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Electromagnetic Multipole Transitions in the Recoupling Picture, or Electron Scattering Without Curls

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Electromagnetic Multipole Transitions in the Recoupling Picture, or Electron Scattering Without Curls

J. S. O'Connell

The formation of the multipole operators from the fundamental charge, current, and moment operators of electron-nucleus scattering is carried through as an angular momentum recoupling problem rather than the usual vector algebra derivation. This point of view allows the electric and magnetic transition operators to be obtained in a simple and intuitive manner. The single particle reduced matrix elements of the charge, current, and moment operators are then calculated in the j-j coupling scheme using a flow chart technique developed by Danos.

Key Words: Electric transitions; electromagnetic operators; electron scattering; magnetic transitions; multipoles; recoupling.

I. Introduction

The theory of electromagnetic multipole transitions is usually developed in the language of vector algebra: gradients, divergences, and curls. This approach evolved from the historic connection between 19th century mathematics and the study of the classical electric and magnetic fields. An alternative approach to the multipole transitions in quantum systems is to regard them as essentially angular momentum recouplings. The quantized rotation theory of Wigner, Racah, and Fano is then brought to bear on not just the electromagnetic operators, but upon the whole matrix element, i.e., the spatial integral over operator and initial and final states of the absorbing or emitting quantum system. This type of analysis is most easily performed using a diagramatic method recently introduced by Danos [1].[1]

In the next section the electron scattering cross section for a discrete nuclear transition is developed in terms of the single particle operator. The reduced matrix elements of the charge, current, and moment operators are calculated in Section III in the j-j coupling scheme. In Section IV a simple example of an M1 transition in ^{12}C is presented. Section V summarizes the advantages of the recoupling approach to multipole transitions.

II. Multipole Expansion of the Fundamental Operators

Electrons interact with the nucleus through the charge, current, and moment operators on each nucleon $(e, e\vec{p}/M, i\mu\vec{\sigma} \times \vec{q}/2M)e^{i\vec{q}\cdot\vec{r}}$, where e=1 for a proton and 0 for a neutron, μ is the nucleon magnetic moment ($\mu_p = 2.79$, $\mu_n = -1.91$), $\vec{p}$ is the moment operator, and $\vec{\sigma}$ the Pauli spin operator acting on the nucleon at position $\vec{r}$. The three momentum transferred to the nucleus is $\vec{q}$. For a nuclear transition from an initial state of angular momentum and parity J_i^π to a final state of J_f^π the electron scattering cross section for one photon exchange neglecting the mass of the electron is given by

$$\frac{d\sigma}{d\Omega} = \sigma_o \frac{1}{2J_i+1} \sum_{M_f, M_i}$$

$$\left\{ \frac{q_\mu^4}{\vec{q}^4} \left|\langle J_f M_f \right| \sum_j e_j e^{i\vec{q}\cdot\vec{r}_j} \left| J_i M_i \rangle \right|^2 + \left(\frac{q^2}{2\vec{q}^2} + \tan^2\theta/2 \right) \right.$$

$$\left. \times \sum_{\lambda = \pm 1} \left| \langle J_f M_f \right| \sum_j \left[e_j e^{i\vec{q}\cdot\vec{r}_j} \frac{(p_j)_\lambda}{M} + \frac{i\mu_j}{2M} e^{i\vec{q}\cdot\vec{r}_j} (\vec{\sigma}_j \times \vec{q})_\lambda \right] \left| J_i M_i \rangle \right|^2 \right\}$$

[1]Figures in brackets indicate the literature references at the end of this paper.

where

$$\sigma_o = \left[\frac{e^2 \cos\theta/2}{2\mathcal{E}_i \sin^2\theta/2}\right]^2 \left(1 + \frac{2\mathcal{E}_i \sin^2\theta/2}{M_T}\right)^{-1} \quad .$$

We present this formula without proof, but give the following explanation of the origin of some of the terms. The σ_o is the familiar Mott cross section for the scattering from a point charge by an electron of energy $\mathcal{E}_i$ times a recoil factor associated with the density of final nuclear states. The $(2J_i+1)^{-1} \sum_{M_f M_i}$ is the sum over final and average over initial nuclear orientations (with the axis of quantization taken in the $\vec{q}$ direction) since they are unobserved. The $\sum_j$ sums the single particle operators over all the nucleons in the nucleus. The first term in the $\{\ \}$ is the square of the longitudinal form factor, the second an angular function times the square of the transverse. The longitudinal component is composed of the Coulomb operator and the longitudinal component of the current operator which, since they are related by current conservation, can be written as the square of the Coulomb matrix element times $\left(1 - \frac{q_o^2}{\vec{q}^2}\right)^2 = \frac{q_\mu^4}{\vec{q}^4}$ where the four vector q_μ is related to the three momentum $\vec{q}$ and the electron energy loss $q_o = \mathcal{E}_i - \mathcal{E}_f$ by $q_\mu^2 = \vec{q}^2 - q_o^2$.

The transverse term uses only the $\lambda = \pm 1$ components of the current and moment operators decomposed along the $\vec{q}$ direction. The angular factor in front of the transverse term comes from the one photon exchange analysis.

Because we are interested in the case of a discrete transition with definite spin and parity change (and eventually definite isospin change), we are forced to make a multipole decomposition of the fundamental operators into terms of definite total angular moment J where

$|J_i - J_f| \le J \le J_i + J_f$ and furthermore to combine terms of the current and moment operators which have the same parity. For the longitudinal operator this is easily accomplished by the usual expansion formula for a plane wave into spherical Bessel functions and spherical harmonics

$$e^{i\vec{q}\cdot\vec{r}} = (4\pi)^{\frac{1}{2}} \sum_L (i)^L \hat{L}\, j_L(qr)\, Y_{L0}(\Omega_r)$$

where $\hat{L} = (2L+1)^{\frac{1}{2}}$ and the z-axis has been taken in the direction of $\vec{q}$.

In this paper we will work with the contrastandard element $Y_M^{[L]}$ which is related to the ordinary spherical harmonic by

$$Y_M^{[L]} \equiv (-i)^L\, Y_{LM} \;.$$

Thus the Coulomb matrix element becomes

$$\langle J_f M_f |\, e^{i\vec{q}\cdot\vec{r}} \,| J_i M_i \rangle$$

$$= (4\pi)^{\frac{1}{2}} \sum_J (-)^J \hat{J} \langle J_f M_f |\, j_J\, Y_0^{[J]} \,| J_i M_i \rangle \;.$$

Using the Wigner-Eckart theorem

$$= (4\pi)^{\frac{1}{2}} \sum_J (-)^{J+J_f-M_f} \hat{J} \begin{pmatrix} J_f & J & J_i \\ -M_f & 0 & M_i \end{pmatrix}$$

$$\times \langle J_f \,\|\, j_J\, Y^{[J]} \,\|\, J_i \rangle \;.$$

The current and moment operators are more complicated. They involve the coupling of an angular momentum L (from the plane wave) with one unit of angular momentum (from the vectors $\vec{p}$ or $\vec{\sigma} \times \vec{q}$) to form the total angular momentum J.

$$e^{i\vec{q}\cdot\vec{r}}\ \frac{p_\lambda}{M} = \frac{(4\pi)^{\frac{1}{2}}}{M} \sum_L (i)^L\ \hat{L}\ j_L\ Y_{L0}\ p_\lambda$$

$$= \frac{(4\pi)^{\frac{1}{2}}}{M} \sum_L (-)^{L+\frac{1}{2}}\ \hat{L}\ j_L\ Y_0^{[L]}\ p_\lambda^{[1]}$$

$$= \frac{(4\pi)^{\frac{1}{2}}}{M}\ i \sum_{L\ J} \hat{L}\ \hat{J} \begin{pmatrix} L & 1 & J \\ 0 & \lambda & -\lambda \end{pmatrix} j_L \left[Y^{[L]} \times p^{[1]}\right]_\lambda^{[J]}$$

In the last step the product of the two contrastandard elements was replaced by its expansion in terms of elements coupled to a definite angular momentum J.

Similarly, for the moment operator

$$e^{i\vec{q}\cdot\vec{r}}\ (\vec{\sigma} \times \vec{q})_\lambda = (4\pi)^{\frac{1}{2}}\ q\lambda \sum_{J\ L} \hat{L}\ \hat{J} \begin{pmatrix} L & 1 & J \\ 0 & \lambda & -\lambda \end{pmatrix} j_L \left[Y^{[L]} \times \sigma^{[1]}\right]_\lambda^{[J]}$$

where we have used $(\vec{\sigma} \times \vec{q})_\lambda = \sqrt{2}\left[\sigma^{[1]} \times q^{[1]}\right]_\lambda^{[1]} = \lambda q \sigma_\lambda^{[1]}$.

If we hold J fixed, L takes on three values: J+1, J, J-1. The parity of each current term is $\pi = (-)^{L+1}$ but each moment term is $\pi = (-)^L$ since $\vec{\sigma}$ is an axial vector. Therefore, to form a current plus moment operator of definite parity we combine the L = J±1 current terms with the L=J moment term to form an operator of parity $\pi = (-)^J$. This combination is called an electric operator $\mathcal{E}_\lambda^{[J]}$ because the classical electric multipole field has this parity. The $\pi = (-)^{J+1}$ or magnetic combination $\mathcal{M}_\lambda^{[J]}$ is formed by the sum of the L = J±1 moment terms with the L=J current term.

$$\mathcal{E}_\lambda^{[J]} = \frac{i\ (4\pi)^{\frac{1}{2}}}{M} \Big\{ \widehat{(J+1)}\ \hat{J} \begin{pmatrix} J+1 & 1 & J \\ 0 & \lambda & -\lambda \end{pmatrix} j_{J+1} \left[Y^{[J+1]} \times p^{[1]} \right]_\lambda^{[J]}$$

$$+ \quad \widehat{(J-1)}\ \hat{J} \begin{pmatrix} J-1 & 1 & J \\ 0 & \lambda & -\lambda \end{pmatrix} j_{J-1} \left[Y^{[J-1]} \times p^{[1]} \right]_\lambda^{[J]}$$

$$+ \ \frac{\mu q}{2}\ \hat{J}\ \hat{J}\ \lambda \begin{pmatrix} J & 1 & J \\ 0 & \lambda & -\lambda \end{pmatrix} j_J \qquad \left[Y^{[J]} \times \sigma^{[1]} \right]_\lambda^{[J]} \Big\}$$

$$\mathcal{M}_\lambda^{[J]} = \frac{i\ (4\pi)^{\frac{1}{2}}}{M} \Big\{ \frac{\mu q}{2}\ \widehat{(J+1)}\ \hat{J}\ \lambda \begin{pmatrix} J+1 & 1 & J \\ 0 & \lambda & -\lambda \end{pmatrix} j_{J+1} \left[Y^{[J+1]} \times \sigma^{[1]} \right]_\lambda^{[J]}$$

$$+ \quad \frac{\mu q}{2}\ \widehat{(J-1)}\ \hat{J}\ \lambda \begin{pmatrix} J-1 & 1 & J \\ 0 & \lambda & -\lambda \end{pmatrix} j_{J-1} \left[Y^{[J-1]} \times \sigma^{[1]} \right]_\lambda^{[J]}$$

$$+ \quad \hat{J}\ \hat{J} \quad \begin{pmatrix} J & 1 & J \\ 0 & \lambda & -\lambda \end{pmatrix} j_J \left[Y^{[J]} \times p^{[1]} \right]_\lambda^{[J]} \Big\}$$

Evaluating the 3-j coefficients we obtain

$$\mathcal{E}_\lambda^{[J]} = i(-)^{J-1}\ \frac{(4\pi)^{\frac{1}{2}}}{M} \Big\{ \left(\frac{J}{2}\right)^{\frac{1}{2}} j_{J+1} \left[Y^{[J+1]} \times p^{[1]} \right]_\lambda^{[J]}$$

$$+ \quad \left(\frac{J+1}{2}\right)^{\frac{1}{2}} j_{J-1} \left[Y^{[J-1]} \times p^{[1]} \right]_\lambda^{[J]}$$

$$+ \quad \frac{\mu q}{2} \quad \left(\frac{2J+1}{2}\right)^{\frac{1}{2}} j_J \left[Y^{[J]} \times \sigma^{[1]} \right]_\lambda^{[J]} \Big\}$$

$$\mathcal{M}_\lambda^{[J]} = i(-)^{(J-1)} \frac{(4\pi)^{\frac{1}{2}}}{M} \lambda \left\{ \frac{\mu q}{2} \left(\frac{J}{2}\right)^{\frac{1}{2}} j_{J+1} \left[Y^{[J+1]} \times \sigma^{[1]}\right]_\lambda^{[J]} \right.$$

$$+ \frac{\mu q}{2} \left(\frac{J+1}{2}\right)^{\frac{1}{2}} j_{J-1} \left[Y^{[J-1]} \times \sigma^{[1]}\right]_\lambda^{[J]}$$

$$\left. + \left(\frac{2J+1}{2}\right) j_J \left[Y^{[J]} \times p^{[1]}\right]_\lambda^{[J]} \right\}$$

When the electric and magnetic operators are sandwiched between nuclear states

$$\langle J_f M_f | \mathcal{E}_\lambda^{[J]} + \mathcal{M}_\lambda^{[J]} | J_i M_i \rangle = (-)^{J_f - M_f} \begin{pmatrix} J_f & J & J_i \\ -M_f & \lambda & M_i \end{pmatrix} \langle J_f \| \mathcal{E}^{[J]} + \mathcal{M}^{[J]} \| J_i \rangle .$$

Therefore, the electron scattering cross section in terms of the multipole operators can be written as

$$\frac{d\sigma}{d\Omega} = \frac{\sigma_o}{2J_i+1} \sum_{M_f M_i, J} \left\{ \frac{q_\mu^4}{\vec{q}^4} \left| \langle J_f \| \sum_j j_J Y_{(j)}^{[J]} \| J_i \rangle \right|^2 (\hat{J})^2 \begin{pmatrix} J_f & J & J_i \\ -M_f & 0 & M_i \end{pmatrix}^2 \right.$$

$$\left. + \left(\frac{q_\mu^2}{2\vec{q}^2} + \tan^2\theta/2\right) \sum_{\lambda = \pm 1} \left| \langle J_f \| \sum_j \left[\mathcal{E}_{(j)}^{[J]} + \mathcal{M}_{(j)}^{[J]}\right] \| J_i \rangle \right|^2 \begin{pmatrix} J_f & J & J_i \\ -M_f & \lambda & M_i \end{pmatrix}^2 \right\} .$$

The completeness relation for 3-j symbols allows us to perform the $M_f M_i$ sums

$$\sum_{M_f, M_i} \begin{pmatrix} J_f & J & J_i \\ -M_f & \lambda & M_i \end{pmatrix}^2 = (\hat{J})^{-2}$$

The sum over $\lambda = \pm 1$ simply gives a factor of 2 since the λ dependence was removed by the previous step. For a given J_f^{π} and J_i^{π} either the electric or the magnetic operator will connect the nuclear states for each allowed value of J since the parity requirement must be satisfied. We come to the final expression for the cross section in terms of the reduced matrix elements of the Coulomb, electric, and magnetic operators.

$$\frac{d\sigma}{d\Omega} = \frac{\sigma_o}{2J_i+1} \sum_J \left\{ \frac{q_\mu^4}{\vec{q}^4} \left| \langle J_f^{\pi} \| \sum_j j_J \, Y_{(j)}^{[J]} \| J_i^{\pi} \rangle \right|^2 \right.$$

$$\left. + \frac{2}{2J+1} \left(\frac{q^2}{2\vec{q}^2} + \tan^2\theta/2 \right) \left[\left| \langle J_f^{\pi} \| \sum_j \mathcal{E}_{(j)}^{[J]} \| J_i^{\pi} \rangle \right|^2 + \left| \langle J_f^{\pi} \| \sum_j \mathcal{M}_{(j)}^{[J]} \| J_i^{\pi} \rangle \right|^2 \right] \right\}$$

We are now required to compute the reduced matrix elements of the operators

$$j_J(qr)\, Y^{[J]}, \; j_L(qr) \left[Y^{[L]} \times p^{[1]} \right]^{[J]}, \; j_L(qr) \left[Y^{[L]} \times \sigma^{[1]} \right]^{[J]}$$

This type of analysis is most easily performed using the diagramatic recoupling technique of Danos [1].

III. Single Particle Reduced Matrix Elements

The single particle states will be taken in a spherical basis in which the orbital angular momentum ℓ (in the $Y^{[\ell]}$ representation) is coupled to the spin 1/2 to a total j.

$$|\psi_m^{[j]}\rangle = R_\ell(r)\left[Y^{[\ell]} \times \chi^{[\frac{1}{2}]}\right]_m^{[j]}$$

with the Hermitian adjoint given by [2]

$$\langle\psi_m^{[j]}| = (-)^{j+m}\, R_\ell(r)\left[Y^{[\ell]} \times \tilde{\chi}^{[\frac{1}{2}]}\right]_{-m}^{[j]}$$

A convenient representation for the spin functions and their adjoint is

$$|\chi_{\frac{1}{2}}^{[\frac{1}{2}]}\rangle = \begin{pmatrix}1\\0\end{pmatrix} \;,\; |\chi_{-\frac{1}{2}}^{[\frac{1}{2}]}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\langle\chi_{\frac{1}{2}}^{[\frac{1}{2}]}| = (1\;\;0) \;,\; \langle\chi_{-\frac{1}{2}}^{[\frac{1}{2}]}| = (0\;\;1)$$

The reduced matrix element of an operator $O^{[J]}$ is related to what Danos calls a projection integral

$$\langle\psi^{[j']}\| O^{[J]} \| \psi^{[j]}\rangle = (-)^{j'+J-j}\left[\tilde{\psi}^{[j']}|O^{[J]}|\psi^{[j]}\right]$$

where the tilde means transpose which affects only the spin part of our wave functions.

A. Coulomb Operator

The projection integral of the Coulomb operator is evaluated with the aid of the recoupling diagram shown in Fig. 1 with the result

$$\left[\tilde{\psi}^{[j']}\middle| j_J Y^{[J]} \middle| \psi^{[j]}\right] = \left[R_{\ell'}[Y^{[\ell']} \times \tilde{\chi}^{[\frac{1}{2}]}]^{[j']} \middle| j_J Y^{[J]} \middle| R_\ell [Y^{[\ell]} \times \chi^{[\frac{1}{2}]}]^{[j]}\right]$$

$$\begin{bmatrix} \ell' & \frac{1}{2} & j' \\ J & 0 & J \\ \ell & \frac{1}{2} & j \end{bmatrix} \begin{bmatrix} \ell & \frac{1}{2} & j \\ \ell & \frac{1}{2} & j \\ 0 & 0 & 0 \end{bmatrix} \left[\tilde{\chi}^{[\frac{1}{2}]} \middle| \chi^{[\frac{1}{2}]}\right] \left[Y^{[\ell']} \middle| Y^{[J]} \middle| Y^{[\ell]}\right] \langle R_{\ell'} | j_J(qr) | R_\ell \rangle$$

The recoupling boxes are related to the usual 9-j symbol by

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \hat{c}\ \hat{f}\ \hat{g}\ \hat{h} \quad \begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix}$$

where $\hat{c} = (2c+1)^{\frac{1}{2}}$. The projection integral for three contrastandard spherical harmonics is given by Danos as

$$\left[Y^{[\ell_1]} \middle| Y^{[\ell_2]} \middle| Y^{[\ell_3]}\right] =$$

$$= \quad (-)^{\frac{1}{2}(\ell_1+\ell_2+\ell_3)} \frac{\hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3}{(4\pi)^{\frac{1}{2}}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}$$

The spin overlap is

$$\left[\tilde{\chi}^{[\frac{1}{2}]} \middle| \chi^{[\frac{1}{2}]}\right] = + \hat{\tfrac{1}{2}} = + 2^{\frac{1}{2}}$$

The final result is [3]

$$(i)^{\ell'-J-\ell} \langle \psi^{[j']} \| j_J Y^{[J]} \| \psi^{[j]} \rangle = (-)^{j+J+\frac{1}{2}}$$

$$\frac{\hat{J}\ \hat{j}'\ \hat{j}\ \hat{\ell}'\ \hat{\ell}}{(4\pi)^{\frac{1}{2}}} \begin{Bmatrix} \ell' & j' & \frac{1}{2} \\ j & \ell & L \end{Bmatrix} \begin{pmatrix} \ell' & J & \ell \\ 0 & 0 & 0 \end{pmatrix} \langle R_{\ell'} | j_J(qr) | R_\ell \rangle$$

which may be further simplified by use of the identity

$$\hat{\ell}\,\hat{\ell}' \begin{Bmatrix} \ell' & j' & \frac{1}{2} \\ j & \ell & J \end{Bmatrix} \begin{pmatrix} \ell' & J & \ell \\ 0 & 0 & 0 \end{pmatrix} = - \begin{pmatrix} j' & J & j \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

when $\ell' + J + \ell$ = an even integer.

As a special case the electric dipole reduced matrix element is

$$(i)^{\ell'-1-\ell} \langle \psi^{[j']} \| r^{[1]} \| \psi^{[j]} \rangle = (-)^{j+\frac{1}{2}}\, \hat{j}'\hat{j} \begin{pmatrix} j' & J & j \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \langle R_{\ell'} | r | R_{\ell} \rangle$$

B. Moment Operator

Fig. 2 shows how the spin and orbital functions are recoupled to give

$$\langle \psi^{[j']} \| j_L \left[Y^{[L]} \times \sigma^{[1]} \right]^{[J]} \| \psi^{[j]} \rangle$$

$$= \quad (-)^{j'-J+j} \begin{bmatrix} \ell' & \frac{1}{2} & j' \\ L & 1 & J \\ \ell & \frac{1}{2} & j \end{bmatrix} \begin{bmatrix} \ell & \frac{1}{2} & j \\ \ell & \frac{1}{2} & j \\ 0 & 0 & 0 \end{bmatrix}$$

$$\times \quad \left[\tilde{X}^{[\frac{1}{2}]} | \sigma^{[1]} | X^{[\frac{1}{2}]} \right] \left[Y^{[\ell']} | Y^{[L]} | Y^{[\ell]} \right]$$

$$\times \quad \langle R_{\ell'} | j_L(qr) | R_{\ell} \rangle$$

The spin projection integral is $(-)^{+\frac{1}{2}}\, \hat{\frac{1}{2}}\, \hat{1} = + i\ 6^{\frac{1}{2}}$ therefore the reduced moment matrix element is [2]

$$(i)^{(\ell' - L - 1 - \ell)} \langle \psi^{[j']} \| j_L \left[Y^{[L]} \times \sigma^{[1]} \right]^{[J]} \| \psi^{[j]} \rangle$$

$$(-)^{\ell'} \frac{6^{\frac{1}{2}} \hat{J}\ \hat{L}\ \hat{j}'\hat{j}\hat{\ell}'\ \hat{\ell}}{(4\pi)^{\frac{1}{2}}} \begin{Bmatrix} \ell' & \ell & L \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j' & j & J \end{Bmatrix} \begin{pmatrix} \ell' & L & \ell \\ 0 & 0 & 0 \end{pmatrix} \langle R_{\ell'} | j_L(qr) | R_{\ell} \rangle$$

C. Current Operator

Evaluation of the current operator requires one trick shown in Fig. 3. A complete set of intermediate angular states is introduced through the unit operator

$$\sum_k \hat{k} \left[Y^{[k]}(\Omega) \times Y^{[k]}(\Omega')\right]^{[o]} = \delta(\Omega - \Omega')$$

in order to separate the $p^{[1]}$ and $Y^{[L]}$ operators. The triangularity condition between ℓ, 1, and k in the final result limits the sum to two terms $k = \ell \pm 1$.

$$\left[\tilde{\psi}^{[j']} \middle| j_L \left[Y^{[L]} \times p^{[1]}\right]^{[J]} \middle| \psi^{[j]}\right]$$

$$= \begin{bmatrix} J & 0 & J \\ \ell & \frac{1}{2} & j \\ \ell' & \frac{1}{2} & j' \end{bmatrix} \begin{bmatrix} \ell' & \frac{1}{2} & j' \\ \ell' & \frac{1}{2} & j' \\ 0 & 0 & 0 \end{bmatrix} \left[\tilde{\chi}^{[\frac{1}{2}]} \middle| \chi^{[\frac{1}{2}]}\right]$$

$$\times \quad \sum_k \hat{k} \begin{bmatrix} k & k & 0 \\ 0 & 1 & 1 \\ k & \ell & 1 \end{bmatrix} \begin{bmatrix} L & 0 & L \\ k & \ell & 1 \\ \ell' & \ell & J \end{bmatrix} \begin{bmatrix} \ell' & \ell & J \\ 0 & \ell & \ell \\ \ell' & 0 & \ell' \end{bmatrix}$$

$$\times \quad \left[Y^{[\ell']} \middle| Y^{[L]} \middle| Y^{[k]}\right] \left[R_{\ell'} Y^{[k]} \middle| j_L p^{[1]} \middle| Y^{[\ell]} R_\ell\right]$$

The projection integral of the momentum operator is given by Danos as

$$\left[Y^{[\ell+1]} \middle| p^{[1]} \middle| Y^{[\ell]}\right] = -i(\ell+1)^{\frac{1}{2}}\left(\frac{\partial}{\partial r} - \frac{\ell}{r}\right) = -i(-)^{\ell+1}\left(\frac{\ell+1}{\widehat{\ell(\ell+1)}}\right) \frac{\left(\frac{\partial}{\partial r} - \frac{\ell}{r}\right)}{\begin{pmatrix} \ell+1 & 1 & \ell \\ 0 & 0 & 0 \end{pmatrix}}$$

$$\left[Y^{[\ell-1]} \middle| p^{[1]} \middle| Y^{[\ell]}\right] = -i\ \ell^{\frac{1}{2}}\left(\frac{\partial}{\partial r} + \frac{\ell+1}{r}\right) = -i(-)^{\ell}\left(\frac{\ell}{\widehat{\ell(\ell-1)}}\right) \frac{\left(\frac{\partial}{\partial r} + \frac{\ell+1}{r}\right)}{\begin{pmatrix} \ell-1 & 1 & \ell \\ 0 & 0 & 0 \end{pmatrix}}$$

We will use the second form to more easily compare our results with the standard treatment [4].

$$(i)^{(\ell'-L-1-\ell)}\langle\psi^{[j']}\|j_L\left[Y^{[L]}\times p^{[1]}\right]^{[J]}\|\psi^{[j]}\rangle$$

$$= \quad (-)^{\ell'+j-1}\quad \frac{\hat{J}\ \hat{L}\ \hat{j}'\ \hat{j}\ \hat{\ell}'\ \hat{\ell}}{(4\pi)^{\frac{1}{2}}}\begin{Bmatrix}\ell' & j' & \frac{1}{2}\\ j & \ell & J\end{Bmatrix}$$

$$\times\left[\begin{Bmatrix}L & 1 & J\\ \ell & \ell' & \ell+1\end{Bmatrix}\begin{pmatrix}\ell' & L & \ell+1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell+1 & 1 & \ell\\ 0 & 0 & 0\end{pmatrix}^{-1}\left(\frac{\ell+1}{2\ell+1}\right)\langle R_{\ell'}|j_L\left(\frac{\partial}{\partial r}-\frac{\ell}{r}\right)|R_\ell\rangle\right.$$

$$\left.+\begin{Bmatrix}L & 1 & J\\ \ell & \ell' & \ell-1\end{Bmatrix}\begin{pmatrix}\ell' & L & \ell-1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell-1 & 1 & \ell\\ 0 & 0 & 0\end{pmatrix}^{-1}\left(\frac{\ell}{2\ell+1}\right)\langle R_{\ell'}|j_L\left(\frac{\partial}{\partial r}+\frac{\ell+1}{r}\right)|R_\ell\rangle\right]$$

IV. An Example

We illustrate this formalism with an example to show how the sum over particles $\sum_j$ works and to show the effect of considering a transition between states of definite isospin T. ^{12}C has a ground state with $J^\pi = 0^+$ and T=0 and an excited state at 15.1 MeV with $J^\pi = 1^+$ and T=1. The only electromagnetic transition operator that can connect these two states is one with J=1, $\pi = +$ which is the magnetic dipole operator $\mathcal{M}^{[1]}$.

If the ^{12}C ground state is taken as the configuration $(1s_{\frac{1}{2}})^4\ (1p_{\frac{3}{2}})^8$ and the 1^+ excited state as $(1s_{\frac{1}{2}})^4(1p_{\frac{3}{2}})^7(1p_{\frac{1}{2}})^1$, then the transition can be pictured as simply one nucleon in the p-shell recoupling its orbital and spin from $j = 1 + \frac{1}{2}$ to $j' = 1 - \frac{1}{2}$. Both neutrons and protons in the p-shell can make this transition. Because of the way single particle states are normalized (total state functions normalized to 1 and single particle wave function also normalized to 1) we have the simple relation for closed shell nuclei

$$\langle \Psi^{[J_f]} \| \sum_i \overset{[J]}{O(i)} \| \Psi^{[J_i]} \rangle$$

$$= \sum_{jj'} \langle \Psi^{[j']} \| O^{[J]} \| \Psi^{[j]} \rangle$$

i.e., the sum over single particle operators between the total state wave functions is the same as the sum of possible single particle transitions. In our example $j = \frac{3}{2}$ and $j' = \frac{1}{2}$ so that only one reduced matrix element is involved $\langle \left[Y^{[1]} \times \tilde{\chi}^{[\frac{1}{2}]} \right]^{[\frac{1}{2}]} \| \mathcal{M}^{[1]} \| \left[Y^{[1]} \times \chi^{[\frac{1}{2}]} \right]^{[\frac{3}{2}]} \rangle$.

Electromagnetic transitions can, in general, change the isospin quantum number of a nuclear state by 0 or 1, an isoscalar or isovector transition. The fundamental interaction operators can be divided into their isoscalar and isovector components by the replacement of

$e_j = \frac{1}{2}(1 + \tau_j^{z})$ where $\tau^z|p\rangle = +|p\rangle$ in the Coulomb and current operators and $\mu_j = \frac{1}{2}(\mu_s + \tau_j^{z}\mu_v)$ where $\mu_s = \mu_p + \mu_n = .88$ and $\mu_v = \mu_p - \mu_n = 4.70$ in the moment operator. In the present example only the isovector parts of the operators will contribute to the transition. The single particle operators are therefore multiplied by the factor $\frac{1}{2}\tau^z$ and μ_j is replaced by the μ_v for all particles. The isospin part of the single particle matrix element is always of the form

$$\sum_j M_{(j)} \langle 1\ 0 | \tfrac{1}{2}\tau_j^{z} | 0\ 0 \rangle = \frac{1}{2}\frac{1}{3^{\frac{1}{2}}} \sum_j M_{(j)} \langle 1 \| \vec{\tau}_j \| 0 \rangle$$

$$= \frac{1}{2}\frac{1}{3^{\frac{1}{2}}} M \langle \tfrac{1}{2} \| \vec{\tau} \| \tfrac{1}{2} \rangle = \frac{M}{2^{\frac{1}{2}}}$$

where the $M_{(j)}$ are the space-spin part of the matrix element. Thus, the additional specification of the isospin quantum numbers of the initial and final states reduces the cross section by a factor of 2. Explicit evaluation of the 3-, 6-, and 9-j coefficients and the use of harmonic oscillator radial wave functions gives (with $\eta = qb$, b is the oscillator parameter)

$$\langle 1p_{\frac{1}{2}} \| j_0(qr)\left[Y^{[0]} \times \sigma^{[1]}\right]^{[1]} \| 1p_{\frac{3}{2}}\rangle = -i\frac{4}{(3)^{\frac{1}{2}}}\left(1 - \frac{1}{6}\eta^2\right)\frac{e^{-\eta^2/4}}{(4\pi)^{\frac{1}{2}}}$$

$$\langle 1p_{\frac{1}{2}} \| j_2(qr)\left[Y^{[2]} \times \sigma^{[1]}\right]^{[1]} \| 1p_{\frac{3}{2}}\rangle = +i\frac{\eta^2}{3(6)^{\frac{1}{2}}}\frac{e^{-\eta^2/4}}{(4\pi)^{\frac{1}{2}}}$$

$$\langle 1p_{\frac{1}{2}} \| j_1(qr)\left[Y^{[1]} \times p^{[1]}\right]^{[1]} \| 1p_{\frac{3}{2}}\rangle = -i\frac{(2)^{\frac{1}{2}}}{3}\, q\, \frac{e^{-\eta^2/4}}{(4\pi)^{\frac{1}{2}}}$$

$$|\langle 1p_{\frac{1}{2}} \| \mathcal{M}^{[1]} \| 1p_{\frac{3}{2}}\rangle|^2 = \frac{q^2}{M^2}\frac{1}{3}\left[1 - \mu_v\left(2 - \frac{q^2b^2}{4}\right)\right]^2 e^{-q^2b^2/2}$$

Taking the isospin factor into account

$$\frac{d\sigma}{d\Omega}(0^+ \rightarrow 1^+) = \sigma_0\left(\frac{q_\mu^2}{2\vec{q}^2} + \tan^2\theta/2\right)\frac{q^2}{M^2}\frac{1}{9}\left[1 - \mu_v\left(2 - \frac{q^2b^2}{4}\right)\right]^2 e^{-q^2b^2/2}$$

This equation neglects the momentum dependence of the nucleon form factor and a center-of-mass correction for the oscillator model. When compared to experiment this expression is found to overestimate the data by a factor of 4. This is thought to be due to the naivé assumption that the ^{12}C ground state is a filled $1p_{\frac{3}{2}}$ subshell. More realistic wave functions put some particles into the $1p_{\frac{1}{2}}$ subshell. The magnitude (but not the shape) of the form factor of the $0^+ \rightarrow 1^+$ transition is very sensitive to this effect.

V. Conclusions

We have tried to make two points in this paper. One that the formation of the electric and magnetic multipole operators is more transparent when expressed as an angular momentum coupling rather than the classical method using the curl of the vector spherical harmonic and other obscure vector algebra tricks. The second point was to show the relative ease with which one can calculate complex reduced matrix elements using the flowchart technique of Danos. An important advantage of this method is that no phases appear in the evaluation of the recoupling boxes. The diagrams also often make selection rules that are operating in a specific problem more apparent. The recoupling diagrams serve the same function in angular momentum calculations that Feynman diagrams do in quantum electrodynamics.

The author thanks Dr. Don Lehman for many useful discussions about the recoupling technique and the treatment of spin $\frac{1}{2}$ in particular.

VI. References

[1] M. Danos, Ann. Physics 63, 319 (1971).

[2] The phase convention has been changed from that of Eq. (2) of [1] on the advice of M. Danos.

[3] The phase factor in front of the reduced matrix element comes from changing from spherical harmonics to contrastandard elements. This phase will disappear when absolute magnitudes of the matrix elements are taken in the cross section expression. The single particle reduced matrix elements are written this way in order to simplify the comparison with the standard [4] results.

[4] T. de Forest, Jr. and J. D. Walecka, Adv. in Physics 15, 1 (1966). The electric and magnetic operators given in Eqs. (4.85 and 4.86) in this reference are related to those derived here by

$$\mathcal{E}_\lambda^{[J]} = i(-)^{J-1} \left[\frac{4\pi(2J+1)}{2}\right]^{\frac{1}{2}} T_{J\lambda}^{el}$$

$$\mathcal{M}_\lambda^{[J]} = i(-)^{J-1} \left[\frac{4\pi(2J+1)}{2}\right]^{\frac{1}{2}} T_{J\lambda}^{mag}$$

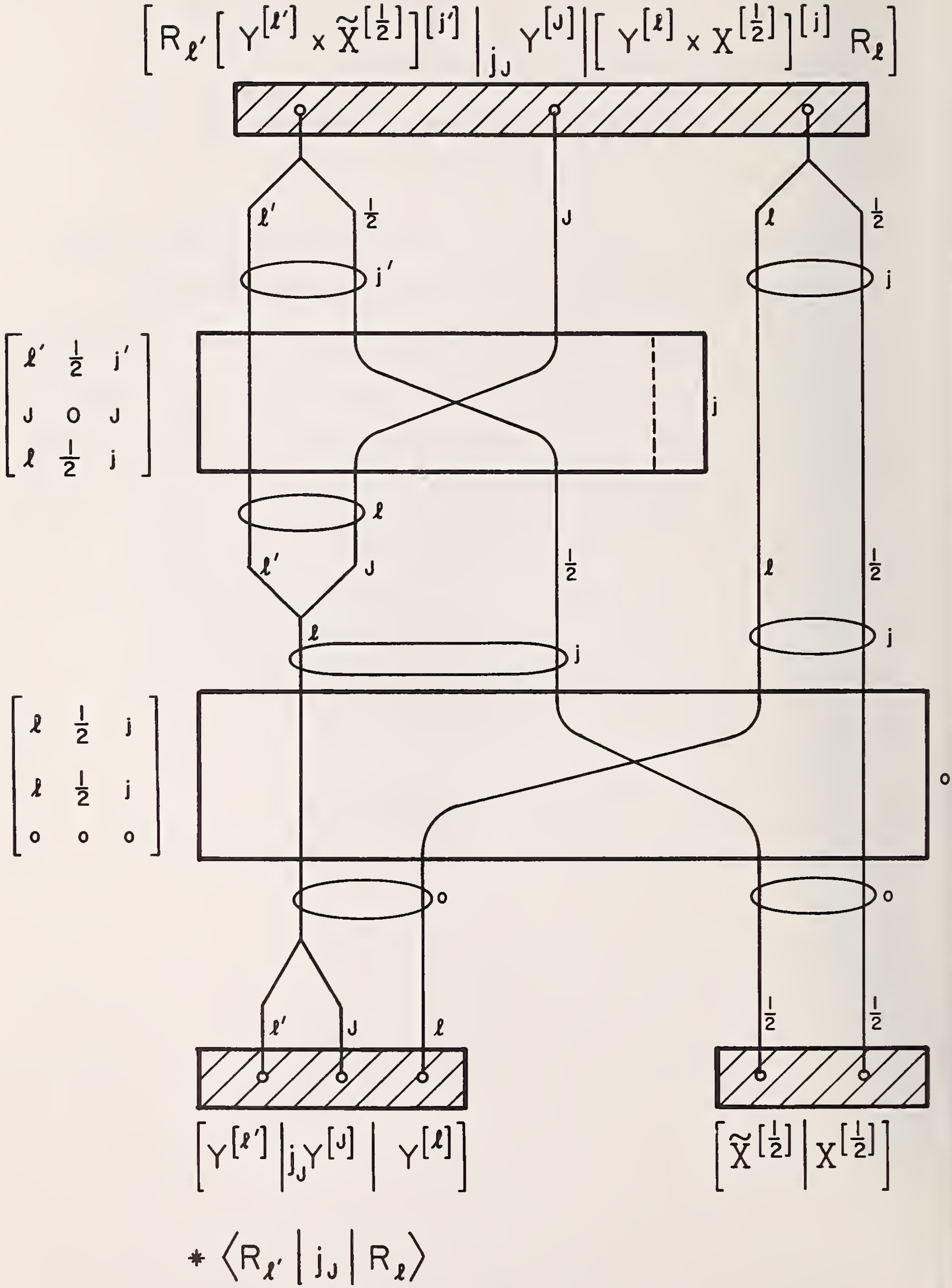

Fig. 1. Coulomb Projection Integral

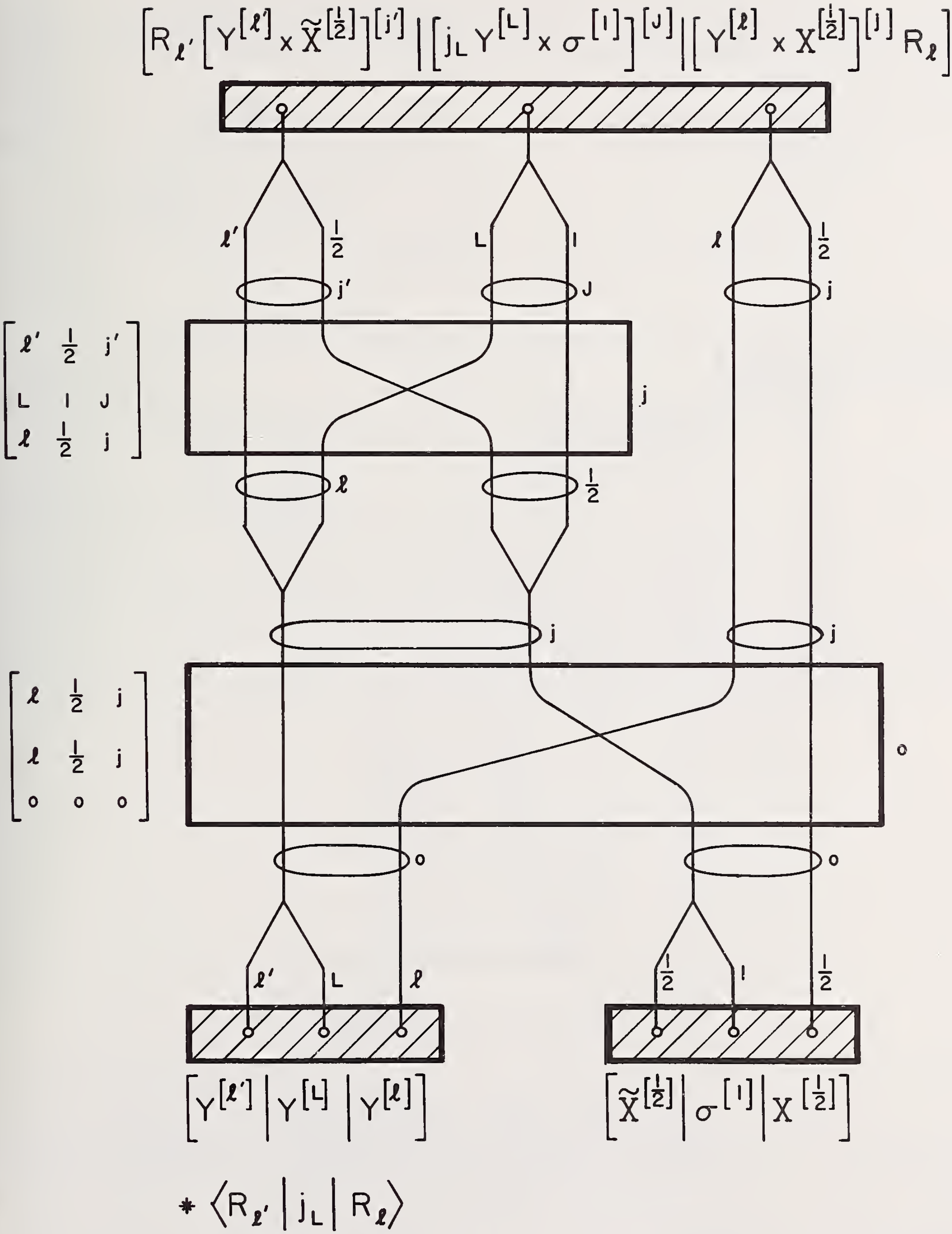

Fig. 2. Moment Projection Integral

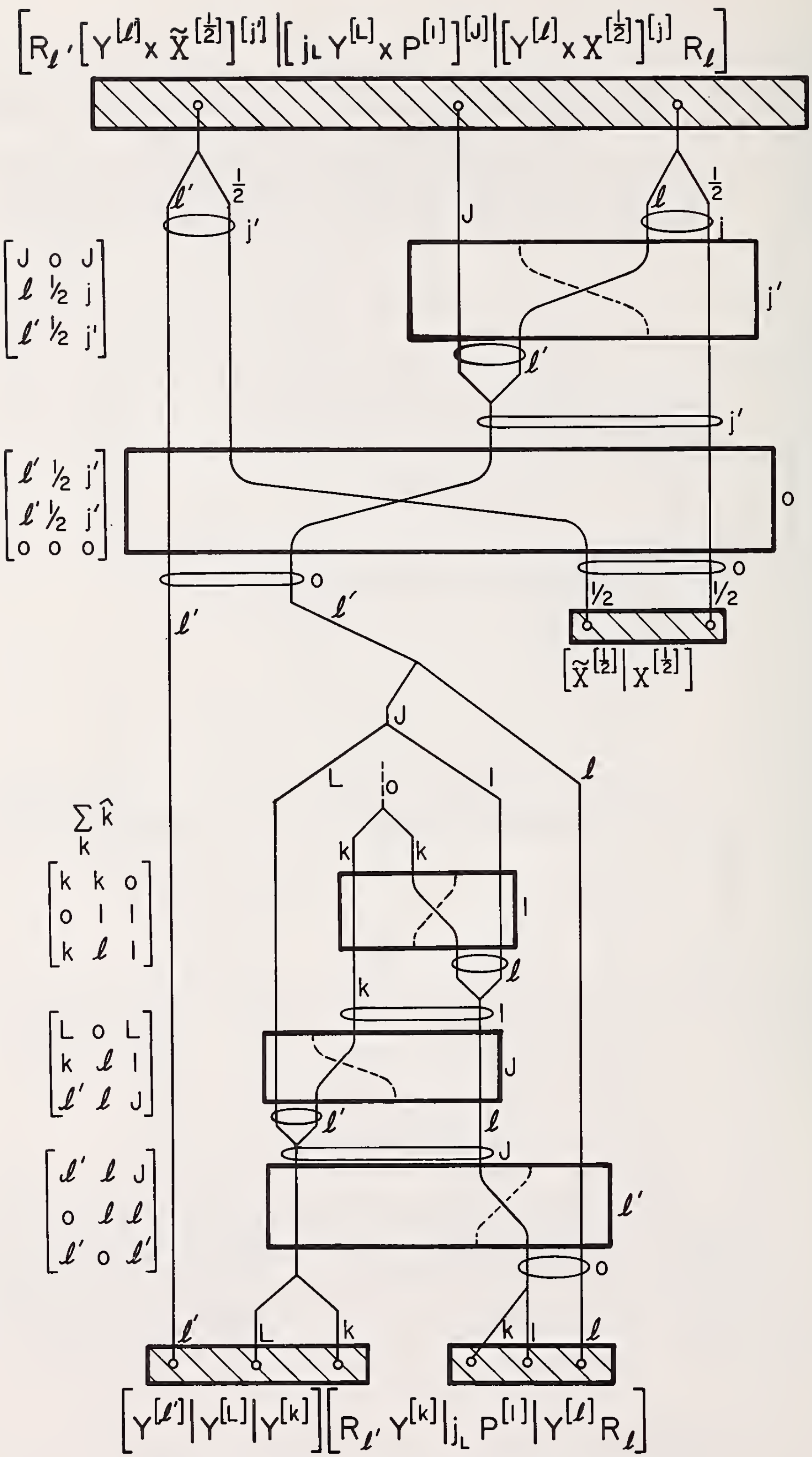

Fig. 3. Current Projection Integral

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The formation of the multipole operators from the fundamental charge, current, and moment operators of electron-nucleus scattering is carried through as an angular momentum recoupling problem rather than the usual vector algebra derivation. This point of view allows the electric and magnetic transition operators to be obtained in a simple and intuitive manner. The single particle reduced matrix elements of the charge, current, and moment operators are then calculated in the j-j coupling scheme using a flow chart technique developed by Danos.

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