PB 161582

NBS

Technical Note 81

AN EVALUATION OF KACSER'S SECOND ORDER BORN APPROXIMATION TO THE BREMSSTRAHLUNG DIFFERENTIAL CROSS SECTION

U. S. DEPARTMENT OF COMMERCE
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JUNE 1961

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An Evaluation of Kacser's Second Order Born Approximation to the Bremsstrahlung Differential Cross Section

George S. Ofelt

The second order term, as derived by C. Kacser [Proc. Roy. Soc. A253 (1959)], of the Born approximation series for the bremsstrahlung cross section differential with respect to photon energy, photon direction, and final electron direction averaged over initial and summed over final polarization states has been put in a form that admits to numerical evaluation for the coplanar case. The results are valid for relativistic as well as non-relativistic incident electrons.

Completely general (non-coplanar) expressions are included for the first order (Bethe-Heitler) and second order terms. However, the latter has not been given in a form admissible to numerical evaluation.

Tabulated values are given for the case of the incident electron kinetic energy = 500 kev, photon energy = 450 kev, and photon direction 20° from the incident electron direction. For this case the second order term increases the differential cross section by about 50 percent for Al and about 200 percent for Au for the various angles of electron emission. A rough approximation to the integrated (over final electron direction) cross section is included and a comparison is made with experimental values. The inclusion of the second order term brings the theoretical cross section closer to the experimental values.

1. Introduction

Measurements of the bremsstrahlung cross section, differential with respect to photon energy and direction in the region where the incident electron's kinetic energy is of the order of the electron rest energy, show the predictions of the first order Born approximation theory to be too low by a considerable factor [1,2]. This discrepancy increases with increasing atomic number of the scattering material and also with increase of photon energy for a given incident electron energy. The inclusion of screening effects does not remove the discrepancy. At extreme relativistic incident electron energies the effect is reversed--Born approximation theory overestimates the experimental values. Therefore, the second order term of the Born approximation series should be evaluated.

The Born approximation is assumed to be valid for the following simultaneous conditions on the initial, β_o, and final, β, electron velocity ratio to the velocity of light:

$$\frac{2\pi\alpha Z}{\beta_o} << 1 \text{ and } \frac{2\pi\alpha Z}{\beta} << 1$$

which are not satisfied for high values of Z. However, the first order Born theory gives at least order of magnitude estimates of the cross section for high values of Z. Therefore a second order calculation might be expected to give reasonable results.

Kacser [3] has calculated the second order term of the Born approximation series for the differential cross section with no screening corrections which is averaged over initial and summed over final polarization states.

The purpose of this paper is to give the second order term, as derived by Kacser, in a form that is readily calculable. In particular only the coplanar case--final electron momentum vector lying in the plane defined by the initial electron momentum and the photon momentum vectors-is considered.

The cross section obtained is differential with respect to photon energy, photon direction, and final electron direction. Therefore an integration over the final electron direction's solid angle would have to be performed in order to compare the contribution of this second order term with existing experimental results. However, this cannot be done easily since only the coplanar case is reduced to a reasonable form. A rough indication of the magnitude of this is given in section 8 (for the tabulated case of section 7).

2. Symbols, Constants, and Energy-Momentum Relations

The system of units and symbols used follow those of Koch and Motz [2] with new symbols being introduced as needed.

T_o = incident electron kinetic energy in mc^2 units.

E_o = incident electron total energy in mc^2 units.

E = final electron total energy in mc^2 units.

k = photon energy in mc^2 units.

$\vec{P}_o$ = incident electron momentum in mc units.

$\vec{P}$ = final electron momentum in mc units.

$\vec{k}$ = photon momentum in mc units.

$\vec{\tau} \equiv \vec{P}_o - \vec{k}$

$\vec{q} \equiv \vec{\tau} - \vec{p}$ momentum transfer to nucleus.

θ_o = angle between $\vec{P}_o$ and $\vec{k}$ in general case.

φ_o = angle between " and " in coplanar case.

θ = angle between $\vec{P}$ and " in general case.

φ = angle between " and " in coplanar case.

ϕ = angle between the planes $(\vec{P}_o, \vec{k})$ and $(\vec{P}, \vec{k})$.

$\chi \equiv \varphi_o - \varphi$.

$d\Omega_k$ = element of solid angle in direction of $\vec{k}$.

$d\Omega_p$ = element of solid angle in direction of $\vec{P}$.

$r_o = 2.82 \times 10^{-13}$ cm classical electron radius.

$\alpha = \frac{1}{137}$ fine structure constant.

Z = atomic number of target material.

$m_o c^2$ = .511 Mev electron rest energy.

$E_o = T_o + 1$.

$E = E_o - k$.

$P_o = (E_o^2 - 1)^{1/2}$.

$P = (E^2 - 1)^{1/2}$.

$\beta_o = \frac{P_o}{E_o}$ initial electron velocity's ratio to velocity of light.

$\beta = \frac{P}{E}$ final electron velocity's ratio to velocity of light.

Symbols Used Only in Section 5-A

n^μ = unit time like 4 vector (0, 0, 0, 1).

$$g^{\mu\nu} = \begin{cases} 1 & \mu=\nu=1,2,3. \\ 0 & \mu\neq\nu \\ -1 & \mu=\nu=4. \end{cases}$$

$P_o = (\vec{P}_o, E_o)$

P_o^i = ith component of $\vec{P}_o$.

$(P_o P) = (\vec{P}_o \cdot \vec{P} - E_o E)$ scalar product.

$(\widehat{P_o P}) = (\vec{P}_o \cdot \vec{P} + E_o E)$.

$(P_o n) = -E_o$.

Symbols Defined in Text	Equation No.
ψ_1	3.2
ψ_2	5.1
χ	5.2
T^{ij}	5.3
J_{ij}	5.4
T_1^{ij}, T_2^{ij}	5.5, 5.6
J_{ij}^1, J_{ij}^2, J_{ij}^3	5.7, 5.8, 5.9
$\vec{R}$	5.10
$\vec{S}$	5.11
$\vec{X}$	5.12
Y	5.13
ω_1, ω_2, ω_3,	5.15
A_{ij}, B_{ij}	5.17, 5.18
a, b, c, d	5.23
w, x, y, z; $\overline{w}$, $\overline{x}$, $\overline{y}$, $\overline{z}$; L, M; $\overline{L}$, $\overline{M}$	5.29
C_1, C_2, C_3	5.33-5.35
C^1, C^2, C^3	5.37-5.39

3. General Formulation for the Differential Cross Section Formula

The Born approximation development of the differential cross section in a power series of αZ can be expressed by the following:

$$d\sigma = [\,\psi_1 + Z\psi_2 + \cdots \text{H.T.}\,]\,Z^2 \frac{dk}{k}\, d\Omega_k\, d\Omega_P \tag{3.1}$$

where the ψ's are defined to be independent of Z. Therefore the ψ's are functions of E_o, k, θ_o, θ, and $\emptyset$

$$\psi_1 = \frac{d\sigma_1}{Z^2 \frac{dk}{k}\, d\Omega_k\, d\Omega_p} \tag{3.2}$$

where $d\sigma_1$ is the Bethe-Heitler cross section (first order) term, then ψ_2 is second order in relation to ψ_1 and is the quantity to be calculated in this paper.

4. The First Order Term (Bethe-Heitler)

A. General Case

This term is given in the same form as in reference (2).

$$\psi_1 = \frac{\alpha r_o^2}{(2\pi)^2}\frac{P}{P_o q^4}\left\{\frac{P^2\sin^2\theta}{(E-P\cos\theta)^2}(4E_o^2-q^2)\right.$$

$$+\frac{P_o^2\sin^2\theta_o}{(E_o-P_o\cos\theta_o)^2}(4E^2-q^2)$$

$$+\frac{2}{(E-P\cos\theta)(E_o-P_o\cos\theta_o)}\Big[k^2(P^2\sin^2\theta+P_o^2\sin^2\theta_o)$$

$$\left.-PP_o\sin\theta\sin\theta_o((4EE_o-q^2)+2k^2)\cos\phi\Big]\right\} \qquad (4.1)$$

where

$$q^2=\{P_o^2+P^2+k^2-2P_ok\cos\theta_o+2Pk\cos\theta$$

$$-2P_oP(\cos\theta_o\cos\theta+\sin\theta_o\sin\theta\cos\phi)\} \qquad (4.2)$$

$\vec{k}$ is taken as the polar axis and the angle ϕ is determined by which side of $\vec{k}$ the projection of $\vec{P}$ on the $\vec{P}_o$, $\vec{k}$ plane lies. $\frac{\pi}{2} > \phi \geq 0$ if the projection of $\vec{P}$ lies on the same side of $\vec{k}$ as $\vec{P}_o$ and $\pi \geq \phi > \frac{\pi}{2}$ if on the opposite side; $(\pi \geq \theta_o \geq 0,\ \pi \geq \theta \geq 0)$.

B. The Coplanar Case, $\vec{P}_o \cdot (\vec{P}\times\vec{k})=0$

In this case ϕ is either zero or π. We define the angles φ_o and φ in the photon emission plane as the angles between $\vec{P}_o$ and $\vec{k}$, and $\vec{P}$ and $\vec{k}$ respectively $(2\pi \geq \varphi_o \geq 0,\ 2\pi \geq \varphi \geq 0)$. Therefore equation 4.1 becomes:

$$\psi_1 = \frac{\alpha}{(2\pi)^2} r_o^2 \frac{P}{P_o q^4}\left\{\frac{P^2\sin^2\varphi}{(E-P\cos\varphi)^2}(4E_o^2-q^2)\right.$$

$$+\frac{P_o^2\sin^2\varphi_o}{(E_o-P_o\cos\varphi_o)^2}(4E^2-q^2)$$

$$+\frac{2}{(E-P\cos\varphi)(E_o-P_o\cos\varphi_o)}\Big[k^2(P^2\sin^2\varphi+P_o^2\sin^2\varphi_o)$$

$$\left.-P_oP\sin\varphi_o\sin\varphi((4E_oE-q^2)+2k^2)\Big]\right\} \qquad (4.3)$$

and 4.2 becomes:

$$q^2 = \left\{P_o^2+P^2+k^2-2P_o k\cos\varphi_o+2Pk\cos\varphi-2P_oP\cos\chi\right\} \tag{4.4}$$

5. The Second Order Term

A. General Case

In the units used here Kacser's second order term* is given by

$$\psi_2 = \pi\alpha\left(\frac{E}{P}-\frac{E_o}{P_o}\right)\psi_1+\frac{\alpha^2}{2\pi}r_o^2\frac{Pk^2}{P_oq^2}\chi \tag{5.1}$$

where ψ_1 is the first order quantity defined in section 4 and χ is defined

$$\chi = \left[\frac{2}{\pi^3}T^{ij}J_{ij}+\frac{1}{(Pk)}\vec{R}\cdot\vec{X}-\frac{1}{(P_ok)}\vec{S}\cdot\vec{Y}\right] \tag{5.2}$$

In order to facilitate calculations in the coplanar case T^{ij} and J_{ij} are defined in several parts

$$T^{ij}=T_1^{ij}+T_2^{ij} \tag{5.3}$$

$$J_{ij}=J_{ij}^1+J_{ij}^2+J_{ij}^3 \tag{5.4}$$

where,

$$T_1^{ij}=g^{ij}\left\{(\widehat{PP_o})\left[\frac{(nP)}{(Pk)}-\frac{(nP_o)}{(P_ok)}\right]-(nk)\left[\frac{1}{(Pk)}+\frac{1}{(P_ok)}\right]\right\} \tag{5.5}$$

$$T_2^{ij}=P_o^ik^j\frac{(Pn)}{(Pk)}+P^ik^j\frac{(P_on)}{(P_ok)}-P^iP^j\frac{(nP_o)}{(Pk)}$$

$$+P_o^iP_o^j\frac{(Pn)}{(P_ok)}-P^iP_o^j\left[\frac{(nP_o)}{(Pk)}-\frac{(Pn)}{(P_ok)}\right] \tag{5.6}$$

$$J_{ij}^1=-\frac{\pi^3}{4(\vec{P}_o\times\vec{P})^2}\Bigg[\Bigg\{\left(\frac{((\vec{P}_o\times\vec{P})\times\vec{P}_o)_i}{P_o}\;\frac{(P_o\tau_j+\tau P_{oj})}{\tau(P_o\tau+\vec{P}_o\cdot\vec{\tau})}+i\leftrightarrow j\right)$$

$$+\frac{(\vec{P}_o\times\vec{P})\cdot(\vec{P}_o\times\vec{\tau})}{P_o}\left(\frac{P_o(\tau^2\delta_{ij}-\tau_i\tau_j)}{\tau^3(P_o\tau+\vec{P}_o\cdot\vec{\tau})}-\frac{(P_o\tau_i+\tau P_{oi})(P_o\tau_j+\tau P_{oj})}{\tau^2(P_o\tau+\vec{P}_o\cdot\vec{\tau})^2}\right)\Bigg\}$$

$$+\{\text{with } \vec{P}_o\leftrightarrow\vec{P}\}-\{\vec{\tau}\rightarrow\vec{q}\}-\{\vec{P}_o\leftrightarrow\vec{P};\vec{\tau}\rightarrow\vec{q}\}\Bigg] \tag{5.7}$$

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*c f. the equations following equation 50 of reference (3).

$$J^2_{ij} = -\frac{\pi^3}{4(\vec{P}_o x\vec{P})^2}\left[\left\{\frac{(\vec{\tau}\cdot\vec{P}_o x\vec{P})^2(\tau^2\delta_{ij}-\tau_i\tau_j)}{2\tau^3(P\tau+\vec{P}\cdot\vec{\tau})(P_o\tau+\vec{P}_o\cdot\vec{\tau})}\right\}\right.$$

$$\left.+\{\vec{P}_o\leftrightarrow\vec{P}\}-\{\vec{\tau}\rightarrow\vec{q}\}-\{\vec{P}_o\leftrightarrow\vec{P};\vec{\tau}\rightarrow\vec{q}\}\right] \qquad (5.8)$$

$$J^3_{ij} = -\frac{\pi^3}{4(\vec{P}_o x\vec{P})^2}\left[\left\{\frac{-(P_oP\tau+P_o\vec{P}\cdot\vec{\tau}+P\vec{P}_o\cdot\vec{\tau}+\tau\vec{P}_o\cdot\vec{P})}{2(P_oP+\vec{P}_o\cdot\vec{P})^2(P\tau+\vec{P}\cdot\vec{\tau})^2(\tau P_o+\vec{\tau}\cdot\vec{P}_o)^2}\right.\right.$$

$$x\left[\frac{1}{\tau}(P_oP+\vec{P}_o\cdot\vec{P})(\vec{\tau}x((\vec{P}_o x\vec{P})x\,\vec{\tau}))_i+(\vec{\tau}x((\vec{P}_o x\vec{P})x(P_o\vec{P}+P\vec{P}_o)))_i\right]$$

$$\left.\left. x\left[i\rightarrow j\right]\right\}+\left\{\vec{P}_o\leftrightarrow\vec{P}\right\}-\left\{\vec{\tau}\rightarrow\vec{q}\right\}-\left\{\vec{P}_o\leftrightarrow\vec{P};\vec{\tau}\rightarrow\vec{q}\right\}\right] \quad (5.9)$$

$$\vec{R} = \frac{1}{2(P_oq+\vec{P}_o\cdot\vec{q})}\left(\frac{\vec{P}_o}{P_o}+\frac{\vec{q}}{q}\right) \qquad (5.10)$$

$$\vec{S} = -\frac{1}{2(\vec{P}x\vec{\tau})^2}\left\{\frac{1}{\tau}\vec{\tau}x(\vec{P}x\vec{\tau})-\frac{1}{q}\vec{q}x(\vec{P}x\vec{q})\right\} \qquad (5.11)$$

$$\vec{X}=\vec{P}\left[\frac{(P_on)}{(Pk)}+\frac{(kn)}{(P_ok)}+2(P_on)+\frac{(P_oP)(P_on)}{(P_ok)}+(Pn)\right]$$

$$+\vec{P}_o\left[\frac{(P_on)}{(Pk)}+\frac{(kn)}{(P_ok)}+(P_on)+\frac{(P_oP)(P_on)}{(P_ok)}-\frac{(kP)(kn)}{(P_ok)}\right]$$

$$+\vec{k}\left[\frac{(P_on)}{(Pk)}-2\frac{(Pn)}{(P_ok)}+(P_on)-\frac{(P_oP)(Pn)}{(P_ok)}\right] \qquad (5.12)$$

$$\vec{Y}=\vec{X}\,\Big|_{P_\mu\leftrightarrow P_{o\mu}\ k_\mu\leftrightarrow -k_\mu} \qquad (5.13)$$

B. The Coplanar Case, $\vec{P}_o\cdot(\vec{P}x\vec{k})=0$

The notation used here is the same as that for the coplanar case of the first order term (sec. 4-B).

It can be shown that

$$T^{ij}_1 J^1_{ij} = 0$$

$$J^2_{ij} = 0$$

$$(T^{ij}_1 + T^{ij}_2)J^3_{ij} = 0$$

Therefore equation 5.2 becomes

$$\chi = \left[\frac{2}{\pi^3} T_2^{ij} J_{ij}^1 + \frac{1}{(Pk)} \vec{R} \cdot \vec{X} - \frac{1}{(P_o k)} \vec{S} \cdot \vec{Y} \right] \tag{5.14}$$

For convenience the following are defined

$$\omega_1 = \frac{2}{\pi^3} T_2^{ij} J_{ij}^1$$

$$\omega_2 = \frac{1}{(Pk)} \vec{R} \cdot \vec{X}$$

$$\omega_3 = - \frac{1}{(P_o k)} \vec{S} \cdot \vec{Y} \tag{5.15}$$

then

$$\chi = \omega_1 + \omega_2 + \omega_3 \tag{5.16}$$

i. Formulae for ω_1

Two new terms A_{ij} and B_{ij} are defined from equation 5.7:

$$A_{ij} = \frac{1}{(\vec{P}_o x \vec{P})^2} \left[\left\{ \frac{1}{P_o} \left[(\vec{P}_o x \vec{P}) x \vec{P}_o \right]_i \frac{(\tau P_{oj} + \tau_j P_o)}{\tau (P_o \tau + \vec{P}_o \cdot \vec{\tau})} + i \leftrightarrow j \right\} \right.$$

$$\left. + \{ \vec{P}_o \leftrightarrow \vec{P} \} - \{ \vec{\tau} \rightarrow \vec{q} \} - \{ \vec{P}_o \leftrightarrow \vec{P}; \vec{\tau} \rightarrow \vec{q} \} \right] \tag{5.17}$$

$$B_{ij} = \frac{1}{(\vec{P}_o x \vec{P})^2} \left[\left\{ (\vec{P}_o x \vec{P}) \cdot (\vec{P}_o x \vec{\tau}) \left[\frac{(\tau^2 \delta_{ij} - \tau_i \tau_j)}{\tau^3 (P_o \tau + \vec{P}_o \cdot \vec{\tau})} \right. \right. \right.$$

$$\left. \left. - \frac{(P_o \tau_i + \tau P_{oi})(P_o \tau_j + \tau P_{oj})}{P_o \tau^2 (P_o \tau + \vec{P}_o \cdot \vec{\tau})^2} \right] \right\}$$

$$\left. + \{ \vec{P}_o \leftrightarrow \vec{P} \} - \{ \vec{\tau} \rightarrow \vec{q} \} - \{ \vec{P}_o \leftrightarrow \vec{P}; \vec{\tau} \rightarrow \vec{q} \} \right] \tag{5.18}$$

Then ω_1 can be expanded, noticing that A_{ij} and B_{ij} are symmetric where as T_2^{ij} is not.

$$\omega_1 = - \frac{1}{2} \left[T_2^{11} (A_{11} + B_{11}) + (T_2^{12} + T_2^{21})(A_{12} + B_{12}) \right.$$

$$\left. + T_2^{22} (A_{22} + B_{22}) \right] \tag{5.19}$$

Equation 5.6 becomes after some algebra

$$T_2^{11} = \left[\frac{EP_o\cos\varphi_o}{E-P\cos\varphi} + \frac{E_oP\cos\varphi}{E_o-P_o\cos\varphi_o} + \frac{(P_o\cos\varphi_o+P\cos\varphi)}{K} \left\{ \frac{EP_o\cos\varphi_o}{E_o-P_o\cos\varphi_o} - \frac{E_oP\cos\varphi}{E-P\cos\varphi} \right\} \right] \tag{5.20}$$

$$T_2^{12}+T_2^{21} = \left[\frac{EP_o\sin\varphi_o}{E-P\cos\varphi} + \frac{E_oP\sin\varphi}{E_o-P_o\cos\varphi_o} + \frac{(P_o\cos\varphi_o+P\cos\varphi)}{K} \left\{ \frac{EP_o\sin\varphi_o}{E_o-P_o\cos\varphi_o} - \frac{E_oP\sin\varphi}{E-P\cos\varphi} \right\} + \frac{(P_o\sin\varphi_o+P\sin\varphi)}{K} \left\{ \frac{EP_o\cos\varphi_o}{E_o-P_o\cos\varphi_o} - \frac{E_oP\cos\varphi}{E-P\cos\varphi} \right\} \right] \tag{5.21}$$

$$T_2^{22} = \left[\frac{(P_o\sin\varphi_o+P\sin\varphi)}{K} \left\{ \frac{EP_o\sin\varphi_o}{E_o-P_o\cos\varphi_o} - \frac{E_oP\sin\varphi}{E-P\cos\varphi} \right\} \right] \tag{5.22}$$

In order to simplify the equations for the A_{ij}'s the following are defined

$$a = [P_o\tau(\tau+P_o-K\cos\varphi_o)]^{-1}$$

$$b = [P\tau(\tau+P_o\cos\chi-K\cos\varphi)]^{-1}$$

$$c = [P_oq(q+P_o-K\cos\varphi_o-P\cos\chi)]^{-1}$$

$$d = [Pq(q-P+P_o\cos\chi-K\cos\varphi)]^{-1} \tag{5.23}$$

And

$$\tau^2 = (P_o^2+K^2-2P_oK\cos\varphi_o) \tag{5.24}$$

$$q^2 = (\tau^2+P^2+2PK\cos\varphi-2P_oP\cos\chi) \tag{5.25}$$

where equation 5.25 is the same as 4.4.

Then equation 5.17 becomes after some algebra

$$A_{11} = 2\left[\frac{(\cos\varphi_o\cos\varphi - \sin\varphi_o\sin\varphi)}{P_oP}\left(\frac{1}{\tau} - \frac{1}{q}\right)\right.$$

$$+ \sin^2\varphi_o\left\{c\left(1+\frac{K\sin\varphi_o}{P\sin\chi}\right) - a\frac{K\sin\varphi_o}{P\sin\chi}\right\}$$

$$\left. - \sin^2\varphi\left\{(d-b)\left(1+\frac{K\sin\varphi}{P_o\sin\chi}\right)\right\}\right] \qquad (5.26)$$

$$A_{12} = 2\left[\frac{(\cos\varphi\sin\varphi_o + \cos\varphi_o\sin\varphi)}{P_oP}\left(\frac{1}{\tau} - \frac{1}{q}\right)\right.$$

$$- \sin\varphi_o\cos\varphi_o\left\{c\left(1+\frac{K\sin\varphi_o}{P\sin\chi}\right) - a\frac{K\sin\varphi_o}{P\sin\chi}\right\}$$

$$\left. + \sin\varphi\cos\varphi\left\{(d-b)\left(1+\frac{K\sin\varphi}{P_o\sin\chi}\right)\right\}\right] \qquad (5.27)$$

The equation for A_{22} need not be given as it combines with B_{22} in a rather simple manner

$$(A_{22}+B_{22}) = \left[-(A_{11}+B_{11})\right.$$
$$\left. + \left\{c\left(1+\frac{K\sin\varphi_o}{P\sin\chi}\right) - a\frac{K\sin\varphi_o}{P\sin\chi}\right\} - \left\{(d-b)\left(1+\frac{K\sin\varphi}{P_o\sin\chi}\right)\right\}\right] \qquad (5.28)$$

In order to simplify the equations for the B_{ij}'s the following are defined:

$$w = (P_o+\tau)\cos\varphi_o - K$$
$$x = P_o\cos\varphi_o + \tau\cos\varphi - K$$
$$y = (P_o+q)\cos\varphi_o - P\cos\varphi - K$$
$$z = P_o\cos\varphi_o - (P-q)\cos\varphi - K$$
$$\overline{w} = (P_o+\tau)\sin\varphi_o$$
$$\overline{x} = P_o\sin\varphi_o + \tau\sin\varphi$$
$$\overline{y} = (P_o+q)\sin\varphi_o - P\sin\varphi$$
$$\overline{z} = P_o\sin\varphi_o - (P-q)\sin\varphi$$
$$L = \frac{P_o^2\sin^2\varphi_o}{\tau^2}$$

$$M = \frac{(P_o \sin\varphi_o - P\sin\varphi)^2}{q^2}$$

$$\overline{L} = \frac{P_o \sin\varphi_o (P_o \cos\varphi_o - K)}{\tau^2}$$

$$\overline{M} = \frac{(P_o \sin\varphi_o - P\sin\varphi)(P_o \cos\varphi_o - P\cos\varphi - K)}{q^2} \qquad (5.29)$$

By use of equation 5.28, only the B_{11} and B_{12} terms need be given.

$$B_{11} = \Big[a \frac{K\sin\varphi_o}{P\sin\chi} \left\{ P_o a w^2 - L \right\}$$

$$- b \left(1 + \frac{K\sin\varphi}{P_o \sin\chi} \right) \left\{ P b x^2 - L \right\}$$

$$- c \left(1 + \frac{K\sin\varphi_o}{P\sin\chi} \right) \left\{ P_o c y^2 - M \right\}$$

$$+ d \left(1 + \frac{K\sin\varphi}{P_o \sin\chi} \right) \left\{ P d z^2 - M \right\} \Big] \qquad (5.30)$$

$$B_{12} = \Big[a \frac{K\sin\varphi_o}{P\sin\chi} \left\{ P_o a w\overline{w} + \overline{L} \right\}$$

$$- b \left(1 + \frac{K\sin\varphi}{P_o \sin\chi} \right) \left\{ P b x\overline{x} + \overline{L} \right\}$$

$$- c \left(1 + \frac{K\sin\varphi_o}{P\sin\chi} \right) \left\{ P_o c y\overline{y} + \overline{M} \right\}$$

$$+ d \left(1 + \frac{K\sin\varphi}{P_o \sin\chi} \right) \left\{ P d z\overline{z} + \overline{M} \right\} \Big] \qquad (5.31)$$

ii. Formulae for ω_2

$$\omega_2 = -\frac{1}{2}\,\frac{C}{(E-P\cos\varphi)}\left\{\frac{P}{K}\left[(P_o+q)\cos\chi - P - K\cos\varphi\right]C_1\right.$$

$$+\frac{P_o}{K}\left[(P_o+q)-P\cos\chi-K\cos\varphi_o\right]C_2$$

$$\left.+\left[(P_o+q)\cos\varphi_o-P\cos\varphi-K\right]C_3\right\} \qquad (5.32)$$

where

$$C_1 = \left\{-E_o\left[3+\frac{(E_oE-P_oP\cos\chi)}{K(E_o-P_o\cos\varphi_o)}-\frac{1}{K(E-P\cos\varphi)}\right]\right.$$

$$\left.+K+\frac{1}{(E_o-P_o\cos\varphi_o)}\right\} \qquad (5.33)$$

$$C_2 = C_1+E_o+E+K\,\frac{(E-P\cos\varphi)}{(E_o-P_o\cos\varphi_o)} \qquad (5.34)$$

$$C_3 = \left\{E_o\left[\frac{1}{K(E-P\cos\varphi)}-1\right]-E\left[\frac{2-(E_oE-P_oP\cos\chi)}{K(E_o-P_o\cos\varphi_o)}\right]\right\} \qquad (5.35)$$

The c in equation 5.32 is the c defined in equations 5.23.

iii. Formulae for ω_3

$$\omega_3 = -\frac{1}{2}\,\frac{1}{K(E_o-P_o\cos\varphi_o)}\left\{\frac{\left[\left(\frac{1}{\tau}-\frac{1}{q}\right)\frac{P_oK\sin\varphi_o}{P}-\frac{P_o}{q}\sin\chi\right]C^1}{(P_o\sin\chi+K\sin\varphi)}\right.$$

$$\left.+\left(\frac{1}{\tau}-\frac{1}{q}\right)C^2-\frac{\left[\left(\frac{1}{\tau}-\frac{1}{q}\right)\frac{P_oK\sin\varphi_o}{P}+\frac{K}{q}\sin\varphi\right]C^3}{(P_o\sin\chi+K\sin\varphi)}\right\} \qquad (5.36)$$

where

$$C^1 = \left[-E\left\{3+\frac{1}{K(E_o-P_o\cos\varphi_o)}-\frac{(E_oE-P_oP\cos\chi)}{K(E-P\cos\varphi)}\right\}-K\right.$$

$$\left.+\frac{1}{(E-P\cos\varphi)}\right] \qquad (5.37)$$

$$C^2 = C^1 + E_o + E - \frac{K(E_o - P_o \cos\varphi_o)}{(E - P\cos\varphi)} \tag{5.38}$$

$$C^3 = \left[-E \left\{ 1 + \frac{1}{K(E_o - P_o \cos\varphi_o)} \right\} + E_o \left\{ \frac{2 - (E_o E - P_o P\cos\chi)}{K(E - P\cos\varphi)} \right\} \right] \tag{5.39}$$

6. Discussion of the Second Order Coplanar Equation

It should be noted that there occur several points in the equations for the second order term that are of the form $\frac{0}{0}$ (for example, when $\sin\chi = 0$ in the equation for ω_1 (5.19) and when $P_o \sin\chi + k \sin\varphi = 0$ in the equation for ω_3 (5.36)). However, the value of ψ_2 is finite at each of these points.

The calculation of the limiting values is rather involved and will not be given here.

Tabulated Values for a Specific Case

We consider the following case where there is a considerable difference between the first order theory and experimental values. Thus the second order term would be expected to be significant.

$$T_o = 500 \text{ kev}$$

$$k = 450 \text{ kev}$$

$$\varphi_o = 20°$$

TABLE I

φ	$\psi_1 \frac{mb}{ster/ster}$	$\psi_2 \frac{mb}{ster/ster}$	$\frac{kd\sigma \text{ Al}}{Z^2 dk d\Omega_p d\Omega_k}$	$\frac{kd\sigma \text{ Au}}{Z^2 dk d\Omega_p d\Omega_k}$
80°	.050	.0016	.071	.180
60°	.125	.0043	.178	.449
40°	.280	.0090	.397	.991
25°	.357	.0115	.506	1.26
15°	.355	.0114	.503	1.26
0°	.295	.0095	.419	1.05
-20°	.195	.0062	.276	.687
-60°	.075	.0025	.108	.273

Values for ψ_1 and ψ_2 are plotted in Figure 1.

8. Summary

The contribution to the differential cross section by the second order term for the case calculated in section 7 leads to an increase of about 50% for Al and about 200% for Au as would be expected as discussed in section 1. However, these values cannot be compared with existing experimental data since these data are for the integrated (over finial electron direction) cross sections.

Since the addition from the second order term (at least for the above case) is so large, it appears that the third or even higher order terms should be calculated or estimated as to magnitude and algebraic sign. The case might be that the first order term of the series gives a closer approximation to the limiting value of the series than the sum of the first several terms. Or, conversely, the third and higher order terms might make a negligible contribution.

Therefore, since at the present time a third order term has not been calculated and no experimental data for the cross section differential with respect to both final electron and photon directions are available, it cannot be said exactly what the limitations on this second order contribution are.

Although, as pointed out above, the values obtained in section 7 cannot be compared to experimental data, a rough approximation to the integrated cross-section for this case <u>only</u> can be made. It is noticed that even in the non-coplanar case ψ_2 is the sum of two parts (see equation 5.1), one involving X(dependent on the final electron direction), the other, a multiplicative factor of ψ_1, independent of angles i.e. $\pi\alpha\left(\frac{E}{P} - \frac{E_o}{P_o}\right)$ which is always positive. In the case tabulated the former is approximately 1/10 of the latter for all final electron angles. Therefore <u>if</u> it is assumed that the relationship between the two does not change much as the plane containing $\vec{P}$ is rotated about $\vec{P}_o$ then the integral of ψ_2 over final electron direction

$$\int \psi_2 d\Omega_p = \left\{ \pi \alpha \left(\frac{E}{P} - \frac{E_o}{P_o} \right) \int \psi_1 d\Omega_p + \right.$$

$$\left. + \frac{\alpha^2}{2\pi} r_o^2 \frac{Pk^2}{P_o} \int \frac{X}{q^2} d\Omega_p \right\} \tag{8.1}$$

is the sum of two positive quantities that are approximately in the ratio of 10 to 1.

Therefore $\int \psi_2 d\Omega_p > \pi \alpha \left(\frac{E}{P} - \frac{E_o}{P_o} \right) \int \psi_1 d\Omega_p$ (8.2)

Then from equation 3.1

$$\int d\sigma \, d\Omega_p = Z^2 \frac{dk}{k} d\Omega_k \int (\psi_1 + Z\psi_2) d\Omega_p \tag{8.3}$$

Substitution of 8.2 into 8.3 gives

$$\int d\sigma d\Omega_p > \left\{ Z^2 \frac{dk}{k} d\Omega_k \int \psi_1 d\Omega_p \right\} \left\{ 1 + \pi \alpha \, Z \left(\frac{E}{P} - \frac{E_o}{P_o} \right) \right\} \tag{8.4}$$

Let

$$d\sigma(1st) = Z^2 \frac{dk}{k} d\Omega_k \int \psi_1 d\Omega_p$$

$$d\sigma(1st + 2d) = Z^2 \frac{dk}{k} d\Omega_k \int (\psi_1 + Z\psi_2) d\Omega_p$$

$d\sigma_{exp}$ = the experimental value for the integrated cross section.

Then 8.4 becomes upon substitution of numerical values

$$d\sigma(1st + 2d) > 1.029 \, Z \, d\sigma(1st) \tag{8.5}$$

In reference (1) experimental values are given for T_o = 500 kev, k = 450 kev; Θ_o = 0°, 30°, 60°, and 90° for Z = 13 and 79, with and estimated error within 20%. Interpolation for Θ_o = 20° gives approximately:

TABLE II

	Z = 13	Z = 79
$\frac{k}{Z^2} \frac{d\sigma}{dk d\Omega_k}$ exp	$2.0 \pm 0.4 \frac{mb}{ster}$	$3.2 \pm 0.6 \frac{mb}{ster}$
$\frac{d\sigma}{d\sigma\,(1st)}$ exp	4.0 ± 0.8	6.4 ± 1.3
$\frac{d\sigma(1st + 2d)}{d\sigma(1st)}$	> 1.38	> 3.29
$\frac{d\sigma}{d\sigma(1st + 2d)}$ exp	$< 2.9 \pm 0.6$	$< 1.9 \pm 0.4$

therefore, even with this crude approximation of the integrated cross section, the second order term brings the Born approximation into better agreement with existing experimental data, (at least for the particular case as tabulated in section 7).

References

(1). J. W. Motz, Phys. Rev. 100, pg. 1560 (1955).

(2). H. W. Koch and J. W. Motz, Rev. M. Physics, 31, pg. 920 (1959).

(3). C. Kacser, Proc. Roy. Soc., A253, pg. 103 (1959).

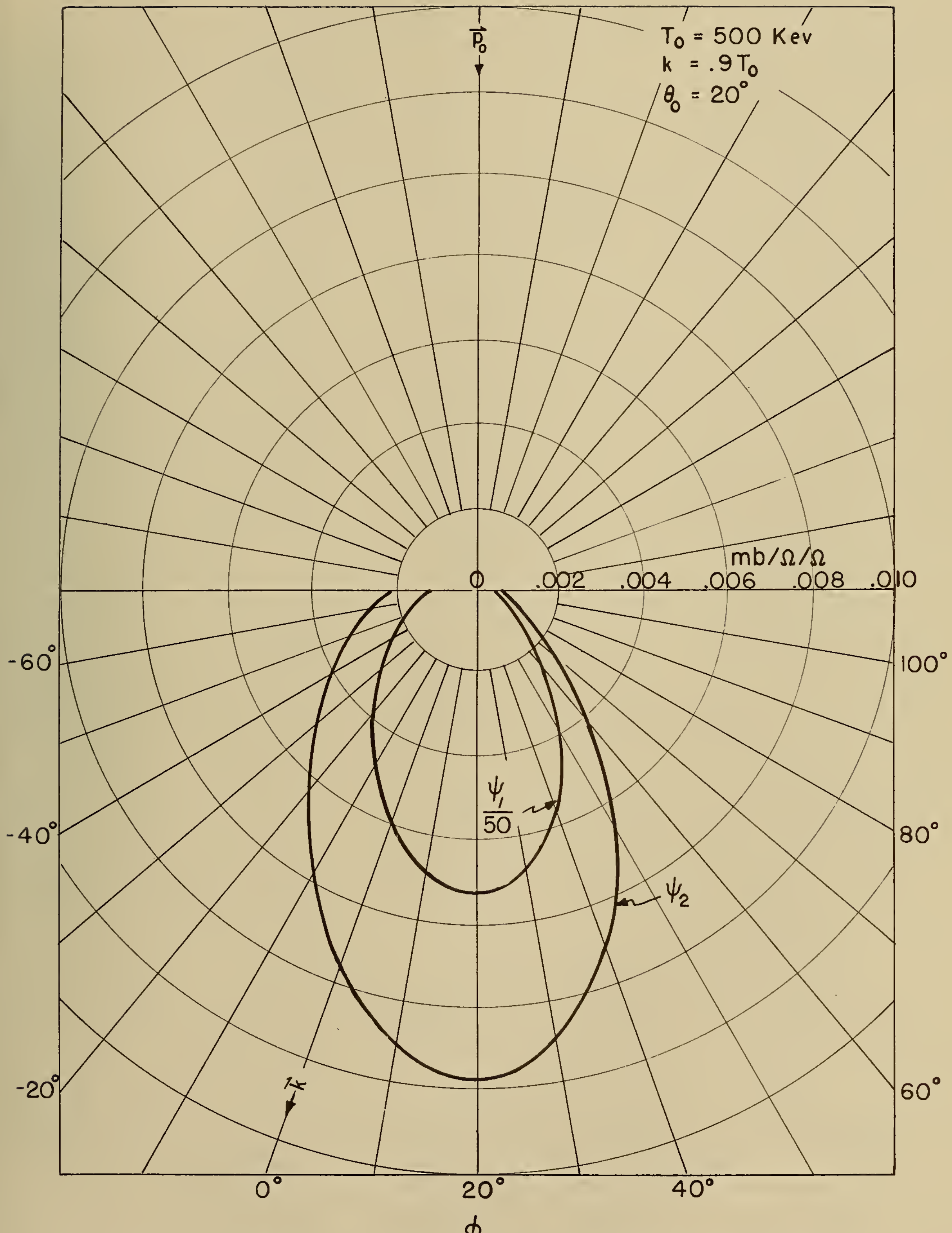

Fig. 1. Variation of Ψ_1 and Ψ_2 with angle, φ, from the data of TABLE I.

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