An Empirical Formula of the Coherent Scattering Cross Section of Gamma Rays
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An empirical formula has been developed for estimating coherent scattering cross sections of γ-rays of energies below 1.5 MeV and in the range of 0 - 2.5 mc units of momentum transfer. The formula has been compared with the experimental data available in the literature in the case of Pb, Sn and Cu scatterers for different photon energies between 0.279 - 1.33 MeV. Experimental data, in general, have been found to be in good agreement with the results of the empirical formula, the deviation being within ±10% in most of the cases. The empirical formula has also been compared with the form factor results of Nelms and Oppenheim and with the accurate theoretical data of the Birmingham group. The latter is in excellent agreement with the empirical formula.

Key Words: coherent scattering, Delbrück, empirical formula, gamma rays, nuclear Thomson, Rayleigh

1. Introduction

Coherent scattering of gamma rays occurs mainly through the non-resonant processes of Rayleigh scattering, nuclear Thomson scattering and Delbrück scattering. Nuclear resonance scattering, though a coherent process, is usually a very rare event and can be observed only under specially adopted experimental situations. The intensity of the nuclear Thomson scattering process can be easily calculated by extending Thomson's classical relation for the electron to the case of the nucleus. Unless q, the momentum transfer involved during the coherent encounters, is rather large (q = 2mc or more) and the atomic number of the scatterer is low, nuclear Thomson scattering constitutes a negligible fraction of the total coherent scattering intensity. The lower limit of the energy of the photon, for which the Delbrück scattering makes barely perceptible contribution to coherent scattering is believed to be at least 1 MeV. Therefore, for all practical purposes Rayleigh scattering is the most important coherent scattering process, especially for γ-rays obtainable from normally available radioactive sources. It is therefore generally sufficient to compute the coherent scattering intensity by a consideration of the Rayleigh scattering process alone.

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Theoretical computation of the Rayleigh scattering intensity is, however, a complex problem. In spite of considerable theoretical development in the last decade, it is as yet not possible to estimate Rayleigh scattering intensity for all values of γ-ray energies, atomic number of scatterer and of the momenta transferred. Although the formalism used by the Birmingham group [1] yields accurate results, up to now it has been used only for the K-electrons of Hg for only five specific γ-ray energies. Even for these limited cases considered, the scattering cross section can be accurately predicted only when \( q \) exceeds the intrinsic momentum of the K-electrons of the atom (a Zmc), in which cases K-shell electrons account for the bulk of the Rayleigh scattering and the contribution of L-shell electrons is approximately known. Of the several form-factor calculations of Rayleigh scattering, only those due to Nelms and Oppenheim [2] in the non-relativistic (up to \( q \approx 0.5mc \)), predict the cross section fairly well in a few particular cases viz. high \( Z \) elements, low energy γ-rays and low momentum transfer (\( q < 0.2mc \)). Above \( q = 0.2mc \), measured Rayleigh scattering cross sections deviate from the values given by Nelms and Oppenheim due to the onset of relativistic effects as well as to the effects of binding in the intermediate state [3]. It is therefore clear that except for a few isolated cases, the Rayleigh scattering cross section cannot be estimated accurately from theoretical considerations.

Rayleigh scattering is not only a fundamental mode of interaction of photons with matter demanding a careful investigation in its own right, but the estimate of coherent scattering intensity is also essential under many practical situations, e.g., in the measurement of nuclear resonance scattering and Delbrück scattering, in shielding calculations, etc. In the last few years a large number of experimental cross section values of reasonable accuracy have been accumulated (for a complete reference see [3]). These cross sections, as well as the accurate theoretical values wherever available, might be used as the basis for the development of an empirical formula for the total coherent scattering cross section. In this report we have presented an empirical formula, valid up to momentum transfer \( q = 2.5mc \), which can be used to estimate the Rayleigh scattering cross section in a simple and straightforward manner. We have shown how a suitable form of the empirical formula was obtained from an analysis of the available data especially by studying the dependence of the cross sections on the atomic number of absorber, energy of the photon and finally on the momenta

\(^1\)Figures in brackets indicate the literature references at the end of this paper.
transferred. A procedure for the evaluation of the empirical constants in the formula has also been indicated. The formula has then been compared with the experimental data and with the accurate theoretical estimates available. We have limited the range of validity of our formula from \( q = 0 \) to \( q = 2.5mc \), with the energy of the photon remaining below 2 MeV.

2. Procedure for the Derivation of the Empirical Formula

Form factor approximations of Rayleigh scattering cross section \( \sigma_R(\theta) \) can be written in the form

\[
\sigma_R(\theta) = \frac{1}{2}(1 + \cos^2 \theta)Zn(q) f(q) \tag{1}
\]

where \( \theta \) is the angle of scattering, \( q \) is the momentum transferred in units of \( mc \), \( Z \) is the atomic number of the scatterer; \( n(q) \) and \( f(q) \) are functions of \( q \) only. If equation (1) is universally valid then the reduced cross section defined by

\[
\sigma'_R(q) = \sigma_R(\theta)/\frac{1}{2}(1 + \cos^2 \theta) \tag{2}
\]

should be a function of \( q \) only and its functional dependence on the energy \( E \) of the photon will be implicit through \( q \) and not explicit at all. In Fig. 1 we have plotted experimental values of \( \sigma'_R(q) \) for various values of \( q \) and \( E \). It will be seen that the trend predicted by the form factor formalism is observable only for low energy \( \gamma \)-rays (up to 0.411 MeV as shown in Fig. 1). For Sn and Pb, data for 0.279 and 0.411 MeV may be represented by a single smooth curve but in both of these cases higher energy results fall systematically below this curve, the deviations increasing with \( E \) and \( q \) monotonically. Examination of the data for Sn and Pb further reveals the fact that the relative deviations behave approximately in the same manner in both the cases. It therefore follows that the accuracy of (1) can be improved by the inclusion of an explicitly energy dependent factor \( \varphi(E, q) \) which becomes important when \( E \) exceeds 0.4 MeV. On the assumption that \( \varphi(E, q) \) is independent of \( Z \) and by examining the relative deviation of the cross section data from the mean curve, it was found that if a factor of the form

\[
\varphi(E, q) = (13.5E - 4.4)^{-0.35q} \quad \text{for } E \geq 0.4 \text{ MeV}
= 1 \quad \text{for } E \leq 0.4 \text{ MeV} \tag{3}
\]
is included on the right hand side of Equation (1) then the values of $\sigma_{R}(q) / \varphi(E,q)$ for different values of $E$ considered in this report, can be made to lie on a smooth curve. The quantity

$$\sigma_{R}^{t}(q) = \sigma_{R}(q) / \varphi(E,q)$$

might be termed the normalized cross section. Its value is

$$\sigma_{R}^{t}(q) = Z_{n}(q)f(q) = \left(\frac{Z}{Z_{o}}\right)^{n(q)f(q)}Z_{o}^{n(q)}$$

$$= Z_{r}^{n(q)f_{r}(q)}$$

where $Z_{o}$ is the atomic number of a standard absorber,

$$Z_{r} = Z/Z_{o}$$
and

\[ f_r(q) = f(q)Z_o^n(q) \]  

(6b)

are respectively the relative Z of the absorber and the reduced value of f(q). The standard absorber chosen in the present investigation is Pb, which being a high Z element, has relatively high coherent scattering cross sections, which have been measured within small error limits by several workers. To find the form of \( f_r(q) \) we have first plotted the normalized cross sections of Pb, obtained both from experimental data as well as from the theoretical computations of the Birmingham group. Then a smooth curve was drawn through the points taking into account the errors of the experimental points and attaching larger weight factors in favor of the accurate theoretical values. It was found by trial that for \( q \geq 0.1mc \), the curve can be represented by a simple relation of the type

\[ f_r(q) = a q^{-m} \]  

(7a)

while for \( 0 \leq q \leq 0.1mc \), a suitable form for \( f_r(q) \) is

\[ f_r(q) = (b+cq)^{-p}, \ 0 \leq q \leq 0.1mc \]  

(7b)

The constants \( a \) and \( m \) were determined by the method of least squares. The constant \( b \) was found by plotting \( f_r(q) \) against \( q \) in a log-log graph and displacing each of the points by constant amount along q axis until the resultant curve was a straight line. Having obtained \( b \), the constants \( c \) and \( p \) were found by the method of least squares. The final expression for \( f_r(q) \) was derived as

\[ f_r(q) = (3.15q)^{-2.50}, \ q > 0.1mc \]  

(8a)

and

\[ f_r(q) = (0.14 + 2.65q)^{-3.20} \text{ for } 0 \leq q \leq 0.1mc \]  

(8b)

The function \( f_r(q) \) is shown in Fig. 2.

The value of \( n(q) \), the exponent of \( Z_r \) in Equation (5) has been measured by several workers for various values of \( q \). In Fig. 3, we have shown these results from \( q \) near zero to 2.5mc. It will be seen that \( n(q) \) gradually increase with \( q \), the rate of increase being faster near \( q = 0 \) but slows down around \( q = 0.1mc \). At \( q = 0 \), we have assumed that \( n(0) \) takes the value 2, as predicted by the form factor formalism. A close examination of the experi-
mental data on \( n(q) \) shows that there exists a wide difference in the trends of the values of \( n(q) \) in the range extending from approximately \( q = 0.5mc \) to \( q = 0.8mc \), relative to their values elsewhere. This range constitutes, from the experimental point of view, the most

Fig. 2. Variation of the function \( f_r(q) \) with \( q \).

Fig. 3. Fit of the experimental \( n(q) \) data with Equation 10.

\[ \text{(in the UNIT)} \]
difficult region for accurate measurement of the coherent scattering cross section. In this region, the hard noncoherent part of the scattered photons poses a serious problem to the experimental measurements. The conventional procedure of subtracting the noncoherent fraction from the scattered spectrum by using a low Z scatterer is not valid for correcting the hard component of the noncoherent radiation. There is therefore a tendency to overestimate the coherent scattering cross section in this momentum transfer region [3]. The necessary correction is relatively more important for low Z elements than for high Z elements due to the fact that the Rayleigh scattering cross section falls off more rapidly with decreasing Z than the cross section corresponding to the hard noncoherent component. In Fig. 3 this tendency is reflected in the large dispersion in the measured values of \( n(q) \) as well as their grouping well below the values expected from the trend of values on both sides of this region. In fitting the Z-dependence curve, therefore, greater weight was attached to the low q data as well as to data for \( q > 0.8mc \). It was found that the observed data could be satisfactorily expressed by a simple polynomial in q for values of \( q > 0.1mc \)

\[
n(q) = 2.90 + 0.08q + 1.27q^2 - 0.36q^3 \tag{9}
\]

The coefficients of the above polynomial have been obtained again by the method of least squares. To account for the sharp drop of \( n(q) \) below \( q = 0.1mc \), an additional term \(-0.90 \exp(-500q^2)\) was found necessary. This term has negligible value above \( q = 0.1mc \), so that, the value of \( n(q) \) for the entire range of momentum transfer can be written as

\[
n(q) = 2.90 + 0.08q + 1.27q^2 - 0.36q^3 - 0.90 \exp(-500q^2) \tag{10}
\]

The final expression for the Rayleigh scattering cross section is, therefore

\[
\sigma_R(\theta) = \frac{3}{8}(1 + \cos^2\theta) \varphi(E,q)(Z/E_2)^n(q)f_r(q) \tag{11}
\]

where \( \varphi(E,q) \) is given by Equation (3), \( n(q) \) by Equation (10) and \( f_r(q) \) by Equation (8). \( \sigma_R(\theta) \) is expressed in b/sr.

3. Comparison with the Experimental Data

In Figs. 4-9 we have compared the experimental coherent scattering cross section data for the scatterers Pb, Sn and Cu with the values predicted by the empirical formula given in
Equation (11).

In the case of Cu (Fig. 4) coherent scattering cross section is small except at small q and the experimental data are scarce and we have presented here only the measurements carried out with 0.662 MeV photons [8]. Within the limits of experimental uncertainty the experimental results are in agreement with the empirical formula, as can be seen in Fig. 4.

Fig. 4. Comparison of the empirical formula with the experimental data in the case of copper and 0.662 MeV photons.

Fig. 5. Comparison of the empirical formula with the experimental data in the case of Pb and Sn scatterers for 0.279 and 0.411 MeV photons. Also shown in the diagram are the modified form factor results after Brown and Mayers [1].
In the case of Pb we have considered the cross sections for 0.279, 0.411 (Fig. 5-Curve 1), 0.662 (Fig. 5), and 1.33 MeV (Fig. 7) photons. The experimental data agree quite well with the values predicted by Equation (11) throughout the range of $q$ considered here. As $q$ tends to zero, the formula (11) gives results which are practically identical to those given by form factor formalisms.

Experimental data considered in the case of low momentum transfer ($q$ up to about 0.5mc) are mainly due to [3]. In the earlier measurements ([11], [12], [13], [10]), the effect of the energy degeneracy of the Compton scattered photons on the observed cross sections [14] was not taken into account and hence, accuracy of some of these data is questionable. In the case of 1.33 MeV photons, a disquieting fact reveals itself, namely, the large number of
experimental results reported in the literature differ from each other by factors which are several times their error limits [3]. Of all the measurements in this field, we consider those by Standing and Jovanovich [7] to be the most accurate set, since in this investigation the hard noncoherent component has been properly taken into account by adopting an ingenious technique. For large values of \( q \), therefore, we have considered only the data given by these authors. These experimental results generally agree with the Equation (11) within ± 10\%.

For 0.411 and 0.662 MeV photons we have included the data given by Mann [4], Bernstein and Mann [5], and by Anand and Sood [9]. For several values of \( q \), the set of values given by the first two measurements differ from those given by the third measurements, by several times the quoted error limits despite the fact that all the three measurements were carried out by the same technique. In these cases the empirical formula gives values which are generally intermediate between the two sets of measurements. We may conclude, therefore, that the agreement of the experimental data for Pb in the energy range of 0.279 to 1.33 MeV with the empirical formula is entirely satisfactory and within the limits of experimental uncertainty.

To test the accuracy of the formula for intermediate \( Z \) elements, we have presented in Figs. 5 II, 8, and 9, the predicted and measured values of \( \sigma_R(q) \) for Sn, which were corrected

Fig. 8. Comparison of the empirical formula with the experimental data for Sn and 0.662 MeV photons. Also shown in the diagram are the modified form factor results after Brown and Mayers [1].
for nuclear Thomson scattering where necessary. Here it is to be noted that although there is general agreement of the experimental data with Equation (11) within the range of experimental uncertainty, there is a systematic trend of the cross section data in the range \( q = 0.5mc \) to about \( q = 1mc \) to be higher than the values given by the formula. If we bear in mind the experimental difficulty of measuring the coherent scattering cross sections in this range of \( q \), due to the presence of relatively large amounts of harder noncoherent radiation in the beam scattered by elements with not too high \( Z \) as described in the previous section, such a trend is to be expected. Elsewhere in the entire range of momentum transfer, the agreement between the predicted and the measured cross sections is quite satisfactory. It is possible to restore the agreement also in the intermediate region by a suitable choice of \( n(q) \), but such values will be rather unrealistic. It is expected that more careful experiments will remove this anomaly in the future.
4. Comparison of the Empirical Formula with the Form Factor
Results of Nelms and Oppenheim

We have compared the form factor results of Nelms and Oppenheim in Figs. 10 and 11 in the case of mercury and arsenic with our formula. In the case of Hg, the two results agree well up to \( q = 0.2mc \), although values given by Equation (11) are systematically lower than the Nelms-Oppenheim values in the region of very low \( q \). This difference is, however, well within 10\%. Above 0.2mc, the difference of the two results increases systematically. This is expected because we have placed greater emphasis on the experimental data, which, for this case, are known to give this type of deviation.

Fig. 10. Comparison of the empirical formula with the form factor results of Nelms and Oppenheim in the case of Hg.

Fig. 11. Comparison of the empirical formula with the form factor results of Nelms and Oppenheim in the case of As.

In the case of As (Fig. 11), also, the agreement is quite good up to \( q = 0.1mc \), the maximum deviation being less than 8\%. After this, Nelms and Oppenheim values fall systematically above the values given by our formula. This trend also confirms the experimental trend of the low Z data.
5. Comparison with the Results of Refined Formalism

A comparison of the two results has been presented in Figs. 6 and 7 in the case of Pb for 0.662 and 1.33 MeV γ-rays. The results of the refined formalism presented here have been obtained from the results of Brown and Mayers [1] for the K-electrons of Hg at 1.28 and 2.56\(mc^2\). L-shell contribution to these results has been estimated after Bernstein and Mann [5]. Minor correction has also been included to account for the difference of the atomic number of Pb and Hg and for the small difference in the energies of 1.28\(mc^2\) and 0.662 MeV photons and of 2.56\(mc^2\) and 1.33 MeV photons. All these corrections should not, however, introduce an uncertainty of more than 3%. We can see from Figs. 6 and 7 that in both the cases, the agreement between the two results is quite satisfactory. In the case of 1.33 MeV, the agreement up to \(q = 1.2mc\) is excellent; above \(q = 1.2mc\), calculated values of Equation (11) tend to be systematically higher than the accurate theoretical data. In the case of 0.662 MeV, however, the two results are quite close to each other right up to \(q = 2.5mc\). The good agreement between the two results is particularly satisfying, because the refined formalism is expected to give Rayleigh scattering cross section quite accurately.

In a number of cases (e.g. Figs. 5 II and 8) we have also presented the theoretical results calculated on the basis of the modified form factor as suggested by Brown and Mayers [1]. Although the accuracy of the modified form factor formalism is not yet clear, the data presented here show that the modified form factor results are generally not appreciably different from the results given in Equation (11) although the agreement is certainly poorer than in the case of the refined formalism.

6. Discussion

We thus find that except in the case of Sn for a small region of \(q\) near 0.5\(mc\) (where the experimental data themselves are considerably uncertain) there is, in general, good agreement of the formula developed here both with the experimental data up to 1.33 MeV in the range of momentum transfer considered here and with the available accurate theoretical data. A few experimental data above 1.33 MeV, e.g., 2.62 and 2.76 MeV, have been reported in the literature. We have not considered these data here for two reasons. Firstly, in most of these measurements, the hard noncoherent component (which is relatively more important for
harder photons) has not been adequately corrected for and hence these data cannot be relied upon [15]. Secondly, the Delbrück scattering contribution is not known, but in all probability it cannot be neglected at the higher energies. Also, as can be seen from Equation (1), we have retained the form factor expression for the polarization factor, i.e., \( 1 + \cos^2 \theta \).

We have thus 100\% of polarization at \( \theta = 90^\circ \). Measurements on the polarization of Rayleigh scattering at 90\(^\circ\) as well as the theoretical data on the refined formalism show that for 0.662 MeV, polarization is indeed close to 100\%. Also, at 1.25 MeV, deviation of the polarization observed at 90\(^\circ\) from 100\% is not very large although certainly finite. Theoretical results at 5.12mc\(^2\), however, predict a polarization which is considerably different from the result expected on the basis of the present formula. In the case of high energy \( \gamma \)-rays (\( E \gg mc^2 \)), the present formula can therefore be used only at smaller angles. Up to 1.33 MeV the present formula has been adequately tested and can therefore be safely used for \( q \) up to 2.5mc to estimate the coherent scattering cross section within \( \pm 10\% \).

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