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DYNAMIC MEASUREMENTS OF THE MAGNETOELASTIC PROPERTIES OF FERRITES

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U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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ABSTRACT

The relations between the mechanical and magnetic properties of a ferrite are derived for the small signal or reversible condition using thermodynamic principles. The equation of motion of a ferrite ring driven in its fundamental mode is set up and solved leading to the equivalent electrical circuit of the magnetostriction resonator. From mechanical measurements of the density and dimensions of the ring and electrical and frequency measurements of the resonator, the elastic modulus, permeability, magnetostriction coefficients and loss factors in the ferrite can be determined. Apparatus is proposed for performing the above measurements.

TABLE OF SYMBOLS

Symbol	Definition	Units
A	Area of cross section of ring	meter^2
A_o	Area of air space between ring and toroid	meter^2
a	Mean radius of ring	meter
B_o	Flux density in air = $\mu_o H$	weber/meter^2
B	Flux density or magnetic induction	weber/meter^2
$C = C_m$	Motional capacitance of equivalent network	farad
$d = \left(\frac{\partial \varepsilon}{\partial H}\right)_\sigma = \left(\frac{\partial B}{\partial \sigma}\right)_H$	Magnetostriction strain coefficient	weber/newton
$d' = \left(\frac{\partial \varepsilon}{\partial B}\right)_\sigma = \left(\frac{\partial H}{\partial \sigma}\right)_B$	Magnetostriction strain coefficient	ampere meter/newton
e	Instantaneous voltage	volt
f_o	Frequency at resonance (no damping)	cycles/second
f_a	Frequency at parallel resonance (reactance zero)	cycles/second
f_s	Frequency at series resonance (reactance zero)	cycles/second
H	Magnetic field intensity	ampere/meter
I_o	Maximum instantaneous current	ampere
j	$\sqrt{-1}$	(none)
k	Magnetoelastic coupling coefficient	(none)
$K_m = \frac{\mu}{\mu_o}$	Relative permeability	(none)
$L = L_m$	Motional inductance of equivalent circuit	henry

TABLE OF SYMBOLS
(Cont.)

Symbol	Definition	Units
L_c	Inductance with ferrite clamped	henry
N	Number of turns on toroid	---
$q = \left(\frac{\partial\sigma}{\partial\varepsilon}\right)_H$	Elastic (Young's) modulus at constant field	newton/meter^2
$q' = \left(\frac{\partial\sigma}{\partial\varepsilon}\right)_B$	Elastic modulus at constant induction	newton/meter^2
Q	Quality factor	(none)
r	Damping coefficient	kg/meter^2 second
$R = R_m$	Motional resistance of equivalent network	ohms
R_c	Resistance of circuit with ferrite clamped	ohms
t	Time	second
W	Work or energy	joule = newton meter
Z_a	Impedance of equivalent circuit at parallel resonance	ohms
Z_s	Impedance of equivalent circuit at series resonance	ohms
Z_m	Motional impedance of equivalent circuit	ohms
Z_c	Impedance of circuit with ferrite clamped	ohms
ε	Strain	(none)
ε_o	Maximum strain amplitude	(none)
ϕ	Magnetic flux	weber
γ	$\left(\frac{q}{a} - \rho a\omega^2 + j\omega r\right)$	newton/meter^3

TABLE OF SYMBOLS
(Cont.)

Symbol	Definition	Units
$\lambda = \left(\frac{\partial\sigma}{\partial H}\right)_\varepsilon = \left(\frac{\partial B}{\partial \varepsilon}\right)_H$	Magnetostriction stress coefficient	newton/ ampere meter
$\lambda' = \left(\frac{\partial\sigma}{\partial B}\right)_\varepsilon = \left(\frac{\partial H}{\partial \varepsilon}\right)_B$	Magnetostriction stress coefficient	newton/ weber
μ_o	Permeability of space (air) (12.57×10^{-7})	weber/ ampere meter
μ^s	Static permeability	weber/ ampere meter
μ^m	Motional permeability	weber/ ampere meter
$\mu' = \left(\frac{\partial B}{\partial H}\right)_\sigma$	Permeability at constant stress	weber/ ampere meter
$\mu = \left(\frac{\partial B}{\partial H}\right)_\varepsilon$	Permeability at constant strain	weber/ ampere meter
μ^*	Complex permeability of ferrite	weber/ ampere meter
σ	Stress	newton/meter^2
θ	Angle	radian
ρ	Density	kg/meter^3
ψ	Instantaneous displacement	meter
ψ_o	Displacement amplitude	meter
ω_o	Angular frequency of resonance	radian/sec
ω_a	Angular frequency at parallel resonance (zero reactance)	radian/sec
ω_s	Angular frequency at series resonance	radian/sec
$\Delta\omega = 2\pi\Delta f$	$(\omega_s - \omega_a)$ difference between series and parallel resonance frequencies	radian/sec

Rationalized M. K. S. System

All quantities in the above table are expressed in the rationalized M. K. S. system. In this system

$$B = \mu_o H + \chi H$$

where χ is called the susceptibility and χH is the intensity of magnetization or magnetic moment per unit volume.

$$B = (\mu_o + \chi)H$$

$$B = \mu H$$

$$\mu = \mu_o + \chi$$

$$\frac{\mu}{\mu_o} = 1 + \frac{\chi}{\mu_o}$$

$K_m = \frac{\mu}{\mu_o}$ is called the relative permeability.

CONVERSION TABLE

Each of the four magnetostriction coefficients, λ, λ', d and d′ may be expressed in terms of the other and the appropriate modulus of elasticity and permeability. From the definitions of the quantities on page 5 we have

$$\lambda = \left(\frac{\partial\sigma}{\partial H}\right)_{\varepsilon} = \left(\frac{\partial\sigma}{\partial B}\right)_{\varepsilon}\left(\frac{\partial B}{\partial H}\right)_{\varepsilon} = \lambda'\mu$$

$$\lambda' = \left(\frac{\partial H}{\partial \varepsilon}\right)_{B} = \left(\frac{\partial H}{\partial\sigma}\right)_{B}\left(\frac{\partial\sigma}{\partial\varepsilon}\right)_{B} = d'q'$$

$$d = \left(\frac{\partial\varepsilon}{\partial H}\right)_{\sigma} = \left(\frac{\partial\varepsilon}{\partial B}\right)_{\sigma}\left(\frac{\partial B}{\partial H}\right)_{\sigma} = d'\mu'$$

$$d' = \left(\frac{\partial H}{\partial\sigma}\right)_{B} = \left(\frac{\partial H}{\partial\varepsilon}\right)_{B}\left(\frac{\partial\varepsilon}{\partial\sigma}\right)_{B} = \frac{\lambda'}{q'}$$

from which the following conversion table can be constructed with the aid of the relation $\frac{\mu}{\mu'} = \frac{q}{q'}$ (eq. 32, page 31).

$\lambda = \lambda'\mu$	$d = \frac{\lambda}{q}$
$\lambda = dq$	$d = d'\mu'$
$\lambda = d'q'\mu = d'q\mu'$	$d = \lambda'\frac{\mu}{q} = \lambda'\frac{\mu'}{q'}$
$\lambda' = \frac{\lambda}{\mu}$	$d' = \frac{\lambda'}{q'}$
$\lambda' = d'q'$	$d' = \frac{d}{\mu'}$

$$\lambda' = d\frac{q}{\mu} = d\frac{q'}{\mu'} \qquad d' = \frac{\lambda}{q'\mu} = \frac{\lambda}{q\mu'}$$

Use of the above table permits comparison of data expressed in different ways so long as the definitions used in obtaining the data are consistent.

The older data on ferrites are usually expressed in c. g. s. units and later data in m. k. s. units. The conversion table is, of course, applicable to either system. To convert from c. g. s. units to m. k. s. units the following conversion factors are required.

Force	1 newton = 10^5 dynes
Mass	1 kilogram = 10^3 grams
Induction	1 weber/square meter = 10^4 gauss
Field	1 ampere/meter = $4\pi \times 10^{-3}$ oersteds
Area	1 meter^2 = 10^4 cm^2

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DYNAMIC MEASUREMENTS OF THE MAGNETOELASTIC PROPERTIES OF FERRITES

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1. INTRODUCTION

In common with other ferromagnetic materials, the ferrites exhibit the property of magnetostriction or changes of dimensions upon magnetization. The simplest form of the phenomenon is the Joule effect or longitudinal magnetostriction discovered in 1847 by J. P. Joule. Although magnetostriction in general involves other forms of stresses and strains, the term is commonly taken to mean the Joule effect and it is in this sense that the term is used in this paper. The fractional change in length or strain in ferrites at saturation magnetization is reported to range from about -0.5 to $+40 \times 10^{-6}$ (Popper[10], 1953) depending upon the composition and method of preparation.

Numerous static methods have been devised for measuring the magnetostriction strain. These include optical and mechanical levers, strain gages, interferometer and capacitance techniques and combinations of these methods.

For various reasons, dynamic measurements are desired and a number of experimenters have devised methods of measuring the strain, the magnetostriction coefficients, the elastic coefficients and permeability by observations made on vibrating rods and rings.

As pointed out by Rowland, many years ago, magnetic measurements are much more simply made using rings than rods due to the absence of demagnetization due to the poles at the ends of the rod. The theory of the vibrating ring driven by magnetostriction is also much simpler than that of the driven rod because the strain is uniform in the ring whereas it varies along the length of the rod in a complex manner.

One difficulty which is encountered in the use of the vibrating ring is the absence of nodal points for supporting it. The ring vibrates, in the fundamental mode, in such a way that the displacement of every point in the ring is radial leaving no nodal points for support. However the losses are not very great if the ring is supported on thin radially stretched fibres. Another difficulty with the use of ring oscillators is that of winding accurately dimensioned toroidal coils on the specimen. One solution to this problem is to construct demountable half toroids which can be assembled over the specimen.

The theory derived in this paper is based upon the assumption that the ferrite is isotropic. The magnetic and mechanical properties are treated as scalars instead of tensors as will be required when single crystalline ferrites are to be investigated. The extension of the theory to include the effects of tensor properties can readily be made.

Much confusion exists in the literature concerning the relationships between the magnetic and mechanical properties. In a number of cases, authors have not been careful to define quantities completely by specifying the conditions under which the measurements were made. Analytical expressions relating the various magnetic and mechanical coefficients exist, in general, only for the small signal or reversible case, under which condition the relationships can be derived by the use of the methods of thermodynamics. There are many ways of setting up the relationships, each being equally logical. The choice, therefore, is determined by practical consideration and in particular by the ease with which the conditions can be realized experimentally.

The coefficients obtained by a dynamic method are the adiabatic coefficients whereas those obtained by static methods are the isothermal coefficients. Generally the difference is less than the error of measurement and this will probably be the case with ferrites.

Equivalent circuits have been used to represent electromechanical transducers for many years. One of the first investigations of the theory of equivalent circuits was made by Butterworth[1] in 1915 in connection with his studies on the vibration galvanometer. Butterworth showed quite generally that any mechanical vibrating system with one degree of freedom driven by the interaction between a current and a magnetic field behaves as if it were an electrical circuit consisting of an inductance, a resistance and a capacitance in parallel. He also showed that a mechanical system excited by the interaction between charged bodies and an electric field is equivalent to an electrical circuit consisting of the same three elements in series. The latter equivalent circuit has been shown to represent the piezoelectric resonator with very high precision and has been of great value in the development of filter and oscillator circuits employing quartz crystal units.

The first paper on the magnetostriction resonator was published by Pierce[2] in 1928. In his paper Pierce derived the equations for the electrical impedance of the magnetostriction resonator but he did not interpret the equations in terms of an equivalent circuit. In 1931 Butterworth and Smith[5] published a paper in which they used a series network of a capacitance, an inductance and a resistance to represent the motional impedance of the magnetostriction resonator. It can readily be shown that the equivalent network used by Butterworth and Smith is a limited equivalent network, applicable at only one frequency whereas the absolute equivalent circuit of a transducer is applicable at all frequencies.

A number of authors[4, 8, 9, 11, 12] have proposed limited equivalent circuits for the magnetostriction resonator and at least two authors[3, 10] have referred to the absolute equivalent network but the literature on the subject does not appear to provide a rigorous derivation of the absolute equivalent network. It is the purpose of this paper to derive the absolute equivalent circuit of the magnetostriction resonator and to relate the constants of the equivalent circuit to the physical properties and the dimensions of the resonator. The equivalent circuit is essential to the design of filters and oscillators which employ ferrite elements as resonators and in determining the values of the coefficients of the physical properties of ferrites by dynamic methods.

2. DEFINITIONS OF THE ELASTIC, MAGNETIC AND MAGNETOELASTIC COEFFICIENTS

The relationships between the magnetic and elastic properties of an isotropic ferrite may be expressed in a number of different ways depending upon whether the induction or the field is expressed in terms of the stress or the strain and vice versa. Since the elastic modulus depends upon the state of magnetization in a ferrite, it is necessary to define two values of the modulus, i.e., the modulus at constant field and the modulus at constant induction. In the same way the permeability depends upon the elastic state so that it becomes necessary to distinguish between the permeability at constant stress and the permeability at constant strain.

The magnetostriction coefficient may be defined in four ways depending upon whether the stress or the strain is related to the field or the induction. In addition there are four inverse coefficients relating the stress and the strain to the field and the induction. These eight coefficients are not independent and by the use of thermodynamic relations the number may be reduced to four independent coefficients.

In order, therefore, to completely describe the magnetoelastic relations in a ferrite it is necessary to define the following quantities:

ε = the strain. (Change in length per unit length)

σ = the stress. (Force per unit area)

H = the magnetic field.

B = the magnetic induction.

In each of these definitions it is assumed that the change in the quantity is small, corresponding to the reversible or small signal condition. It is further understood that the quantity is defined for a particular magnetoelastic state of the ferrite. For example, the stress is defined for a particular value of the strain and the field or of the strain and the induction. It should be noted that none of the quantities is a constant and therefore the word "coefficient" is used instead of the word "constant".

The following coefficients relating the magnetoelastic properties may now be defined.

$$q = \left(\frac{\partial \sigma}{\partial \varepsilon}\right)_H$$ Young's modulus at constant field.

$$q' = \left(\frac{\partial \sigma}{\partial \varepsilon}\right)_B$$ Young's modulus at constant induction.

$$\mu = \left(\frac{\partial B}{\partial H}\right)_\varepsilon$$ Permeability at constant strain.

$$\mu' = \left(\frac{\partial B}{\partial H}\right)_\sigma$$ Permeability at constant stress.

$$\lambda = \left(\frac{\partial \sigma}{\partial H}\right)_\varepsilon = \left(\frac{\partial B}{\partial \varepsilon}\right)_H$$
$$\lambda' = \left(\frac{\partial \sigma}{\partial B}\right)_\varepsilon = \left(\frac{\partial H}{\partial \varepsilon}\right)_B$$
Magnetostriction stress coefficients.

$$d = \left(\frac{\partial \varepsilon}{\partial H}\right)_\sigma = \left(\frac{\partial B}{\partial \sigma}\right)_H$$
$$d' = \left(\frac{\partial \varepsilon}{\partial B}\right)_\sigma = \left(\frac{\partial H}{\partial \sigma}\right)_B$$
Magnetostriction strain coefficients.

The equalities in the last four definitions follow from the thermodynamic relations

$$dU = BdH + \sigma\, d\varepsilon$$

$$dU = HdB + \sigma\, d\varepsilon$$

$$dU = BdH + \varepsilon d\sigma$$

$$dU = HdB + \varepsilon d\sigma$$

To prove the first of the four equalities we use the first thermodynamic equation to write

$$\left(\frac{\partial U}{\partial H}\right)_\varepsilon = B \quad \text{and} \quad \left(\frac{\partial U}{\partial \varepsilon}\right)_H = \sigma \ .$$

Taking a second derivative

$$\frac{\partial^2 U}{\partial H \partial \varepsilon} = \left(\frac{\partial B}{\partial \varepsilon}\right)_H \quad \text{and} \quad \frac{\partial^2 U}{\partial \varepsilon \partial H} = \left(\frac{\partial \sigma}{\partial H}\right)_\varepsilon \ .$$

Therefore

$$\left(\frac{\partial B}{\partial \varepsilon}\right)_H = \left(\frac{\partial \sigma}{\partial H}\right)_\varepsilon \ .$$

The remaining equalities may be proved in the same way using the other three thermodynamic relations.

The stress in a ferrite may be expressed as a function of the strain and the magnetic field intensity $\sigma = \sigma(\varepsilon, H)$ or as a function of the strain and the induction $\sigma = \sigma(\varepsilon, B)$. Similarly we have

$$\varepsilon = \varepsilon(\sigma, H) \quad \text{and} \quad \varepsilon = \varepsilon(\sigma, B) \ .$$

The induction and the field may also be expressed as functions of the stress and strain leading to the following relationships:

$$B = B(\sigma, H)$$

$$B = B(\varepsilon, H)$$

$$H = H(\sigma, B)$$

$$H = H(\varepsilon, B)$$

From these relationships and the definitions above we obtain the four direct relations

(a) $d\sigma = q d\varepsilon + \lambda dH$

(b) $d\sigma = q' d\varepsilon + \lambda' dB$

(c) $d\varepsilon = \frac{1}{q} d\sigma + d\, dH$

(d) $d\varepsilon = \frac{1}{q'} d\sigma + d' dB$

and from the inverse relations

(e) $dB = d\, d\sigma + \mu' dH$

(f) $dB = \lambda d\varepsilon + \mu dH$

(g) $dH = d' d\sigma + \frac{1}{\mu'} dB$

(h) $dH = \lambda' d\varepsilon + \frac{1}{\mu} dB$

In all the relationships it is understood that the changes are reversible, i.e., infinitesimally small and that the values are for the particular magnetoelastic state specified.

It is, of course, never possible to achieve perfect conditions of constant stress, strain, field or induction. However, the conditions may be approximated experimentally in the case of a vibrating ferrite ring in the following ways:

Constant stress. The condition of constant stress is approximated by mounting the ring so that it is free of all external constraints. The ring may be supported on three points or on fine threads in vacuum. Unfortunately, in the case of the vibrating ring, no nodal points are available for support.

Constant strain. The condition of constant strain implies that the ring is not free to move. In theory the ring could be clamped in such a way that no displacement is possible. In practice it is sufficient to drive the ring at a high but non-resonant frequency. For small driving forces, the displacement and the strain are negligible at a non-resonant frequency.

Constant field. The condition of constant field implies that the magnetic field due to a coil surrounding the ring is unaffected by changes in the induction in the ring. Changes in the elastic state of the ferrite produce changes in the induction which, in general, produce currents in the coil, thereby changing the field. However, if the impedance of the circuit is very high, the induced current is small and the field is approximately constant.

Constant induction. The condition of constant induction implies that the induction in the ferrite is not affected by changes in the elastic state. This condition can be approximated by surrounding the ferrite with a shorted coil of very low resistance. Then any changes in the induction set up currents in the coil which are in the direction to oppose the changes of induction. If the resistance of the circuit is zero, no change of induction is possible.

3. DERIVATION OF THE EQUIVALENT CIRCUIT OF THE VIBRATING RING DRIVEN BY MAGNETOSTRICTION

We consider a uniform, homogeneous polycrystalline, isotropic ferrite ring of mean radius a with a uniform toroidal winding having N turns. The area of the cross section of the ring is A and the radial dimension of the cross section is small compared with a, the mean radius of the ring.

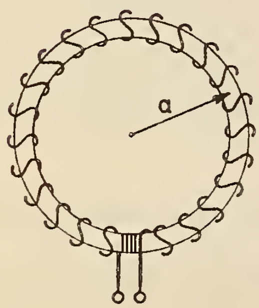

Fig. 1
Magnetostriction Ring Resonator

Consider an element of the ring $a\,d\theta$, fig. 2, having a mass $\rho A a d\theta$ where ρ is the density of the ferrite. An alternating current in the coil causes the ring to expand radially by an amount ψ.

The kinetic reaction of the element is $\rho A a d\theta \frac{\partial^2 \psi}{\partial t^2}$. As the ring expands, the length of the arc is increased from $a d\theta$ to $(a + \psi)d\theta$ and the strain

$$\varepsilon = \frac{d\ell}{\ell o} = \frac{\psi d\theta}{a d\theta} = \frac{\psi}{a} . \qquad (1)$$

The stress normal to the end of the element required to produce the strain ε is, from eq. a, page 7,

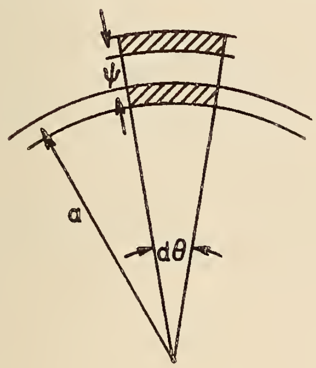

Fig. 2
Displacement in Ring Resonator

$$\sigma = q\varepsilon + \lambda H = q\frac{\psi}{a} + \lambda H .$$

The tangential force on the end of the element is σA and the radial component of the resultant of the tangential forces on the ends of the element is (fig. 3)

$$F_r = \sigma A\, d\theta = (q\frac{\psi}{a} + \lambda H)\, A\, d\theta .$$

If we assume that H is produced by a sinusoidal current $I = I_o \exp j\omega t$ then

$$H = \frac{NI_o}{2\pi a} \exp j\omega t .$$

We further assume the existence of a damping or frictional force proportional to the velocity having the form $rAd\theta \frac{d\psi}{dt}$.

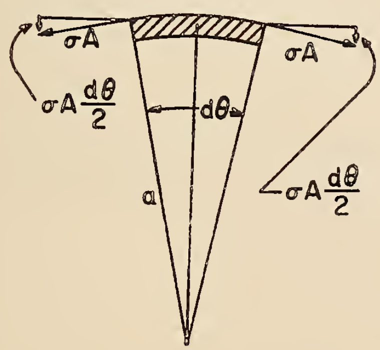

Fig. 3
Forces on Element of Vibrating Ring

The equation of motion thus becomes

$$\rho a \frac{\partial^2 \psi}{\partial t^2} + r \frac{\partial \psi}{\partial t} + \frac{q}{a} \psi = \frac{\lambda N I_o}{2\pi a} \exp j\omega t \ . \tag{2}$$

Each term in (2) has the dimensions of force/area, i.e., of stress. The damping coefficient r has the dimensions of mass/area per unit time. The assumption that the damping is proportional to the velocity is, of course, arbitrary, and other assumptions might be used. It is found, however, for a large variety of damped vibrating systems, that the assumption leads to results which can be verified experimentally. The justification for the present assumption must rest upon the consistency of the results obtained. Fortunately, most of the results do not depend at all upon the nature of the damping force so long as it is small, a condition which is readily satisfied in the case of many mechanical vibrating systems.

. . .

To solve the equation of motion in the absence of a driving force we set $I_o = 0$ and assume

$$\psi = \psi_o \exp j\omega t$$

which gives for the resonance frequency

$$\omega_o = j \frac{r}{2\rho a} \pm \frac{1}{2\rho a} \sqrt{4\rho q - r^2}$$

and for the displacement

$$\psi = \psi \exp\left(- \frac{rt}{2\rho a} \right) \cdot \exp \frac{j}{2\rho a} \sqrt{4\rho q - r^2} \quad t \tag{3}$$

which is, of course, a damped harmonic motion. The frequency in the absence of damping is

$$\omega_o = \frac{1}{a}\sqrt{\frac{q}{\rho}} \qquad \text{or} \qquad f_o = \frac{1}{2\pi a}\sqrt{\frac{q}{\rho}} \;. \tag{3a}$$

Just as in any other mechanical system, the actual resonance frequency is slightly reduced by frictional damping. From (3a) it follows that the velocity of sound in the ring is $\sqrt{\frac{q}{\rho}}$ and the period of the vibration is therefore the time required for a sound wave to travel around the ring, a distance $2\pi a$. Since the value of q depends upon the state of magnetization and upon whether B or H is fixed, the velocity of sound and the period are not constant except under fixed conditions.

We are interested, however, in the ring driven in the steady state condition. As a solution we try

$$\psi = \psi' \exp j\omega t$$

$$\dot{\psi} = j\omega\psi \qquad \text{and} \qquad \ddot{\psi} = -\omega^2\psi$$

where the dots signify time derivatives. Substituting into (2) we obtain

$$-\rho a\omega^2\psi + j\omega r\psi + \frac{q}{a}\psi = \frac{\lambda NI_o}{2\pi a}\,\frac{\psi}{\psi'}$$

from which

$$\psi' = \frac{\lambda NI_o}{2\pi a\gamma} \tag{4}$$

where

$$\gamma = \frac{q}{a} - \rho a \omega^2 + j\omega r \tag{4a}$$

and the solution of (2) is

$$\psi = \frac{\lambda N I_o}{2\pi a \gamma} \exp j\omega t \ . \tag{4b}$$

The motion of the ring is such that all points in the ring experience sinusoidal, radial displacement with the same phase and (if the ring is thin) the same amplitude. The strain and the induction are uniform at all points in the ring. It is for this reason that the vibrating ring is to be preferred over vibrating rods in which both the strain and the induction vary along the length of the rod. The problem is further complicated, in the case of the rod, by the demagnetizing effect produced by the poles formed at the ends of the rod.

If $r = 0$ corresponding to the ideal case of zero damping, the amplitude becomes infinite if

$$\frac{q}{a} - \rho a \omega^2 = 0$$

or when

$$\omega = \omega_o = \frac{1}{a}\sqrt{\frac{q}{\rho}}$$

which is the same as the undamped resonance frequency.

The strain, ε, which for the vibrating ring is $\frac{\psi}{a}$, is given by

$$\varepsilon = \frac{\lambda N}{2\pi a^2 \gamma} I_o \exp j\omega t \quad . \tag{5}$$

From eq. f, page 7, the reversible induction in the ferrite is given by

$$B = \lambda\varepsilon + \mu H \tag{6}$$

where ε and H are the small signal strain and field respectively. Substituting the values of ε and H into (6) gives for the induction in the ferrite

$$B = \left(\frac{\lambda^2 N}{2\pi a^2 \gamma} + \frac{\mu N}{2\pi a} \right) I_o \exp j\omega t \quad .$$

If the area of the cross section A, of the ring, is equal to that of the toroidal coil, the flux through the coil is BA. It is not practical, however, to make the coil fit so closely to the ferrite ring as to make the areas sensibly equal without introducing damping in the ring. It is therefore preferable to make the area of the cross section of the coil larger than that of the ring.

If A is the area of the cross section of the ring and A_o is the area of the air space between the ferrite ring and the toroid, then the flux through the toroid is

$$\phi = B_o A_o + BA$$

where B_o is the induction in the air and B the induction in the ferrite. The changing flux associated with the vibration of the ferrite and the alternating current in the coil results in a back emf in the coil, the magnitude of which is given by

$$e = N \frac{d\phi}{dt}$$

$$= NA_o \frac{dB_o}{dt} + NA \frac{dB}{dt} \quad .$$

But

$$B_o = \mu_o H \qquad \text{so} \qquad \frac{dB_o}{dt} = \mu_o \frac{dH}{dt}$$

and

$$B = \lambda\varepsilon + \mu H \qquad \text{so} \qquad \frac{dB}{dt} = \lambda \frac{d\varepsilon}{dt} + \mu \frac{dH}{dt} \quad .$$

Therefore

$$e = NA_o\mu_o \frac{dH}{dt} + NA\lambda \frac{d\varepsilon}{dt} + NA\mu \frac{dH}{dt} \quad . \tag{7}$$

Substituting for $\frac{d\varepsilon}{dt}$ and $\frac{dH}{dt}$ we have

$$e = \left[j\omega \frac{N^2 A}{2\pi a} \left(\frac{A_o}{A} \mu_o + \mu \right) + j\omega \frac{N^2 \lambda^2 A}{2\pi a^2 \gamma} \right] I_o \exp j\omega t \tag{8}$$

which may be written as

$$e = j\omega \left[\mu_1 + \frac{\lambda^2}{\gamma a} \right] \frac{N^2 A}{2\pi a} I_o \exp j\omega t \tag{8a}$$

where

$$\mu_1 = \frac{A_o}{A} \mu_o + \mu = \left(\frac{A_o}{A} + K_m \right) \mu_o \tag{8b}$$

where $K_m = \frac{\mu}{\mu_o}$ is the relative permeability of the ferrite.

If A_o is made zero by fitting the coil closely to the ferrite ring then $\mu_1 = \mu$, the permeability of the ferrite at constant strain, $\left(\frac{\partial B}{\partial H} \right)_\varepsilon$. Eq. (8a) shows that the effect of magnetostriction is to increase the permeability and to make it complex since λ^2 is necessarily positive and γ is in general complex.

The term in the bracket of (8) has the dimensions of an impedance. The first term is the inductive reactance of the coil assuming the ferrite to be non-vibrating or clamped. The second term is due to the motion of the ferrite ring excited through the magnetostriction effect. We may therefore write (8) as

$$e = [Z_c + Z_m] I_o \exp j\omega t$$

where

$$Z_c = jX_c$$

$$X_c = \omega \frac{N^2 A \mu_1}{2\pi a} \quad \text{or} \quad L_c = \frac{N^2 A \mu_1}{2\pi a} \,. \tag{9}$$

For certain purposes it may be desirable to include the resistance R_c of the coil and to write

$$Z_c = R_c + j\omega L_c$$

for the impedance of the coil with the ferrite clamped. The resistance R_c is principally due to the ohmic resistance (copper loss) in the winding of the toroid since eddy current losses should be negligible in most ferrites, although residual losses may still be present.

The motional impedance Z_m is from (8) and (4a)

$$Z_m = j\omega \frac{\lambda^2 N^2 A/2\pi aq}{\left(1 - \frac{\rho a^2 \omega^2}{q} + \frac{j\omega r a}{q}\right)} \tag{10}$$

Z_m has the same form as that of the impedance of the electrical circuit of fig. 4 consisting of an inductance, capacitance and a resistance in parallel the impedance of which is

$$Z_{AB} = j\omega \frac{L_m}{1 - \omega^2 L_m C_m + \frac{j\omega L_m}{R_m}} \quad . \tag{10a}$$

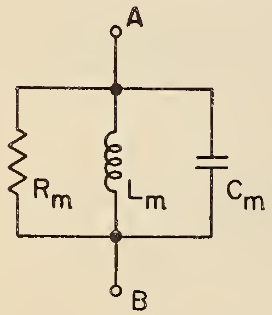

Fig. 4
Motional Part of
Equivalent Circuit

Hence the equivalent motional inductance is

$$L_m = \frac{\lambda^2 N^2 A}{2\pi qa} \quad . \tag{11}$$

The equivalent motional capacitance is

$$C_m = \frac{2\pi\rho a^3}{\lambda^2 N^2 A} \qquad (12)$$

and the equivalent motional resistance is

$$R_m = \frac{\lambda^2 N^2 A}{2\pi r a^2} \, . \qquad (13)$$

The equivalent circuit of the resonator as seen at the terminal of the resonator is that of the circuit of fig. 5.

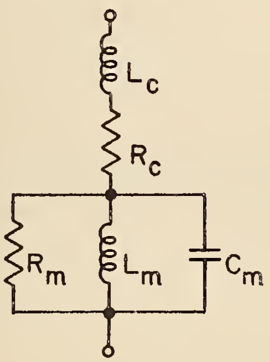

Fig. 5
Complete Equivalent Circuit

It is of interest to note that R_m, the motional resistance of the equivalent circuit is very large if the damping coefficient r is small. This somewhat unusual relationship is due to the presence of R_m in parallel with C_m and L_m.

4. THE EQUIVALENT ELECTRICAL CIRCUIT

The impedance of the equivalent circuit of fig. 5 is

$$Z = R_c + j\omega L_c + \frac{j\omega L}{1 - \omega^2 LC + \frac{j\omega L}{R}} \qquad (14)$$

where for convenience the subscripts have been omitted from the terms representing the motional parameters. After writing (14) over a common denominator and multiplying the numerator and denominator by the complex conjugate of the latter, we obtain

$$Z = \frac{M + jM'}{(1 - \omega^2 LC)^2 + \frac{\omega^2 L^2}{R^2}} \tag{14a}$$

where the real part of the numerator is

$$M = R_c + \omega^2 \left(\frac{L^2}{R} + \frac{L^2 R_c}{R^2} - 2LCR_c \right) + \omega^4 L^2 C^2 R_c \tag{14b}$$

and the imaginary part of the numerator is

$$M' = \omega^5 (L^2 L_c C^2) - \omega^3 \left(L^2 C + 2LL_c C - \frac{L^2 L_c}{R^2} \right) + \omega (L + L_c) \tag{14c}$$

The frequencies at which the reactance is zero, which are very close to the resonant frequencies, are found by setting $M' = 0$ and solving for ω. If the Q of the circuit is high enough the term involving R may be neglected. (The low Q case is treated later). Neglecting the term in R in (14c) we have

$$\omega^4 L^2 L_c C^2 - \omega^2 (L^2 C + 2LL_c C) + (L + L_c) = 0 \tag{14d}$$

from which, by use of the quadratic formula

$$\omega^2 = \frac{1}{LC} \qquad \text{and} \qquad \omega^2 = \frac{1}{LC} + \frac{1}{L_c C} \quad .$$

The first frequency is identified with the mechanical resonance of the ring and may be termed, from the nature of the equivalent circuit, the parallel resonant frequency. The second and slightly higher frequency occurs at the frequency at which the motional reactance is capacitive and numerically equal to the inductive reactance of the coil L_c. This frequency may be termed the series resonant frequency. We may therefore write

$$\omega_a^2 = \frac{1}{LC} \qquad \text{and} \qquad f_a = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \qquad (15a)$$

and

$$\omega_s^2 = \frac{1}{LC} + \frac{1}{L_c C} \qquad \text{and} \qquad f_s = \frac{1}{2\pi}\sqrt{\frac{1}{LC} + \frac{1}{L_c C}} \qquad (15b)$$

NOTE: Some authors use the opposite designation for the two resonant frequencies.

4.1 Determination of Circuit Parameters (High Q)

From (15a) and (15b) we have

$$\omega_s^2 - \omega_a^2 = \frac{1}{L_c C}$$

$$(\omega_s + \omega_a)(\omega_s - \omega_a) = \frac{1}{L_c C} \quad .$$

But since ω_s and ω_a are very close together and $(\omega_s - \omega_a) = \Delta\omega << \omega_s$, we may write

$$2\omega \Delta\omega = \frac{1}{L_c C}$$

and

$$C = \frac{1}{2\omega\, \Delta\omega\, L_c} = \frac{1}{8\pi^2 f\, \Delta f\, L_c} \tag{16}$$

where L_c is the inductance of the coil measured at a non-resonant frequency and Δf is the difference between the two frequencies of zero reactance f_s and f_a.

The value of L is also readily determined since

$$\omega_a^2 = \frac{1}{LC}$$

$$L = \frac{1}{\omega_a^2 C} = \frac{8\pi^2 f\, \Delta f\, L_c}{4\pi^2 f^2} = 2\,\frac{\Delta f}{f}\, L_c = k^2 L_c \tag{17}$$

where k is called the magnetoelastic coupling coefficient.

The Q of the ferrite may also be determined from the parameters of the equivalent circuit. However, the Q of the circuit of fig. 4 is not $\frac{\omega L}{R}$ as is the usual case with resonant circuits in which the resistance is in series with the coil but is $\frac{R}{\omega L}$. This may be shown as follows: The impedance Z_{AB} of fig. 4 is

$$Z_{AB} = \frac{j\omega L}{1 - \omega^2 LC + j\,\frac{\omega L}{R}} \,.$$

Rationalizing the denominator gives

$$Z_{AB} = \frac{\omega^2 L^2}{RD} + j\left(\frac{\omega L}{D} - \frac{\omega^3 L^2 C}{D}\right)$$

where $D = (1 - \omega^2 LC)^2 + \frac{\omega^2 L^2}{R^2}$. The above equation represents the impedance of the series circuit $Z_{A'B'} = R' + j\,(\omega L' - \frac{1}{\omega C'})$ which has a Q given by $\frac{\omega L'}{R'}$. In terms of the parameters of the original equivalent circuit

$$Q = \frac{R}{\omega L} = R\omega C \quad . \tag{18}$$

This somewhat unusual form of the expression for Q is a consequence of the relationship between $R = R_m$ and r of (13).

In terms of measureable quantities

$$Q = \frac{R}{\omega L} = \frac{R}{4\pi\, L_c\, \Delta f} \tag{18a}$$

where $\Delta f = f_s - f_a$.

In order to find the resistance at the terminals of the toroid at the parallel resonant frequency f_a we substitute $\omega_a^2 = \frac{1}{LC}$ into the real part of (14a) and obtain

$$Z_a = R_c + R \quad . \tag{19}$$

In the same way the impedance at the series resonant frequency is found by substituting $\omega_s^2 = \frac{1}{LC} + \frac{1}{L_c C}$ into the real part of (14a). The result, after a considerable amount of algebra, is found to be

$$Z_s = R_c + \frac{R}{1 + \frac{R^2}{\omega^2 L_c^2}} = R_c + \frac{R}{1 + (k^2 Q)^2} \quad . \tag{20}$$

Taking the difference between (19) and (20) we obtain

$$Z_a - Z_s = \Delta Z = \frac{R^3}{R^2 + \omega^2 L_c^2} = \frac{R}{1 + \left(\frac{1}{k^2 Q} \right)^2} \qquad (21)$$

Unfortunately both terms in the denominator of (21) are often of the same order of magnitude so that the determination of the value of R involves the solution of a cubic equation. The solution can most readily be obtained by numerical methods.

4.2 Determination of Circuit Parameters (Low Q)

Thus far we have considered the damping to have negligible effect upon the frequencies of parallel and series resonance. If, however, as is the case in many ferrite materials, the damping is large and the Q is small it is necessary to further examine the solution for the frequencies of resonance. If the term involving R is reinserted into (14d) the resulting equation has real roots only if the discriminant is equal to or greater than zero. This leads to the condition that for ω^2 to be real

$$Q > 2 \frac{L_c}{L} = \frac{2}{k^2}$$

or that

$$Q k^2 > 2 \quad .$$

Physically this result means that if k^2Q is less than two, the motional part of the equivalent circuit is unable to develop a capacitive reactance numerically equal to the inductive reactance of L_c.

Assuming that $k^2Q > 2$, equation (14c), including the term in R, may be set equal to zero and solved for ω^2. To a second approximation it is found that

$$\omega_a^2 = \frac{1}{LC}\left(1 + \frac{1}{k^2Q^2}\right)$$

and

$$\omega_s^2 = \frac{1}{L_cC} + \frac{1}{LC}\left(1 - \frac{1}{k^2Q^2}\right).$$

Hence the effect of damping is to cause the parallel resonant frequency to be increased and the series resonant frequency to be decreased thereby bringing the two frequencies closer together. If k^2Q is considerably greater than two and if Q is large the effect of damping on the difference between the two frequencies is small compared with the difference between the two and equations (16) through (21) are good approximations. In cases where both k and Q are small these equations may not yield satisfactory results.

If k^2Q is less than two, no frequencies exist (except f = 0) for which the reactance of (14a) is zero. Consequently the impedances at the maximum and minimum points are complex. Fair approximations to the values of the complex impedances may be obtained provided k^2Q is not much less than two and the damping is not too great by substituting (15a) and (15b) into (14a). The results are found to be

$$Z_a = R_c + R + j\omega L_c$$

and
$$Z_s = R_c + \frac{R + j\omega L_c}{1 + (k^2 Q)^2}$$

These equations may be compared with (19) and (20) which are derived on the assumption that damping has negligible effect upon the resonant frequencies.

Fig. 6 illustrates qualitatively the effect of damping on the shape of the resonance curve of the magnetostriction resonator. The effect of damping is to reduce the value of Q which from (18) results in decreasing the value of R. Therefore, the antiresonant impedance is lower in low Q resonators. Similarly the effect of damping is to slightly increase the series resonance impedance. As shown before, damping decreases the interval between the series and parallel resonant frequencies.

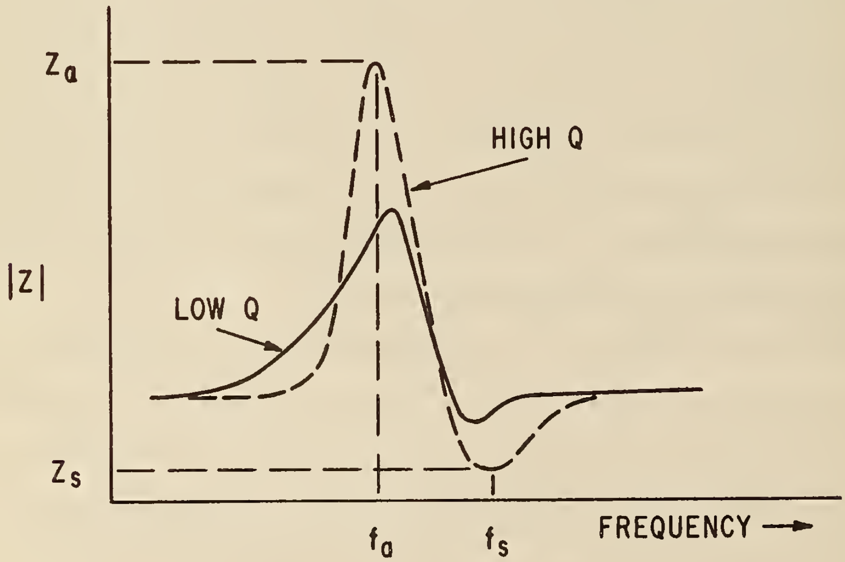

Fig. 6 Resonance Curve of the Magnetostriction Resonator

5. EVALUATION OF THE PHYSICAL COEFFICIENTS OF THE FERRITE

The values of the elastic modulus, the magnetostriction stress and strain coefficients, the permeability, Q, and the magnetoelastical coupling coefficient are all related to the values of the parameters of the equivalent circuit which can be determined by electrical and frequency measurements. In addition, measurements of the density of the ferrite and of the dimensions of the ring and toroid are required. It must be kept in mind that the values of all the parameters are dependent upon the state of magnetization of the ring.

5.1 Young's Modulus

The value of L_c can be determined by the use of a bridge or Q-meter at a frequency near but not too near the resonance frequency. The resonance frequency f_o which, for this purpose may be taken as either f_a or f_s, must be measured. Then from (3a)

$$f_o = \frac{1}{2\pi a}\sqrt{\frac{q}{\rho}}$$

from which the elastic (Young's) modulus

$$q = \left(\frac{\partial\sigma}{\partial\varepsilon}\right)_H = 4\pi^2 a^2 f_o^2 \rho \qquad (22)$$

5.2 Magnetostriction Coefficients

From (11) the motional inductance

$$L_m = L = \frac{\lambda^2 N^2 A}{2\pi q a}$$

and from (18a)

$$L_m = 2L_c \frac{\Delta f}{f_o}$$

so that

$$\lambda^2 = \frac{4\pi q a L_c \Delta f}{N^2 A f_o}$$

which may be written in a different form by the use of (22) as

$$\lambda = \left(\frac{\partial\sigma}{\partial H}\right)_\varepsilon = \frac{2\pi a}{N}\sqrt{\frac{4\pi\rho a L_c f_o \Delta f}{A}} \qquad (23)$$

5.3 Permeability

The effective permeability of the core (ferrite and surrounding air space) of the toroid in the absence of magnetostrictive strain may be determined from a measurement of L_c, made at a nonresonant frequency. From (9)

$$\mu_1 = \frac{2\pi a L_c}{N^2 A} . \qquad (24)$$

Using (8b) and (24) the static permeability

$$\mu^s = \left(\frac{\partial B}{\partial H}\right)_\varepsilon = \frac{2\pi a L_c}{N^2 A} - \frac{A_o}{A}\mu_o . \qquad (25)$$

Eq. (25) gives the permeability of the ferrite in the absence of magnetostriction strain. This is the value of the permeability which is applicable to the ferrite when clamped or at a nonresonant frequency. At or near a resonant frequency the permeability becomes complex.

At the frequency of parallel resonance f_a, $\gamma = jr\omega_a$ and the contribution to the permeability due to the magnetostriction effect is from (8a)

$$\mu^m = \frac{\lambda^2}{\gamma a} = -j\frac{\lambda^2}{r\omega_a a}$$

which using (13) may be written

$$\mu^m = \frac{\lambda^2}{\gamma a} = -j\frac{a R_m}{f_a N^2 A}$$

from which the complex permeability of the ferrite at the frequency of mechanical resonance may be written as

$$\mu^* = \mu^s + j\mu^m = \left(\frac{2\pi a L_c}{N^2 A} - \frac{A_o}{A} \mu_o \right) - j \frac{a R_m}{f_a N^2 A} . \qquad (26)$$

5.4 Q of the Ferrite

The Q of the vibrator is equal to the Q of the circuit of fig. 4 representing the motional impedance of the vibrating ring. Unfortunately the Q includes not only the internal losses in the ferrite but also the losses due to sound radiated in the air and frictional forces at the points of support. The latter may be minimized by placing the vibrating ring in vacuum and by careful design of the supporting mechanism but the Q of the ferrite must always be higher than the measured value.

We have shown that the Q of the equivalent circuit of fig. 4 is $Q = \frac{R_m}{\omega_a L_m}$. Using (11) and (13) we have

$$Q_m = \frac{q}{\omega_a r a} = \frac{\omega \rho a}{r} = \frac{\sqrt{q\rho}}{r} . \qquad (27)$$

The damping coefficient r is therefore given by

$$r = \frac{\sqrt{q\rho}}{Q} .$$

It is now quite easy to show that the damping has negligible effect upon the frequency of mechanical resonance if Q is large. From (3) we have that the frequency, including damping effects is

$$\omega_a \equiv \frac{\sqrt{4\rho q - r^2}}{2\rho a} \quad .$$

Substituting for r from (27) we obtain

$$\omega_a = \frac{1}{a}\sqrt{\frac{q}{\rho}}\left(1 - \frac{1}{4Q^2}\right)^{\frac{1}{2}} \quad .$$

So that with a high degree of approximation

$$\omega_a \approx \omega_o\left(1 - \frac{1}{8Q^2}\right) \; .$$

Hence if $Q > 100$, ω_a differs from ω_o by less than 0.002%.

5.5 The Magnetoelastic Coupling Coefficient (mecc)

Interesting relationships between the elastic moduli at constant field and induction and between the permeabilities at constant stress and strain are obtained in terms of the magnetoelastic coupling coefficient k. The magnetoelastic coupling coefficient is defined as the square root of the ratio of energy stored mechanically to that stored in the magnetic field. The coefficient k must not be confused with r, the damping coefficient which is related to the energy dissipated; k relates only to conservative effects.

The magnetic energy stored in a magnetic field (per unit volume) is

$$W = \frac{1}{2}\mu H^2 \; .$$

If the field is of the type $H = H_o \exp j\omega t$ then

$$W = \frac{1}{2} \mu H_o^2$$

is the maximum instantaneous energy stored during the cycle. If the ferrite is clamped, (constant strain)

$$W = \frac{1}{2} \mu H_o^2$$

but if it is free to vibrate (constant stress)

$$W' = \frac{1}{2} \mu' H_o^2 \quad .$$

The elastic energy stored at constant stress is greater than that stored at constant strain since some work is done against elastic forces in the first case. From the definition of k we then have

$$k^2 = \frac{W' - W}{W'} = \frac{\mu' - \mu}{\mu'} \tag{28}$$

from which

$$\frac{\mu}{\mu'} = 1 - k^2 \quad . \tag{29}$$

It is, of course, also possible to drive the ring mechanically by applying a sinusoidal stress of the form $\sigma = \sigma_o \exp j\omega t$. The maximum instantaneous energy stored (per unit volume) at constant magnetic field is

$$W = \frac{1}{2} \frac{\sigma_o^2}{q} \quad .$$

But if the induction is held constant by winding the ring with a shorted coil of negligible resistance

$$W' = \frac{1}{2} \frac{\sigma_o^2}{q'} \quad .$$

$W' < W$ because the contribution of the magnetostriction effect to the strain is eliminated. Therefore, from the definition of k

$$k^2 = \frac{W - W''}{W} = \frac{q' - q}{q'} \tag{30}$$

from which

$$\frac{q}{q'} = 1 - k^2 \quad . \tag{31}$$

We thus have the interesting relation that

$$\frac{q}{q'} = \frac{\mu}{\mu'} = 1 - k^2 \quad . \tag{32}$$

The mecc can also be related to the parameters of the equivalent circuit. The power into the circuit is given by

$$I^2 \omega L_c$$

while the power into the resonator is

$$I^2 \omega L_m \quad .$$

From the definition of k we have

$$k^2 = \frac{L_m}{L_c}$$

which agrees with (17)

$$k^2 = \frac{L_m}{L_c} = 2 \frac{\Delta f}{f} \quad .$$

Therefore

$$q = q' \left(1 - 2 \frac{\Delta f}{f_o} \right) \tag{33}$$

and

$$\mu = \mu' \left(1 - 2 \frac{\Delta f}{f_o} \right) \tag{34}$$

where f_o may be taken as either f_a or f_s, the difference being ordinarily negligibly small.

5.6 The Strain and Displacement at the Resonant Frequency

From (4), (4a) and (4b) the displacement in a ring vibrating in a simple radial mode is

$$\psi = \frac{\lambda N}{2\pi a \gamma} I_o \exp j\omega t \quad .$$

At the resonant frequency f_a, $\gamma = j\omega_a r$.

The strain ε in the ring is $\frac{\psi}{a}$ so that at the frequency f_a

$$\varepsilon = \frac{\psi}{a} = -j \frac{\lambda N}{2\pi a^2 \omega_a r} I_o \exp j\omega t \quad . \qquad (35)$$

Eq. (35) shows the strain amplitude to be

$$\varepsilon_o = \frac{\lambda N}{2\pi a^2 \omega r} I_o \qquad (36)$$

or using (27) and (13)

$$\varepsilon_o = \frac{\lambda N Q}{2\pi a^2 \omega \sqrt{q\rho}} I_o = \frac{R_m I_o}{\omega \lambda N A} \quad . \qquad (37)$$

The radial displacement amplitude

$$\psi_o = \varepsilon_o a = \frac{\lambda N Q}{2\pi a \omega \sqrt{q\rho}} I_o = \frac{a R_m I_o}{\omega \lambda N A} \quad . \qquad (38)$$

6. EXPERIMENTAL PROCEDURE

One advantage of the dynamic method is that a number of coefficients can be evaluated from a relatively small number of measurements. In order to evaluate the magnetoelastic coupling coefficient k, the reversible complex permeability μ^*, the magnetostrictive coefficient λ, Young's modulus q, and the elastic quality factor Q of the material, the following mechanical measurements are required:

A = the area of the cross section of the ring.

A_o = the area of the air space between the toroid and the ring.

N = the number of turns on the toroid.

a = the mean radius of the ring.

ρ = the density of the ferrite.

In addition, the following measurements must be made electrically.

L_c = the inductance of the coil at a non-resonant frequency.

f_a = the antiresonant frequency.

f_s = the series resonant frequency.

Z_a = the resistance at the antiresonant frequency.

Z_s = the resistance at the series resonant frequency.

6.1 Dimensions

It is desirable that the thickness of the ring in the radial direction be made as thin as possible in order that the mean radius may be more precisely defined. The mean radius may be taken to a first approximation to be one-fourth the sum of the inside and outside diameters. The area of the cross section should be determined with an accuracy of 1% or better. To permit a measurement of this accuracy, the radii and thickness of the ring must be uniform to better than 0.5%.

6.2 Density

The density of the ferrite may be determined by taking the ratio of the mass to the volume computed from the relationships $\rho = \frac{m}{V}$ and

$$V = \frac{\pi}{4}(D_2^2 - D_1^2)h$$

where D_2 and D_1 are the outside and inside diameters respectively and h is the thickness of the ring. Alternatively the density may be determined by immersing the ring in a fluid of density ρ_ℓ and using the relationship

$$\rho = \rho_\ell \frac{W_a}{W_a - W_\ell}$$

where W_a is the weight in air (vacuum) and W_ℓ is the weight immersed in the liquid.

6.3 Inductance

The inductance L_c is measured at a non-resonant frequency close but not too close to the resonant frequency. It is desirable to measure L_c at several frequencies to insure that the measured value is not being influenced by mechanical resonance in the ring. L_c may be measured by means of a Q-meter or better by means of a suitable radio frequency bridge. Fig. 7 shows the basic circuit diagram of a radio frequency bridge with the magnetostriction apparatus and the biasing circuit.

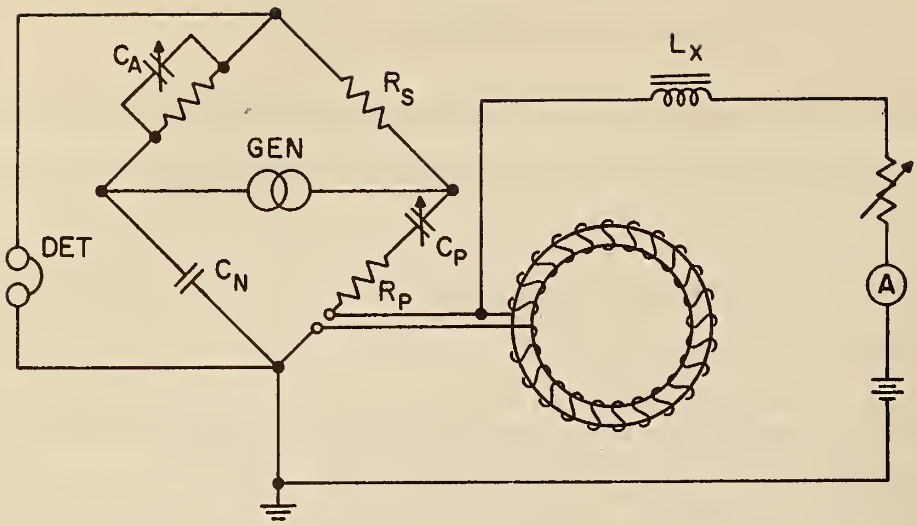

Fig. 7. Bridge and Biasing Circuit

The circuit of the bridge permits the exciting winding of the resonator to be used also for supplying the biasing field since one terminal of the coil is grounded and the other is in series with capacitor C_p which isolates it from the rest of the bridge for direct currents. It is therefore possible to pass a direct current through the coil without affecting the bridge. It is necessary, however, to provide an isolating inductance L_x having an impedance which is high compared with that of the resonator.

6.4 Measurement of Series and Parallel Impedances and Frequencies

Fig. 8 shows the apparatus used for measuring the series and parallel resonant frequencies and impedances.

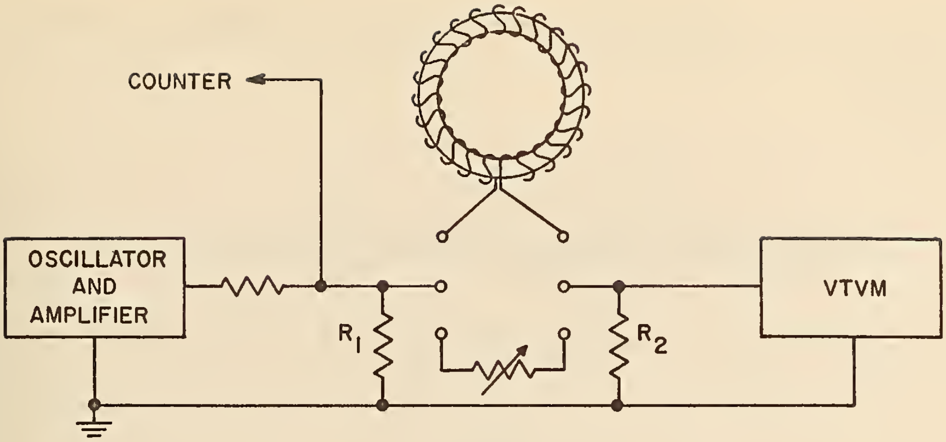

Fig. 8. Magnetostriction Resonator Apparatus

The oscillator must be adjustable and stable to one cycle per second or better. The amplifier must be capable of developing from 1 to 10 volts across R_1 which is small compared with the series resonant resistance of the resonator. R_2 is small compared with R_1. Typical values for R_1 and R_2 are 50 ohms and 1 ohm respectively but the actual values are determined by the series resonant impedance of the resonator. The current through the resonator is measured by the vacuum tube voltmeter and the frequency by the frequency counter.

The values of Z_a and Z_s are measured by the substitution method using a calibrated variable carbon resistor or a decade resistance box of suitable frequency range.

The frequency of parallel resonance is, by definition, the frequency at which the reactive component of the motional impedance of the equivalent network is zero. This frequency differs only very slightly from the frequency of mechanical resonance in the absence of damping and from the frequency of maximum impedance. If the Q is high, the differences between the three frequencies are very small and may be neglected. To a very high degree of approximation, even when the Q is not high, the frequency of series resonance is equal to the frequency of minimum impedance. It is therefore usually adequate to consider

Δf to be the difference between the frequencies of maximum and minimum impedance.

6.5 The Magnetoelastic Coupling Coefficient

The magnetoelastic coupling coefficient (mecc) k is given by

$$k^2 = 2\frac{\Delta f}{f}$$

where Δf is the difference between the frequencies of series and parallel resonance and approximately the difference between the frequencies of maximum and minimum impedance. To a sufficient degree of accuracy f may be taken as either frequency.

6.6 The Parameters of the Equivalent Circuit

The inductance in the motional part of the equivalent circuit is given by

$$L_m = k^2 L_c \quad .$$

The motional capacitance is

$$C_m = \frac{1}{8\pi^2 f \Delta f L_c} \quad .$$

The motional resistance R_m is found from the relationship

$$|Z_a| - |Z_s| = \frac{R_m^3}{R_m^2 + \omega^2 L_c^2} .$$

6.7 The Quality Factor

The mechanical Q or elastic quality factor is determined from the relation

$$Q_m = \frac{R_m}{4\pi L_c \Delta f} .$$

The measured value of Q_m will always be somewhat lower than the true Q_m of the ferrite because of the effects of air damping and frictional losses at the support points. These may be minimized by careful design of the mounting system and by placing the ring in vacuum. Rings with chips or cracks may be expected to exhibit abnormally low Q's because of frictional losses at these points.

6.8 Young's Modulus

Young's modulus is determined from the relationship for the resonant frequency giving

$$q = 4\pi^2 f^2 a^2 \rho$$

where a is the mean radius of the ring. The above relationship is derived on the assumption that the radial thickness of the ring is negligible compared with the mean radius of the ring. If the ring is not thin, it may be necessary to consider the effects of lateral inertia. It can be shown that the error made in the determination of q as a result of neglecting the radial thickness of the ring is of the order of

$$\frac{\tau^2}{16\pi^2 a^2}$$

where τ is the radial thickness and a the mean radius of the ring. For a small ring having an inside radius of 0.6 cm and an outside radius of 1.0 cm the error introduced into the determination of q by neglecting the radial thickness of the ring is about 0.2%.

6.9 The Magnetostriction Coefficient

The magnetostriction stress coefficient is given by

$$\lambda = \sqrt{\frac{4\pi q a L_c \Delta f}{N^2 A \quad f}} = \frac{k}{N}\sqrt{\frac{2\pi q a L_c}{A}} .$$

The other magnetostriction stress and strain coefficients may be calculated using the conversion table on page vi.

6. 10 The Permeability

The permeability at the resonant frequency is complex. The real part is the ordinary permeability of the non-vibrating ring. The imaginary part is due to the motion of the ring. The complex permeability is given by

$$\mu^* = \left(\frac{2\pi a L_c}{N^2 A} - \frac{A_o}{A} \mu_o \right) - j \frac{a R_m}{f N^2 A} \quad .$$

6. 11 The Biasing Field

In order to obtain a complete picture of the behaviour of a ferrite it is necessary to measure the permeability, coupling coefficient, magnetostriction coefficient, Q, etc., under different magnetic and elastic conditions. In particular it is desirable to determine the values of the various parameters at different values of the magnetic field or induction from zero to saturation. For this purpose it is necessary to provide a biasing field of known intensity. The biasing field may be provided in various ways; by use of a second winding over the exciting winding, by a heavy conductor along the axis of the ring or by passing a polarizing current through the exciting winding. In every case it is necessary to suppress the currents induced in the conductor by the exciting current and by the change of induction resulting from the mechanical motion of the ferrite if constant field conditions are desired. It is necessary, therefore, to include in the biasing circuit an inductive reactance which offers high impedance at the resonant frequency of the ring.

On the other hand, if conditions of constant induction are desired, the biasing circuit must include substantially zero impedance at the resonant frequency. This requires that the direct current supply be shunted with a capacitor having an extremely low impedance at the resonant frequency. Neither of the two conditions is completely realizable but the condition of constant field appears to be more readily approximated.

The magnetic field around a long straight conductor at a distance a from the conductor is

$$H = \frac{I}{2\pi a}$$

where I is the current in the conductor. For a toroid of N turns on a core of mean radius a

$$H = \frac{NI}{2\pi a} \quad .$$

For either method of furnishing the biasing field the number of ampere turns required to produce induction B is

$$NI = \frac{aB}{2K_m} \cdot 10^7 \quad .$$

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