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U.S. DEPARTMENT OF COMMERCE/National Bureau of Standards

Near-Field Antenna Measurements On A Cylindrical Surface: A Source Scattering-Matrix Formulation

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Near-Field Antenna Measurements On A Cylindrical Surface:

A Source Scattering-Matrix Formulation

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NEAR-FIELD ANTENNA MEASUREMENTS ON A CYLINDRICAL SURFACE: A SOURCE SCATTERING-MATRIX FORMULATION[1]

Arthur D. Yaghjian

The theory for probe-corrected measurement of antennas by scanning on a circular cylindrical surface enclosing the test antenna in the near-field is formulated from a source scattering matrix description of the test and probe antennas. The basic transmission formula is derived without recourse to reciprocity, and from a common center approach which separates as an isolated problem the probe characterization and transformation. Moreover, it is shown how an experimental technique can, in principle, determine the required transformed probe coefficients without the use of addition theorems. Computer inversion of the transmission formula is accomplished accurately and efficiently with the aid of the sampling theorem and FFT algorithm.

Key words: Cylindrical scanning; near-field measurements; source scattering matrix.

1. INTRODUCTION

The ease with which many antennas can be scanned on a cylindrical surface by moving a probe along a single axis for different azimuthal orientations of the test antenna about a parallel axis makes cylindrical scanning a very attractive near-field measurement technique. In addition to its operational simplicity, cylindrical near-field-far-field deconvolution utilizes the FFT in a straightforward fashion without the requirement of special measurement probes, and the far-field emerges in conventional elevation and azimuth angles, the latter of which covers a full 360°. And, in fact, both the 2-dimensional and 3-dimensional formulation of the probe-corrected technique for measuring antennas on a cylindrical surface have been derived and verified through experiment by Brown and Jull [1][2] (2-D case) and Leach and Paris [2] (3-D case). In both papers the EM fields of the test antenna were expressed in terms of cylindrical waves and the Lorentz reciprocity theorem was used (assuming reciprocal probes) to derive the basic "transmission formula" needed to apply the cylindrical scanning technique. Brown and Jull expressed the receiving characteristics of the probe in terms of a plane-wave spectrum, while Leach and Paris used cylindrical waves to express the receiving characteristics of the probe. In the present work, the basic transmission formula (25) is derived from the "source scattering matrix" equations for the test and probe antennas referred to a common-center cylindrical coordinate system. This cylindrical scattering matrix approach, which is analogous to the plane-wave scattering matrix approach used by Kerns [3], proves simple and straightforward, and does not require the probe or test antenna to be reciprocal, although reciprocity is expressed conveniently as a relationship between elements of the scattering matrix (see eqs. (15 and 19)). The parallels between the cylindrical source scattering matrix formulation used throughout these notes and the plane-wave scattering matrix formulation introduced by Kerns is striking and is emphasized to facilitate the reader's understanding of the cylindrical formulation.

Another distinct advantage of the source scattering matrix approach is that it clearly and completely separates the problem of determining the receiving characteristics

[1] This Technical Note was originally written as a tutorial set of notes for the NBS short course (July 7-11, 1975 - Boulder, Colorado).

[2] Figures in brackets indicate the literature references at the end of this paper.

of the probe from the derivation of the transmission formula. The characterization of the probe can be dealt with as an isolated topic. It is irrelevant to the basic formulation whether the required receiving functions for the probe are, e.g., measured directly, transformed from its known far-field or plane-wave spectrum [1], or transformed from its known cylindrical mode expansion [2]. Of course, the accurate characterization of the probe is a requirement of any reliable probe-corrected theory and is discussed as a separate topic in section 4 below.

2. THE RADIATING FIELDS OF AN ANTENNA IN CYLINDRICAL WAVES

This work addresses the problem of measuring the radiating fields of a test antenna by scanning with a probe antenna on an imaginary cylinder surrounding the test antenna. The problem of measuring the receiving characteristics of the test antenna is very closely related, but for the sake of brevity it will be left as an exercise for the reader. Of course, most antennas are reciprocal so once the radiating or transmitting characteristics are determined the receiving properties follow immediately from the reciprocity relationship (15a).

As a first step in formulating the cylindrical scanning technique, the EM fields outside an arbitrary test antenna radiating into free space will be expanded in a complete set of cylindrical waves (eigenfunctions or modes). The test antenna and its coordinate system fixed to the test antenna are shown schematically in figure 1. In cylindrical coordinates the point P is designated by (ρ, ϕ, z), in spherical coordinates by (r, ϕ, θ).

It has been shown [4] that in cylindrical coordinates the $(\overline{E}, \overline{H})$ fields in free space (in this case outside the radius (a) of the smallest cylinder circumscribing the test antenna) can be divided into a TE and TM part with respect to the z-direction. Specifically,

$$\overline{E} = \nabla \times \hat{e}_z \psi_1 + \frac{1}{k} \nabla \times (\nabla \times \hat{e}_z \psi_2) \tag{1a}$$

$$i\omega\mu_o \overline{H} = k \nabla \times \hat{e}_z \psi_2 + \nabla \times (\nabla \times \hat{e}_z \psi_1), \tag{1b}$$

where ψ_1 and ψ_2, which give rise to the TE and TM parts of the field respectively, satisfy the scalar wave equations

$$\nabla^2 \psi_s + k^2 \psi_s = 0, \quad s = 1,2. \tag{2}$$

All quantities with hats (^) over them denote unit vectors, $e^{-i\omega t}$ $(\omega > o)$ time dependence and the rationalized mks system are used throughout, and the free-space wave number k is defined as usual by $k = \omega\sqrt{\mu_o \varepsilon_o} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$.

The solution to the scalar wave equation (2) in cylindrical coordinates outside the antenna $\left(\begin{smallmatrix} \rho > a \\ -\infty < z < \infty \end{smallmatrix}\right)$ is written in terms of the well-known scalar cylindrical waves

$$\psi_s = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} B_n^s(\gamma)\, H_n^{(1)}(\kappa\rho) e^{in\phi} e^{i\gamma z}\, d\gamma. \tag{3}$$

$$\kappa = (k^2 - \gamma^2)^{1/2}$$

The $B_n^s(\gamma)$ are arbitrary constant coefficients and $H_n^{(1)}(\kappa\rho)$ are the Hankel functions of the first kind. The Hankel functions of the first kind are the only cylindrical functions which possess the proper behavior as $\rho\to\infty$ for κ chosen positive real or imaginary. (Later, when the probe enters the picture giving rise to sources outside ρ, Bessel functions $J_n(\kappa\rho)$ are also included in the complete set of cylindrical modes.) When the solutions (3) for ψ_s are substituted into eqs. (1) and the various curls are taken in cylindrical coordinates, the $(\overline{E}, \overline{H})$ fields can be written

$$\overline{E}(\rho,\phi,z) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} [B_n^1(\gamma)\ \overline{M}_{n\gamma}^{(1)} + B_n^2(\gamma)\ \overline{N}_{n\gamma}^{(1)}]d\gamma \tag{4a}$$

$$\overline{H}(\rho,\phi,z) = \frac{1}{iZ_o} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} [B_n^1(\gamma)\ \overline{N}_{n\gamma}^{(1)} + B_n^2(\gamma)\ \overline{M}_{n\gamma}^{(1)}]d\gamma\ , \tag{4b}$$

$$(Z_o = \sqrt{\mu_o/\varepsilon_o} \approx 377 \text{ ohms})$$

where the $\overline{M}$ and $\overline{N}$ functions are given by

$$\overline{M}_{n\gamma}^{(1)}(\rho,\phi,z) = \nabla\times(H_n^{(1)}(\kappa\rho)e^{in\phi}e^{i\gamma z}\hat{e}_z) = [\frac{in}{\rho}H_n^{(1)}(\kappa\rho)\hat{e}_\rho - \kappa\ H_n^{(1)'}\ \hat{e}_\phi]e^{in\phi}e^{i\gamma z} \tag{5a}$$

$$\overline{N}_{n\gamma}^{(1)}(\rho,\phi,z) = \frac{1}{k}\nabla\times\overline{M}_{n\gamma}^{(1)} = \frac{1}{k}[i\gamma\kappa\ H_n^{(1)'}\hat{e}_\rho - \frac{n\gamma}{\rho}H_n^{(1)}\hat{e}_\phi + \kappa^2 H_n^{(1)}e_z]e^{in\phi}e^{i\gamma z}\ . \tag{5b}$$

If the $B_n^s(\gamma)$ coefficients were known for the test antenna, eqs. (4) and (5) would give the electromagnetic field of the test antenna everywhere outside the radius of the smallest cylinder circumscribing the test antenna. Essentially the main purpose of this work is to formulate a procedure for finding the $B_n^s(\gamma)$ for an unknown test antenna, by scanning on a cylindrical surface with an arbitrary but known probe antenna. The only quantities in eqs. (4) which change from antenna to antenna are the coefficients $B_n^s(\gamma)$.

If one is fortunate enough to have a probe at his disposal which approximates an ideal dipole, i.e., a probe which measures the transverse components of the electric field, then the $B_n^s(\gamma)$ can be found directly and almost trivially from eq. (4a) alone. To show this, the following orthogonality relationships for the $\overline{M}$ and $\overline{N}$ functions are required

$$\int_{-\infty}^{\infty}\int_0^{2\pi} (\overline{M}_{n\gamma}^{(1)} \times \overline{M}_{n'\gamma'}^{(1)})\cdot\hat{e}_\rho d\phi\ dz = 0 \tag{6a}$$

$$\int_{-\infty}^{\infty}\int_0^{2\pi} (\overline{N}_{n\gamma}^{(1)} \times N_{n'\gamma'}^{(1)})\cdot\hat{e}_\rho d\phi\ dz = 0 \tag{6b}$$

$$\int_{-\infty}^{\infty}\int_0^{2\pi} (\overline{N}_{n\gamma}^{(1)} \times \overline{M}_{n'\gamma'}^{(1)})\cdot\hat{e}_\rho d\phi\ dz = \frac{4\pi^2\kappa^3}{k} H_n^{(1)}(\kappa\rho)H_n^{(1)'}(\kappa\rho)\ \delta_{-nn'}\delta(\gamma+\gamma')\ . \tag{6c}$$

These relationships derive from the definitions of $\overline{M}$ and $\overline{N}$ in eqs. (5).

Now cross $\overline{N}$ into eq. (4a) and use the orthogonalities (6) to yield the $B_n^1(\gamma)$,

$$B_n^1(\gamma) = \frac{k}{4\pi^2\kappa^3 H_{-n}^{(1)}(\kappa\rho_o)H_n^{(1)'}(\kappa\rho_o)} \int_{-\infty}^{\infty}\int_0^{2\pi} \left((\overline{N}_{-n,-\gamma} \times \overline{E}(\rho_o,\phi,z)\right)\cdot\hat{e}_\rho d\phi\ dz\ ,$$

or written out in component form,

$$B_n^1(\gamma) = \frac{-1}{4\pi^2\kappa^3 H_n^{(1)\prime}(\kappa\rho_o)} \int_{-\infty}^{\infty}\int_{o}^{2\pi} \left[\frac{n\gamma}{\rho_o} E_z(\rho_o,\phi,z) + \kappa^2 E_\phi(\rho_o,\phi,z)\right] e^{-in\phi} e^{-i\gamma z} d\phi dz \quad . \tag{7a}$$

ρ_o is the radius of the imaginary scanning cylinder on which the transverse electric field is measured. Similarly cross $\overline{M}$ into eq. (4a) to yield the $B_n^2(\gamma)$,

$$B_n^2(\gamma) = \frac{k}{4\pi^2\kappa^2 H_n^{(1)}(\kappa\rho_o)} \int_{-\infty}^{\infty}\int_{o}^{2\pi} E_z(\rho_o,\phi,z) e^{-in\phi} e^{-i\gamma z}\, d\phi dz \quad . \tag{7b}$$

Equations (7) give the required modal coefficients in terms of the measured transverse electric field on a cylinder. Moreover, the sampling theorem and fast Fourier transform (see section 5) can be utilized for the efficient computation of the double integrals in eqs. (7). Thus for an ideal dipole probe the deconvolution problem reduces to a trivial application of the orthogonality relationships for the $\overline{M}$ and $\overline{N}$ functions. This is true for planar and spherical scanning as well--and as a matter of fact, for any of the six separable coordinate systems for which $\overline{M}$ and $\overline{N}$ vector wave solutions exist [4].

2.1. Expressions for the Far-Field, Power Gain Function, and Complex Polarization Ratio

Equations (4) and (5) give the electromagnetic fields everywhere outside the test antenna radiating into free space. Often, however, only the far-fields of the test antenna are of interest. Thus, it is helpful to derive explicit expressions for the far-field from eqs. (4) and (5). This has been done in appendix A by a straightforward manipulation of the Fourier transforms without using asymptotic methods such as the method of stationary phase or steepest descent. The final results written in the spherical coordinates of figure 1 are:

$$\underset{r\to\infty}{\overline{E}(r,\phi,\theta)} = -\,2\,k\,\sin\theta\,\frac{e^{ikr}}{r}\sum_{n=-\infty}^{\infty}(-i)^n[B_n^1(k\cos\theta)\hat{e}_\phi - iB_n^2(k\cos\theta)\hat{e}_\theta]e^{in\phi} \tag{8a}$$

$$\underset{r\to\infty}{\overline{H}(r,\phi,\theta)} = \frac{e_r\times\overline{E}_{r\to\infty}}{Z_o} = \frac{2\,k\,\sin\theta\, e^{ikr}}{Z_o\, r}\sum_{n=-\infty}^{\infty}(-i)^n[B_n^1(k\cos\theta)\hat{e}_\theta + iB_n^2\,(k\cos\theta)\hat{e}_\phi]e^{in\phi} \quad . \tag{8b}$$

The far-field equations (8) do not involve an integration, only a summation, and the values of $B_n^s(\gamma)$ for $|\gamma|>k$ do not affect the far-fields. Note also that at least some of the $B_n^s(\gamma)$ must behave as $1/\sin\theta$ as $\theta\to o$ if the far-field patterns are finite there.

The power gain function is defined as

$$G(\theta,\phi) = \frac{2\pi r^2\, |\overline{E}|^2_{r\to\infty}}{Z_o\, P_{input}} \;, \tag{9a}$$

where P_{input} is the net power input to the antenna. The input power can be written in terms of the ingoing (a_o) and outgoing (b_o) modes of the waveguide which feeds the antenna (see figure 1 and Kerns [3]),

$$P_{input} = \frac{1}{2}\eta_o(|a_o|^2-|b_o|^2) = \frac{1}{2}\eta_o|a_o|^2(1-|\Gamma_o|^2) ,$$

where η_o is the characteristic admittance of the waveguide and Γ_o is the reflection coefficient at the reference plane S_o ($b_o = \Gamma_o a_o$). Substitution of P_{input} and the far-electric-field from eq. (8a) into eq. (9a) yields the final expression for the gain function,

$$G(\theta,\phi) = \frac{16\pi k^2 \sin^2\theta}{Z_o\eta_o|a_o|^2(1-|\Gamma_o|^2)}\{|\sum_{n=-\infty}^{\infty}(-i)^n B_n^1(k\cos\theta)e^{in\phi}|^2 + |\sum_{n=-\infty}^{\infty}(-i)^n B_n^2(k\cos\theta)e^{in\phi}|^2\} . \quad (9b)$$

The complex polarization ratio defined here as E_θ/E_ϕ may be written directly from eq. (8a),

$$\frac{E_\theta}{E_\phi} = \frac{-i\sum_{n=-\infty}^{\infty}(-i)^n B_n^2(k\cos\theta)e^{in\phi}}{\sum_{n=-\infty}^{\infty}(-i)^n B_n^1(k\cos\theta)e^{in\phi}} . \quad (10)$$

2.2 Total Power Radiated

Also, from the far-field equations (8) it is a fairly simple matter to find the total power P_r radiated by the test antenna,

$$P_r \equiv \frac{1}{2}\oint_{r\to\infty}(\bar{E}\times\bar{H}^*)\cdot\hat{e}_r da = \frac{1}{2Z_o}\int_0^\pi\int_0^{2\pi}|\bar{E}|^2 r^2 \sin\theta\, d\phi d\theta . \quad (11a)$$

After substituting the electric field from eq. (8a) into eq. (11a) and making use of the orthogonality of the $e^{in\phi}$ functions, P_r can be written

$$P_r = \frac{4\pi k^2}{Z_o}\sum_{n=-\infty}^{\infty}\int_0^\pi [|B_n^1(k\cos\theta)|^2 + |B_n^2(k\cos\theta)|^2]\sin^3\theta\, d\theta. \quad (11b)$$

A change of the variable of integration from $k \cos\theta$ to γ transforms eq. (11b) into the final expression for the total radiated power,

$$P_r = \frac{4\pi}{kZ_o}\sum_{n=-\infty}^{\infty}\int_{-k}^{k}\kappa^2[|B_n^1(\gamma)|^2 + |B_n^2(\gamma)|^2]d\gamma . \quad (12)$$

Observe that no power is carried in the waves for $|\gamma|>k$.

3. DERIVATION OF THE TRANSMISSION FORMULA

In section 2 we derived the $\bar{E}$ and $\bar{H}$ fields everywhere outside the test antenna in terms of its cylindrical wave coefficients $B_n^s(\gamma)$. Explicit expressions were written for the far-fields, power gain function, complex polarization ratio, and total power radiated.

In this section we will derive the basic transmission formula needed to compute the coefficients $B_n^s(\gamma)$ for an unknown test antenna by scanning on a cylinder with a probe of known receiving characteristics. The approach that is taken is to first write "source scattering matrix equations" for the test antenna with respect to the coordinate system (previously defined in figure 1) fixed in the test antenna. Secondly, the source scattering matrix equations for the probe are written for a cylindrical coordinate system (ρ',ϕ',z') fixed to the probe but with its z'-axis lying along the z-axis of the test antenna coordinate system. Thirdly, the relationship is derived between the modal coefficients defined in the coordinate system of the probe and the modal coefficients defined in the coordinate system of the test antenna. This relationship is entered into the source scattering matrix equations and the resulting equations are combined, neglecting multiple reflections, to give the desired transmission formula. Finally, the transmission formula is inverted to yield the $B_n^s(\gamma)$.

3.1. The Source Scattering Matrix Equations for the Test Antenna

Consider the cylindrical coordinate system fixed in the test antenna as shown in figure 1. While measurements are being taken with the probe antenna, it will act as a source of EM fields outside the test antenna. Thus the cylindrical mode expansion in the coordinate system of the test antenna for radii between the test antenna and probe must include two linearly independent radial solutions--not just the $H_n^{(1)}(\kappa\rho)$ solutions as was the case for the test antenna in free space. As we shall see, the present derivation is greatly facilitated by choosing $J_n(\kappa\rho)$ as the second linearly independent solution.

The $\overline{E}$ and $\overline{H}$ fields between the test and probe antennas may be written as in eqs. (4), but with the $J_n(\kappa\rho)$ solutions included as well. Specifically,

$$\overline{E}(\rho,\phi,z) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \{[b_n^1(\gamma)\overline{M}_{n\gamma}^{(1)} + b_n^2(\gamma)\overline{N}_{n\gamma}^{(1)}] + [a_n^1(\gamma)\overline{M}_{n\gamma} + a_n^2(\gamma)\overline{N}_{n\gamma}]\}d\gamma \tag{13a}$$

$$\overline{H}(\rho,\phi,z) = \frac{1}{i\omega\mu_o} \nabla\times\overline{E} \quad , \tag{13b}$$

where $\overline{M}_{n\gamma}^{(1)}$, $\overline{N}_{n\gamma}^{(1)}$ have the first Hankel functions $H_n^{(1)}(\kappa\rho)$ for their radial dependence, and $\overline{M}_{n\gamma}$, $\overline{N}_{n\gamma}$ have the Bessel functions $J_n(\kappa\rho)$ for their radial dependence. The modes with coefficients $b_n^s(\gamma)$ represent the fields from sources (induced as well as applied) inside or on the surface of the test antenna. The modes with coefficients $a_n^s(\gamma)$ represent the fields from sources (induced as well as applied) inside the probe antenna, i.e., outside the test antenna.

Also, the waveguide modes with coefficients a_o and b_o represent fields from sources inside and outside the generator (assumed shielded) of the test antenna, or equivalently, outside and inside the test antenna proper.

In brief then, the modal coefficients (a_o, $a_n^s(\gamma)$) and (b_o, $b_n^s(\gamma)$) represent fields from sources outside and inside the test antenna respectively. The inside coefficients (b_o, $b_n^s(\gamma)$) are related to the outside coefficients (a_o, $a_n^s(\gamma)$) by a linear matrix transformation which will be called the source scattering matrix for the test antenna, to distinguish it from the scattering matrix for the $H_n^{(1)}$, $H_n^{(2)}$ representation of the fields instead of the $H_n^{(1)}$, J_n representation used here. Specifically, these source scattering matrix equations are written

$$b_o = \Gamma_o a_o + \sum_{s=1}^{2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} R_n^s(\gamma) a_n^s(\gamma)\, d\gamma \tag{14a}$$

$$b_n^s(\gamma) = T_n^s(\gamma) a_o + \sum_{q=1}^{2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} S_{nm}^{sq}(\gamma,\beta) a_m^q(\beta) d\beta \quad . \tag{14b}$$

The functions $R_n^s(\gamma)$, $T_n^s(\gamma)$ and $S_{nm}^{sq}(\gamma,\beta)$ embody the receiving, transmitting, and scattering properties of the test antenna. Γ_o is the waveguide reflection coefficient defined at the surface S_o. For a given cylindrical coordinate system fixed in the test antenna, these four quantities depend only on the character of the test antenna. In Kern's notation [3] S_{01}, S_{10}, S_{11}, and S_{00} correspond to R, T, S, and Γ_o, respectively.

The source scattering matrix equations (14) can be taken as a definition of linearity for the test antenna or they can be derived from Maxwell's equations through a "source adjoint reciprocity lemma." The derivation of this source adjoint lemma and the scattering matrix equations (14) may be found in appendix B. It is also proven in appendix B that the receiving and transmitting functions of a reciprocal test antenna (or between a nonreciprocal antenna and its adjoint) satisfy the relationship

$$R_{-n}^s(-\gamma) = (-1)^n \frac{4\pi\kappa^2}{\eta_o Z_o k} T_n^s(\gamma) \ , \tag{15a}$$

and the scattering function satisfies the relationship

$$S_{-m,-n}^{qs}(-\beta,-\gamma) = S_{nm}^{sq}(\gamma,\beta). \tag{15b}$$

When the probe antenna is removed so that the test antenna is radiating into free space, eq. (14b) becomes

$$b_n^s(\gamma) = B_n^s(\gamma) = T_n^s(\gamma) a_o. \tag{16}$$

Thus if we can determine $T_n^{(s)}(\gamma)$ for the test antenna we can compute the free-space EM fields of the test antenna by means of eqs. (4). And, in particular, the far-fields, power gain function, complex polarization ratio, and total power radiated could be computed from eqs. (8), (9b), (10), and (12), respectively. Thus the problem of measuring the radiating properties of the test antenna reduces to that of determining $T_n^s(\gamma)$. The next step toward this end is to write the source scattering matrix equations for the probe antenna.

3.2. The Source Scattering Matrix Equations for the Probe Antenna

Consider a cylindrical coordinate system fixed in the probe antenna but with its z'-axis lying outside the probe antenna along the z-axis of the cylindrical coordinate system fixed in the test antenna (as shown in figure 2). a'_o and b'_o are the coefficients of the ingoing and outgoing waveguide mode of the probe antenna, and, in particular, the primes on the cylindrical coordinate system emphasize the fact that this coordinate system is fixed in the probe. Of course, the probe coordinate system is nothing more than the test coordinate system rotated about the z-axis through an angle ϕ_o and translated along the z-axis through a displacement z_o. (ϕ_o and z_o vary while scanning.)

The EM fields on a cylinder between the test and probe antennas can be written in terms of the prime system fixed in the probe:

$$\overline{E}'(\rho',\phi',z') = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \{[a'^{1}_{n}(\gamma)\overline{M}^{(1)}_{n\gamma} + a'^{2}_{n}(\gamma)\overline{N}^{(1)}_{n\gamma}] + [b'^{1}_{n}(\gamma)\overline{M}_{n\gamma} + b'^{2}_{n}(\gamma)\overline{N}_{n\gamma}]\}d\gamma \tag{17a}$$

$$\overline{H}'(\rho',\phi',z') = \frac{1}{i\omega\mu_o} \nabla'\times\overline{E}'. \tag{17b}$$

These equations are identical to eqs. (13) except for the primed coordinates (ρ',ϕ',z') and primed coefficients (a',b') replacing the unprimed. The a' and b' coefficients have deliberately replaced the b and a coefficients respectively, so that the a' and b' coefficients would now be associated with sources outside and inside the probe antenna respectively. In other words, the a' now refers to the $H^{(1)}_n(\kappa\rho')$ modes, and the b' to the $J_n(\kappa\rho')$ modes.

Just as was done for the test antenna, the inside probe coefficients $(b'_o, b'^{s}_{n}(\gamma))$ can be related to the outside probe coefficients $(a'_o, a'^{s}_{n}(\gamma))$ by a source scattering matrix for the probe:

$$b'_o = \Gamma'_o a'_o + \sum_{s-1}^{2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} R'^{s}_{n}(\gamma)\; a'^{s}_{n}(\gamma)d\gamma \tag{18a}$$

$$b'^{s}_{n}(\gamma) = T'^{s}_{n} a'_o + \sum_{q=1}^{2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} S'^{sq}_{nm}(\gamma,\beta)a'^{q}_{m}\, d\beta \quad . \tag{18b}$$

The functions $R'^{s}_{n}(\gamma)$, $T'^{s}_{n}(\gamma)$, and $S'^{sq}_{nm}(\gamma,\beta)$ represent receiving, transmitting, and scattering properties of the probe antenna. Γ'_o is merely the reflection coefficient for the probe waveguide referred to the surface S'_o. For a given cylindrical coordinate system fixed in the probe antenna, these four quantities depend only on the character of the probe antenna.

Like eqs. (14), the probe eqs. (18) are derived from Maxwell's equations in appendix B. Appendix B also proves that the source scattering matrix for a reciprocal probe (or for a non-reciprocal probe and its adjoint) obeys the following relationships analogous to eqs. (15) which apply to the test antenna,

$$R'^{s}_{-n}(-\gamma) = (-1)^{n} \frac{4\pi\kappa^2}{\eta_o Z_o k} T'^{s}_{n}(\gamma) \tag{19a}$$

$$S'^{qs}_{-m,-n}(-\beta,-\gamma) = S'^{sq}_{nm}(\gamma,\beta) \quad . \tag{19b}$$

The probe scattering equations (18) can be linked directly to the corresponding test antenna scattering equations (14) by relating the coefficients (a'^{s}_{n}, b'^{s}_{n}) of the cylindrical modes in the coordinate system fixed in the probe to the coefficients (a^{s}_{n}, b^{s}_{n}) of the cylindrical modes in the coordinate system fixed in the test antenna. This relationship between (a'^{s}_{n}, b'^{s}_{n}) and (a^{s}_{n}, b^{s}_{n}) is surprisingly simple and can be found in a surprisingly simple fashion.

3.3. Relationship Between the Probe and Test Antenna Coefficients in the Common-Center Systems

We know that, regardless of whether the probe or test antenna coordinate system is used, the EM fields at every point in space must be the same, i.e.,

$$\overline{E}'(\rho', \phi', z') = \overline{E}(\rho, \phi, z) \tag{20a}$$

$$\overline{H}'(\rho', \phi', z') = \overline{H}(\rho, \phi, z) \ , \tag{20b}$$

when (ρ', ϕ', z') and (ρ, ϕ, z) refer to the same point in space. But as the probe scans the test antenna, the probe coordinate system is merely rotated through an angle ϕ_o about the z-axis and translated a distance z_o along the z-axis, i.e.,

$$\rho' = \rho, \ \phi' = \phi - \phi_o, \text{ and } z' = z - z_o$$

and eqs. (20) become

$$\overline{E}'(\rho, \phi - \phi_o, z - z_o) = \overline{E}(\rho, \phi, z) \tag{21a}$$

$$\overline{H}'(\rho, \phi - \phi_o, z - z_o) = \overline{H}(\rho, \phi, z) \ . \tag{21b}$$

Now when ($\overline{E}$, $\overline{H}$) and ($\overline{E}'$, $\overline{H}'$) are written explicitly in terms of the linearly independent cylindrical waves in eqs. (13) and (17), the only way eqs. (21) can be satisfied is if

$$a'^{s}_{n}(\gamma) = b^{s}_{n}(\gamma)\, e^{in\phi_o}\, e^{i\gamma z_o} \tag{22a}$$

$$b'^{s}_{n}(\gamma) = a^{s}_{n}(\gamma)\, e^{in\phi_o}\, e^{i\gamma z_o} \ . \tag{22b}$$

These are the desired relationships linking the probe and test antenna scattering equations (18) and (14).

3.4. The Transmission Formula

Equations (22a) recast the probe receiving equation (18a) into the form

$$b'_o(\phi_o,z_o) = \Gamma'_o a'_o + \sum_{s=1}^{2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} R'^s_n(\gamma) b^s_n(\gamma)\, e^{in\phi_o} e^{i\gamma z_o} d\gamma. \tag{23}$$

Moreover, $b^s_n(\gamma)$ from eq. (14b) can be substitued in this eq. (23) to yield

$$b'_o(\phi_o,z_o) = \Gamma'_o a'_o + \sum_{s=1}^{2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \{R'^s_n(\gamma)[T^s_n(\gamma)a_o + \sum_{q=1}^{2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} S^{sq}_{nm}(\gamma,\beta) a^q_m(\beta) d\beta] e^{in\phi_o} e^{i\gamma z_o}\} d\gamma. \tag{24}$$

The contribution to $b'_o(\phi_o, z_o)$ from the term in eq. (24) containing $S^{sq}_{nm}(\gamma,\beta)$ represents multiple reflections. That is, it is that part of the output of the probe caused by fields which have left the test antenna, reflected from the probe, re-reflected from the test antenna, and returned to excite the probe. This process repeats itself ad infinitum. Assuming these multiple reflections are negligible, and the input a'_o to the probe is kept zero, eq. (24) reduces to

$$b'_o(\phi_o,z_o) = a_o \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} [\sum_{s=1}^{2} R'^s_n(\gamma) T^s_n(\gamma)] e^{in\phi_o} e^{i\gamma z_o} d\gamma \quad . \tag{25}$$

(If a'_o is not kept zero but the probe is terminated in a load with reflection coefficient Γ'_L, the right side of eq. (25) is merely divided by $(1-\Gamma'_o\Gamma'_L)$.)

Here, it can be noted that if $(H^{(1)}_n, H^{(2)}_n)$ functions had been used instead of $(H^{(1)}_n, J_n)$ for the expansion of the fields in cylindrical waves, it would be impossible without going through an extremely tedious procedure to show the effect of neglecting multiple reflections since $H^{(1)}_n$ waves would be excited by the probe as well as test antenna.

This "transmission formula" (25) relating the output of the probe and the "coupling product" ($\sum_{s=1}^{2} R'^s_n(\gamma)\, T^s_n(\gamma)$ where $R'^s_n(\gamma)$ is the receiving function of the probe and $T^s_n(\gamma)$ is the transmitting function of the test antenna) is analogous to the "transmission integral" of the planar formulation [3], and forms the central relationship on which the cylindrical scanning technique is based. In essence, it is the same formula as eq. (23) in Leach and Paris [2], and the same formula as the 2-dimensional equation (18) in Brown and Jull [1] if the polarization index s, z_o - dependence, and γ integration are deleted from eq. (25).

The transmission formula (25) has been derived in a simple, straightforward manner from the source scattering matrix equations for the test and probe antennas under the assumption of negligible multiple reflections. No other restrictive assumptions were made - the probe and test antenna may even be non-reciprocal. (The derivations of Leach and Paris [2] and Brown and Jull [1] inherently assume a reciprocal probe although both their derivations can be modified slightly, using adjoint reciprocity, to apply to non-reciprocal probes as well.)

3.5. Inversion of the Transmission Formula

Let's take a closer look at the transmission formula (25). In principle, as the probe scans on the surface of an imaginary cylinder surrounding the test antenna, the output $b_o'(\phi_o, z_o)$ of the probe is recorded for

$$0 \leq \phi_o < 2\pi$$

$$-\infty < z_o < \infty$$

That is, the amplitude and phase of $b_o'(\phi_o, z_o)$ is measured for all values of ϕ_o and z_o. (In practice, the z_o scan is limited to some finite scan length and $b_o'(\phi_o, z_o)$ is assumed negligible outside this region. Also, data need be sampled and recorded only at a finite number of measurement points. These topics of data collection and computations will be discussed in section 5 below.) Having measured $b_o'(\phi_o, z_o)$, the Fourier series and integral of eq. (25) can immediately be inverted to yield the solution for the coupling product in terms of the measured data $b_o'(\phi_o, z_o)$:

$$\sum_{s=1}^{2} R_n'^{s}(\gamma)\, T_n^s(\gamma) = \frac{1}{4\pi^2 a_o} \int_{-\infty}^{\infty} \int_{o}^{2\pi} b_o'(\phi_o, z_o) e^{-in\phi_o} e^{-i\gamma z_o} d\phi_o dz_o \quad . \tag{26a}$$

By computing the double integral on the right side of eq. (26a), the value of the coupling product,

$$R_n'^{1}(\gamma) T_n^1(\gamma) + R_n'^{2}(\gamma) T_n^2(\gamma) \ ,$$

can be determined for as many n and γ as desired. However, even if the receiving characteristics ($R_n'^1$, $R_n'^2$) of the probe are known, the transmitting characteristics (T_n^1, T_n^2) of the test antenna are not determined uniquely (2 unknowns for each equation). Fortunately, all that is needed is another scan with a different probe, or the same probe in a different orientation (say rotated about its baseline axis by 90°). In other words, two linearly independent scans are necessary to account for the polarization of the EM fields. The second scan produces a second equation to complement eq. (26a). Specifically,

$$\sum_{s=1}^{2} R_n''^{s}(\gamma) T_n^s(\gamma) = \frac{1}{4\pi^2 a_o} \int_{-\infty}^{\infty} \int_{o}^{2\pi} b_o''(\phi_o, z_o) e^{-in\phi_o} e^{-i\gamma z_o} d\phi_o \, dz_o \quad , \tag{26b}$$

where $R_n''^{s}(\gamma)$ and $b_o''(\phi, z_o)$ are the receiving characteristics and output of the second or reoriented probe respectively.

Assuming for the moment that the receiving functions (R', R'') of the probe(s) are known, eqs. (26) can be solved immediately for the transmitting functions $T_n^s(\gamma)$ of the test antenna:

$$T_n^1(\gamma) = [R_n''^{2}(\gamma) I_n'(\gamma) - R_n'^{2}(\gamma) I_n''(\gamma)]/\Delta_n(\gamma) \tag{27a}$$

$$T_n^2(\gamma) = [R_n'^{2}(\gamma) I_n''(\gamma) - R_n''^{1}(\gamma) I_n'(\gamma)]/\Delta_n(\gamma), \tag{27b}$$

where $I'_n(\gamma)$ and $I''_n(\gamma)$ are the integrals on the right side of eqs. (26a) and (26b) respectively; and $\Delta_n(\gamma)$ is the determinant

$$\Delta_n(\gamma) = R''^2_n(\gamma)R'^1_n(\gamma) - R'^2_n(\gamma)R''^1_n(\gamma) \quad . \tag{28}$$

Provided the receiving characteristics of the probe(s) are known, eqs. (26), (27), and (28) determine the complete transmitting characteristics of the test antenna in terms of the measured probe outputs $b'_o(\phi_o, z_o)$ and $b''_o(\phi_o, z_o)$.

4. THE RECEIVING CHARACTERISTICS OF THE PROBE ANTENNA

Naturally the question arises as to how the receiving characteristics of the probe can be determined. Specifically, how do we find the $R'^s_n(\gamma)$ and $R''^s_n(\gamma)$ which are required in eqs. (27) and (28). (In the subsequent discussion, I will mention only R'^s_n since R''^s_n can be found by the same procedure.)

Two methods will be explained for determining R'^s_n. The first involves no coordinate transformations but requires two cylindrical scans and two test antennas (or one test antenna in two orientations) with known far-fields. The second method, which is closely related to the approach used by Brown and Jull [1] and Leach and Paris [2] for cylindrical scanning, and by Jensen [5] and Wacker [6] for spherical scanning, involves a coordinate-like transformation between the known receiving functions of a probe in a coordinate system centered on the probe and the required receiving characteristics $R'^s_n(\gamma)$ which have been defined with respect to a coordinate system centered outside the probe.

The chief advantage of the first method is that it requires no coordinate transformations, which can be rather cumbersome expressions. Its chief disadvantage, besides the fact that it requires the a priori knowledge of the far-fields of a test antenna, may be that the given antenna must have non-negligible transmitting functions $T^s_n(\gamma)$ for all values of n and γ required of the probe. This will be explained specifically below. The second method consisting of transforming the receiving functions of a known probe does not exhibit this limitation. It should be mentioned that the two-identical-antenna or generalized-three-antenna techniques sometimes applicable in planar near-field scanning [3] do not appear feasible for cylindrical scanning because of the more complicated coordinate transformations (such as eqs. (37) and (38) below) which are involved in the cylindrical case.

In any case, it is emphasized that the common-center, source scattering matrix approach applied in the present paper to the problem of cylindrical scanning results in a transmission formula (25) which embodies the essence of non-planar scanning in a simple form (analogous to the planar transmission integral) whose derivation was accomplished without the introduction of cumbersome coordinate transformations. Questions related to the characterization of the measuring probe and the associated transformations can be concentrated on and discussed as a separate topic.

4.1. Measuring the Probe Receiving Functions Using a Known Test Antenna

This first method for determining $R'^{s}_{n}(\gamma)$ is quite easy to explain. The unknown probe whose receiving characteristics are wanted scans two test antennas (or one test antenna in two orientations) whose far-fields (amplitude, phase, and polarization), say, are known. Since the far-fields are given, eqs. (4) (with the asymptotic form of the Hankel function for large ρ inserted) or eqs. (8) for $|\gamma| \leq k$ can be inverted using the orthogonality relationships (6) between $\overline{M}$ and $\overline{N}$ to yield the transmitting functions $T^{s}_{n}(\gamma)$ of the known test antenna(s). With the T^{s}_{n} of the test antenna(s) known, eqs. (27) with the T's and R's interchanged can be used to determine the receiving functions R'^{s}_{n} of the probe. To use eqs. (27), however, the determinant $\Delta_{n}(\gamma)$ of eq. (28), now containing the T's, should not be too small. This implies, as mentioned above, that the T's of the known antenna(s) must not be negligible for a given n and γ.

4.2. Transformation of the Receiving Functions of a Known Probe

This second method for determining $R'^{s}_{n}(\gamma)$ assumes a probe of known receiving characteristics (or transmitting characteristics, if reciprocal) but known in a coordinate system centered at the probe, whereas $R'^{s}_{n}(\gamma)$ was defined with respect to a cylindrical coordinate system fixed to the probe but centered outside the probe. Here it will be assumed that the known receiving characteristics are given in terms of a cylindrical wave expansion, in order to conform with the work of Leach and Paris [2]. The transformation to $R'^{s}_{n}(\gamma)$ from known receiving characteristics given in terms of a plane wave spectrum has been derived by the author but the derivation will not be included in these notes. This latter transformation is similar to that derived by Brown and Jull [1] for the 2-dimensional cylindrical problem, and recently by Borgiotti [11] for the 3-D problem. As Borgiotti points out, one advantage of the plane-wave transformation is that the integrals involved lend themselves to asymptotic evaluation.

Consider a probe whose receiving functions $R'^{s}_{1n}(\gamma)$ are known with respect to a cylindrical coordinate system fixed in the probe and <u>centered</u> <u>on</u> <u>itself</u> (as opposed to $R'^{s}_{n}(\gamma)$ which is defined with respect to a cylindrical coordinate system fixed in the probe but centered outside the probe). Let the z-coordinates (but, of course, not the z-axes which are separated by a distance d) be the same for each system. Also choose the baseline from which the angles in each system are measured to coincide. The coordinate system centered on the probe will be called the C_1 system and will have cylindrical coordinates labelled (ρ_1, ϕ_1, z_1). The coordinate system centered outside the probe at the test antenna but still fixed in the probe will be called the C system,and as before, its coordinates will be labelled (ρ',ϕ',z'). Refer to figure 3 for a schematic of the two systems.

The relationship between the output of the probe (for $a'_o = 0$) and the receiving functions $R'^s_{1n}(\gamma)$ in the C_1 system is by definition (compare with eqs. (14)),

$$b'_o = \sum_{s=1}^{2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} R'^s_{1n}(\gamma) a'^s_{1n}(\gamma) d\gamma \ , \tag{29}$$

where a'^s_{1n} are the modal coefficients of the $J_n(\kappa\rho_1)$ modes which are excited by sources existing outside the probes, i.e., by sources on the test antenna. The values of these a'^s_{1n} coefficients are independent of sources within ρ_1, i.e., sources on the probe.

The output of the probe b'_o for $a'_o = 0$ is given alternatively by eq. (18a) in terms of the required probe receiving functions $R'^s_n(\gamma)$ in the C system,

$$b'_o = \sum_{s=1}^{2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} R'^s_n(\gamma) a'^s_n(\gamma) d\gamma \ . \tag{30}$$

A comparison of eqs. (30) and (29) shows that the required $R'^s_n(\gamma)$ can be found in terms of the given $R'^s_{1n}(\gamma)$ if the a'^s_{1n} can be written in terms of the a'^s_n, i.e., the $J_n(\kappa\rho_1)$ modal coefficients in terms of the $H^{(1)}_n(\kappa\rho')$ modal coefficients. Fortunately, this can be accomplished in a straightforward manner by equating the EM fields written in the C_1 and C coordinate systems.

In the C_1 coordinate system, the EM fields can be expanded in its complete set of cylindrical modes. Likewise, in the C coordinate system the same EM fields can be expanded in its complete set of cylindrical modes. The fields, regardless of which expansion is used, are the same. So the $\overline{E}$ and $\overline{H}$ fields written in each coordinate system can be equated. Before doing this, however, a subtle point should be explained.

The C cylindrical mode expansion is valid only up to a radius ρ' touching the probe antenna, and the C_1 cylindrical expansion is valid only up to a radius ρ_1 touching the test antenna. Thus, at first sight, we might conclude that the EM fields cannot be equated on any complete circle of either the C_1 or C system. Fortunately, the problem disappears under closer scrutiny. Since both the a'^s_n and a'^s_{1n} refer to modes whose sources lie at the test antenna, <u>the part of</u> the EM fields expressed by both expansions are valid for all (ρ_1, ϕ_1, z_1) within any cylinders of radius ρ_1 up to the radius which touches the test antenna. Again, it is noted that this would not be true if $(H^{(1)}_n, H^{(2)}_n)$ functions had been used instead of $(H^{(1)}_n, J_n)$ and the derivation would have become unnecessarily complicated, if performed correctly.

Specifically, the a'^s_n field of system C is given from eq. (17a) as

$$\overline{E}'(\rho', \phi', z') = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} [a'^1_n(\gamma)\overline{M}^{(1)}_{n\gamma}(\rho', \phi', z') + a'^2_n(\gamma)\overline{N}^{(1)}_{n\gamma}(\rho', \phi', z')] d\gamma \ . \tag{31a}$$

Similarly the a'^s_{1n} field of system C_1 is given by

$$\overline{E}_1(\rho_1, \phi_1, z_1) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} [a'^1_{1n}(\gamma)\overline{M}_{n\gamma}(\rho_1, \phi_1, z') + a'^2_{1n}(\gamma)\overline{N}_{n\gamma}(\rho_1, \phi_1, z')] d\gamma \ . \tag{31b}$$

$$(z_1 = z')$$

Now $\overline{M}^{(1)}_{n\gamma}$ is defined in eqs. (5) as

$$\overline{M}^{(1)}_{n\gamma}(\rho', \phi', z') = \nabla'\times \hat{e}_z\, H^{(1)}_n(\kappa\rho')e^{in\phi'}e^{i\gamma z'} \quad . \tag{32}$$

From Graf's addition theorem ([7], formula (9.1.79)) for cylindrical waves, we have the identity

$$H^{(1)}_n(\kappa\rho')e^{in\phi'} = \sum_{m=-\infty}^{\infty} H^{(1)}_{n-m}(\kappa d)\, J_m(\kappa\rho_1)e^{im\phi_1} \ , \tag{33}$$

which allows $\overline{M}^{(1)}_{n\gamma}$ in eq. (32) to be written

$$\overline{M}^{(1)}_{n\gamma}(\rho', \phi', z') = \sum_{m=-\infty}^{\infty} H^{(1)}_{n-m}(\kappa d)\overline{M}_{m\gamma}(\rho_1, \phi_1, z_1) \quad , \tag{34a}$$

since $\nabla'\times = \nabla_1\times$ and $z' = z$. (That $\nabla'\times = \nabla_1\times$ follows from the fact that the curl operator by its integral definition remains invariant under coordinate transformations.) Similarly

$$\overline{N}^{(1)}_{n\gamma}(\rho', \phi', z') = \sum_{m=-\infty}^{\infty} H^{(1)}_{n-m}(\kappa d)\overline{N}_{m\gamma}(\rho_1, \phi_1, z_1) \quad . \tag{34b}$$

Substitution of eqs. (34) into eq. (31a) yields through comparison with eq. (31b) (after interchanging m and n, and using orthogonality (6)) the relationship between the $J_m(\kappa\rho_1)$ and $H^{(1)}_m(\kappa\rho')$ modal coefficients,

$$a'^{s}_{1n} = \sum_{m=-\infty}^{\infty} H^{(1)}_{m-n}(\kappa d)a'^{s}_m \ . \tag{35}$$

Further substitution of a'^{s}_{1n} of eq. (35) into eq. (29) yields (after m and n are again interchanged),

$$b'_o = \sum_{s=1}^{2}\ \sum_{n=-\infty}^{\infty}\ \int_{-\infty}^{\infty} [\sum_{m=-\infty}^{\infty} R'^{s}_{1m}(\gamma)H^{(1)}_{n-m}(\kappa d)]a'^{s}_n(\gamma). \tag{36}$$

Comparing eq. (36) with eq. (30) while remembering that the a'^{s}_n are linearly independent allows us to write the final relationship between the desired receiving functions R'^{s}_n in the cylindrical coordinate system centered outside the probe and the known receiving functions R'^{s}_{1m} in the cylindrical coordinate system centered on the probe:

$$R'^{s}_n(\gamma) = \sum_{m=-\infty}^{\infty} R'^{s}_{1m}(\gamma)H^{(1)}_{n-m}(\kappa d) \ . \tag{37}$$

Equation (37) displays a very interesting and useful property. Even though the probe receiving functions R'^{s}_{1m} in the system centered on itself may be non-negligible for only a few values of m, eq. (37) will give the receiving coefficients $R'^{s}_n(\gamma)$ for much larger values of n. This means that the transformation expressed by eq. (37) can be applied even to very small probes which are described by a small number of cylindrical modes. Also note from eq. (37) that as d gets very small $R'^{s}_n(\gamma)$ may get very large.

Finally, it should be mentioned that most probes used in practice will be reciprocal (see eq. (15a)) so that eq. (37) may be rewritten

$$R'^{s}_{n}(\gamma) = \frac{4\pi\kappa^2}{\eta_o Z_o k} \sum_{m=-\infty}^{\infty} (-1)^m T'^{s}_{1,-m}(-\gamma) H^{(1)}_{n-m}(\kappa d) , \tag{38}$$

where the $T'^{s}_{1m}(\gamma)$ times a'_o are merely the modal coefficients of a probe-centered cylindrical wave expansion of the radiated field of the probe. Of course, the $T'^{s}_{1m} a'_o = B'^{s}_{m\gamma}$ can be computed from the far-electric-field by inverting eq. (4a) or eq. (8a) for $|\gamma| \leq k$ using the orthogonality (6) of the $\overline{M}$ and $\overline{N}$ functions.

Note that if the probe is omni in the ϕ direction $T'^{s}_{1m} = 0$ for $m \neq 0$, and $R'^{s}_{n}(\gamma) = f(\gamma) H^{(1)}_{n}(\kappa d)$ in eq. (38), which is compatible with the ideal dipole probe case, eqs. (7). ($f(\gamma)$ denotes an arbitrary function of γ.)

5. APPLICATIONS OF THE SAMPLING THEOREM AND FAST FOURIER TRANSFORM

Let's assume the receiving characteristics of the probe(s) have been determined by either the "direct measurement method" of section 4.1., or the "transformation method" using eq. (37) or eq. (38). Then after scanning the test antenna with two linearly independent probes (or one probe in two orientations), eqs. (27) yield immediately the transmitting functions $T^{s}_{n}(\gamma)$ of the test antenna. From the values of T^{s}_{n}, the far-field, for example, of the test antenna can be computed by summing eqs. (8) with

$$B^{s}_{n}(k \cos \theta) = T^{s}_{n}(k \cos \theta)\, a_o . \tag{39}$$

There still remains an important question to be answered. How closely must the scan points be spaced, at which the amplitude and phase of the probe outputs $b'_o(\phi_o, z_o)$ and $b''_o(\phi_o, z_o)$ are measured, in order for the integrals $I'_n(\gamma)$ and $I''_o(\gamma)$ in eqs. (27) to be evaluated accurately. This question is answered by looking at the definition of $I'_n(\gamma)$ (or $I''_n(\gamma)$) given in eqs. (26):

$$I'_n(\gamma) = \sum_{s=1}^{2} R'^{s}_{n}(\gamma) T^{s}_{n}(\gamma) = \frac{1}{4\pi^2 a_o} \int_{z_o^{min}}^{z_o^{max}} \int_{0}^{2\pi} b'_o(\phi_o, z_o) e^{-in\phi_o} e^{-i\gamma z_o}\, d\phi_o dz_o . \tag{40}$$

(The $\pm\infty$ limits of integration for z_o have been replaced by (z_o^{min}, z_o^{max}) because in practice the probe output $b'_o(\phi_o, z_o)$ must be assumed negligible outside a certain finite scan length.)

Consider the transmitting functions $T^{s}_{n}(\gamma)$ of the test antenna. They represent the coefficients of the cylindrical waves emanating from the test antenna (with $a_o = 1$) transmitting into free space. The subscript n refers to the n^{th} order Hankel function,

$$H^{(1)}_{n}(\kappa\rho) \qquad \kappa^2 = k^2 - \gamma^2 .$$

It can be shown [8] that the reactive energy in the cylindrical modes $\overline{M}^{(1)}_{n\gamma}$, $\overline{N}^{(1)}_{n\gamma}$ containing the Hankel function $H^{(1)}_{n}(\kappa\rho)$ grows extraordinarily large whenever n^2 becomes much greater than $(\kappa\rho)^2$. This result implies that the reactive fields or, equivalently, the Q of the antenna is extremely large unless the $T_n^s(\gamma)$ grow extremely small for $n^2>(\kappa a)^2$, where "a" is the radius of the smallest cylinder circumscribing the test antenna. Ordinarily, antennas do not have a high Q, and thus the $T_n^s(\gamma)$ will become negligible not far outside the circle shown in figure 4 and defined by

$$n^2 = (\kappa a)^2 = (k^2 - \gamma^2)a^2 \ , \tag{41}$$

or

$$n^2 + (\gamma a)^2 = (ka)^2 \ .$$

There is a reasonable physical explanation accompanying the result (41). Equations (8) show that the values of γ nearer to k correspond to the far-field on a circle ($o\leq\phi<2\pi$) nearer to the z-axis ($\theta = \cos^{-1}\frac{\gamma}{k}$). Assuming the number of far-field fluctuations per solid angle are roughly the same in all directions, the far-field variation with ϕ will be lesser for circles nearer the z-axis (larger γ). Thus the number of cylindrical modes needed to expand the field will grow less and less as γ approaches k, since the higher order modes, which represent the more rapidly varying parts of the field, will not be needed to expand the slower variations with ϕ. Equation (41) merely expresses quantitatively this intuited result. Specifically, eq. (41) says that the maximum number of modes needed to represent the far-field at the angle θ is approximately ka sin θ.

Applying this information about $T_n^s(\gamma)$ to eq. (40) reveals that $b_o'(\phi_o,z_o)$ can be considered the double Fourier transform of a band-limited function - provided the separation distance d between test and probe antenna is large enough to insure the values of $R_n'^s$ do not grow extraordinarily large for $n^2>(\kappa a)^2$ (see eq. (37) or eq. (38)). Consequently, the sampling theorem [9] can be utilized to convert the integration on the right side of eq. (40) to the following summation:

$$I_n'(\gamma) = \frac{1}{4k_1k_2aa_o} \sum_{j,\ell} b_o'(\phi_o^j, z_o^\ell)e^{-in\phi_o^j} e^{i\gamma z_o^\ell} \ , \tag{42a}$$

where ϕ_o^j and z_o^ℓ refer to points taken at increments of $\lambda_1/2a$ and $\lambda_2/2$ radians, respectively. Similarly, $I_n''(\gamma)$ may be written

$$I_n''(\gamma) = \frac{1}{4k_1k_2aa_o} \sum_{j,\ell} b_o''(\phi_o^j, z_o^\ell)e^{-in\phi_o^j} e^{-i\gamma z_o^\ell} \ . \tag{42b}$$

The values of $k_1 = 2\pi/\lambda_1$ and $k_2 = 2\pi/\lambda_2$ need not be chosen much larger than the free-space wave number $k = 2\pi/\lambda$, as long as the separation distance between test and probe antenna remains large enough to keep the $R_n'^s(\gamma)$ of eq. (37) or eq. (38) from growing extraordinarily large for $|n| > |\kappa a|$. It is difficult to get a reasonable estimate from eq. (37) or eq. (38) of how large the separation distance d should be. However, experience with planar near-field scanning as well as with hypothetical problems which are solvable analytically suggests that k_1, k_2 can be chosen approximately equal to k provided the probe and test antennas remain more than a few wavelengths apart (i.e., outside the reactive fields) as the probe scans on the surface of the cylinder enclosing the test antenna.

In summary, eqs. (42) show that the amplitude and phase of the output of the probe(s) need be sampled only at discrete points on the surface of the scan cylinder. As long as the probe does not get within a few wavelengths of the test antenna, the data point spacing need be no closer than about $\lambda/2a$ for the ϕ_o angular separation and $\lambda/2$ for the z_o increments. After the output of the probe(s) is recorded for the entire cylindrical grid of data points, the "fast Fourier transform" (FFT) can be utilized to compute $I'_n(\gamma)$ and $I''_n(\gamma)$ from eqs. (42). From the values of $(I'_n(\gamma), I''_n(\gamma))$ and the receiving functions of the probe, the transmitting function $T^s_n(\gamma)$ of the test antenna can be evaluated immediately from eqs. (27). Finally the entire electromagnetic field outside the test antenna is determined by substituting $B^s_n(\gamma) = T^s_n(\gamma)\, a_o$ into eqs. (4) or into eqs. (8) for just the far-fields.

6. SUMMARY

The basic theory and formulas needed to implement the probe-corrected measurement of antennas by scanning on a cylinder have been derived by a systematic approach based on the cylindrical-wave source scattering-matrix description of antennas. This approach, which parallels the plane-wave scattering matrix approach used by Kerns [3], proves simple and straightforward and does not require the probe or test antenna to be reciprocal.

The derivation began by expressing the EM fields radiating from the test antenna in a complete set of cylindrical vector wave solutions $(\overline{M}, \overline{N})$ with first Hankel functions $(H^{(1)}_n)$ for the radial dependence. The radiation characteristics of the test antenna are determined by the weighting coefficients of these cylindrical waves, which in themselves remain independent of the particular antenna. Expressions were derived for the far-fields (a technique was introduced in appendix A for deriving the far-fields by direct manipulation of the Fourier transforms rather than by the more involved method of stationary phase), power gain function, complex polarization ratio, and total radiating power of the test antenna in terms of the cylindrical waves and the weighting coefficients $B^s_n(\gamma)$. Thus the problem of determining the radiation characteristics of antennas by cylindrical scanning reduces to that of determining the modal coefficients $B^s_n(\gamma)$.

If an ideal dipole antenna were available to measure the transverse electric field on a cylinder about the test antenna, the coefficients $B^s_n(\gamma)$ were shown to emerge immediately through a trivial application of the orthogonality relationships for the vector wave solutions $(\overline{M}, \overline{N})$. When an arbitrary probe antenna was inserted into the picture, a second set of cylindrical waves identical to the first, except for Bessel functions (J_n) replacing $H^{(1)}_n$, was required to express the EM fields excited by the sources on the probe. The coefficients of this second set of J_n cylindrical modes along with the input to the test antenna were related to the coefficients of the first set of $H^{(1)}_n$ modes and the output of the test antenna through the "source scattering matrix" for the test antenna.

The source scattering matrix contains a complete and convenient description of the transmitting, receiving, scattering, and reflection properties of the test antenna. The transmitting elements of the matrix essentially equals the required modal coefficients $B_n^s(\gamma)$. Reciprocity was also shown to display itself conveniently as a relationship between the elements of the scattering matrix.

A similar source scattering matrix was written for the probe antenna but with reference to the origin or center of the coordinate system situated in the test antenna. This "common-center" approach along with the use of $(H_n^{(1)}, J_n)$ scattering matrices greatly facilitated the subsequent derivation of the transmission formula, which contains the required coupling product between the transmitting coefficients of the test antenna and receiving coefficients of the probe antenna. Another advantage of the common-center source scattering-matrix approach was that it clearly and completely separated the derivation of the transmission formula from the problem of determining the receiving characteristics of the probe.

The characterization of the probe was dealt with as an isolated topic in section 4. It was found that the receiving coefficients of the probe could be either measured directly, or transformed from its measured far-field. The specific transformation involved depends on the choice of coordinate system (not necessarily cylindrical) in which the far-field of the probe is expressed. To conform to the work of Leach and Paris [2], however, cylindrical coordinates were chosen to express the far-field of the probe and the transformation was accomplished with the help of Graf's addition theorem.

Finally, it was shown that the transmission formula could be inverted to yield the transmitting coefficients of the test antenna, and that the inversion could be accomplished efficiently through the use of the sampling theorem and FFT algorithm. For reasonably low Q antennas the number of cylindrical modes needed to expand the nonreactive field was shown to depend on the elevation angle $(90-\theta)$, and was given approximately as $ka \sin \theta$ where "a" is the radius of the smallest cylinder circumscribing the test antenna.

APPENDIX A. DERIVATIÓN OF THE FAR-FIELDS IN TERMS OF THE CYLINDRICAL WAVE COEFFICIENTS

In this appendix the far-fields radiated by a test antenna into free space (eqs. (8) of the main text) are derived in terms of the coefficients of the cylindrical wave expansion (eqs. (4) of the main text). Although Leach and Paris [2] have derived the results using the method of steepest descent, the alternative derivation shown here is interesting enough in itself to warrant a separate appendix.

The starting point for the derivation is eq. (4a) for the $\overline{E}$ field outside the test antenna expressed in a cylindrical wave expansion. If $\overline{M}$ and $\overline{N}$ from eqs. (5) are substituted into eq. (4a), and the large argument value of the Hankel function,

$$H_n^{(1)}(\kappa\rho) \sim (-i)^n \sqrt{\frac{2}{\pi\kappa\rho}}\, e^{i(\kappa\rho - \frac{\pi}{4})} \quad ,$$

is used in $\overline{M}$ and $\overline{N}$, the far-electric-field can be written

$$\underset{r\to\infty}{\overline{E}(\rho,\phi,z)} = \int_{-\infty}^{\infty} \kappa\overline{\varepsilon}_\gamma(\rho,\phi)\, \frac{e^{i\kappa\rho}}{\kappa}\, e^{i\gamma z} d\gamma \quad , \tag{A1}$$

where

$$\overline{\varepsilon}_\gamma(\rho,\phi) = \sqrt{\frac{2}{\pi\kappa\rho}}\, e^{-\frac{\pi}{4}i} \sum_{n=-\infty}^{\infty} (-i)^n [B_n^1(\gamma)\overline{M}_{n\gamma} + B_n^2(\gamma)\overline{N}_{n\gamma}] e^{in\phi} \quad , \tag{A2}$$

with

$$\overline{M}_{n\gamma} \equiv i(\frac{n}{\rho}\,\hat{e}_\rho - \kappa\hat{e}_\phi)$$

$$\overline{N}_{n\gamma} \equiv \frac{1}{k}(-r\kappa\hat{e}_\rho - \frac{n\gamma}{\rho}\,\hat{e}_\phi + \kappa^2\hat{e}_z) \quad . \tag{A3}$$

Symbolically, eq. (A1) may be written

$$\overline{E}_{r\to\infty} = F^{-1}[(\kappa\overline{\varepsilon}_\gamma)\, \frac{e^{i\kappa\rho}}{\kappa}] \quad , \tag{A4}$$

where F and F^{-1} will be used to denote the Fourier transform and its inverse, respectively. The convolution theorem transforms eq. (A4) to

$$\underset{r\to\infty}{\overline{E}(\overline{r})} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_\xi^{-1}[\kappa\overline{\varepsilon}_\gamma] F_{\xi-z}^{-1}[\frac{e^{i\kappa\rho}}{\kappa}] d\xi \quad . \tag{A5}$$

The second inverse Fourier transform in eq. (A5) is nothing more than an integral representation of the zeroth order Hankel function of the first kind, i.e.

$$F_{\xi-z}^{-1}[\frac{e^{i\kappa\rho}}{\kappa}] = \int_{-\infty}^{\infty} \frac{e^{i\kappa\rho}}{\kappa}\, e^{i\gamma(\xi-z)} d\gamma = \pi H_0^{(1)}(k\sqrt{(\xi-z)^2+\rho^2}) \underset{r\to\infty}{\sim} \sqrt{\frac{2\pi}{kr}}\, e^{-\frac{\pi}{4}i} e^{ikr} e^{-ik\xi\cos\theta} \quad .$$

Thus, the far-electric-field may be written

$$\bar{E}(\bar{r})_{r\to\infty} = \sqrt{\frac{1}{2\pi kr}}\, e^{-\frac{\pi}{4}i} e^{ikr} \int_{-\infty}^{\infty} F_{\xi}^{-1}[\kappa\bar{\varepsilon}_{\gamma}]e^{-ik\xi \cos\theta} \quad . \tag{A6}$$

But this last integral is the Fourier transform of an inverse Fourier transform. Thus, the result is the original function (multiplied by 2π) evaluated at $k \cos \theta$, i.e.,

$$\bar{E}(\bar{r})_{r\to\infty} = \sqrt{\frac{2\pi}{kr}}\, e^{-\frac{\pi}{4}i} e^{ikr} k \sin\theta\, \bar{\varepsilon}_{\gamma}(\gamma=k \cos\theta)$$

$$(\kappa = \sqrt{k^2-\gamma^2} = k \sin\theta \text{ when } \gamma=k\cos\theta) \quad . \tag{A7}$$

Substitution of $\bar{\varepsilon}_{\gamma}(\gamma=k \cos \theta)$ from its definition in eq. (A2) gives the required far-field expression for $\bar{E}$ shown in eqs. (8) of the main text,

$$\bar{E}(r,\phi,\theta)_{r\to\infty} = -2k \sin\theta \frac{e^{ikr}}{r} \sum_{n=-\infty}^{\infty} (-i)^n [B_n^1(k \cos\theta)\hat{e}_{\phi} - iB_n^2(k\cos\theta)\hat{e}_{\theta}]e^{in\phi} \tag{A8a}$$

$$\bar{H}(r,\phi,\theta)_{r\to\infty} = \frac{\hat{e}_r \times \bar{E}_{r\to\infty}}{Z_o} \quad . \tag{A8b}$$

Finally it is mentioned that the same technique can be utilized to find the far-fields of a radiator in terms of its plane-wave spectrum, i.e., to derive eqs. (1.2-16) of reference [3b].

APPENDIX B. DERIVATION OF THE SOURCE SCATTERING MATRIX EQUATIONS AND RECIPROCITY RELATIONS FOR THE PROBE AND TEST ANTENNAS

Refer first to the test antenna of fig. 1. The reference surface S_o is a dividing plane between the "test antenna proper" and the <u>shielded</u> "generator" of the test antenna. Let S_ρ be the surface of the cylinder of radius ρ surrounding the test antenna.

In the region between the generator and the surface S_ρ assume that the EM fields satisfy the following very general form of Maxwell's equations

$$
\begin{aligned}
\nabla x\overline{E} - i\omega\overline{B} &= 0 \\
\nabla x\overline{H} + i\omega\overline{D} - \overline{\overline{\sigma}}\cdot\overline{E} &= 0 \\
\overline{D} &= \overline{\overline{\varepsilon}}\cdot\overline{E} + \overline{\overline{\tau}}\cdot\overline{H} \\
\overline{B} &= \overline{\overline{\nu}}\cdot\overline{E} + \overline{\overline{\mu}}\cdot\overline{H} \quad .
\end{aligned} \tag{B1}
$$

Consider the "adjoint fields" which satisfy the adjoint set of equations,

$$
\begin{aligned}
\nabla x\overline{E}^a - i\omega\overline{B}^a &= 0 \\
\nabla x\overline{H}^a - i\omega\overline{D}^a - \overline{\overline{\sigma}}_t\cdot\overline{E}^a &= 0 \\
\overline{D}^a &= \overline{\overline{\varepsilon}}_t\cdot\overline{E}^a - \overline{\overline{\nu}}_t\cdot\overline{H}^a \\
\overline{B}^a &= -\overline{\overline{\tau}}_t\cdot\overline{E}^a + \overline{\overline{\mu}}_t\cdot\overline{H}^a \quad ,
\end{aligned} \tag{B2}
$$

where the subscript t designates the transposed dyadic. A straightforward manipulation of eqs. (B1) and (B2) yields an "adjoint reciprocity lemma" [3b]

$$
\oint_{S_o+S_\rho} (\overline{E}^a x\overline{H} - \overline{E}x\overline{H}^a)\cdot\hat{n}da = 0 \quad , \tag{B3}
$$

where $(\overline{E},\overline{H})$ are <u>any</u> fields satisfying eqs. (B1) and $(\overline{E}^a, \overline{H}^a)$ are <u>any</u> fields satisfying eqs. (B2).

On the waveguide reference surface S_o

$$
\overline{E}_t = (a_o+b_o)\hat{e}_o \quad , \quad \overline{H}_t = \eta_o(a_o-b_o)\hat{h}_o
$$

$$
\overline{E}^a_t = (a^a_o+b^a_o)\hat{e}_o \quad , \quad \overline{H}^a_t = \eta_o(a^a_o-b^a_o)\hat{h}_o
$$

$$
\int_{S_o} \hat{e}_o x\hat{h}_o\cdot\hat{n}da = 1 \quad ,
$$

and thus eq. (B3) becomes

$$
2\eta_o(b^a_o a_o - b_o a^a_o) = \int_{S_\rho} (\overline{E}^a x\overline{H} - \overline{E}x\overline{H}^a)\cdot\hat{e}_\rho da \quad . \tag{B4}
$$

Divide the fields into fields emanating from sources at the test antenna (denoted by subscript "o") and fields emanating from the probe (denoted by subscript "1"). Then

$$\int_{S_\rho} (\overline{E}^a x\overline{H}-\overline{E}x\overline{H}^a)\cdot\hat{e}_\rho da = \int_{S_\rho} (\overline{E}^a_o x\overline{H}_1-\overline{E}_o x\overline{H}^a_1+\overline{E}^a_1 x\overline{H}_o-\overline{E}_1 x\overline{H}^a_o)\cdot\hat{e}_\rho da \quad . \tag{B5}$$

The integrals over the "o" fields alone and over the "1" fields alone vanish because they satisfy the adjoint reciprocity lemma separately. Substitution of eq. (B5) into eq. (B4) gives an expression which will be called the <u>source adjoint reciprocity lemma</u>,

$$2\eta_o(b^a_o a_o-b_o a^a_o) = \int_{S_\rho} (\overline{E}^a_o x\overline{H}_1-\overline{E}_o xH^a_1+\overline{E}^a_1 x\overline{H}_o-\overline{E}_1 x\overline{H}^a_o)\cdot\hat{e}_\rho da \quad . \tag{B6}$$

If the $\overline{E}$, $\overline{H}$ fields in the coordinate system of the test antenna given by eqs. (13) are substituted into the right side of eq. (B6), it becomes

$$\eta_o(b^a_o a_o-b_o a^a_o) = \frac{4\pi\kappa^2}{Z_o k}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}(-1)^n\sum_{S=1}^{2}[b^s_n(\gamma)a^{as}_{-n}(-\gamma)-b^{as}_{-n}(-\gamma)a^s_n(\gamma)]d\gamma \ . \tag{B7}$$

To get eq. (B7) in the form shown, the orthogonality relations (6) between the $\overline{M}$ and $\overline{N}$ functions and the Wronskian for the Bessel functions were used ([7], Formulas 9.1.15–9.1.17)

Recall that $a^s_n(\gamma)$ and $a^{as}_n(\gamma)$ are coefficients of modes with arbitrary sources outside the test antenna and its adjoint, respectively. Thus, $a^s_n(\gamma)$ and $a^{as}_n(\gamma)$ may be chosen independently and arbitrarily. Similarly, a_o and a^a, the inputs to the test antenna and its adjoint, may be chosen independently and arbitrarily.

In particular, if we choose

$$a^a_o = 1 \qquad\qquad a^{as}_n = 0$$

eq. (B7) yields

$$b_o = b^a_o a_o + \frac{4\pi\kappa^2}{\eta_o Z_o k}\sum_{s=1}^{2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}(-1)^n b^{as}_{-n}(-\gamma)a^s_n(\gamma)d\gamma \quad . \tag{B8}$$

But this is nothing more than the source scattering matrix equation (14b) for the test antenna with

$$\Gamma_o = b^a_o$$

$$R^s_n(\gamma) = \frac{4\pi\kappa^2}{\eta_o Z_o k}(-1)^n\, b^{as}_{-n}(-\gamma) \quad .$$

In a similar manner the second source scattering matrix equation (14b) emerges if we choose

$$a^a_o = 0 \quad , \quad a^{as}_n = \delta_{sq}\delta_{nm}\delta(\gamma-\beta) \quad .$$

In other words, the source scattering matrix equations (14) derived from Maxwell's equations through the source adjoint reciprocity lemma (B6).

Equation (B6) also reveals the reciprocity relationships between the source scattering matrix of the test antenna and it adjoint. When eqs. (14) are inserted into eq. (B7) the necessary and sufficient conditions for eq. (B7) to be satisfied for all values of the "a" coefficients are

$$\Gamma_o^a = \Gamma_o \qquad \text{(a)}$$

$$R_{-n}^{as}(-\gamma) = (-1)^n \frac{4\pi\kappa^2}{\eta_o Z_o k} T_n^s(\gamma) \qquad \text{(b)}$$

$$T_{-n}^{as}(-\gamma) = (-1)^n \frac{\eta_o Z_o k}{4\pi\kappa^2} R_n^s(\gamma) \qquad \text{(c)}$$

$$S_{-m,-n}^{aqs}(-\beta,-\gamma) = S_{nm}^{sq}(\gamma,\breve{\beta}) \quad . \qquad \text{(d)} \qquad \text{(B9)}$$

By definition, a reciprocal antenna is one in which the antenna and its adjoint are identical. Thus for a reciprocal antenna the adjoint superscript "a" in eqs. (B9) are removed and eq. (B9b) becomes identical to eq. (B9c). (Also, eq. (B9a) becomes a trivial identity.)

We have just completed the derivation of the source scattering matrix equations (14) and their reciprocity relationships (15) for the test antenna directly from Maxwell's equations through the source adjoint reciprocity lemma (B6). For the probe antenna we can proceed in the same fashion as for the test antenna, but this time the region of interest lies between S_ρ and the probe generator which is separated from the probe antenna proper by the waveguide reference surface S_o'. Applying the adjoint reciprocity lemma to this external region yields a surface integral similar to eq. (B3),

$$\oint_{S_o'+S_\rho+S_\infty} (\overline{E}'^a \times \overline{H}' - \overline{E}' \times \overline{H}'^a)\cdot\hat{n}'da' = 0 \quad , \qquad \text{(B10)}$$

where the primes have been used to emphasize the fact that we are dealing with the probe antenna. Immediately, the radiation condition at infinity eradicates the integral over S_∞, and eq. (B10) becomes identical in form to eq. (B3).

Substitution of the EM fields of the probe waveguide on S_o' into eq. (B10), and division of the fields on S_ρ into those emanating from the test and probe antennas yield an equation identical in form (except for a negative sign) to eq. (B6), i.e., a source adjoint reciprocity lemma for the probe antenna is derived. Moreover, substitution of the $\overline{E}'$ $\overline{H}'$ fields from eq. (17) gives an expression identical in form to eq. (B7) but with the primed coefficients of the probe system. Continuing the derivation as for the test antenna gives rise to the source scattering matrix eqs. (18) for the probe, and reciprocity relationships for the probe identical in form to eqs. (B9). That is, the reciprocity relationships for a reciprocal probe are

$$R'^{s}_{-n}(-\gamma) = (-1)^n \frac{4\pi\kappa^2}{\eta_o Z_o k} T'_n(\gamma) \tag{B11a}$$

$$S'^{qs}_{-m,-n}(-\beta,-\gamma) = S'^{sq}_{nm}(\gamma,\beta) \quad . \tag{B11b}$$

Finally it is mentioned that the derivations for the plane-wave scattering matrices and their reciprocity relationships for antennas have been extended to electroacoustic transducers [10]. The analysis in reference [10] can be utilized immediately to extend the results of this appendix to the cylindrical scanning of electroacoustic transducers.

GLOSSARY OF SYMBOLS

a: Radius of smallest cylinder circumscribing the test antenna.

$\left\{\begin{matrix} a_o, a'_o \text{ or } a''_o \\ \\ b_o, b'_o \text{ or } b''_o \end{matrix}\right\} : \left\{\begin{matrix} \text{Ingoing} \\ \\ \text{Outgoing} \end{matrix}\right\}$ wave-coefficients in the test and probe antenna feeds, respectively.

$\left\{\begin{matrix} a_n^s, a_n'^s, a_{1n}'^s \\ \\ b_n^s, b_n'^s, b_{1n}'^s \end{matrix}\right\} : \left\{\begin{matrix} \text{Outside} \\ \\ \text{Inside} \end{matrix}\right\}$ source cylindrical mode coefficients for the test antenna, for the probe with respect to the C coordinate system and for the probe with respect to the C_1 coordinate system, respectively.

B_n^s : Radiated cylindrical mode coefficients for the test antenna.

$C(\rho', \phi', z')$, $C_1(\rho_1, \phi_1, z_1)$: The two cylindrical coordinate systems fixed in the probe; the first (C) centered on the test antenna and the second (C_1) centered on the probe (see fig. 3).

d : Distance between the z-axes of the C and C_1 system (see Fig. 3).

$(\overline{E}, \overline{H}), (\overline{E}', \overline{H}'), (\overline{E}_1, \overline{H}_1)$: Complex electric and magnetic fields referred to in the test antenna, the probe-C, and the probe-C_1 coordinate systems, respectively.

$\hat{e}_\alpha$: Unit vector for coordinate α.

γ : Fourier transform parameter for the z-part of the cylindrical modes ($-\infty < \gamma < \infty$).

Γ_o, Γ'_o : The test and probe antenna input reflection coefficients defined at the surface S_o and S'_o, respectively.

$H_n^{(1)}$, $H_n^{(2)}$: Cylindrical Hankel functions of the first and second kind.

$H_n^{(1)'}$: Derivative with respect to argument of the Hankel function of the first kind.

I'_n, I''_n : Inversion integrals of the transmission formula (defined by the right side of eq. (26a) and (26b), respectively).

J_n : Cylindrical Bessel functions.

k : Free-space wave number = $2\pi/\lambda$.

κ : $(k^2-\gamma^2)^{1/2}$, taken positive real when $\gamma < k$, and positive imaginary when $\gamma > k$.

λ : Free-space wavelength.

$(\overline{M}_{n\gamma}, \overline{N}_{n\gamma})$, $(\overline{M}^{(1)}_{n\gamma}, \overline{N}^{(1)}_{n\gamma})$: Cylindrical vector wave solutions with J_n and $H^{(1)}_n$ radial dependence, respectively.

η_o : Characteristic admittance of the propagated mode in the test antenna feed.

(ϕ_o, z_o) : Cylindrical coordinates describing the position of the probe as it scans the test antenna (see fig. 2).

(ρ,ϕ,z), (r,ϕ,θ) : Cylindrical and spherical coordinates, respectively, fixed in the test antenna (see fig. 1).

R^s_n, R'^s_n or R''^s_n, R'^s_{1n} : Cylindrical receiving functions for the test antenna, for the probe with respect to the C coordinate system, and for the probe with respect to the C_1 coordinate system, respectively.

T^s_n, T'^s_n or T''^s_n, T'^s_{1n} : Cylindrical transmitting functions for the test antenna, for the probe with respect to the C coordinate system, and for the probe with respect to the C_1 coordinate system, respectively.

S^{sq}_{nm}, S'^{sq}_{nm} : Scattering functions for the test and probe antenna, respectively.

S_o, S'_o : The waveguide reference surfaces for the test and probe antenna, respectively.

ω : Angular velocity in the suppressed time factor $e^{-i\omega t}$ $(\omega > 0)$.

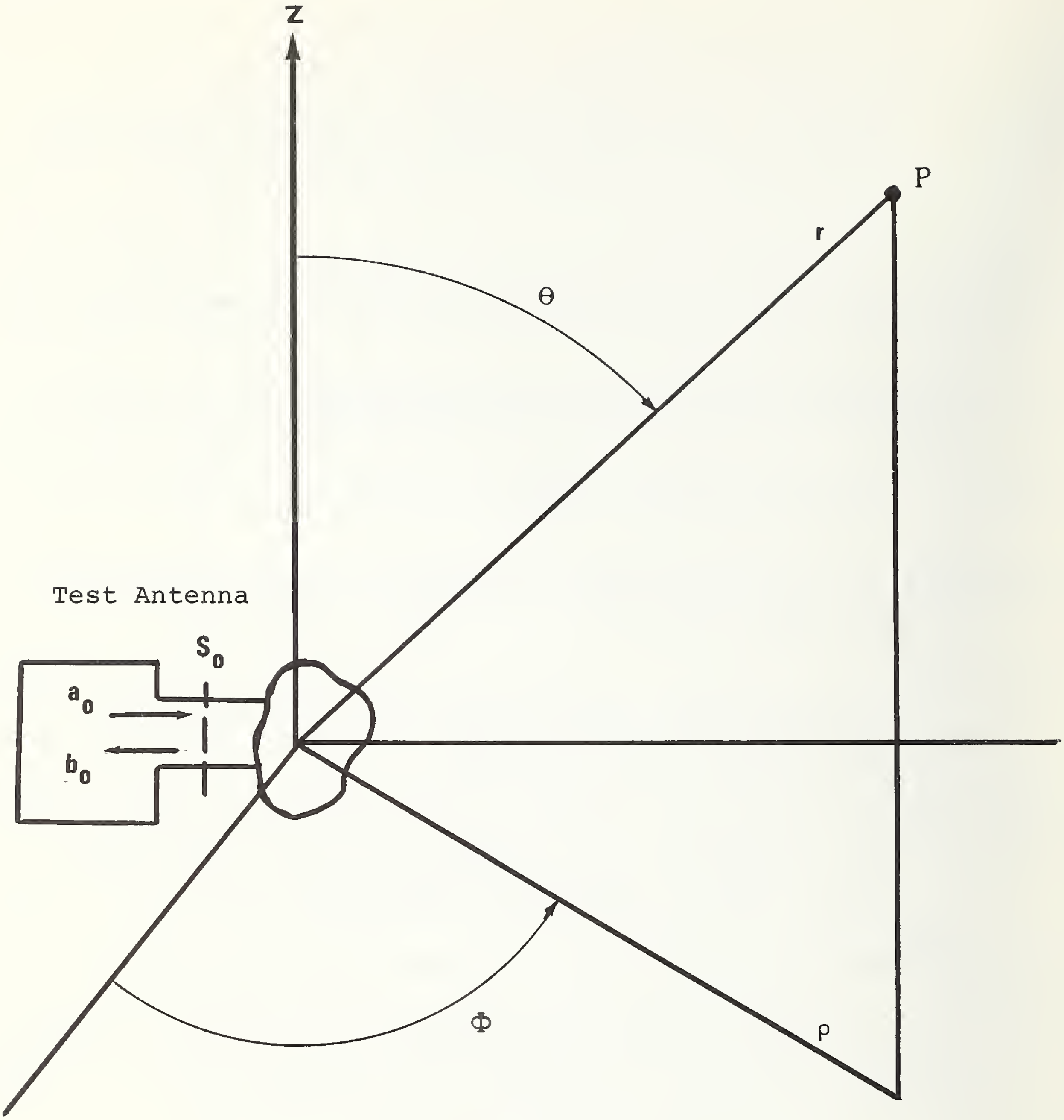

Figure 1. Schematic of test antenna and its coordinate system.

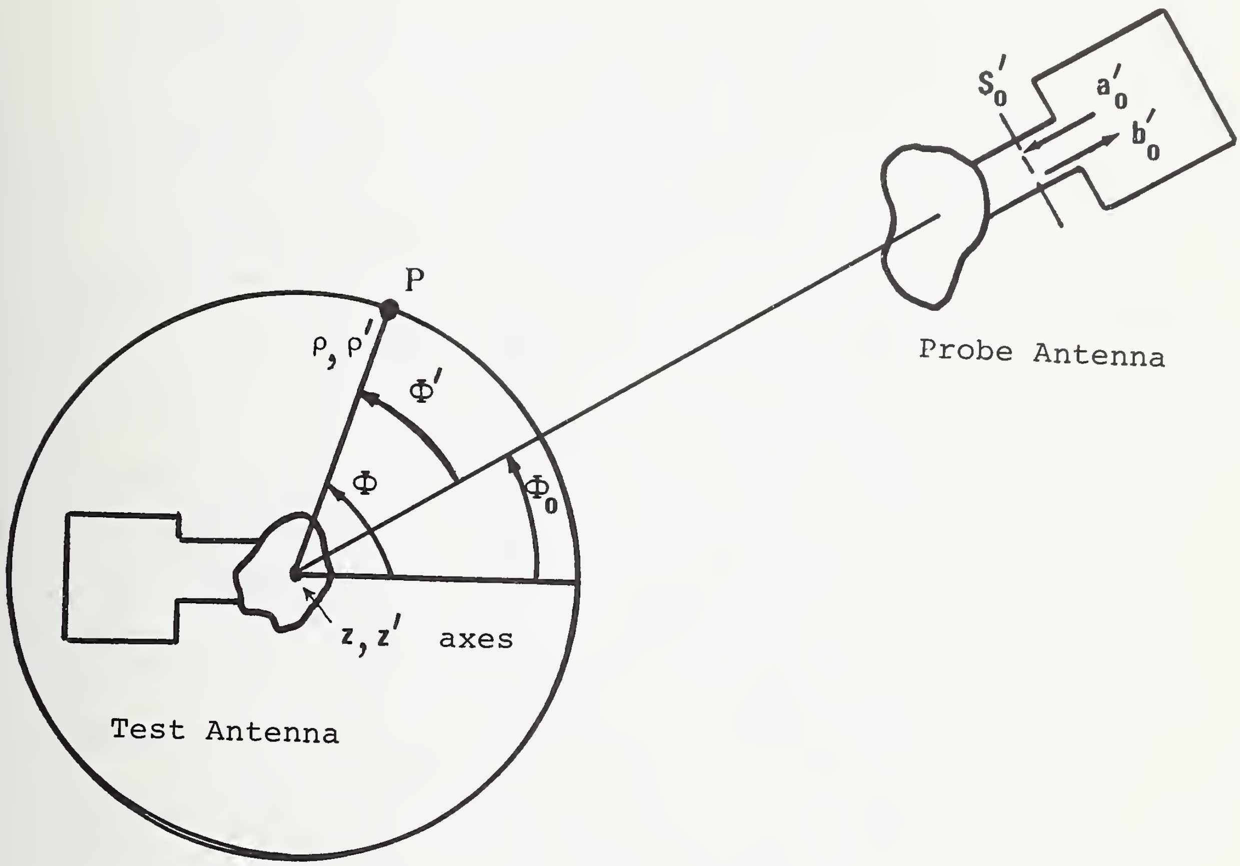

Figure 2. Schematic of probe antenna and its coordinate system.

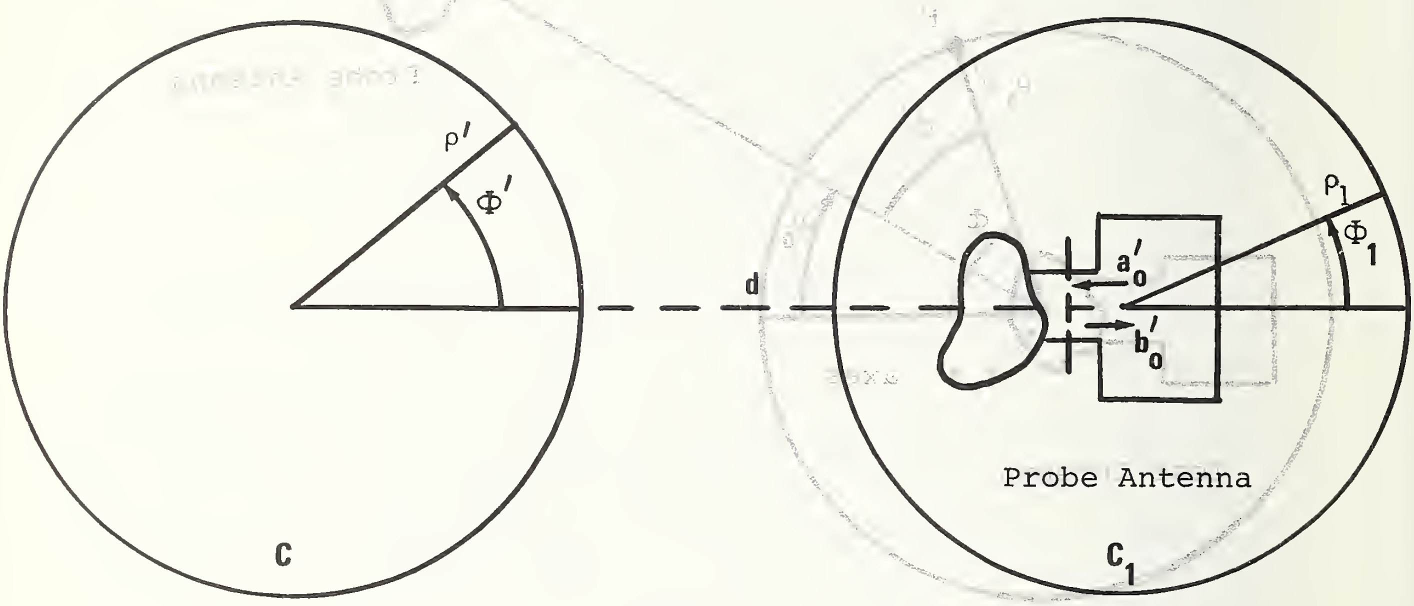

Figure 3. Schematic of the two cylindrical coordinate systems fixed in the probe.

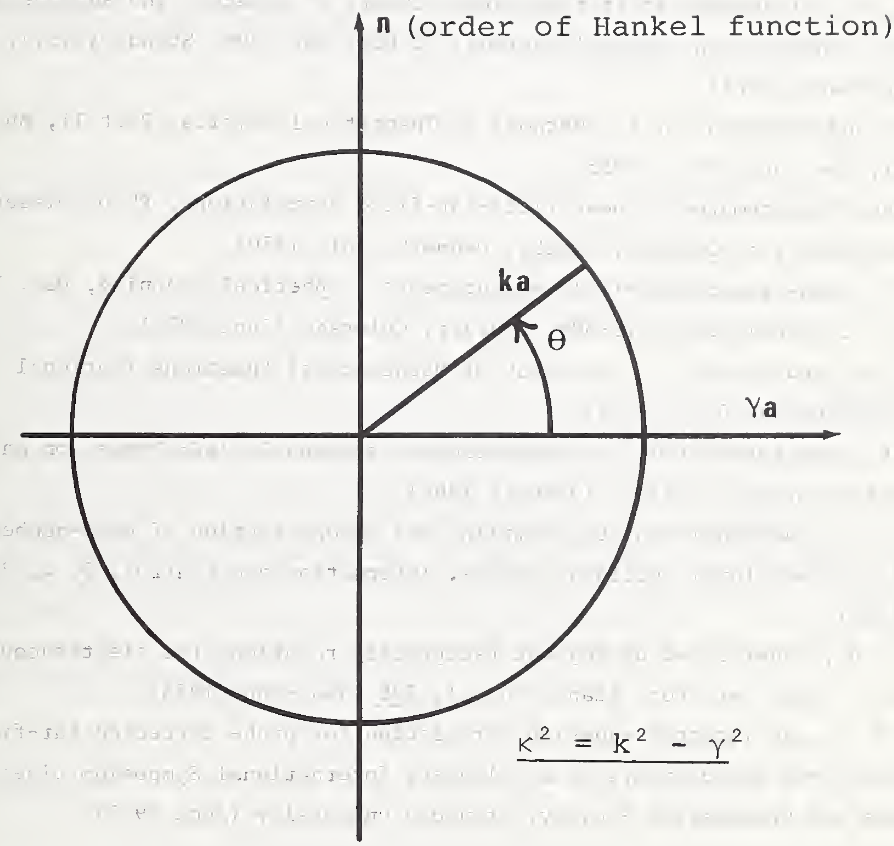

Figure 4. Transmitting coefficient becomes negligible not far outside the circle of radius ka.

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15. SUPPLEMENTARY NOTES

16. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

The theory for probe-corrected measurement of antennas by scanning on a circular cylindrical surface enclosing the test antenna in the near-field is formulated from a source scattering matrix description of the test and probe antennas. The basic transmission formula is derived without recourse to reciprocity, and from a common center approach which separates as an isolated problem the probe characterization and transformation. Moreover, it is shown how an experimental technique can, in principle, determine the required transformed probe coefficients without the use of addition theorems. Computer inversion of the transmission formula is accomplished accurately and efficiently with the aid of the sampling theorem and FFT algorithm.

17. KEY WORDS *(six to twelve entries; alphabetical order; capitalize only the first letter of the first key word unless a proper name; separated by semicolons)* Cylindrical scanning; near-field measurements; source-scattering matrix.

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