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435

The Hyperbolic Character of Certain Experimental Results Which Tend Toward Limiting Values

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THE HYPERBOLIC CHARACTER OF CERTAIN EXPERIMENTAL RESULTS WHICH TEND TOWARD LIMITING VALUES

by Arthur F. Kirstein

ABSTRACT

The hyperbolic character of certain experimental data obtained from structural and mechanical testing provides an extremely simple graphical method for use in approximating the limiting values associated with the data. The method, developed by Southwell to explain the behavior of eccentrically loaded thin elastic struts, is shown to have much wider application. A limited number of illustrations are given to describe the method and to show how it might be used as a research tool.

Key words: Experimental, graphical solution, hyperbolic character, limiting values.

1. Introduction

It is a common occurrence in experimental investigations to find that plotted test results may form a curve which passes through the origin and tends toward limiting values in much the same manner as a rectangular hyperbola having horizontal and vertical asymptotes. Experimental data possessing this hyperbolic character will approach limiting values asymptotically. Thus an extremely simple graphical method is provided for use in approximating the limiting values of asymptotes. The equation for the rectangular hyperbola used in this method of approximation was given by Ayrton and Perry[1]* and later developed by Southwell[2] to explain the behavior of eccentrically loaded thin elastic struts. This method was also used by Ross[3] in a slightly modified form to represent the creep of concrete under load.

This investigation explored a number of different instances where the hyperbolic character of experimental data was used to illustrate

*Figures in brackets indicate the literature references at the end of this paper.

the method of approximation as a useful research tool. The basic method was explained and some of the variations or extensions were also shown.

2. Background

To the writer's knowledge the first published account recognizing this hyperbolic character in the load vs deflection data obtained from eccentrically loaded struts was given by Ayrton and Perry. They found that when the reciprocals of deflection and stress from tests of oak struts were plotted on rectangular coordinates the points fell very nearly on a straight line. This was expressed as

$$\delta = \frac{\alpha\sigma}{\beta - \sigma} \qquad (1)$$

which can be written in the form used by Southwell nearly one half century later

$$\delta - \beta\frac{\delta}{\sigma} + \alpha = 0 \qquad (2)$$

where δ = deflection

σ = stress

α = initial eccentricity

β = critical elastic stress.

When equation (2) is divided by δ we recognize that a linear relationship exists between the reciprocals of σ and δ.

The use of this graphical method of approximating limiting values can be explained more fully by examining the properties of the hyperbola. Consider the rectangular hyperbola passing through the origin of the cartesian coordinate system shown in figure 1 having as asymptotes

$$x + \alpha = 0 \qquad (3)$$

and

$$y - \beta = 0;$$

the equation of this hyperbola is

$$xy - \beta x + \alpha y = 0. \qquad (4)$$

Equation (4) may be written as

$$x - \beta v + \alpha = 0 \qquad (5)$$

where

$$v = \frac{x}{y} \quad . \qquad (6)$$

Equation (5) is recognized as the equation for the straight line shown in figure 2. The inverse slope and the intercept on the x axis of this line are measures of β and α, respectively, which are the limiting values or asymptotes of the original hyperbola shown in figure 1.

The application of this method to experimental data is accomplished graphically by plotting the data in the form shown in figure 2. The linearity of the resulting curve is a measure of how well the data approximates the rectangular hyperbola and the limiting values or asymptotes may be determined as described previously.

Southwell found that the load-deflection relationship for an eccentrically loaded thin elastic column approximated this hyperbola and compared the critical load of the column to the predicted limiting value, β. The results of this comparison were very good. No doubt, this along with other factors accounts for the assumption that the method would apply only so long as the system remained elastic.

Conversely, Ross used this method in a slightly modified form to represent the creep of concrete under load. Thus Ross indicated that the method could be used to predict limiting values other than critical elastic loads as supposed by Southwell.

Certain other experimental data which may be observed from mechanical or structural testing may be typified by the curves shown in figure 3. The curves shown in figure 3a were modified as indicated to produce the curves representing stress vs nonlinear strain in figure 3b. This procedure is similar to that used by Galletly and Reynolds[4] which produces curves that resemble Southwell's rectangular hyperbola of figure 1. The modified results shown in figure 3b can be used in the graphical method of approximating the limiting values of the curves given in figure 3a. The success or failure of this method in predicting the limiting value is not dependent upon elastic action, but is dependent on how well the experimental results approximate the hyperbola.

3. Limiting Values and the Hyperbolic Character of Experimental Results

It was indeed unfortunate that Southwell arbitrarily restricted the use of his method to the examination of the critical elastic buckling loads. Had he continued to examine this method when applied to Kármán's medium and thick struts[5] he would have found that it would have served to predict with a fair degree of accuracy the ultimate load carrying capacities of the struts. Table 1 contains the data from these tests of medium and thick struts. Column three of this table contains the values predicted from Southwell's rectangular hyperbola and column four gives the ratio of these predicted limiting values to those observed by Kármán. The good agreement shown in table 1 has little or nothing to do with the elasticity of the medium and thick struts. It merely shows that the load vs deflection curves possess this hyperbolic character which is useful for the purpose of approximating the values of the horizontal asymtotes or limiting values which in this case correspond to the ultimate loads.

Using the previously discussed modification of typical stress vs strain data as it pertains to figures 2 and 3, consider the stress vs strain curves for compressive tests on standard 6 x 12 inch concrete cylinders shown in figure 4. Applying the linearity test which is inherent in the graphical method shown in figure 2 to these data produced the results given in figure 5. Note that the curves in figure 5 are linear for the larger values of nonlinear strain, which indicates that the data does at least approximate the rectangular hyperbola. It should be noted that in the graphical method illustrated the straight line is selected so that it coincides more closely with data associated with the larger values of nonlinear strain, and no attempt was made to apply the method of least squares or other mathematic method of curve fitting. It has been found from experience[3] that the graphical method produces results which more closely represent the limiting values. A comparison of the limiting values obtained from these curves and the compressive strengths determined from the test results is made in table 2. These results show that the method can be used to make reasonable predictions of the compressive strength.

Ross found that the creep in concrete when plotted as the ordinate against the corresponding time as the abscissa produced a curve similar in appearance to the rectangular hyperbola shown in figure 1. Further, he plotted a curve of time/creep vs time (t/c vs t) which he referred to as the "reduction of creep curve". This curve was similar to that shown in figure 2, however, Ross elected to write the linear equation for the curve as

$$\frac{t}{c} = a + bt \qquad (7)$$

where c = creep per unit length in time, t

a = a constant (t/c intercept)

b = a coefficient (slope)

Ross found it useful to correlate a and b of equation (7) with factors or conditions which affected the results of his studies of creep.

4. Summary

It has been observed that the graphical method of predicting the limiting values or asymptotes for experimental data that exhibit this hyperbolic character has a wider application than was recognized in the past. It is obvious from the results presented in table 1 that the usefulness of the method extends beyond that ascribed to it by Southwell when dealing with the stability of struts. Further, it was shown in table 2 that the method has some degree of usefulness in predicting mechanical properties associated with materials testing. Ross also found this method to be of use in predicting the limiting values of creep in concrete.

As noted it is fairly common in experimental investigations involving structural and mechanical testing to obtain test results which tend toward limiting values asymptotically. Furthermore, from the previously cited examples it would appear that the method described could be applied as a very useful tool in the structural or mechanical testing laboratory to predict results from limited investigation and to plan testing programs which would yield a greater amount of useful results with a lesser amount of testing. It should also be recognized that the graphical method can be used to write empirical equations which show the relationship of the variables involved.

5. Acknowledgments

Acknowledgment is made to Mr. R. G. Mathey of the NBS Structures Section for furnishing the data presented in figure 4.

6. References

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Table 1. - Comparison of Kármán's Observed Maximum Stress and the Predicted Limiting Value

Strut No.	Maximum stress Observed	Predicted	Predicted stress / Observed stress
	kg/cm^2	kg/cm^2	
	Medium struts		
7A	2760	2794	1.012
7B	2685	2796	1.041
8	2740	2724	0.994
9A	3030	3058	1.009
9B*	--	--	--
10A	3185	3255	1.022
10B	3080	3120	1.013
11	3165	3185	1.006
12A*	--	--	--
12B	2960	3009	1.017
13	3060	3105	1.015
	Thick struts		
14B	3320	3439	1.036
15A	3395	3440	1.015
15B	3485	3360	0.964
16	3890	3989	1.025
17	4330	4510	1.042

*Insufficient data recorded for determination of predicted value

Table 2.- Observed and Predicted Compressive Strength of 6 by 12 in. Concrete Cylinders

Cylinder	Compressive strength		Predicted strength / Observed strength
	Observed (psi)	Predicted (psi)	
1	3360	3566	1.061
2	3077	3271	1.063
3	2911	3090	1.061
4	2844	2922	1.027
5	2741	2960	1.080

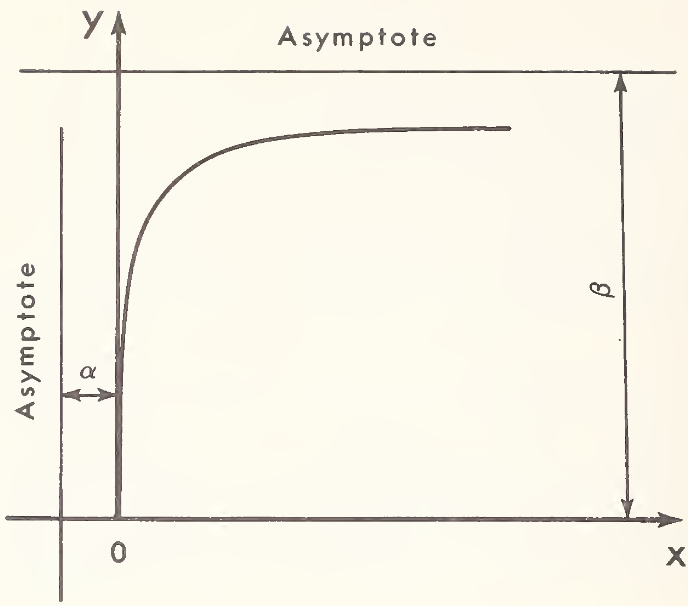

Figure 1. Rectangular hyperbola

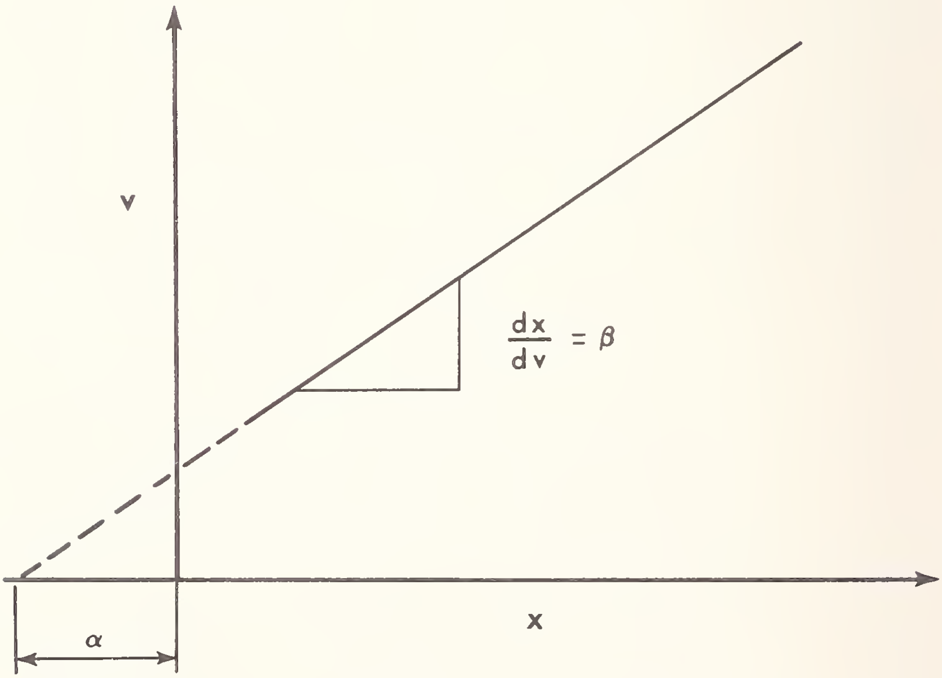

Figure 2. Relationship between v and x of equation (5).

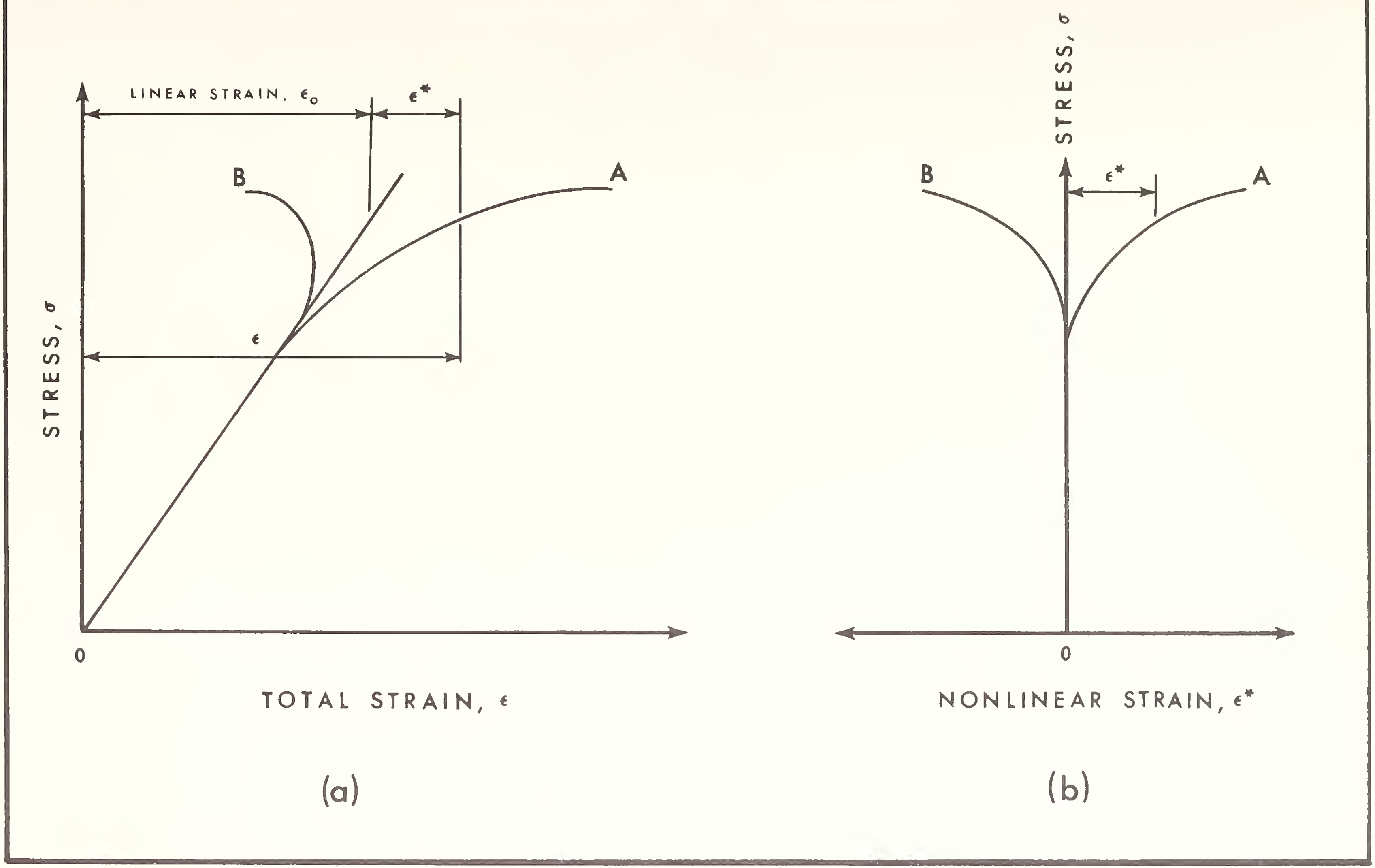

Figure 3. Typical stress-strain relationship observed in mechanical testing.

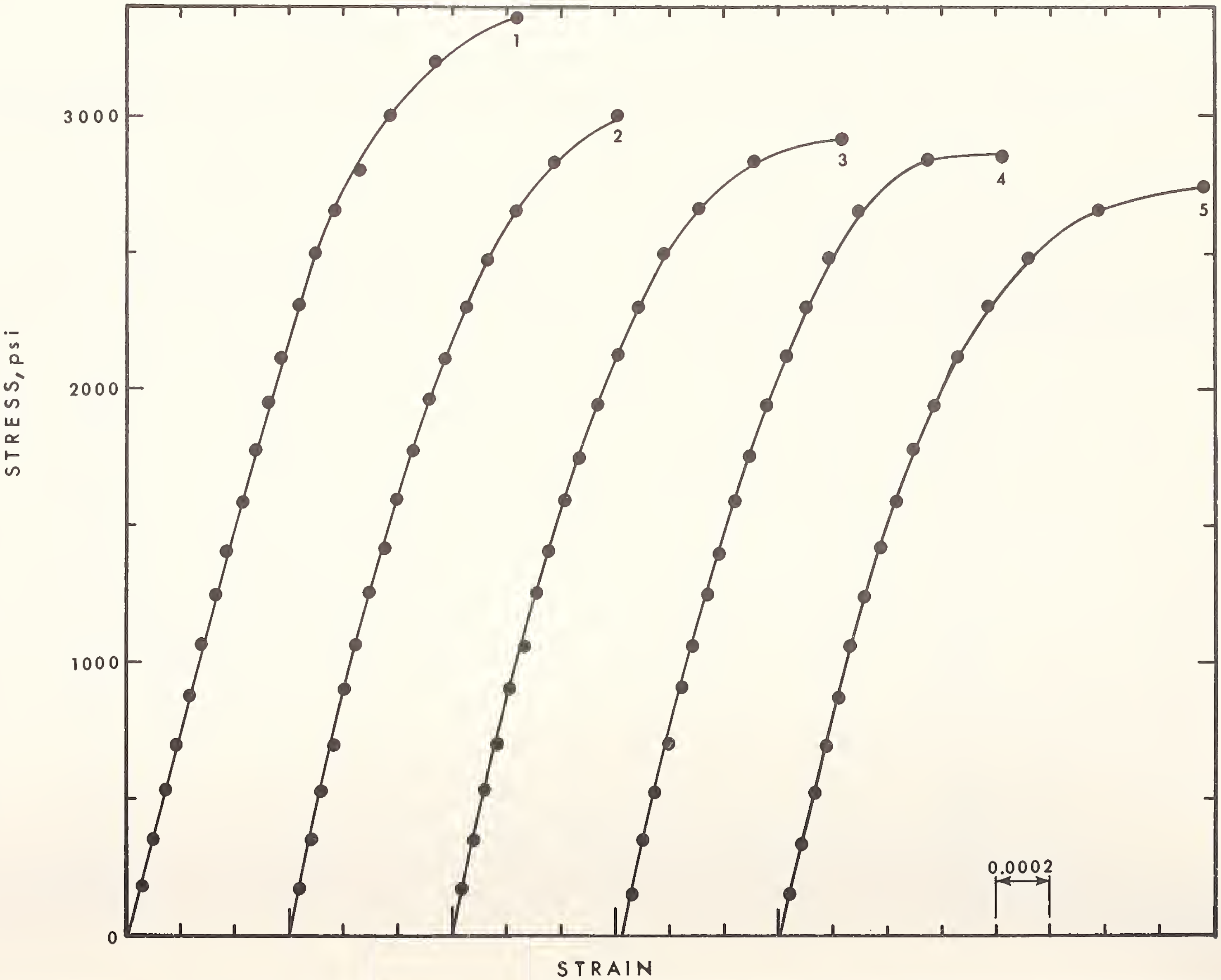

Figure 4. Results of compression tests of 6 by 12 in. concrete cylinders.

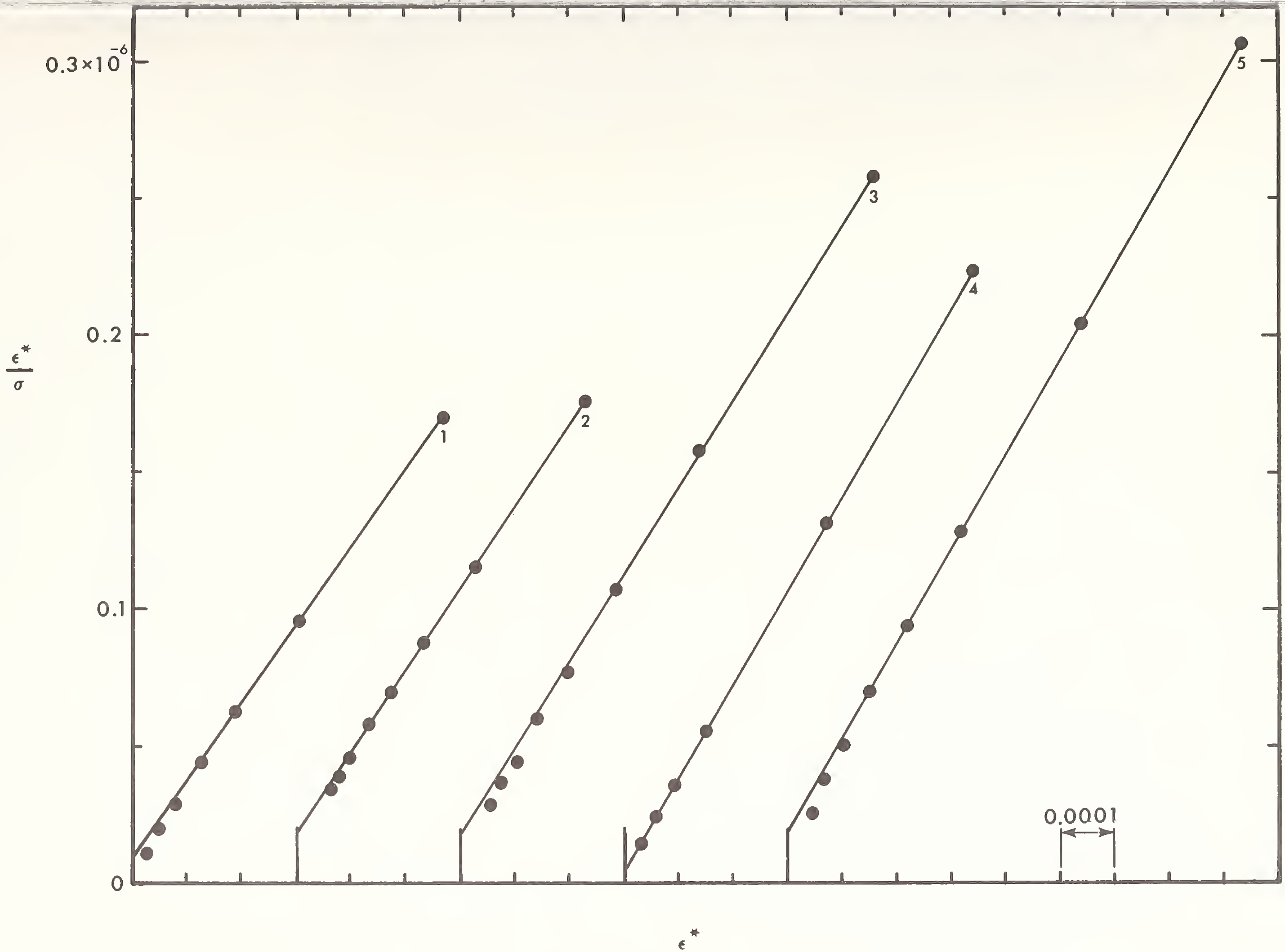

Figure 5. Graphical solution for predicting compressive strength.

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