

NBS

TECHNICAL NOTE

368

**Solution of the Abel Integral
Transform for a Cylindrical
Luminous Region with Optical
Distortions at its Boundary**



**U.S. DEPARTMENT OF COMMERCE
National Bureau of Standards**

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UNITED STATES DEPARTMENT OF COMMERCE
C.R. Smith, Secretary
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SOLUTION OF THE ABEL INTEGRAL TRANSFORM FOR A CYLINDRICAL LUMINOUS REGION WITH OPTICAL DISTORTIONS AT ITS BOUNDARY

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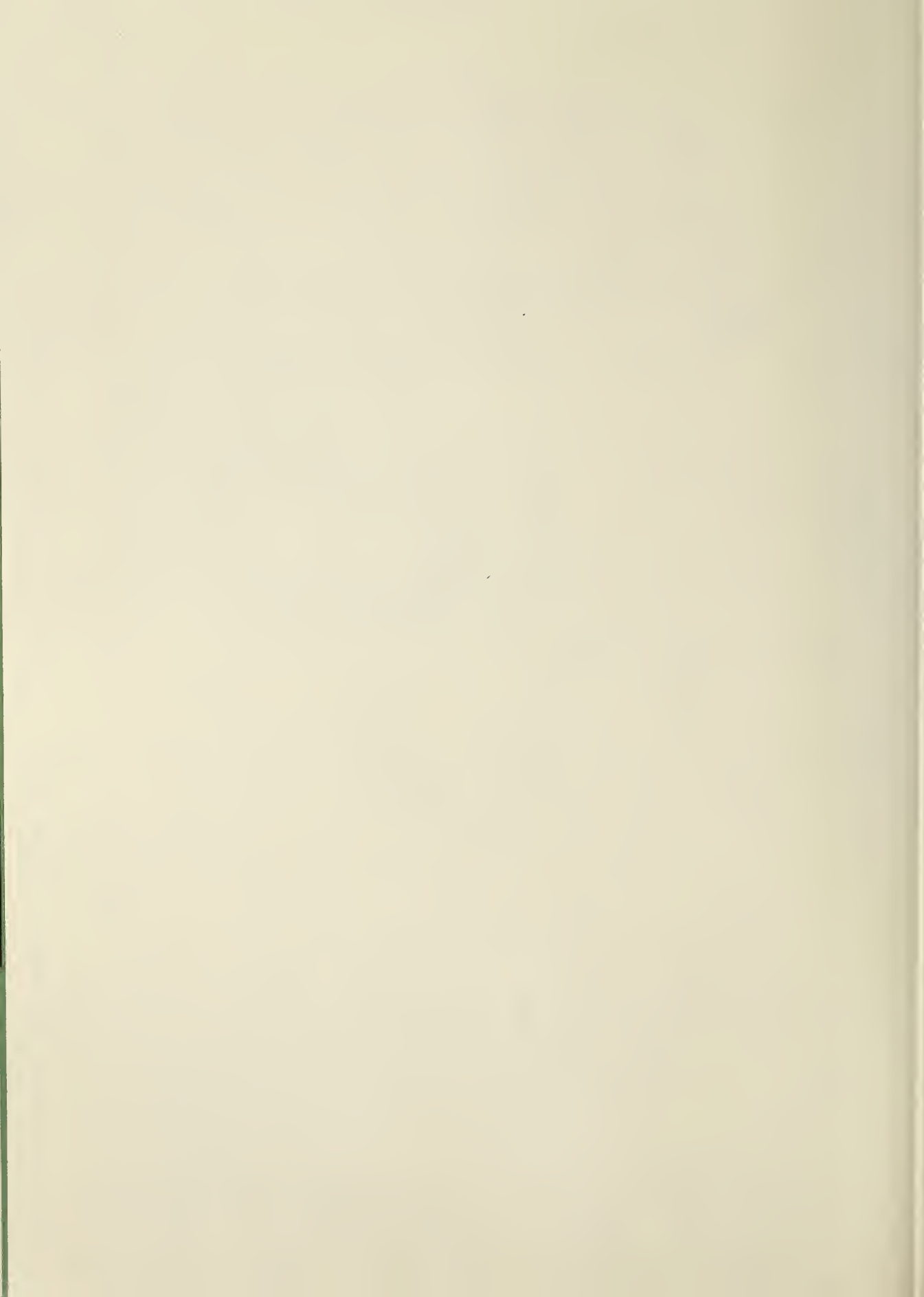


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SOLUTION OF THE ABEL INTEGRAL TRANSFORM FOR A
CYLINDRICAL LUMINOUS REGION WITH OPTICAL
DISTORTIONS AT ITS BOUNDARY*

Earl R. Mosburg, Jr. and Matthew S. Lojko

The use of orthogonal polynomial expansions in the calculation of the Abel integral transform is discussed. Particular attention is directed to the effects of optical and instrumental distortions when the luminous region is contained by a cylindrical glass tube. An easily calculable solution of the Abel integral is presented which reduces the effect of such distortions by employing a weighting function which has a maximum at the center and vanishes at the boundary. This approach results in a more accurate solution of the Abel integral transform in the case where significant optical and instrumental distortions are present near the boundary of the luminous region.

Key Words: Abel transform, Abel inversion, plasma diagnostics, emissivity profile, radiance profile.

INTRODUCTION

In order to obtain the radial distribution of volume light emissivity within a cylindrical, non-absorbing luminous region, we must solve the Abel integral transform using the projected brightness profile, measured by scanning the detector in a direction perpendicular to the axis of the tube. If the projected brightness profile is $f(x)$, where x is the ratio of the distance off axis to the radius of the luminous region, and if $g(r)$ is the corresponding volume emissivity distribution, where r is the normalized radius, then the Abel integral transforms can be written as

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$$g(r) = -\frac{2}{\pi} \frac{d}{r dr} \int_r^1 \frac{f(x)x dx}{\sqrt{x^2 - r^2}} \quad (1a)$$

and

$$f(x) = \int_x^1 \frac{g(r) r dr}{\sqrt{r^2 - x^2}} \quad (1b)$$

or alternatively in the forms

$$g(r) = -\frac{2}{\pi} - \int_r^1 \frac{f'(x) dx}{\sqrt{x^2 - r^2}} \quad (1c)$$

and

$$f(x) = g(1) \sqrt{1-x^2} - \int_x^1 \sqrt{r^2 - x^2} g'(r) dr \quad (1d)$$

where the primes indicate differentiation. A direct numerical solution for $g(r)$ using measured values of $f(x)$ in Eq. (1a) or Eq. (1c) is subject to considerable error due to the behavior of the denominator in the integrand and to the necessity for numerical differentiation. These difficulties are considerably alleviated by first making a least square fit of $f(x)$ to a power series expansion as described by Freeman and Katz.¹

A more convenient expansion in terms of orthogonal polynomials has been reported by Herlitz² using Tchebycheff polynomials of the second kind. Popenoe and Shumaker³ have used Herlitz's method as well as an expansion in terms of Legendre polynomials. The use of orthogonal polynomial expansions is equivalent to a weighted least squares analysis. But here, because of the orthogonality of the basis functions, the coefficients can be independently calculated. Each coefficient can then be tested for statistical significance and the

expansion appropriately truncated without any prior, ad hoc decision about the number of terms to be used. The weighting function $w(x)/v(x)$ of Eq. (5) is determined once a particular series of orthogonal polynomials is chosen for the expansion.

Sufficient attention has not, however, been given to the use of orthogonal polynomial expansions in the case where optical or instrumental distortions are introduced at the boundary of the luminous region, as for example, by the presence of a glass container. In this case, distortions due to the scattering and uneven refraction⁴ of light in the tube walls may become important. These effects are a maximum near $x = 1$ where a near grazing angle is involved in the measurement. Furthermore, the finite size of the spectrometer slit introduces an averaging over the normalized spacial resolution function of the instrument, $R(x-\zeta)$, such that the measured curve becomes a function, $h(\zeta)$, where

$$h(\zeta) = \int_{\zeta - D/2}^{\zeta + D/2} f(x)R(x-\zeta)dx \quad (1e)$$

and $\pm D/2$ are the limiting values of $(x-\zeta)$ for which there is appreciable contribution to the integral. The projected brightness profile, $f(x)$, can, in principle, be recovered from $h(\zeta)$ by an appropriate inversion of Eq. (1e), but residual errors will be present. These errors will also be larger near $x = 1$ where the differences between functions $h(\zeta)$ and $f(x)$ are largest, i. e., where the second derivative of $h(\zeta)$ is more important. These distortions are particularly large when it is desired to invert projected profiles approximating $f(x) = \sqrt{1-x^2}$, which corresponds to $g(r) = \text{constant}$. Here the second derivative of $h(\zeta)$ is even larger near the boundary and the luminosity is now high in the region of maximum distortion.

We wish to stress at this point that, in contrast to the distortions, most projected brightness profiles of experimental interest vanish at $x = 1$ and exhibit maxima at or near the center of the light source. It is now clear that in order to reduce the effect of the distortions, we would like a weighting function in Eq. (5) which vanishes at $x = 1$ and exhibits a maximum at $x = 0$.

In this paper we restrict our choice of polynomial to the general class of Ultraspherical or Gegenbauer polynomials, which includes Legendre and Tchebycheff polynomials as special cases. In what follows we will use the notation of the Handbook of Mathematical Functions.⁵ We expand $f(x)$ in terms of general Gegenbauer polynomials as

$$f(x) = \sum_{n=0}^N a_n v(x) C_n^{(\alpha)}(u(x)) \quad (2)$$

where $v(x)$ is some shape function to be chosen and $u(x)$ is some function of x . Substituting Eq. (2) into Eq. (1a) we arrive at the expression

$$g(r) = -\frac{2}{\pi} \sum_{n=0}^N a_n \frac{d}{rdr} \int_r^1 \frac{v(x) C_n^{(\alpha)}(u(x)) x dx}{\sqrt{x^2 - r^2}}. \quad (3)$$

When the orthonormalization integral for the Gegenbauer polynomials⁵ is written in the form

$$\int_a^b w(x) C_n^{(\alpha)}(u(x)) C_m^{(\alpha)}(u(x)) dx = h_{n\alpha} \delta_{nm}, \quad (4)$$

then multiplying Eq. (2) by $\frac{w(x)}{v(x)} C_m^{(\alpha)}(u(x))$ and using Eq. (4), we obtain

$$a_n = \frac{1}{h_{n\alpha}} \int_a^b f(x) \frac{w(x)}{v(x)} C_n^{(\alpha)}(u(x)) dx \quad (5)$$

which allows the calculation of the coefficients a_n needed in Eq. (3). It is clear that the function $f(x)$ can be written as the sum of an experimentally significant part, $f_e(x)$, and a part due to optical and instrumental distortions, $f_d(x)$; that is,

$$f(x) = f_e(x) + f_d(x). \quad (6)$$

In many cases it may be convenient to further split the experimental part into an easily soluble approximate form, $f_a(x)$, and a relatively small perturbation to this form, $f_p(x)$, so that

$$f(x) = f_a(x) + [f_p(x) + f_d(x)] = f_a(x) + f_c(x). \quad (7)$$

Experimentally the terms are not, of course, separable from prior knowledge, but we may arbitrarily separate out the approximate function $f_a(x)$. The same final result is obtained by performing the Abel inversion of these terms separately and then summing. Note that the polynomial expansion used for $f_a(x)$ need not be the same as that used for the two other terms of Eq. (7).

The problem then becomes one of reducing the effect of the distortion contribution, $f_d(x)$, in treating the combined contribution, $f(x)$ of Eq. (6), or $f_c(x)$ of Eq. (7). When choosing a specific form of Eq. (2) we wish therefore to satisfy two requirements:

(A) Proper weighting factor. The weighting factor in Eq. (5) should be such that the contribution to the calculation of a_n is reduced where the distortion contribution, $f_d(x)$ is largest. One would therefore like the weighting function $w(x)/v(x)$ to approach zero as $x \rightarrow 1$ and be a maximum at the center.

(B) Ease of calculation. In principle, Eq. (3) can always be evaluated numerically, but the number of integrals that must be calculated can be very large. To illustrate this point, if we wish to calculate $g(r)$ for S different values of r , then the number of integrals becomes $N(S+1)$ where N is the number of terms in the polynomial expansion. For convenience, then, Eq. (3) should be directly integrable in some closed form, or, this failing, it should be easily calculable as, for example, by a recurrence relation between the integrals of order $n+2$, $n+1$, and n . In this paper we have settled for the latter condition in order to satisfy requirement A in full.

Once the form of Eq. (2) has been set, the weighting factor of requirement A and the integrability or non-integrability of Eq. (3) in closed form are fully determined. Thus the simultaneous satisfaction of requirements A and B must be somewhat fortuitous. The number of solutions of the Abel integral equation in terms of well-known polynomials is severely limited. In rows 1 thru 4 of Table 1, we show the weighting factors and solutions of the integrals of Eq. (3) for several choices of the form of Eq. (2) where the function $v(x)$ has been chosen to allow evaluation of these integrals in closed form. All of these solutions can be derived by proper manipulation of the general equation

Table I. Solutions of the Abel Transform for some specific forms of the polynomial expansion.

Name	$v(x)C_n^{(\alpha)}(u(x))$	$w(x)/v(x)$	$1/h_{n\alpha}$	a, b	$\frac{1}{r} \frac{d}{dr} \int_r^1 \frac{v(x) C_n^{(\alpha)}(u(x)) x dx}{\sqrt{x^2 - r^2}}$
Tchebycheff of 1st kind ($\alpha=0$)	$T_{2n+1}(x)/x \sqrt{1-x^2}$	x	$2/\pi$	-1, +1	$+2\pi C_{n-1}^{(3/2)} (2r^2 - 1)$
Legendre ($\alpha=1/2$)	$P_n(2x^2 - 1)$	x	$4n+2$	0, 1	$-T_{2n+1}(r/r \sqrt{1-r^2})$
Tchebycheff of 2nd kind ($\alpha=1$)	$\sqrt{1-x^2} U_{2n}(x)$	1	$2/\pi$	-1, +1	$-\pi(n + \frac{1}{2}) P_n(2r^2 - 1)$
Gegenbauer ($\alpha = \frac{3}{2}$)	$C_n^{(3/2)}(2x^2 - 1)$	$x^3(1-x^2)$	$\frac{8(2n+3)}{(n+2)(n+1)}$	0, 1	$\frac{[1+2n T_{2n+1}(r/r \sqrt{1-r^2}) - U_{2n}(r)]}{4(1-r^2)^{3/2}}$
Gegenbauer ($\alpha = 2$)	$\sqrt{1-x^2} C_{2n}^{(2)}(x)$	$(1-x^2)$	$\frac{8}{\pi(2n+3)(2n+1)}$	-1, +1	No simple closed form expression *

* Cannot be simply expressed in terms of well-known polynomials.

$$\frac{d}{r dr} \int_r^1 \frac{(1-x^2)^{\alpha-\frac{1}{2}} C_n^{(\alpha)}(2x^2-1) x dx}{\sqrt{x^2-r^2}} = -\sqrt{\pi} \frac{\alpha \Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)} (1-r^2)^{\alpha-1} \left\{ C_n^{(\alpha+\frac{1}{2})}(2r^2-1) - C_{n-1}^{(\alpha+\frac{1}{2})}(2r^2-1) \right\} \quad (8)$$

We are not aware of any closed form solutions which are not specific cases derivable from this equation. The form of $v(x)$ in these solutions is closely related to the normalization function $w(x)$ and therefore the weighting function $w(x)/v(x)$ cannot be arbitrarily chosen. None of the solutions listed has a weighting function of the form we desire. In this paper we present a solution of the Abel integral equation which involves a weighting function of the desired type and consists of an expansion in terms of Gegenbauer ($\alpha=2$) polynomials (Table 1, Row 5).

THE SOLUTION USING GEGENBAUER ($\alpha=2$) POLYNOMIALS

If we choose an expansion of the form

$$f(x) = \sum_n a_n \sqrt{1-x^2} C_{2n}^{(2)}(x) \quad (9)$$

then Eq. (5) becomes

$$a_n = \frac{8}{\pi (2n+3)(2n+1)} \int_{-1}^{+1} f(x) (1-x^2) C_{2n}^{(2)}(x) dx, \quad (10)$$

and we immediately see that criterion A is satisfied. For the sake of completeness, the first few polynomials of interest are given below.

$$C_0^{(2)}(x) = 1 \quad C_2^{(2)}(x) = 12x^2 - 2 \quad C_4^{(2)}(x) = 80x^4 - 48x^2 + 3$$

$$C_6^{(2)}(x) = 448x^6 - 480x^4 + 120x^2 - 4$$

It remains to evaluate the equation

$$g(r) = - \frac{2}{\pi} \sum_{n=0}^N a_n \frac{1}{r} \frac{d}{dr} \int_r^1 \frac{\sqrt{1-x^2} C_{2n}^{(2)}(x) x dx}{\sqrt{x^2-r^2}}. \quad (11)$$

It can be shown that

$$\begin{aligned} \sqrt{1-x^2} C_{2n}^{(2)}(x) &= \frac{1}{4\sqrt{1-x^2}} [(2n+3) U_{2n}(x) \\ &- (2n+1) U_{2n+2}(x)] \end{aligned} \quad (12)$$

and therefore

$$g(r) = + \sum_{n=0}^N a_n [(2n+3) I_n(r) - (2n+1) I_{n+1}(r)], \quad (13)$$

where

$$I_n(r) \equiv \frac{-1}{2\pi} \frac{1}{r} \frac{d}{dr} \int_r^1 \frac{U_{2n}(x) x dx}{\sqrt{1-x^2} \sqrt{x^2-r^2}} \quad (14)$$

$$\text{and thus } I_0(r) = 0 \text{ and } I_1(r) = -1. \quad (15)$$

Substituting the recurrence relation

$$\begin{aligned} U_{2n+4}(x) &= 2(2x^2-1) U_{2n+2}(x) - U_{2n}(x) \\ &= +2 U_{2n+2}(x) - U_{2n}(x) + 4(x^2-1) U_{2n+2}(x) \end{aligned} \quad (16)$$

into Eq.(14), we obtain

$$I_{n+2} = 2I_{n+1} - I_n + \frac{2}{\pi} \frac{1}{r} \frac{d}{dr} \int_r^1 \frac{\sqrt{1-x^2} U_{2n+2}(x) x dx}{\sqrt{x^2-r^2}}. \quad (17)$$

The integral of the last term has a closed form solution (see Table 1) so that Eq. (17) becomes

$$I_{n+2}(r) = 2 I_{n+1}(r) - I_n(r) - (2n+3)P_{n+1}(2r^2 - 1), \quad (18)$$

where $P_{n+1}(2r^2 - 1)$ is also generated by a recurrence relation⁵ given by

$$P_{n+1}(2r^2 - 1) = \frac{2n+1}{n+1} (2r^2 - 1) P_n(2r^2 - 1) - \frac{n}{n+1} P_{n-1}(2r^2 - 1). \quad (19)$$

Using Eqs. (15), (18), and (19), the desired function, $g(r)$, can readily be obtained from Eq. (13).

This approach was readily programmed for a computer solution. The integral of Eq. (10) was evaluated using the trapezoidal rule and a 40 point Gaussian quadrature. The change in the values of the coefficients using the trapezoidal rule was completely negligible. A listing of the Fortran computer program using the trapezoidal rule is given in Appendix II. The series of coefficients so calculated can be terminated by applying the F-distribution significance test⁶ to each coefficient in turn. We start by assuming that a_0 is always significant. We then calculate the mean square deviation, δ , between the input curve $f(x_i)$ and the curve defined by Eq. (2) using the coefficients known to be significant. This calculation, when repeated using one additional coefficient to be tested, yields δ' . The quantity $F^* = (\delta - \delta') \times (\text{number of points in the input curve})/\delta'$ is then compared with the value of F chosen from

Table V of Ref. 6. If $F^* > F$ then the additional coefficient is considered significant and the procedure is repeated for the following coefficient. When a non-significant coefficient is found, all subsequent coefficients are then set equal to zero. This is done on the physical basis that fine structure is not expected.

In treating the input curves we allowed for asymmetries by following the approach of Freeman and Katz,¹ and thus did not force the transformed curves to be symmetric. This was done, in spite of the fact that no large asymmetries were expected, in order to obtain a check on the symmetry of the discharge. In this approach, the symmetric part of $f(x)$ is contained in the even function

$$f_g(x) = \frac{f(+x) + f(-x)}{2} \quad (20)$$

and the asymmetric part of $f(x)$ is described by the even function

$$f_u(x) = \frac{f(+x) - f(-x)}{2x} \quad (21)$$

After the Abel inversion of these functions to obtain $g_g(r)$ and $g_u(r)$, the radial distribution is constructed by using

$$g(+r) = g_g(r) + rg_u(r) \quad (22)$$

and

$$g(-r) = g_g(r) - rg_u(r). \quad (23)$$

CONCLUSION

The use and advantages of orthogonal polynomial expansions in solving the Abel integral transform have been briefly discussed and a new solution in terms of Gegenbauer ($\alpha = 2$) polynomials, which utilizes a weighting function $(1-x^2)$ in the calculation of the expansion coefficients, has been presented. In the presence of significant distortions near the boundary of the luminous region, a condition which is particularly true for projected profiles approximating $f(x) = 1-x^2$, or $g(r)$ approximately constant, the use of the Gegenbauer ($\alpha=2$) polynomial expansion will reduce the contribution of such distortions in the calculation of the expansion coefficients, and in principle allow a more accurate solution of the Abel inversion integral.

ACKNOWLEDGEMENT

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REFERENCES

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2. S. I. Herlitz, Technical Note #5 Institute of Physics, Univ. of Uppsala, Uppsala, Sweden (July 31, 1961); *Arkiv Fysik* 23, 571 (1963).
3. G. H. Popenoe and J. B. Shumaker, *NBS J. Res.* 69A, 495 (1965).
4. For an axially symmetric light source there is no distortion due to refraction in passing thru a glass wall of uniform thickness. The azimuthal angle of a light ray is changed but the "impact parameters" inside and outside of the tube are equal, to the extent that the index of refraction inside is equal to that of air (see Appendix I).
5. M. Abramowitz and Irene A. Stegun, Handbook of Mathematical Functions, NBS Applied Mathematics Series 55 (June 1964), in particular p. 782.
6. Paul G. Hoel, Introduction to Mathematical Statistics (Wiley, 1954).

APPENDIX I

We present here a proof of the statement that the "impact parameter" of a light ray is unchanged by passage thru the wall of a uniform glass tube. Referring to figure 1, we write the law of sines for triangle ABC as $\sin \theta_2 = \sin (180-\theta_3)/(1 + \tau)$, or

$$(1 + \tau) \sin \theta_2 = \sin \theta_3. \quad (\text{A-1})$$

The initial and final impact parameters are given by the relations

$$p_i = (1 + \tau) \sin \theta_1, \text{ and } p_f = \sin \theta_4. \quad (\text{A-2})$$

The angles involved are related by Snell's law as follows:

$$\sin \theta_1 / \sin \theta_2 = \sin \theta_4 / \sin \theta_3 = \eta. \quad (\text{A-3})$$

Here η is the index of refraction of the glass tube relative to that of air and the indices of refraction both inside and outside of the tube are assumed identical. Combining these relations, we obtain $p_f = \sin \theta_4 = \sin \theta_1 (\sin \theta_3 / \sin \theta_2) = \sin \theta_1 (1 + \tau) = p_i$. Thus the impact parameter inside the tube is identical with that of the incident ray. This result is simply another expression of the conservation of angular momentum. There is consequently no distortion due to refraction for the case of a cylindrically symmetric light source, coaxial with a cylindrical glass tube of uniform thickness.

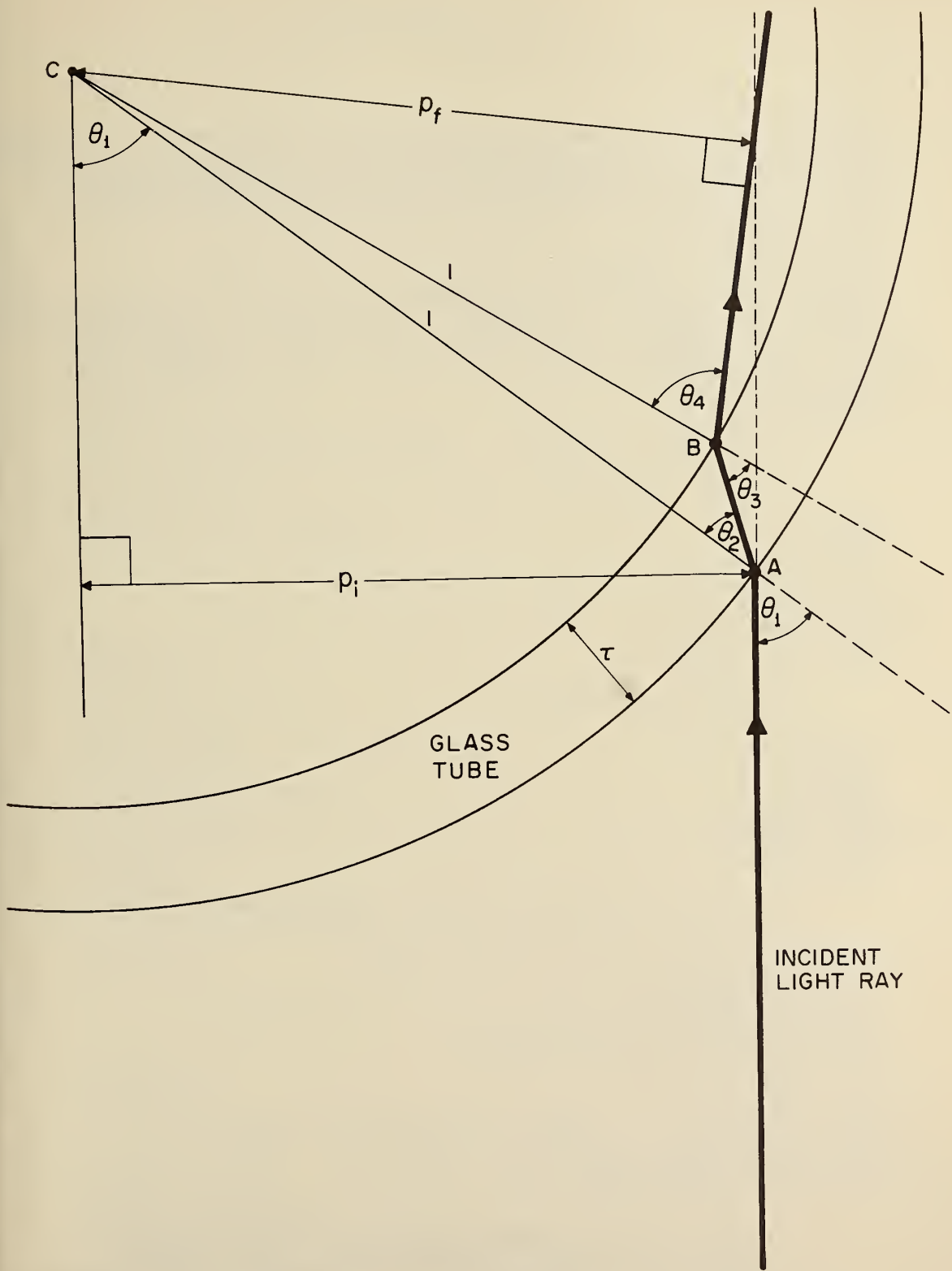


Figure 1. Displacement of an incident beam of light in passing through an optically perfect glass tube. The "impact parameter" is not changed.

APPENDIX II

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PROGRAM ABELTAPE
  DIMENSION HED( 5),X(500),Y(500),RP(100),RM(100),XP(250),YP(250),
  1YM(250),XF(501),FX(501),UR1(501),UR2(501),YKN(251),AN(20),BN(20),
  2ANN(20),BNN(20),RA(20),RB(20),RX(501),CS(501),BS(501),GP(100),
  3GM(100),KIND(3,2),GR(201),XJP(251),XJM(251),XX(251),HEE(10)
  TYPE INTEGER DATE
  COMMON NPOINT
  DATA(IRUNO=0),(ISETO=0)
  DATA(ESPI=5.092958),(FSP=1.2732395),((KIND(1,I),I=1,2)=8H  TCHEB,
  15HICHEF),((KIND(2,I),I=1,2)=8H  LEGE,4HNDRE),((KIND(3,I),I=1,2)=
  28H  GEGE,5HNBAUR)
112 FORMAT(*OSET=*,I2,5X,*RUN=*,I2,*  NOT FOUND ON INPUT TAPE*)
  READ(60,505)HEE
505 FORMAT(10A8)
C
C          GENERATE RHO ARRAY
C
  RZ=0.
  RP(1)=.01
  RM(1)=-.01
  DO 106 I=2,100
  RP(I)=RP(I-1)+.01
106 RM(I)=-RP(I)
  DATE=IDATE(XDUMMY)
C
C          GENFRATE GR ARRAY
C
  DO 500 I=1,100
  IM=101-I
  IP=101+I
  GR(I)=RM(IM)
500 GR(IP)=RP(I)
  GR(101)=0.
C
C          READ INPUT PARAMETERS,DATA, AND HEADING.
C
  99 READ(60,111)ISET,IRUN,NN,NX,HED,INFLAG
111 FORMAT(4I5,5A8,I1)
  IF(EOF,60)98,501
501 IF(INFLAG.EQ.0) GO TO 503
  READ(60,153)(X(I),Y(I),I=1,NX)
153 FORMAT(8(F5.0,F5.1))
  GO TO 9999
503 READ(60,151)(X(I),I=1,NX),(Y(I),I=1,NX)
151 FORMAT(8E10.3)
9999 READ(60,152)NPOINT,NP,FVALUE,THICK,IPPP
152 FORMAT(2I5,2F10.5,I1)
C
C          GENERATE ARRAYS FOR YM AND YP, THE CORRESPONDING Y
C          VALUES FOR -X AND +X RESPECTIVELY AND ASSUME THE
C          VALUES AT THE END POINTS ARE AVERAGFS OF THE MEASURED Y.
C          NH=NX/2 FOR EVEN NX.
C          NH=(NX-1)/2 FOR ODD NX.
C          NT IS THE TOTAL NO. OF POINTS.

```

```

C
C      FIRST FIND THE AVERAGE DELTA X.
C
6 DXA=2./FLOATF(NX-1)
C
C      FIND N= NX  MODULO 2.
C
N=XMODF(NX,2)
IF(N.EQ.0)12,13
C
C      N=0 IMPLIES NX IS EVEN.
C
12 NH=NX/2
DXH=DXA/2.
NHPO=NH+1
DO 14 I=1,NH
NHP=NH+I
YP(I)=Y(NHP)
XP(I)=DXA*FLOATF(I-1)+DXH
NHM=NHPO-I
14 YM(I)=Y(NHM)
YZ=.5*(Y(NH)+Y(NHPO))
GO TO 16
C
C      N NOT 0 IMPLIES NX IS ODD.
C
13 NH=(NX-1)/2
NHPO=NH+1
DO 15 I=1,NH
NHP=NHPO+I
YP(I)=Y(NHP)
NHM=NHPO-I
XP(I)=DXA*FLOATF(I)
15 YM(I)=Y(NHM)
YZ=Y(NHPO)
DXH=DXA
16 XP(NH)=1.
YA=.5*(YP(NH)+YM(NH))
YP(NH)=YA
YM(NH)=YA
C
C      APPLY BACKGROUND NOISE CORRECTION TO THE DATA.
C
TP=THICK+1.
TPS=TP*TP
IF(THICK.EQ.0.) C=0.
IF(THICK.NE.0.) C=YA/SQRT(TPS-1.)
YZ=YZ-C*THICK
DO 17 I=1,NH
XSQ=XP(I)*XP(I)
DXP=SQRTF(TPS-XSQ)-SQRTF(1.-XSQ)
CD=C*DXP
YP(I)=YP(I)-CD
17 YM(I)=YM(I)-CD

```



```

WRITE(61,124)(X(I),Y(I),I=1,NN)
124 FORMAT(5(F13.4,F9.4))
WRITE(61,121)HEE,HFD,DATE,ISFT,IRUN,NN,NX,FVALUE,THICK,NP,KIND(NPO
1INT,1),KIND(NPOINT,2)
WRITE(61,125)
125 FORMAT(* THE LISTING OF THE NORMALIZED INPUT DATA FOLLOWS*//)
WRITE(61,123)
WRITE(61,124)(XF(I),FX(I),I=1,NT)
C GENERATION OF THE COEFFICIENTS FOR THE DESIRED POLYNOMIAL
DXH=DXH/2.
XX(1)=0.
DO 91 I=2,NHPO
91 XX(I)=XP(I-1)
GO TO (100,200,300),NPOINT
C NPOINT=1, TCHEBICHEF CASE
100 DO 101 I=1,NHPO
UR2(I)=1.
UR1(I)=XX(I)+XX(I)
IF(I.FQ.1)198,199
198 AN(1)=XJP(1)*DXH
BN(1)=XJM(1)*DXH
GO TO 101
199 AN(1)=AN(1)+XJP(I)*DXA
BN(1)=BN(1)+XJM(I)*DXA
101 CONTINUE
AN(1)=AN(1)*FSP
BN(1)=BN(1)*FSP
DO 102 J=2,NP
DO 103 I=1,NHPO
TX=XX(I)+XX(I)
UR2(I)=TX*UR1(I)-UR2(I)
UR1(I)=TX*UR2(I)-UR1(I)
IF(I.EQ.1)104,105
104 URR=UR2(I)*DXH
AN(J)=XJP(I)*URR
BN(J)=XJM(I)*URR
GO TO 103
105 URR=UR2(I)*DXA
AN(J)=AN(J)+XJP(I)*URR
BN(J)=BN(J)+XJM(I)*URR
103 CONTINUE
AN(J)=AN(J)*FSP
BN(J)=BN(J)*FSP
102 CONTINUE
GO TO 11
C NPOINT=2, LEGENDRE CASE
200 DO 201 I=1,NHPO
UR2(I)=1.
YKN(I)=2.*XX(I)*XX(I)-1.
UR1(I)=YKN(I)
IF(I.EQ.1)298,299
298 URR=DXH*XX(1)
AN(1)=XJP(1)*URR
BN(1)=XJM(1)*URR

```

```

GO TO 201
299 URR=DXA*XX(I)
   AN(1)=AN(1)+XJP(I)*URR
   BN(1)=BN(1)+XJM(I)*URR
201 CONTINUE
   AN(1)=2.*AN(1)
   BN(1)=2.*BN(1)
   DO 202 J=2,NP
   IF(J.EQ.2)234,235
234 XN=J-1
   GO TO 231
235 XN=J-2
   XN1=J-1
231 XN2=XN+XN+1.
   DO 203 I=1,NHPO
   IF(J.EQ.2)244,245
244 UR=UR1(I)
   GO TO 249
245 UR=(XN2*YKN(I)*UR1(I)-XN*UR2(I))/XN1
   UR2(I)=UR1(I)
   UR1(I)=UR
249 IF(I.EQ.1)204,205
204 URR=UR*DXH*XY(I)
   AN(J)=XJP(I)*URR
   BN(J)=XJM(I)*URR
   GO TO 203
205 URR=UR*DXA*XX(I)
   AN(J)=AN(J)+XJP(I)*URR
   BN(J)=BN(J)+XJM(I)*URR
203 CONTINUE
   IF(J.EQ.2)254,255
255 XN2=XN2+2.
254 TX=XN2*2.
   AN(J)=AN(J)*TX
   BN(J)=BN(J)*TX
202 CONTINUE
   GO TO 11

```

C

NPOINT=3, GEGENBAUR CASE

```

300 DO 301 I=1,NHPO
   YKN(I)=1.-XX(I)*XX(I)
   UR2(I)=1.
   UR1(I)=4.*XX(I)
   IF(I.EQ.1)398,399
398 URR=YKN(I)*DXH
   AN(1)=XJP(1)*URR
   BN(1)=XJM(1)*URR
   GO TO 301
399 URR=YKN(I)*DXA
   AN(1)=AN(1)+XJP(I)*URR
   BN(1)=BN(1)+XJM(I)*URR
301 CONTINUE
   TX=ESPI/3.
   AN(1)=AN(1)*TX
   BN(1)=BN(1)*TX

```



```

XN=0.
DO 302 J=2,NP
XN=XN+1.
XN1=XN+1.
XN2=XN1+1.
XN3=XN2+1.
XN4=XN3+1.
TXRAT=XN2*XN4
DO 303 I=1,NHPO
UR2(I)=(2.*XN2* XX(I)*UR1(I)-XN3*UR2(I))/XN1
UR1(I)=(2.*XN3* XX(I)*UR2(I)-XN4*UR1(I))/XN2
UR=UR2(I)*YKN(I)
IF(I.EQ.1)304,305
304 URR=UR*DXH
AN(J)=XJP(I)*URR
BN(J)=XJM(I)*URR
GO TO 303
305 URR=UR*DXA
AN(J)=AN(J)+XJP(I)*URR
BN(J)=BN(J)+XJM(I)*URR
303 CONTINUE
TX=ESPI/TXRAT
AN(J)=AN(J)*TX
BN(J)=BN(J)*TX
302 CONTINUE

```

C
C
C

TEST FOR SIGNIFICANT COEFFICIENTS.

```

11 DO 31 I=1,NP
ANN(I)=AN(I)
BNN(I)=BN(I)
RA(I)=0.
31 RB(I)=0.
RA(1)=FVALUE+1.
GO TO (32,33,32),NPOINT
33 DO 34 K=1,NT
34 RX(K)=1.
GO TO 35
32 DO 36 K=1,NT
36 RX(K)=SQRTF(1.-XF(K)*XF(K))
35 DO 37 I=1,NP
IF(I.EQ.1)38,39
38 DO 41 J=1,NT
UR1(J)=1.
UR2(J)=1.
41 BS(J)=AN(1)
GO TO 26
39 IF(I.EQ.2)43,44
43 DO 45 J=1,NT
GO TO (51,52,53),NPOINT
51 UR2(J)=4.*XF(J)*XF(J)-1.
UR1(J)=XF(J)+XF(J)
GO TO 54
52 Z=2.*XF(J)*XF(J)-1.

```

```

UR2(J)=Z
GO TO 54
53 UR2(J)=12*XF(J)*XF(J)-2.
UR1(J)=4.*XF(J)
54 CS(J)=BS(J)
45 BS(J)=BS(J)+AN(I)*UR2(J)*RX(J)
GO TO 46
44 DO 47 J=1,NT
GO TO (61,62,63),NPOINT
61 TXX=XF(J)+XF(J)
UR1(J)=TXX*UR2(J)-UR1(J)
UR2(J)=TXX*UR1(J)-UR2(J)
GO TO 64
62 XM2=I
XMN=I-1
XM1=XM2+XMN
Z=2.*XF(J)*XF(J)-1.
TXX=(Z*UR2(J)*XM1-XMN*UR1(J))/XM2
UR1(J)=UR2(J)
UR2(J)=TXX
GO TO 64
63 XM2=I+I-3
XM3=XM2+1.
XM4=XM3+1.
XM5=XM4+1.
UR1(J)=(2.*XM3*XF(J)*UR2(J)-XM4*UR1(J))/XM2
UR2(J)=(2.*XM4*XF(J)*UR1(J)-XM5*UR2(J))/XM3
64 CS(J)=BS(J)
47 BS(J)=BS(J)+AN(I)*UR2(J)*RX(J)
46 AS=0.
DS=0.
DO 48 K=1,NT
ASR=FX(K)-CS(K)
DSR=FX(K)-BS(K)
AS=AS+ASR*ASR
48 DS=DS+DSR*DSR
RA(I)=ABSF(100.*(AS-DS)/DS)
26 AS=0.
DS=0.
DO 68 K=1,NT
CS(K)=BS(K)
BS(K)=BS(K)+BN(I)*UR2(K)*XF(K)*RX(K)
ASR=FX(K)-CS(K)
DSR=FX(K)-BS(K)
AS=AS+ASR*ASR
68 DS=DS+DSR*DSR
RB(I)=ABSF(100.*(AS-DS)/DS)
37 CONTINUE
IPP=1
DO 42 I=1,NP
IF(RA(I).LE.FVALUE)65,69
65 DO 66 J=I,NP
ANN(J)=0.
66 BNN(J)=0.

```

```

GO TO 85
69 GO TO (84,42),IPP
84 IF(RB(I).LE.FVALUE)71,42
71 DO 72 J=I,NP
72 BNN(J)=0.
   IPP=2
42 CONTINUE
85 IPP=1

```

C
C
C
GENERATE F(RHO) VERSUS RHO.

```

67 CALL FGRHO(RZ,ANN,1,GZ,NP)
   DO 73 I=1,100
   CALL FGRHO(RP(I),ANN,1,GP(I),NP)
   CALL FGRHO(RM(I),BNN,2,GM(I),NP)
73 CONTINUE

```

C
C
C
WRITE OUT SECOND PAGE OF OUTPUT.

```

WRITE(61,121)HEE,HED,DATE,ISSET,IRUN,NN,NX,FVALUE,THICK,NP,KIND(NPO
1INT,1),KIND(NPOINT,2)
WRITE(61,126)
126 FORMAT(8X,*SIG COEFF*,21X,*ALL COEFF*,24X,*RATIOS*/4X,*A(I)*,9X,*B
1(I)*,13X,*A(I)*,9X,*B(I)*,13X,*RA(I)*,10X,*RB(I)//)
WRITE(61,127)(ANN(I),BNN(I),AN(I),BN(I),RA(I),RB(I),I,I=1,NP)
127 FORMAT(F10.6,F13.6,F17.6,F13.6,E19.4,E15.4,6X,I2)

```

C
C
C
NORMALIZE F(RHO) AND SET TO ZERO IF IT IS NEGATIVE.

```

81 DO 74 I=1,100
   IM=101-I
   IP=101+I
   FX(I)=GP(IM)-GM(IM)
74 FX(IP)=GP(I)+GM(I)
   FX(101)=GZ
   FMXOUT=FX(1)
   DO 75 I=2,201
   IF(FX(I).GT.FMXOUT)76,75
76 FMXOUT=FX(I)
75 CONTINUE
83 DO 77 I=1,201
   FX(I)=FX(I)/FMXOUT
   IF(FX(I).LT.0.)78,77
78 FX(I)=0.
77 CONTINUE

```

C
C
C
WRITE OUT THIRD PAGE OF OUTPUT.

```

WRITE(61,121)HEE,HED,DATE,ISSET,IRUN,NN,NX,FVALUE,THICK,NP,KIND(NPO
1INT,1),KIND(NPOINT,2)
WRITE(61,128)
128 FORMAT(5(9X,3HRHO,4X,6HF(RHO))//)
WRITE(61,124)(GR(I),FX(I),I=1,201)
WRITE(61,129)YMAXIN,FMXOUT

```

```

129 FORMAT(*0THE NORMALIZATION FACTORS ARE*// * INPUT/*,E11.4,10X,*OUTP
1UT/*,E11.4)
    IF(IPPP.EQ.1)9999,99
98 WRITE(61,131)
131 FORMAT(* END OF THIS RUN*)
    WRITE(61,506)HEE,DATE
506 FORMAT(1H1,10A8,20X,A8)
    CALL EXIT
    END
    SUBROUTINE FGRHO(X,AN,J,FOX,NP)
    DIMENSION AN(20)
    COMMON NPOINT
    DATA(TP=6.2831853)
    Y=2.*X*X-1.
    XX=3.
    GO TO (11,12,13),NPOINT
11 P2=1.
    P1=Y
    S=AN( 1)+3.*AN( 2)*Y
    XN=0.
    DO 1 I=3,NP
    XX=XX+2.
    XN=XN+1.
    PN=((2.*XN+1.)*Y*P1-XN*P2)/(XN+1.)
    S=S+XX*AN( I)*PN
    P2=P1
    P1=PN
1 CONTINUE
    GO TO (2,3),J
2 FOX=.5*S
    RETURN
3 FOX=-.5*S*X
    RETURN
12 XSQ=X*X
    ZSQ=1.-XSQ
    TZSQ=ZSQ+ZSQ
    Z=SQRTF(ZSQ)
    P2=1.
    P1=3.-4.*XSQ
    S=AN( 1)-AN( 2)*P1
    SI=1.
    DO 4 I=3,NP
    PN=TZSQ*P1-P2
    P2=P1
    P1=PN
    S=S+SI*AN( I)*PN
4 SI=-SI
    DEN=1.5707963*Z
    GO TO (5,6),J
5 FOX=S/DEN
    RETURN
6 FOX=-S*X/DEN
    RETURN
13 P2=1.

```

```

P1=Y
T2=-1.
T1=1.-6.*X*X
S=AN( 1)+AN( 2)*(5.*T2-3.*T1)
DO 8 I=3,NP
  XN=I-1
  XNN=XN-1.
  XB=XN+XN+1.
  XA=XB+2.
  PN=((XN+XNN)*Y*P1-XNN*P2)/XN
  TN=T1+T1-XB*PN-T2
  S=S+AN( I)*(XA*T1-XB*TN)
  T2=T1
  T1=TN
  P2=P1
8 P1=PN
  GO TO (7,9),J
9 FOX= S*X
  RETURN
7 FOX=S
  RETURN
END

```



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