$t > \tau_1 = \ell^2/\chi\alpha_1^2 , \tag{6}$$

only the first term is important. The experimental determination of G(t) can in principle be obtained by abruptly changing the temperature of the medium in which the sensor is embedded and then monitoring the reading of the sensor as a function of time. The response function is then found by scaling Eq. (2) by an amount T^e to be

$$G(t) = (T^e - T(t))/T^e. \tag{7}$$

Since the realization of an effective step function change in T^e is not always possible we should consider the possibility of extracting G(t) from the more general result

$$T(t) = T^e(t) - \int_0^t dy\, G(t-y) \frac{dT^e(y)}{dy} . \tag{1}$$

The solution of this integral equation for G(t) is possible with reasonable accuracy if T(t) and $T^e(t)$ are quite accurately known, if the zero of time is accurately known and if the time variation of $T^e(t)$ occurs over about the same time interval as does the variation of G(t). Some examples of this way of obtaining G(t) are discussed in Section III-F.

Once G(t) has been determined, the exponential series form

$$G(t) = \sum_{j=1}^{\infty} a_j f_j(r) \exp(-\chi t\, \alpha_j^2/\ell^2) \tag{7}$$

extracted from Eq. (5) provides a convenient representation. The coefficients a_j and $f_j(r)$ should be computed from the geometry of the sensor and the factors in the arguments of the exponentials adjusted to fit G(t). Of course, the full infinite set of terms will not be determined but a reasonable representation would involve a few terms.

One may also turn the question around to ask; if G(t) and T(t) are known, can one infer $T^e(t)$? Viewing Eq. (1) as an equation for $T^e(t)$ in terms of T(t) and G(t) leads to difficulties. It is not possible, using the procedure described in Section III-F, to construct a function $T^e(t)$ given only a set of values for G(t) and T(t) (unless G(t) = 1 - T(t) so that $T^e(t)$ is a step function). An example illustrating the problem is discussed in Section III-F. It may be possible to construct solutions for $T^e(t)$ in terms of specified functions subject to some sort of least squares error condition. We have not investigated this possibility.

It should not come as a surprise that solutions $T^e(t)$ are difficult to obtain. As the variation of $T^e(t)$ becomes more and more rapid, less information shows up in T(t). As shown in the example in Section III-F, when

$$T^e(t) = T_0 \sin \omega t \tag{8}$$

for a plate, then

$$T(t) = A(\omega) \sin \omega t + B(\omega) \cos \omega t \tag{9}$$
$$+ \text{ decaying terms.}$$

Both $A(\omega)$ and $B(\omega)$ go to zero as $\omega \to \infty$. This is simply a manifestation of the inability of the sensor to follow (or respond to) changes of sufficiently high frequency. Since any $T^e(t)$ which can be represented as a Fourier integral contains some high frequency components, the integral equation for $T^e(t)$ cannot be expected to yield solutions unless these high frequency components are reimposed by means of constraining the solution to be expressed in terms of some set of functions which do possess the necessary Fourier components.

The type of thermal coupling between the sensor and the surroundings can make a large difference in the time response of a sensor. To illustrate this consider a cylindrical sensor of radius *a* which is analysed in Section III-B2. At one extreme, the cylinder is in perfect thermal contact with the surroundings and heat flow into (or out of) the cylinder is unimpeded at the surface. This condition results in the most rapid response. More generally, there is a surface impediment to heat flow and the heat flow at the surface is proportional to the temperature discontinuity at the surface (Newton's law of cooling);

$$\lambda \left. \frac{\partial T}{\partial r} \right|_{r=a} = H[T^e(t) - T(a,t)] \qquad (4)$$

where the proportionality coefficient H is known as the heat transfer coefficient. This coefficient is a measure of the ability of the surroundings to deliver heat to (or carry it away from) the surface of the cylinder. Thus perfect thermal contact corresponds to very large $H(\lambda/H = 0)$ and $T^e(t) = T(a,t)$. In fluid flow problems, H can depend on the details of the flow past the cylinder. H is smaller for laminar flow than for turbulent flow and changes of a factor of 10 or more in H with fluid velocity can be expected.

The time response functions for the solid cylinder for varying thermal coupling conditions are constructed in Section III-B2. The longest relaxation time is found to be

$$\tau_1 = a^2/\beta_1{}^2\chi \qquad (10)$$

where β_1 is the smallest root of the transcendental equation

$$\beta \frac{\lambda}{aH} J_1(\beta) = J_o(\beta) \qquad (11)$$

Here J_o and J_1 are the Bessel functions of order 0 and order 1. The details of the derivation of Eq. (11) are to be found in Section III-B2.

To illustrate the influence of thermal coupling on time response, we show in Fig. 4, the inverse of τ_1 (in units of χ/a^2) as a function of λ/aH. As H decreases (the surroundings become less able to transport heat) $1/\tau_1$ decreases rapidly. This has the result, depicted in Fig. 5, of increasing the response time of the sensor. This indicates the importance of specifying the boundary conditions as a part of a calibration statement of time response.

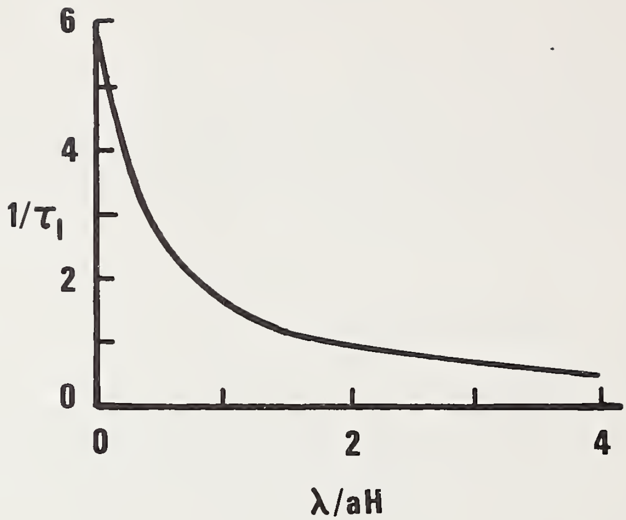

Fig. 4. The inverse of the longest relaxation time for a solid cylinder as a function of the quantity, λ/aH. The inverse time is expressed in units of χ/a^2. Perfect thermal contact corresponds to $\lambda/aH = 0$.

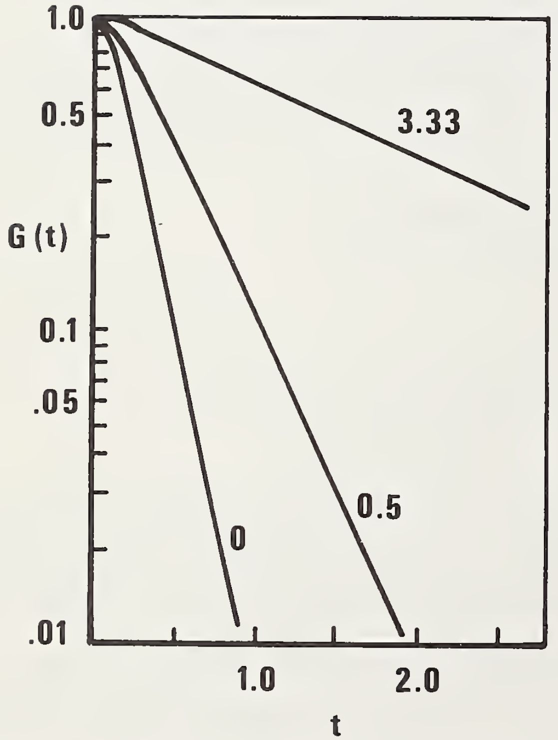

Fig. 5. Response functions for the center of a solid cylinder for different values of λ/aH. The time is expressed in units of a^2/χ. Perfect thermal contact corresponds to $\lambda/aH = 0$.

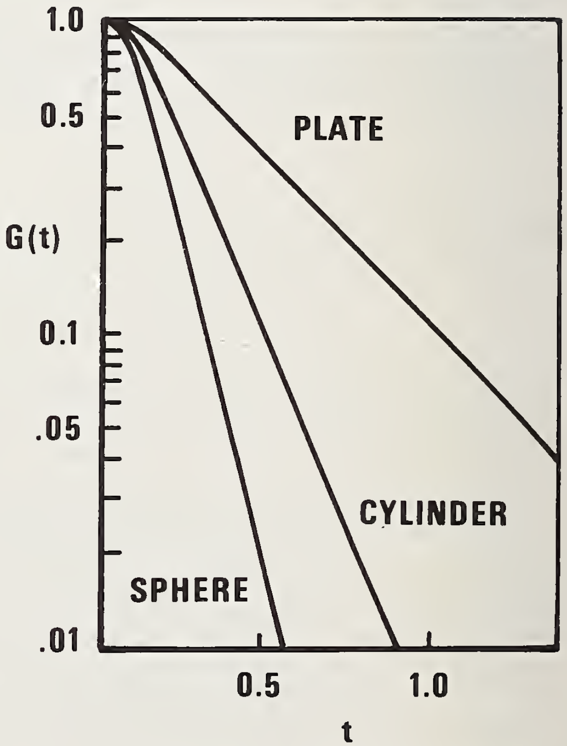

Fig. 6. Comparison of the "center line" response function for a sphere cylinder and plate for a common value of χ/ℓ^2. The time is expressed in units of ℓ^2/χ. Here ℓ is the radius of the sphere or cylinder and one-half the thickness of the plate.

Another feature which influences the response time of a sensor but which is not likely to pose a calibration problem is the geometrical shape of the sensor. For a given value of χ/ℓ^2, where ℓ is the radius for the sphere and the cylinder and the half thickness for a slab, the response time for a solid sphere is less than that for a solid cylinder which in turn is less than that for a slab as shown in Fig. 6. Some illustrative examples including solid and hollow sensors are contained in Sec. III-B.

Changes in the material properties for a uniform sensor change only the parameter λ/aH, which concerns the heat transfer at the sensor-surroundings interface, and the scale factor χ/ℓ^2, which is used in going from a dimensionless response time to the real response time. For layered sensors the situation is more complicated. The addition of an electrically insulating layer outside of the active portion of a sensor increases the response time of the sensor in a way which depends upon both the relative thickness of the layer and the extend to which the thermal impedances of the two materials coincide. This is examined in Sec. III-C.

Finally, we consider the effects of internal heating on the time response. A source term, $Q/(\rho_o C_p)$ where Q is the rate at which heat is added to the sensors of density ρ_o and specific heat at constant pressure C_p, must be added to Eq. (3) with the result,

$$\frac{\partial T(r,t)}{\partial t} = \chi\nabla^2 T(r,t) + Q/(\rho_o C_p). \tag{12}$$

If the heating rate Q is time independent, then there is no effect on the time variation of the temperature inside the sensor although it does affect the temperature profile. If Q does change in time and if Q does not strongly depend on the change in temperature, then the response of the temperature of the sensor initially at a stationary temperature condition $T_0(r)$ is found in Sec. III-E to be represented as

$$T(r,t) = T_0(r) + \int_0^t dy\ G(r,t-y)\ Q(y)/(\rho_0 C_P) \quad . \tag{13}$$

The magnitude of the effect depends on the heat capacity of the sensor and may be estimated for specific cases by means of Eq. (13).

In summary we reiterate the answers to the four questions posed earlier concerning the response function G(t).

1. G(t) can usefully be represented as a series of exponential functions (Eq.(7)).

After very small times only a few terms are significant.

2. The coefficients in the exponential series representation of G(t), Eq. (7), can be determined by matching Eq.(1) to time temperature data. In making this match, it is important that the zero of time be accurately known as well as the temperatures of the sensor and of the medium.
3. Eq. (1), which relates T(t), $T^e(t)$ and G(t), is not easily solved for $T^e(t)$ given T(t) and G(t). The constructive procedure described in III-F is unstable and cannot be used. The use of eq. (1) to infer "real-time" values of $T^e(t)$ will require some input as to the form of $T^e(t)$ in addition to the response function and T(t). This point requires further study.
4. G(t) is independent of how the external temperature changes and of the nature of stationary heat sources within the sensor. The effect on G(t) of changing the material properties or the geometry can be readily analyzed. The effect on G(t) of layer structure in a sensor is more complicated to analyze. Finally, the nature of the thermal contact between the medium and the sensor has a large effect on G(t) and depends on the properties of the sensor as well as those of the fluid.

III. MODEL CALCULATIONS

This section consists of model calculations which provide a basis for our discussion in Sections I and II. Part A contains a derivation of Eq. (1). Part B is concerned with the construction of response functions for plates, cylinders and spheres. The temperature equation, Eq. (3) is solved and the response functions are deduced from the solution. The influence of layer structure on the response function of a sphere is examined in Part C. The effect of thermal coupling of the sensor to the environment is considered in Part D and internal heating transients are examined in Part E. Part F is concerned with applications of the time response equation, Eq. (1), and some limitations on how it may be used are considered. These parts are intended to serve as appendices to the first two sections of this report.

A. DERIVATION OF EQUATION 1.

Equation 1 describes the relation between the external temperature $T^e(t)$ and the temperature of the sensor $T(t)$ in terms of the response function $G(t)$. The response function indicates how the sensor responds to a unit step change in external temperature at time $t=0$. The essential feature used to develop Eq. (1) is the linearity of the system.

An arbitrary variation of external temperature can be viewed as a series of step changes. To see this, consider a plot of $T^e(t)$ vs. t which is represented by a bar graph as in Fig. A-1. The value of T^e at the ith increment may be written as

$$T^e(i) = T^e(i-1) + \Delta T^e(i). \tag{A-1}$$

The response of the system at a time corresponding to increment j, $j > i$ is obtained by scaling Eq. (2) by $\Delta T^e(i)$ to yield

$$\Delta T^e(i) \; [1 - G(t_j-t_i)]. \tag{A-2}$$

The total response of the sensor at a time t is obtained by adding up all of the individual responses to changes occurring prior to t,

$$T(t_j) = \sum_{i=0}^{j} \Delta T^e(i) \; [1 - G(t_j-t_i)]. \tag{A-3}$$

Next we take the limit $t_{i+1}-t_i \to 0$ so that

$$\Delta T^e(i) \to \frac{dT^e(y)}{dy} \, dy \tag{A-4}$$

and we recover Eq. (1),

$$T(t) = T^e(t) - \int_0^t dy \; G(t-y) \frac{dT^e(y)}{dy} \; . \tag{1}$$

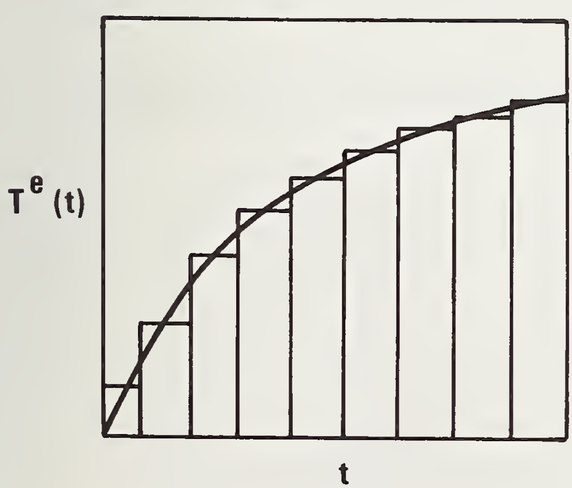

Fig. A-1. Decomposition of an arbitrary temperature change into a series of step changes.

Another way of deriving Eq (1) is to use the Laplace transform technique to solve Eq. (3). The solution to the temperature equation, Eq. (3), is

$$\hat{T}(z) = \hat{\Gamma}(z)\,\hat{T}^e(z) \tag{A-5}$$

where $\hat{T}(z)$ is the Laplace transform of the solution, $T^e(z)$ is the Laplace transform of the temperature on the boundary and $\hat{\Gamma}(z)$ (the Green function for the problem) depends on geometrical factors and material properties. The formal solution is found by using the convolution theorem for Laplace transforms with the result

$$T(t) = \int_0^t dy\, \Gamma(t-y)\, T^e(y). \tag{A-6}$$

Unfortunately $\lim_{x\to 0} \Gamma(x)$ is infinite so it is desirable to integrate by parts to remove this feature. The result is

$$T(t) = T^e(t) - \int_0^t dy\, G(t-y)\, \frac{d\,T^e(y)}{dy} \tag{1}$$

with $T^e(0) = 0$ so that the divergence disappears. Now we see that

$$\hat{G}(z) = \frac{1}{z} - \frac{\hat{\Gamma}(z)}{z} \tag{A-7}$$

or

$$G(t) = 1 - \mathcal{L}^{-1}\left\{\frac{\hat{\Gamma}(z)}{z}\right\} \tag{A-8}$$

If the pole at z=0 in $\hat{\Gamma}(z)/z$ is excluded, then G(t) can be obtained as

$$G(t) = -\frac{1}{2\pi i}\int_C dz e^{zt}\,\frac{\hat{\Gamma}(z)}{z}\ , \tag{A-9}$$

where C is the usual contour for inverting Laplace transforms <u>except</u> that the pole at z=0 is not enclosed by C.

B1. CALCULATION OF THE RESPONSE FUNCTION FOR A SOLID SPHERE AND A SOLID PLATE BY THE SERIES SOLUTION METHOD

The series solution method is used here to derive G(t) for a solid sphere. The problem is formulated as follows. A solid sphere of radius R is located at the origin of a coordinate system. The initial temperature distribution is

$$T(r,0) = 0 \qquad r < R. \tag{B1-1}$$

The temperature for $r > R$ is fixed at T^e. The temperature equation is

$$\frac{\partial T(r,t)}{\partial t} = \chi \left[\frac{\partial^2 T(r,t)}{\partial r^2} + \frac{2}{r} \frac{\partial T(r,t)}{\partial r}\right] \qquad \text{(B1-2)}$$

because of spherical symmetry.

We postulate a solution ($r < R$) of the form

$$T(r,t) = \sum_{j=0}^{\infty} a_j R_j(r)\, T_j(t) \;. \qquad \text{(B1-3)}$$

Substitution of this into Eq. (D1-1) yields

$$\frac{1}{\chi T_j(t)} \frac{dT_j(t)}{dt} = \frac{1}{R_j(r)} \frac{d^2\, R_j(r)}{dr2} + \frac{2}{r} \frac{dR_j(r)}{dr} = C_j \;. \qquad \text{(B1-4)}$$

Solving for $T_j(t)$ we find

$$T_j(t) = T_j(0)\, \exp(-C_j \chi t) \;. \qquad \text{(B1-5)}$$

The equation for $R_j(r)$ is

$$r^2 \frac{d^2R_j(r)}{dr^2} + 2r \frac{dR_j(r)}{dr} + r^2\, C_j\, R_j(r) = 0 \;. \qquad \text{(B1-6)}$$

The solution of this equation which is finite at $r = 0$ is

$$R_j(r) = j_0(C_j^{1/2} r). \qquad \text{(B1-7)}$$

Using the orthogonality relations for spherical Bessel functions (j_o) and the initial conditions, we find

$$T(r,t) = T^e \left\{1 - 2 \sum_{k=1}^{\infty} (-1)^{k+1}\, j_0\!\left(\frac{\pi k r}{R}\right) \exp[-\chi t(\pi k/R)^2\,]\right\} \;. \qquad \text{(B1-8)}$$

The response function for a sphere is thus

$$G(r,t) = 2 \sum_{k=1}^{\infty} (-1)^{k+1}\, j_0 \left(\frac{\pi k r}{R}\right) \exp[-\chi t(\pi k/R)^2] \;. \qquad \text{(B1-9)}$$

The response function for a plate is

$$G(x,t) = \frac{4}{\pi} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{2\ell + 1} \cos\left[\frac{(2\ell + 1)\;\pi x}{2L}\right] \exp \left[-\chi t\left(\frac{2\ell + 1}{2L}\pi\right)^2\right] \qquad \text{(B1-10)}$$

where x is measured from the center of the plate and 2L is the width of the plate. The same technique used in deriving Eq. (B1-9) can be used to derive Eq. (B1-10).

These expressions were evaluated for x=0 and r=0 and the results are shown in Fig. 6 (located in Sect. II).

For short times these series converge very slowly and many terms are needed. However, for longer times, say $\chi t/R^2 > .1$, the convergence is quite good and only a few terms are required. This is important when systems with more than one layer are involved since the C_j's are no longer derivable from simple transcendental equations with closed form solutions. Instead numerical solutions of transcendental equations are required. This is illustrated in Section III-C for the case of a two layer spherical sensor.

B2: CALCULATION OF THE RESPONSE FUNCTION FOR A SOLID CYLINDER BY THE LAPLACE TRANSFORM TECHNIQUE

The Laplace transform technique is widely used for solving heat conduction problems in solids. We illustrate the technique by solving heat conduction problems for an infinite solid cylinder with a "radiation" boundary condition.

Suppose that initially the sensor and surroundings are in thermal equilibrium at an arbitrarily defined T=0. Then, at an initial time, t=0, the surroundings undergo a step change of temperature, T^e. The heat conduction equation of interest and the associated "radiative" boundary condition are given by

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T \qquad r < a \tag{B2-1}$$

$$\lambda \left. \frac{\partial T}{\partial n} \right|_{r=a} + H\,(T(r=a) - T^e) = 0 \tag{B2-2}$$

Making use of cylindrical symmetry, we obtain for Eqs. (B2-1) and (B2-2)

$$\frac{\partial T}{\partial t} = \chi \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \qquad r < a \tag{B2-3}$$

$$\lambda \left. \frac{\partial T}{\partial r} \right|_{r=a} + H(T\,(r=a) - T^e) = 0\ . \tag{B2-4}$$

Taking the Laplace transform of Eqs. (B2-3) and (B2-4) yields

$$\frac{d^2\hat{T}}{dr^2} + \frac{1}{r} \frac{d\hat{T}}{dr} - \frac{p}{\chi} \hat{T} = 0 \tag{B2-5}$$

$$\lambda \frac{d\hat{T}}{dr} + H(\hat{T} - \frac{T^e}{p}) = 0 \quad \text{at } r=a. \tag{B2-6}$$

where $\hat{T} = \int_0^\infty e^{-pt} T dt$

By changing variables, $\omega = \sqrt{\frac{p}{\chi}}\, r$, we see that Eq. (D2-5) is the modified Bessel differential equation for the case of zero eigenvalue.

$$\frac{d^2\hat{T}}{d\omega^2} + \frac{1}{\omega}\frac{d\hat{T}}{d\omega} - \hat{T} = 0 . \qquad \text{(B2-7)}$$

The solution to (B2-7) that satisfies the boundary condition and that remains finite at the origin must be of the form

$$\hat{T} = F(p)\, I_0\left(\sqrt{\frac{p}{\chi}}\, r\right). \qquad \text{(B2-8)}$$

where I_o is the modified Bessel function of the first kind.

The function F(p) is obtained by substituting Eq. (B2-8) into Eq. (B2-6). We thus find

$$\hat{T} = \frac{H\, T^e\, I_0\left(\sqrt{\frac{p}{\chi}}\, r\right)}{\lambda\, p\left[I_0'\left(a\sqrt{\frac{p}{\chi}}\right)\sqrt{\frac{p}{\chi}} + \frac{H}{\lambda}\, I_0\left(\sqrt{\frac{p}{\chi}}\, a\right)\right]} . \qquad \text{(B2-9)}$$

The quantity of interest T can be obtained from $\hat{T}$ by taking the inverse Laplace transform. Complex variable theory tells us that the inverse Laplace transform may be obtained by the following prescription:

$$T = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}\, \hat{T}(p)\, dp , \qquad \text{(B2-10)}$$

where γ is to the right of the poles of $\hat{T}(p)$. The only singularities for the integrand are simple poles along the negative p axis including the point p=0. We thus evaluate the integral above by performing the integration around the contour shown in Fig. B-1. The

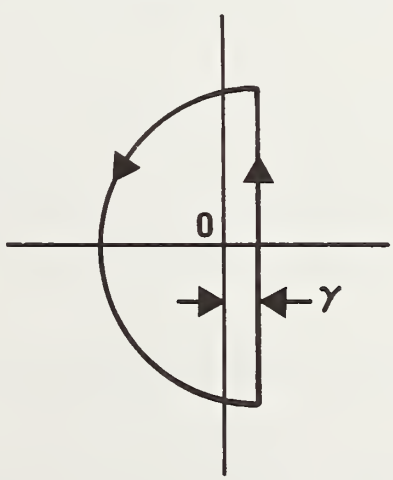

Fig. B-1. Contour used to evaluate Eq. (B2-10).

contour integral can be readily performed by using the residue theorem which states that the integral around a closed loop is equal to $2\pi i$ times the sum of the residues. It can be shown that the integral along the semicircle goes to zero as the radius approaches ∞. Thus the contour integral is equal to the integral in Eq. (B2-10) and

$$T = \sum_{\text{Res.}} [e^{pt}\, \hat{T}(p)] \ . \tag{B2-11}$$

In order to evaluate the residues, we first must find the poles of Eq. (B2-9), or, equivalently, the roots of the equation

$$I_1 \sqrt{\frac{p_n}{\chi}}\ a \sqrt{\frac{p_n}{\chi}} + \frac{H}{\lambda}\, I_0 \sqrt{\frac{p_n}{\chi}}\ a \ = 0 \ , \tag{B2-12}$$

where we have made use of the relation $I_0'(z) = I_1(z)$. The roots of Eq. (B2-12) are along the negative axis so that the argument of the modified Bessel function is imaginary. By making use of the relationship between the Bessel function J and the modified Bessel function I, we can express Eq. (B2-12) in terms of J_0 and J_1 with real arguments.

$$\alpha_n\, J_1(\alpha_n a) - \frac{H}{\lambda}\, J_0(\alpha_n a) = 0 \tag{B2-13}$$

$$\alpha_n = i\sqrt{\frac{p_n}{\chi}} \tag{B2-14}$$

The quantity α_n is the n^{th} root of Eq. (B2-13) with α_0 corresponding to the smallest root in absolute magnitude. Carslaw and Jasger's book contains the first five roots to Eq. (B2-13) for a wide range of values for the parameter H/λ.

Since all the poles are simple, the residue at the n^{th} pole is given by the formula

$$R_n = [(p - p_n)\, e^{pt}\, \hat{T}(p)]\Big|_{p=p_n} \tag{B2-15}$$

We treat the case $p_0=0$ separately from the other values of p_n. For this case we obtain from Eqs. (B2-9) and (B2-15) plus the properties of the modified Bessel functions at zero argument

$$R_0 = T^e \tag{B2-16}$$

For the other residues, we must factor out the n^{th} root from the denominator in Eq. B2-9. Let

$$D(p) = (p - p_n)\, D_1(p) = I_1\left(\sqrt{\frac{p}{\chi}}\, a\right)\sqrt{\frac{p}{\chi}} + \frac{H}{\lambda}\, I_0\left(\sqrt{\frac{p}{\chi}}\, a\right). \tag{B2-17}$$

By differentiation we find

$$D_1(p_n) = \frac{d}{dp} D(p) \Big|_{p_n} \quad . \tag{B2-18}$$

We make use of Eq. B2-12 plus the identity,

$$z\, I'_1(z) + I_1(z) = zI_0(z) \ , \tag{B2-19}$$

to obtain the following expression for $D_1(p_n)$

$$D_1(p_n) = \frac{I_0\left(\sqrt{\frac{p_n}{\chi}}\, a\right)}{2ap} \left[a^2 \frac{p_n}{\chi} - \left(\frac{Ha}{\lambda}\right)^2\right] \tag{B2-20}$$

From this equation plus Eq. (B2-9) and Eq. (B2-15) we obtain the residue,

$$R_n(p_n) = \frac{2\, T^e\, H\, a \exp(p_n t)\, I_0\left(\sqrt{\frac{p_n}{\chi}}\, r\right)}{\lambda\, I_0\left(\sqrt{\frac{p_n}{\chi}}\, a\right)\left[a^2 \frac{p_n}{\chi} - \left(\frac{Ha}{\lambda}\right)^2\right]} \quad . \tag{B2-21}$$

The solution for T can now be obtained from Eqs. (B2-21), (B2-16), and the relation between I_0 and J_0.

$$T = T^e - \frac{2T^e Ha}{\lambda} \sum_{n=1}^{\infty} \frac{\exp(-\alpha_n{}^2\chi t) J_0(\alpha_n r)}{J_0(\alpha_n a)\left[a^2\alpha_n{}^2 + \left(\frac{Ha}{\lambda}\right)^2\right]} . \tag{B2-22}$$

The response function $(1- T/T^e)$ is thus found to be

$$G(t) = \frac{2Ha}{\lambda} \sum_{n=1} \frac{\exp(-\alpha_n{}^2\chi t)\, J_0(\alpha_n r)}{J_0(\alpha_n a)\left[a^2\alpha_n{}^2 + \left(\frac{Ha}{\lambda}\right)^2\right]} \tag{B2-23}$$

B3. HEAT TRANSFER IN A HOLLOW CYLINDER

The problem of heat transfer in a hollow cylinder is of interest to us for two reasons. First, it has an additional parameter, the ratio of the outer to inner radii b/a, which is not found in the solid cylinder. Second, it is expected that a hollow cylinder can be made to more nearly approximate a real platinum-resistance thermometer than can a solid cylinder.

We consider the case of heat transfer in the hollow cylinder with "radiation" boundary condition at the outer surface and with no heat flux across the inner surface. Again, we are interested in the case of a step change in temperature T^e of the surroundings at t=0. The heat equation and the boundary conditions are given below:

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T \qquad b > r > a \tag{B3-1}$$

$$\lambda \left. \frac{\partial T}{\partial n} \right|_{r=b} + H[T(r=b) - T^e] = 0 \tag{B3-2}$$

$$\left. \frac{\partial T}{\partial n} \right|_{r=a} = 0 \ . \tag{B3-3}$$

This problem is solved by the Laplace transform technique described in Section B2. The response function at r=a is given by

$$G(t) = \sum_{n=1}^{\infty} \frac{e^{-\chi \alpha_n^2 t/a^2}}{D(\alpha_n)} \tag{B3-3}$$

$$\begin{aligned} D(\alpha_n) = \frac{-\pi}{4} \alpha_n^2 \{ & J_0(\alpha_n) [Y_0(\alpha_n b/a) - \alpha_n m Y_1(\alpha_n b/a)] + Y_0(\alpha_n) [-J_0(\alpha_n b/a) + \\ & \alpha_n m J_1(\alpha_n b/a)] - \frac{b}{a} J_1(\alpha_n) [Y_1(\alpha_n b/a) + m\alpha_n Y_0(\alpha_n b/a)] + \\ & \frac{b}{a} Y_1(\alpha_n) [J_1(\alpha_n b/a) + m\alpha_n J_0(\alpha_n b/a)] \} \ , \end{aligned} \tag{B3-4}$$

where $m = \frac{\lambda}{aH}$ and Y is the solution to the Bessel differential equation that is infinite at the origin. The α_n are the roots to the transcendental equation

$$m\alpha_n [J_1(\alpha_n b/a) Y_1(\alpha_n) - Y_1(\alpha_n b/a) J_1(\alpha_n)] + [J_1(\alpha_n) Y_0(\alpha_n b/a) - Y_1(\alpha_n) J_0(\alpha_n b/a)] = 0.$$

We show in Fig. B-2 how the response time increases for a hollow cylinder as the outer diameter b increases with the inner diameter fixed. The response function is evaluated at the inner surface and the dimensionless ratio λ/aH equals 3.33 for each curve. The time is expressed in units of a^2/χ.

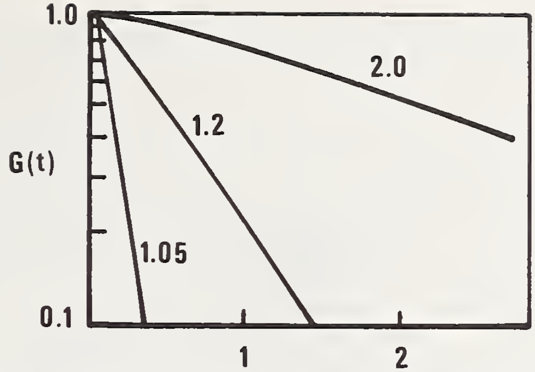

Fig. B-2. Response functions for hollows cylinders for different values of b/a, the ratio of the outer diameter to the inner diameter. The ratio λ/aH is 3.33 for each case and time is expressed in units of a^2/χ.

C. LAYER STRUCTURE

The effect of layers on the time response of a sensor can be illustrated by considering the case of a two-layer, spherical sensor. An idealized version of this object is sketched in Fig. C-1. The inner, active region 1 is surrounded by an outer protective/electrically insulating region 2. The temperature at the center can be related to the temperature at surface 2 quite conveniently by using the Laplace transform method.

The result is

$$T_0 = T_2/D(x_1,\gamma,R,\Lambda) \qquad \text{(C-1)}$$

Here $x_1 = q_1R_1$

$$x_2 = \gamma x_1 = q_2\ell$$

$$\gamma = \frac{\ell}{R_1}\sqrt{\frac{\chi_1}{\chi_2}} \qquad \text{(C-2)}$$

$$q_i = \sqrt{z/\chi_i} \quad i = 1,2$$

$$\Lambda = \lambda_1/\lambda_2; \quad R = R_1/R_2$$

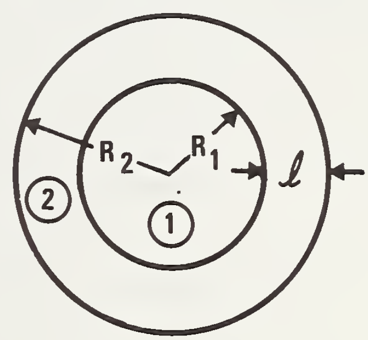

Fig. C-1. Sketch of a two layer spherical sensor. Region 1 is assumed to be the active region and region 2 is a protective layer.

and

$$D(x_1, \gamma, R, \Lambda) =$$

$$\frac{\sinh x_1}{x_1}\left[R \cosh(\gamma x_1) + (1-\Lambda)(1-R)\frac{\sinh(\gamma x_1)}{\gamma x_1}\right. \qquad \text{(C-3)}$$
$$\left. + \Lambda(1-R)\frac{\cosh x_1 \sinh(\gamma x_1)}{\gamma x_1}\right]$$

The zeros of D occur when $x_1 = R_1\sqrt{z/\chi_1}$ is purely imaginary. By excluding the pole at $x_1=0$, G(t) can be constructed using the technique outlined in Section III-A. Fig. C-2 shows G(t) for the case $\Lambda = 0.4$, $\frac{\chi_1}{\chi_2} = 2.25$ for several values of R. As expected, the presence of an outer layer slows down the response of the sensor. Time is expressed in units of R_1^2/χ_1.

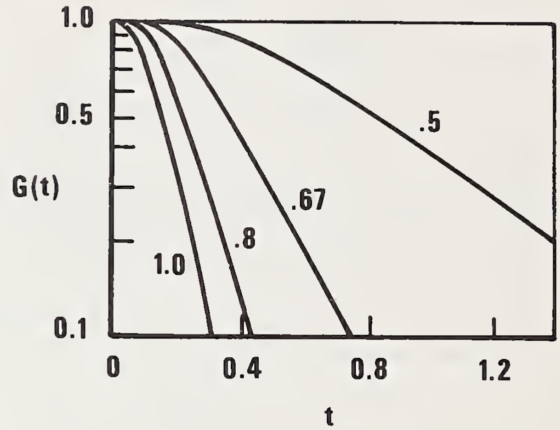

Fig. C-2. Response functions for a two-layer spherical sensor for several values of R_1/R_2. For this calculation $\lambda_1/\lambda_2 = 0.4$ and $\chi_1/\chi_2 = 2.25$. The time is expressed in units of R_1^2/χ_1.

D. THERMAL COUPLING OF SENSOR TO ENVIRONMENT

As pointed out earlier, the type of thermal coupling between the sensor and the surroundings can make a large difference in the response time of a sensor. Below we shall consider two types of boundary conditions for the heat transfer at the interface between a sensor and its surroundings: (1) Prescribed surface temperature or good thermal contact and (2) "radiation" boundary condition. In the first case the temperature at the surface is a specified function of time. Mathematically this is the easiest boundary condition to study, though in practice it is difficult to achieve. The "radiation" boundary condition, also referred to as "Newton's law", states that the heat flux at the surface is proportional to the temperature difference between the surface of the sensor and the medium. Actual conditions more often conform to the "radiation" boundary condition rather than the prescribed surface temperature condition.

Suppose that initially the sensor and surroundings are in thermal equilibrium at t=0. Then, at an initial time, t=0, the surroundings undergo a step change to T^e. The heat equation and the prescribed surface temperature boundary condition for a solid cylinder of radius a are given by:

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T \qquad \text{for } r < a \qquad \text{(D-1)}$$

$$T(r=a) = T^e \qquad t > 0 \qquad \text{(D-2)}$$

Eq. (D-2) means that the temperature at the outer surface of the cylinder changes instantly to T^e at t=0 (good thermal contact). The response function G(t) for this problem, which can be obtained using the techniques outlined in Sec. III-B, is

$$G(t) = 2 \sum_{n=1}^{\infty} \frac{e^{-\alpha_n^2 \chi t} J_0(\alpha_n r)}{a\alpha_n J_1(\alpha_n a)} \qquad \text{(D-3)}$$

where the J's are Bessel functions. The α_n are the roots to the equation

$$J_0(\alpha_n a) = 0 \qquad \text{(D-4)}$$

A plot of the quantity G(t) versus t (time in units of a^2/χ) is given in Fig. 5 (labeled 0) for T evaluated at the center of the cylinder.

For the case of the "radiative" boundary conditions, Eq. (D-2) is replaced by

$$\lambda \left.\frac{\partial T}{\partial n}\right|_{r=a} + H(T(r=a) - T^e) = 0. \qquad \text{(D-5)}$$

The first term is the heat flux from conduction with the $\partial T/\partial n$ being the normal derivative of the temperature at the surface of the cylindrical sensor. The symbol H stands for the heat transfer coefficient. The response function for this problem with the boundary condition Eq. (D-5) and with a unit step change in temperature is given by Eq. (B2-23),

$$G(t) = \frac{2Ha}{\lambda} \sum_{n=1}^{\infty} \frac{\exp(-\alpha_n^2 \chi t)\, J_0(\alpha_n r)}{\left[(a\alpha_n)^2 + \left(\frac{Ha}{\lambda}\right)^2\right] J_0(\alpha_n a)} \qquad \text{(B2-23)}$$

Here the α_r are the roots of the transcendental equation

$$\alpha_n \frac{\lambda}{H} J_1(\alpha_n a) = J_0(\alpha_n a) \,. \qquad \text{(D-6)}$$

For the case of the "radiative" boundary condition we have one more parameter, H, than for the case of the prescribed surface temperature. In the limit λ/H approaches zero, the radiative boundary condition becomes the same as the prescribed surface temperature boundary condition. As the ratio increases the response time of the sensor decreases as shown in Fig. 5, where the plots are parametrized by the dimensionless ratio λ/aH. The increase in the response time with an increase in λ/aH is also indicated by the plot of the inverse of the longest relaxation time, $1/\tau_1$, vs. λ/aH in Fig. 4 where τ_1 is expressed in units of a^2/χ. The longest relaxation time is defined as the time at which the argument of the exponent for the first term in Eq. (B2-23) is unity.

In designing a thermometer, one would like to minimize the response time and this can be done by maximizing the heat transfer coefficient H. Below we give an empirical equation[2] for H in terms of the Reynolds number N_{Re}, the Prandtl number N_{Pr}, the radius of the sensor a, and the thermal conductivity of the fluid λ_f.

$$H = .3 \frac{\lambda_f}{a} (N_{Re})^{.5} (N_{Pr})^{.31} , \tag{D-7}$$

$$N_{Re} = \frac{2av\rho}{\eta} \tag{D-8}$$

$$N_{Pr} = \frac{C_p \eta}{\lambda_f} \tag{D-9}$$

In these equations η is the fluid viscosity, ρ the fluid density, and C_p the constant pressure heat capacity and v is the fluid velocity. Eq. (D-7) is valid for N_{Re} between 50 and 10,000.

We illustrate the use of the equations in this section by predicting the response time of a model platinum-resistance thermometer to a step change in temperature when immersed in H_20 moving 1 m/s. The model consists of an infinite cylinder of powdered Al_20_3, a material used in many commercial platinum resistance thermometers, with a platinum wire of infinitesimal thickness in the center. The temperature of such a thermometer is the temperature at the center of the cylinder. We take the radius of the cylinder to be .001 m. From the data in Table D-1 we calculate N_{Re}, N_{Pr}, and $\frac{\lambda}{aH}$ from Eqs. (D-8), (D-9), and (D-7) respectively: $N_{Re} = 2x10^3$, $N_{Pr} = 7.0$, and $\frac{\lambda}{aH} = 2.4$. Then we find from Fig. 4 that the longest relaxation time $\tau_1 = 1.33$, which from the thermal properties of Al_20_3 corresponds to 0.1 s.

Table D-1 Approximate Values of the Transport Properties for Al_20_3 and H_20 at 20°C

Substance	λ[J/(s-m-k)]	ρ[kg/m^3]	C_p[J/(kg-K)]	η[Pa s]
Al_20_3	30[a]	3800[a]	753[b]	-
H_20	.59[a]	1000[a]	4200[a]	.001[a]

[a] Handbook of Chemistry and Physics, 56th Edition 1975-1976 (CRC Press, Cleveland, 1975).

[b] D. C. Ginnings and R. J. Corruccini, J. Res. NBS 38, 593 (1947).

E. INTERNAL HEATING TRANSIENTS

The value of knowing G(t) is further illustrated by studying the time response of a spherical sensor to changes in internal heating. Suppose the sensor is embedded in a reservoir at T=0 and is in equilibrium with it for $t < 0$. At t=0, an internal heat source is turned on. This problem may be posed as follows:

$$T(r, t \leq 0) = 0 \tag{E-1}$$

$$\frac{\partial T(r,t)}{\partial t} = \chi \nabla^2 T(r,t) + \frac{Q(t)}{\rho C_p} \qquad r < a \tag{E-2}$$

where

$$Q(t) = \begin{cases} 0 & t < 0 \\ Q(t) & t > 0 \end{cases} \qquad r < a \tag{E-3}$$

and

$$T(r \geq a, t) = 0 \ . \tag{E-4}$$

This problem may be conveniently treated using the Laplace transform method. In terms of $V(r,t) = rT(r,t)$, the equation in transform variables becomes

$$\frac{\hat{V}(r,z)}{\partial r^2} - \frac{z}{\chi}\hat{V}(r,z) = \frac{-r\hat{Q}(z)}{\chi \rho C_p} = \frac{-r\hat{Q}(z)}{\lambda} \ . \tag{E-5}$$

The solution finite at the origin is

$$\hat{V}(r,z) = A\sinh r\sqrt{z/\chi} + \frac{r\chi Q(z)}{\lambda z} \tag{E-6}$$

so

$$\hat{T}(r,z) = \frac{A\sinh r\sqrt{z/\chi}}{r} + \frac{\chi\hat{Q}(z)}{\lambda z} \ . \tag{E-7}$$

This solution is a superposition of the homogeneous solution and a particular solution appropriate to the heating pattern. By invoking the boundary condition

$$T(a,t) = 0 \tag{E-8}$$

we obtain

$$\hat{T}(r,z) = \frac{\hat{Q}(z)}{z\rho_0 C_p}\left[1 - \frac{a}{r}\ \frac{\sinh r\sqrt{z/\chi}}{\sinh a\sqrt{z/\chi}}\right] \ . \tag{E-9}$$

The pole at z=0 has zero residue so using the convolution theorem we obtain

$$T(r,t) = \frac{1}{\rho_0 C_p}\int_0^t dy\ G(r,t-y)\ Q(y) \tag{E-10}$$

where

$$G(r,t) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \exp -\chi t \left(\frac{n\pi}{a}\right)^2 \frac{\sin\left(\frac{r}{a} n\pi\right)}{(n\pi\, r/a).} \tag{E-11}$$

is the response function for a sphere.

Equation 13 is obtained by changing the initial condition $T(r,t) = 0$ to

$$T(r,0) = T_0(r)\ . \tag{E-12}$$

F. APPLICATIONS OF EQUATION 1.

For numerical work, the discrete form of Eq. (1)

$$T_j = T^e_j - \delta \sum_{i=0}^{j} \varepsilon_{i,j} G_{j-i} \frac{T^e_{i+1} - T^e_{i-1}}{2\delta} \tag{F-1}$$

is used. The time increment is δ, $G_0 = 1$ and $\varepsilon_{i,j}$ is 1 except if $i = 0$ or $i = j$ when it is 1/2. If we set $T^e_{-1} = T^e_1$, we can solve for G_j in terms of the quantities $\{T_i\}$ and $\{T^e_i\}$. The first step, $j = 1$, is found to be

$$G_1 = 2(T^e_1 - T_1)/T^e_1 - (T^e_2 - T^e_0)/2T^e_1. \tag{F-2}$$

for $j \geq 2$ the result is

$$G_j = 2(T^e_j - T_j)/T^e_1 - (T^e_{j+1} - T^e_{j-1})/2T^e_1$$
$$-2 \sum_{i=1}^{j-1} G_{j-i}(T^e_{i+1} - T^e_{i-1})/(T^e_1 - T^e_{-1}). \tag{F-3}$$

To illustrate the use of Eqs. (F-2) and (F-3) consider the following model calculation. The temperature, $T(t)$, at the center of a plate is determined using Eq. (B1-3) and the postulated external temperature

$$T^e(t) = 1 - \exp(-0.6T) \tag{F-4}$$

in Eq. (1). The calculated values of $T(t)$ and the values of $T^e(t)$ obtained from Eq. (F-4) are then used to estimate $G(t)$ using Eqs. (F-2) and (F-3). Although it does not explicitly appear in the equations, a time increment of $\delta = 0.05$ was used. The results are displayed in Fig. F1. The actual error

$$\Delta G = G_{calc} - G_{exact} \tag{F-5}$$

is shown in the upper curve. This is to be compared with G_{exact} computed using Eq. B1-3. The time interval $0 \leq t \leq 1$ is in the usual reduced units of $(2L/\pi)^2 \chi^{-1}$ where 2L is

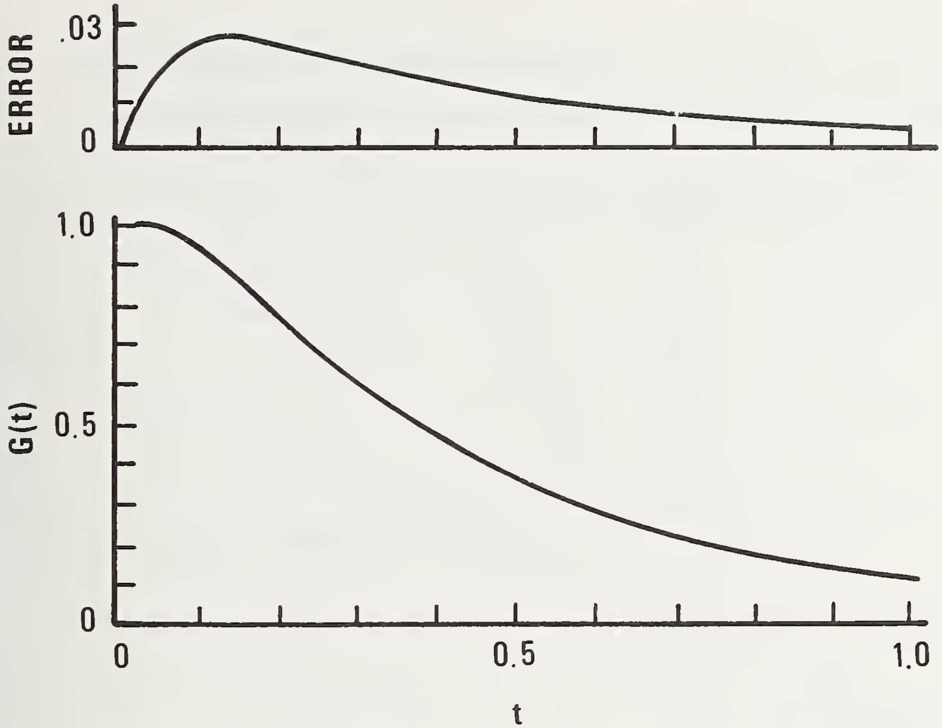

Fig. F1. Lower curve: The response function G(t) for a slab sensor from Eq. B1-3. Upper curve: the error in G(t) determined using Eqs. F-2 and F-3.

the thickness and χ is the thermal diffusivity of the plate. We see that if T(t) and $T^e(t)$ are accurately known, then G(t) can be extracted with reasonable accuracy using the discrete version of Eq. 1 embodied in Eqs. (F-2) and (F-3). If $T^e(t)$ changes slowly compared with G(t), then it's procedure will not be accurate. To see this consider Eq. (F-2); slow changes in $T^e(t)$ mean that the difference $T_1^e - T_1$ will be small with a resulting loss in precision which will lead to an inaccurate estimation of G(t). This is a point which should be checked with model calculations of the type described here before this procedure is applied to other problems.

A further note of caution is in order. The algebraic solution to Eq. (F-1) for T_j^e in terms of the $\{T_i\}$ and $\{G_i\}$ is readily found to be

$$T_{j+1}^e = 4(T_j^e - T_j) + T_{j-1}^e - 2 \sum_{i=1}^{j-1} G_{j-i} (T_{i+1}^e - T_{i-1}^e) \quad j \geq 1 \qquad \text{(F-6)}$$

where $G_0=1$, $T_{-1}^e=0$ and the value of T_1^e is specificied. The quantities $\{T_j\}$ and $\{G_j\}$ are assumed to be known. Unfortunately, this way of estimating T^e is numerically unstable. To illustrate this we use the example considered above. The exact values of the set $\{G_j\}$ for a plate of thickness 2L and of the set $\{T_j\}$ when Eq. (F-4) applies have been used infer a set of $\{T_j^e\}$ values using Eq. (F-6) and a time increment of $\delta = 0.05$. The results are displayed in Fig. B-2. The dashed curves are the estimates of $T^e(t)$ obtained using Eq. (F-6) with T_i^e and with T_1^e, T_2^e and T_e^3 specificied by Eq. (F-4).

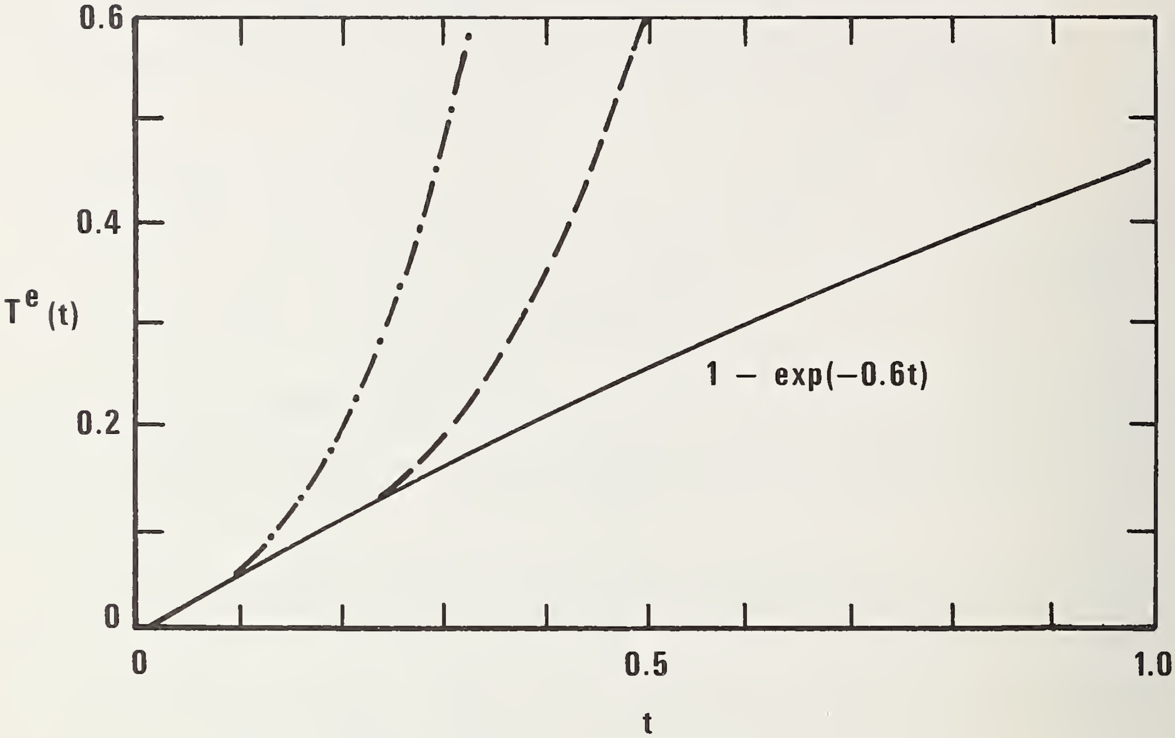

Fig. F-2. An illustration of the unsuitability of Eq. F-6 for inferring external temperatures. The solid curve is the correct result for the example considered and the dashed curves are obtained using F-6 with different numbers of points constrained to be on the correct curve.

These results indicate that $T^e(t)$ cannot be constructed from a knowledge of G(t) and T(t) by this procedure. As suggested in Section I, some additional constraints on the form of $T^e(t)$ are needed.

The response of a sensor to various frequency components is a complicated function of the frequency. This can be seen by considering the case of a sinusoidally varying external temperature

$$T^e(t) = T_0 \sin \omega t \qquad \text{(F-7)}$$

applied to a plate. The temperature T(t) at the center of a sheet of thickness 2L is found by using Eq. (1) and G(t) for the plate which is derived in Section III-B1. The result is

$$\frac{T(t)}{T_0} = A(\omega)\sin\omega t + B(\omega)\cos\omega t + \phi(t) \qquad \text{(F-8)}$$

where $\phi(t)$ is a transient. Written out explicitly

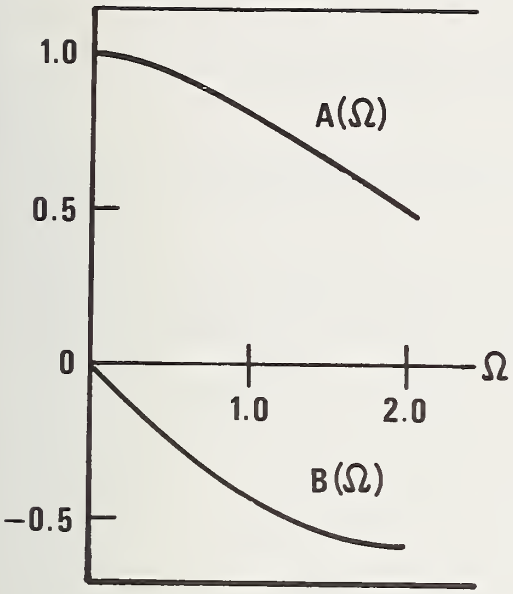

Fig. F-3. The frequency dependent amplitudes A and B in Eq. F-8 in terms of dimensionless frequency $\Omega=\omega L^2/\chi$; 2L is the plate.

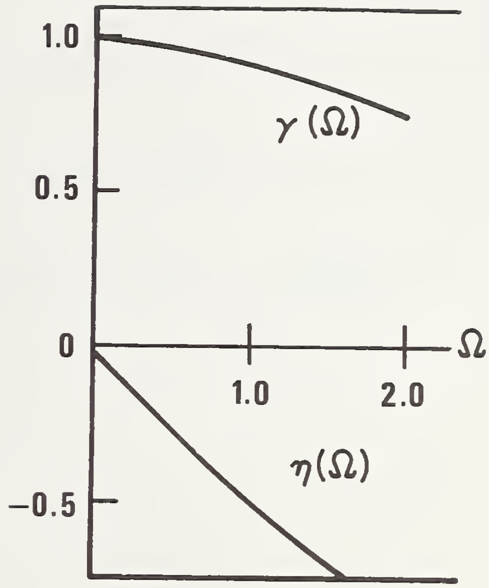

Fig. F-4. The steady state amplitude γ and phase shift η of Eq. F-12 as functions of the dimensionless frequency $\Omega=\omega L^2/\chi$; 2L is the thickness of the plate.

$$A(\omega) = 1 - \frac{4\omega^2}{\pi} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)\left\{\omega^2 + \left[\frac{\chi}{L}\left(\frac{2\ell+1}{2}\pi\right)^2\right]^2\right\}}, \qquad \text{(F-9)}$$

$$B(\omega) = -\frac{\pi\chi\omega}{L^2} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}(2\ell+1)}{\left\{\omega^2 + \left[\frac{\chi}{L^2}\left(\frac{2\ell+1}{2}\pi\right)^2\right]^2\right\}} \qquad \text{(F-10)}$$

and

$$\phi(t) = \frac{\pi\chi\omega}{L^2} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}(2\ell+1)\, e^{-\frac{\chi t}{L^2}\left(\frac{2\ell+1}{2}\pi\right)^2}}{\left\{\omega^2 + \left[\frac{\chi}{L^2}\left(\frac{2\ell+1}{2}\pi\right)^2\right]^2\right\}}. \qquad \text{(F-11)}$$

The frequency dependance for $A(\omega)$ and $B(\omega)$ are shown in Fig. B-3 in terms of the reduced frequency $\Omega = \omega L^2/\chi$ for $0.2 < \Omega < 2.0$. The large ω limits of A and B are both zero. The steady state response can be expressed as

$$\frac{T(t)}{T_0} = \gamma(\omega)\sin(\omega t + \eta(\omega)). \qquad \text{(F-12)}$$

The quantities $\gamma(\omega)$ and $\eta(\omega)$ are shown in Fig. (F-4) as functions of Ω. For $\Omega = 2$, the steady state amplitude has been reduced by over 20% of the input amplitude and the phase has been shifted by 50°.

This example illustrates the difficulties involved in attempting to infer $T^e(t)$ directly from $T(t)$ and $G(t)$. The necessary information is simply not available in those data.

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16. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

Some of the conceptual problems associated with the calibration of the time response of a temperature sensor are examined in this report. The discussion is in terms of a time response function which characterizes the way a given sensor responds to changes in the temperature of its surroundings. A series of model calculations of the response function for idealized sensors are used to investigate the general features of the response functions. Important features are the sensitivity of these functions to (i) material properties of the sensor, (ii) the type of thermal coupling of the sensor with the environment and (iii) the geometry of the sensor. These features must be considered in the design of procedures for calibrating the time response of thermometers.

17. KEY WORDS *(six to twelve entries; alphabetical order; capitalize only the first letter of the first key word unless a proper name; separated by semicolons)* Calibration; response time; temperature; thermal response; thermometer; time response function.

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