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CALCULATION OF THE ADMITTANCE OF A PARALLEL PLATE CAPACITOR CONTAINING A TOROID-SHAPED SAMPLE

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U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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CALCULATION OF THE ADMITTANCE OF A PARALLEL PLATE CAPACITOR CONTAINING A TOROID – SHAPED SAMPLE

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Calculation of the Admittance of a Parallel Plate Capacitor Containing a Toroid-Shaped Sample

Eric G. Johnson, Jr.

The theoretical admittance of a toroid-shaped sample in a parallel plate capacitor system is developed. This calculation represents a first and second order perturbation correction to the usual quasistatic formula. The FORTRAN listing of the machine program is included. The analysis is done under the condition that the sample is not in contact with the capacitor plates and can be described by a complex permeability, μ_M, and complex permittivity, ϵ_M.

Key Words:

Fortran
Lossy materials
Parallel plate capacitor
Permeability
Permittivity
Theoretical admittance

1. Introduction

The purpose of this calculation is to permit an accurate determination of the permittivity and permeability of a material sample for the frequency range in which the capacitor and inductor system are generally applicable.* This particular calculation considers

*This technical note is a companion to the NBS Technical Note #311, "Computation of the Permeability and Permittivity of a Relatively Small Ring Sample in a Toroidal Coil," which contains a similar analysis for the inductor structure of a toroid-shaped sample. Both the capacitor and inductor measurements are necessary if one is to get a complete determination of the permeability and permittivity of a sample.

the parallel plate capacitor system in which there has been placed a toroidal-shaped sample. Our purpose is to correct the usual quasi-static formulae (zero order solution) for characterizing a capacitor to at least second order in perturbation theory. The zero and first order calculation is performed under the assumption that there is no height dependence of the electromagnetic field. Only radial dependence is permitted. However, because we wish to be able to study lossy materials (complex μ_M and ϵ_M) we have chosen the geometry of the experiment to be such that the sample is not in contact with either of the metal plates of the capacitor system. Because of this configuration, it is necessary to make a guess of an effective permeability $\bar{\mu}$ and permittivity $\bar{\epsilon}$ for the zero and first order solution to the problem. The second order term primarily attempts to correct for the fact that there is indeed height dependence for the electromagnetic field. This second order is derived under a relaxation of the requirement that tangential component of the magnetic field is continuous. It is expected that when this second order term is important there is the corresponding circumstance that the sample tends to not have an internal electromagnetic field, i.e., it is as though there is a surface current acting on the sample surface. So to aid in the convergence of the second order term and hopefully to make it a more reliable estimate of the corrections necessary to the zero and first order term, this calculation errs in the direction of assuming that there is a surface current in the second order term.

2. Initialization of the Problem

Assume an experimental system with the following boundary conditions:

1. We have two finite radius parallel plates. Each plate has infinite conductivity.

2. A rectangular cross-sectioned toroid is placed between the plates.

3. Two complex frequency dependent parameters--permeability, μ_M, and permittivity, ϵ_M-- are used to characterize the material in the toroid.

4. The excitation mechanism of the capacitor system is such that no angular field dependence exists and is such the H_φ is the only magnetic variable. This means that the basic electromagnetic field equations namely (1) will be reduced to a single equation, (2) below. Equation (1) assumes the exp $(i\omega t)$ time dependence. Thus,

$$\nabla \times \underset{\sim}{E} = - i\omega \mu \underset{\sim}{H}, \text{ and}$$

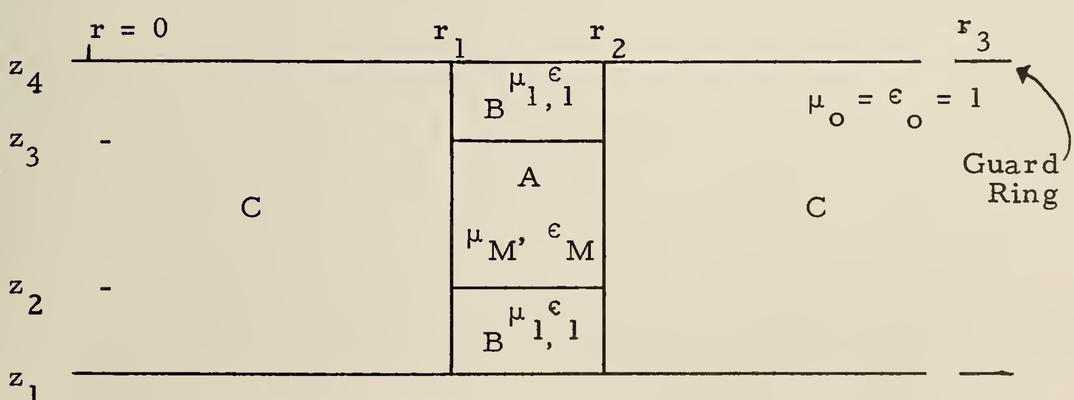
$$\nabla \times \underset{\sim}{H} = i \omega \epsilon \underset{\sim}{E} \qquad (1)$$

$$i = \sqrt{-1}$$

(natural units are used, namely, $c = 1$. Hence all units are powers of cm. The ω is cm^{-1}; μ, ϵ are dimensionless).

5. The coordinates of interest are shown in Figures 1 and 2.

SIDE VIEW



(The system is symmetric around the r = 0 position.)

Figure 1

TOP VIEW OF SYSTEM

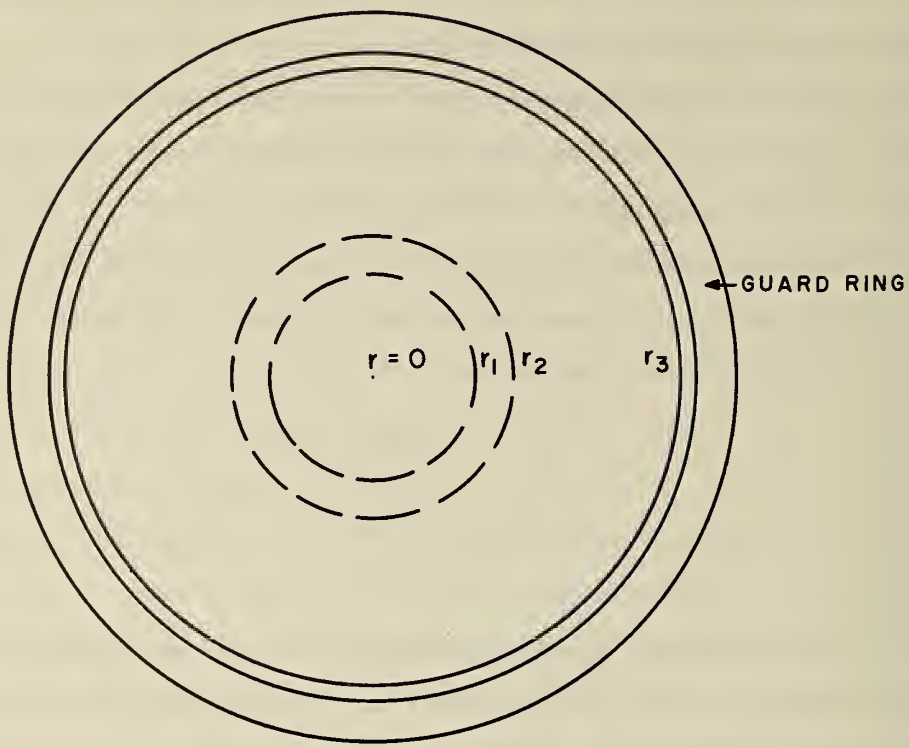


Figure 2

The r_1 and r_2 are the radial coordinates of the toroid sample A. The r_3 is the radius at which the system has its voltage drop prescribed. The z_2 and z_3 are the z axis coordinates of the toroid sample A. The z_4, z_3 and z_1, z_2 are the z axis coordinates of two toroid-shaped material B. They have no loss in their ϵ_1 and μ_1. The relative size of $\frac{z_2 - z_1}{z_3 - z_2}$ and $\frac{z_4 - z_3}{z_3 - z_2}$ is about 2.5%. The structure is used to provide a precise measure of the distances $z_4 - z_3$ and $z_2 - z_1$. We characterize the permeability and permittivity of the space $r_3 \geq r \geq 0$ and $z_1 \leq z \leq z_4$ by:

$\mu = 1$ and $\epsilon = 1$, for region C;

$\mu = \mu_1$ and $\epsilon = \epsilon_1$, for region B;

and $\mu = \mu_M$ and $\epsilon = \epsilon_M$, for region A.

The regions A, B, and C are indicated in Figure 1. A functional form for μ is

$$\mu = P(r)\left[P(z)(\mu_M - \mu_1) + \mu_1 - 1\right] + 1 .$$

The same form for ϵ is possible. Here we have defined

$$P(r) = \theta(r - r_1) - \theta(r - r_2)$$

and

$$P(z) = \theta(z - z_3) - \theta(z - z_2),$$

where θ is the step function.

3. Analysis

Upon reducing the (1) with assumptions 1 to 5 and setting $f \equiv rH_\varphi$ the following differential equation results:

$$\frac{r}{\ } \partial_r \frac{1}{r\epsilon} \partial_r f + \partial_z \frac{1}{\epsilon} \partial_z f + \omega^2 \mu f = 0 \qquad (2)$$

The boundary conditions are $\partial_z f = 0$ at $z = z_1, z_4$; and $\frac{f}{r} \to 0$ as $r \to 0$. The admittance, Y, of this system at $r = r_3$ is defined as the ratio of the peak current to peak driving voltage. Thus,

$$Y = 2\pi i \omega f(r_3) \Big/ \left[\int_z^{z_4} \frac{1}{r} \partial_r f \, dz \right]_{r = r_3} = \frac{2\pi i \omega r_3}{z_4 - z_1} \left[f(r_3)/(\partial_r f)(r_3) \right] \qquad (3)$$

The position r_3 is chosen so that the influence of the toroid sample on the field structure has no resulting z dependence. That is, r_3 is sufficiently far from the sample so that the only type of field that exists at r_3 is pure propagating waves with no z dependence $\left(r_3 > r_2 + 4(z_4 - z_1) \right)$. By fixing the voltage drop point at r_3 we solve a basic problem for making the admittance of a parallel plate capacitor unique. Thus the voltage drop at r_3 is given by the line integral

$$\frac{1}{i\omega} \int_{z_1}^{z_4} \frac{1}{\epsilon r} \partial_r f dz \Big|_{r = r_3}.$$

This means we have chosen a path at $r = r_3$ that is only along the z direction. With this specification and noting that the admittance contribution outside $r = r_3$ is the same with or without the sample, we can subtract the two admittance measurements and equate the result to the corresponding subtraction of the two theoretical admittances.

In order to eliminate mathematical consideration of fringing fields, we assume that the capacitor is infinite. We then arbitrarily choose r_3 as the point where we define the interior of the capacitor system. We must experimentally fix r_3 and still be able to neglect the effects of fringing fields. Practically, this is accomplished by having a capacitor of radius r_3 and by using a guard ring. We now solve the equation for f. A first order guess to f is to assume that the sample completely fills both regions A and B with the weighted average parameters $\bar{\epsilon}$, $\bar{\mu}$, as is defined below. We can solve this f_o exactly. We next assume that f may be written as $f = f_o + \epsilon(r, z)\, g\,(r, z)$, where g is assumed to be a perturbation connection to the f_o solution. By allowing this transformation and making g and its derivative continuous we are granting the existence of surface current on the material sample. We thus have the following equation for g

$$r\partial_r \frac{1}{r}\partial_r g + \partial_z^2 g + \omega^2\mu\epsilon g = H \equiv \omega^2(\bar{\mu}-\mu)f_o + r\partial_r \frac{1}{r}\left(\frac{1}{\bar{\epsilon}} - \frac{1}{\epsilon}\right)\partial_r f_o, \tag{4}$$

where f_o obeys the equation

$$r\partial_r \frac{1}{r\bar{\epsilon}}\partial_r f_o + \omega^2\bar{\mu} f_o = 0. \tag{5}$$

Here, $\bar{\epsilon} = P(r)(\epsilon_2 - 1) + 1$, $\bar{\mu} = P(r)(\mu_2 - 1) + 1$,

where $\epsilon_2 = \epsilon_1 / \left(1 + (\epsilon_1/\epsilon_M - 1)\left(\frac{z_3 - z_2}{z_4 - z_1}\right)\right)$

and
$$\mu_2 = \mu_1 + (\mu_M - \mu_1)\left(\frac{z_3 - z_2}{z_4 - z_1}\right).$$

The solution to (5) is

$f_o = r\left[A I_1(i\omega r\sqrt{\bar{\mu}\bar{\epsilon}}) + B K_1(i\omega\sqrt{\bar{\mu}\bar{\epsilon}}\, r)\right]$ for each region.

Thus,

$$\begin{aligned} f_o &= r I_1(i\omega r) && r \le r_1 \\ f_o &= r\left[\bar{A} I_1(i\omega\sqrt{\mu_2\epsilon_2}\, r) + \bar{B} K_1(i\omega\sqrt{\mu_2\epsilon_2}\, r)\right], && r_1 \le r \le r_2 \\ f_o &= r\left[A I_1(i\omega r) + B K_1(i\omega r)\right], && r_2 \le r \le r_3 \end{aligned} \tag{6}$$

We require f_o and $\frac{1}{\varepsilon}\,\partial_r f_o$ to be continuous across the r_1 and r_2 boundary. This implies with definitions $x_{1,2} = i\omega r_{1,2}$ and $\bar{x}_{1,2} = i\omega r_{1,2}\sqrt{\mu_2 \varepsilon_2}$

that
$$I_1(x_1) = \bar{A}\, I_1(\bar{x}_1) + \bar{B}\, K_1(\bar{x}_1)$$
$$I_0(x_1) = \sqrt{\frac{\mu_2}{\epsilon_2}}\left[\bar{A}\, I_0(\bar{x}_1) - \bar{B}\, K_0(\bar{x}_1)\right],$$
and that
$$C_1 \equiv \bar{A}\, I_1(\bar{x}_2) + \bar{B}\, K_1(\bar{x}_2) = A\, I_1(x_2) + B\, K_1(x_2)$$
$$C_2 \equiv \sqrt{\frac{\mu_2}{\epsilon_2}}\left[\bar{A}\, I_0(\bar{x}_2) - \bar{B}\, K_0(\bar{x}_2)\right] = A\, I_0(x_2) - B\, K_0(x_2) \,.$$
We solve for A, B as follows:
$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} K_0(x_2) & K_1(x_2) \\ I_0(x_2) & -I_1(x_2) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} / D_1, \qquad (7)$$
where $D_1 = \left(I_1(x_2)\, K_0(x_2) + K_1(x_2)\, I_0(x_2) \right)$.

The $\bar{A}$, $\bar{B}$ is given by
$$\begin{pmatrix} \bar{A} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} K_0(\bar{x}_1) & K_1(\bar{x}_1) \\ I_0(\bar{x}_1) & -I_1(\bar{x}_1) \end{pmatrix} \begin{pmatrix} I_1(x_1) \\ \sqrt{\frac{\epsilon_2}{\mu_2}}\, I_0(x_1) \end{pmatrix} / D_2, \qquad (8)$$
where
$$D_2 = \left(I_1(\bar{x}_1)\, K_0(\bar{x}_1) + K_1(\bar{x}_1)\, I_0(\bar{x}_1) \right),$$
The Wronskian is

$$W = I_0(z)'\, K_0(z) - I_0(z)\, K_0(z)' = I_1(z)\, K_0(z) + I_0(z)\, K_1(z) \qquad (9)$$
$$= \frac{1}{z} \quad \text{for all } z \text{ including } z \to 0.$$

Thus we can write $D_1 = \frac{1}{x_2}$ and $D_2 = \frac{1}{\bar{x}_1}$. (10)

Next we reduce the inhomogeneous term as follows:
$$H = \omega^2(\bar{\mu} - \mu)\, f_0 + \left(1 - \frac{\bar{\epsilon}}{\epsilon}\right)\left(-\bar{\mu}\,\omega^2 f_0\right) + \frac{1}{\bar{\epsilon}}\,\partial_n(f_0)\,\partial_n\left(1 - \frac{\bar{\epsilon}}{\epsilon}\right)$$
$$= \omega^2 f_0\left[-\mu + \frac{\bar{\epsilon}\bar{\mu}}{\epsilon}\right] + \frac{1}{\bar{\epsilon}}\,\partial_n(f_0)\,\partial_n\left(1 - \frac{\bar{\epsilon}}{\epsilon}\right). \qquad (11)$$

Equation (4) can be solved in terms of the expansion

$$g = \sum_{n=0}^{\infty} N_n \cos k_n (z - z_1) \; R_n(r),$$

where $k_n = \frac{n\pi}{z_4 - z_1}$ and $N_n = \sqrt{\frac{2 - \delta_{no}}{z_4 - z_1}}$.

The equations to fix $R_n(r)$ are now given as

$$\left[r \partial_r \frac{1}{r} \partial_r - k_n^2 + \omega^2 \right] R_n + \omega^2 \sum_{n'=0}^{\infty} A_{nn'} R_{n'} P(r) = \begin{array}{l} - f_o(r) B_n P(r) \\ - D_n^a \, \delta(r - r_1) \\ - D_n^b \, \delta(r - r_2) \end{array} \quad \cdot \quad (12)$$

Here, $A_{nn'}$, D^a_n, B_n, D^b_n are defined:

$$A_{nn'} = N_n N_{n'} (\mu_M \epsilon_M - \mu_1 \epsilon_1) \int_{z_2}^{z_3} \cos k_n (z - z_1) \cos k_{n'} (z - z_1) dz + (\mu_1 \epsilon_1 - 1) \delta_{nn'} \qquad (13)$$

$$B_n = \omega^2 \left[\left\{ (\mu_M - \mu_1) - \epsilon_2 \mu_2 \left(\frac{1}{\epsilon_M} - \frac{1}{\epsilon_1} \right) \right\} \frac{N_n}{k_n} \sin k_n (z - z_1) \Big|_{z_2}^{z_3} + \frac{\delta_{no}}{N_o} \left[\mu_1 - \frac{\epsilon_2 \mu_2}{\epsilon_1} \right] \qquad (14)$$

$$D^a_n = \frac{1}{\epsilon} \partial_r f_o \Big|_{r = r_1} E_n \qquad D^b_n = -\frac{1}{\epsilon} \partial_r f_o \Big|_{r = r_2} E_n, \text{ where}$$

$$E_n \equiv N_n \left[\epsilon_2 \left(\frac{1}{\epsilon_M} - \frac{1}{\epsilon_1} \right) \frac{\sin k_n (z - z_1)}{k_n} \Big|_{z_2}^{z_3} \right] + \left(\frac{\epsilon_2}{\epsilon_1} - 1 \right) \frac{\delta_{no}}{N_o} \quad \cdot$$

Here $\delta_{no} = 0 \quad n \neq 0$, and

$\delta_{no} = 1 \quad n = 0$.

We further compute $A_{nn'}$ to get

$$A_{nn'} = \frac{N_n N_{n'}}{2} (\mu_M \epsilon_M - \mu_1 \epsilon_1) \left[\frac{\sin (k_n - k_{n'})(z - z_1)}{k_n - k_{n'}} + \frac{\sin (k_n + k_{n'})(z - z_1)}{k_n + k_{n'}} \right] \Bigg|_{z_2}^{z_3} \qquad (15)$$

when $n \neq n'$.

When $n = n'$, we have

$$A_{nn} = \frac{\left(1 - \frac{1}{2} \delta_{no}\right)}{z_4 - z_1} \left[z + \frac{\sin 2k_n (z - z_1)}{2k_n} \right] \Bigg|_{z_2}^{z_3} (\mu_M \epsilon_M - \mu_1 \epsilon_1) + \mu_1 \epsilon_1 - 1 \quad . \qquad (16)$$

Thus we now define: $\lambda_n^2 \equiv k_n^2 - \omega^2, \quad \lambda_n \equiv +\sqrt{\lambda_n^2},$

$$\bar{\lambda}_n^2 \equiv \lambda_n^2 - \omega^2 A_{nn}, \quad \bar{\lambda}_n \equiv +\sqrt{\bar{\lambda}_n^2}, \tag{17}$$

$\lambda_n^2(r) \equiv (\bar{\lambda}_n^2 - \lambda_n^2) P(r) + \lambda_n^2$ and the Green's function equations

$$\left[r \vec{\partial}_r \frac{1}{r} \vec{\partial}_r - \lambda_n^2(r) \right] G_n(r, \bar{r}) = -\delta(r - \bar{r}), \tag{18}$$

$$G_n(r, \bar{r}) \left[\bar{r} \overleftarrow{\partial}_{\bar{r}} \frac{1}{\bar{r}} \overleftarrow{\partial}_{\bar{r}} - \lambda_n^2(\bar{r}) \right] = -\delta(r - \bar{r}). \tag{19}$$

The solution to these equations is given as

$$G_n(r, \bar{r}) = r\, C_n(\bar{r})\, T_n(r), \qquad \bar{r} > r,$$

$$\equiv r\, C_n(r)\, T_n(\bar{r}), \qquad \bar{r} < r.$$

The normalization condition on T_n given by

$$r\left[C_n' T_n - T_n' C_n \right] = -1 \equiv rW,$$

or since the term in the bracket is the Wronskian, W, and its solution is

$W' = -\frac{1}{r} W$ or $W = \frac{C}{r}$, we have the requirement that C = - 1.

To find W, we fix $T_n(r)$ and $C_n(r)$ as follows:

$$\begin{aligned}
C_n(r) &= C_n^a I_1(\lambda_n r) + C_n^b K_1(\lambda_n r), && 0 \le r \le r_1 \\
C_n(r) &= \bar{C}_n^a I_1(\lambda_n r) + \bar{C}_n^b K_1(\bar{\lambda}_n r), && r_1 \le r \le r_2 \\
C_n(r) &= K_1(\lambda_n r), && r_2 \le r \\
T_n(r) &= I_1(\lambda_n r) / C_n^b, && 0 \le r \le r_1 \\
T_n(r) &= \bar{S}_n^a I_1(\bar{\lambda}_n r) + \bar{S}_n^b K_1(\bar{\lambda}_n r), && r_1 \le r \le r_2 \\
T_n(r) &= S_n^a I_1(\lambda_n r) + S_n^b K_1(\lambda_n r), && r_2 \le r .
\end{aligned} \tag{20}$$

To check if rW = -1, we consider the case r → 0. We have

$$\lambda_n \left[(C_n^a I_0 - C_n^b K_0) \frac{1}{C_n^b} I_1 - \frac{1}{C_n^b} I_0 (C_n^a I_1 + C_n^b K_1) \right] \underset{r \to 0}{\longrightarrow} -\frac{1}{r}, \tag{21}$$

hence rW = -1 as desired.

The conditions for connection of the three regions are that T_n, C_n, and their derivatives must be continuous across the r_1, r_2 boundaries. As a consequence, we have the following equations:

$$
\begin{aligned}
&C_n^a I_1(\lambda_n r_1) + C_n^b K_1(\lambda_n r_1) = \bar{C}_n^a I_1(\bar{\lambda}_n r_1) + \bar{C}_n^b K_1(\bar{\lambda}_n r_1) \\
&\bar{C}_n^a I_1(\bar{\lambda}_n r_2) + \bar{C}_n^b K_1(\bar{\lambda}_n r_2) = K_1(\lambda_n r_2) \\
&C_n^a I_0(\lambda_n r_1) - C_n^b K_0(\lambda_n r_1) = \frac{\bar{\lambda}_n}{\lambda_n}\left[\bar{C}_n^a I_0(\bar{\lambda}_n r_1) - \bar{C}_n^b K_0(\bar{\lambda}_n r_1)\right] \qquad (22) \\
&\bar{C}_n^a I_0(\bar{\lambda}_n r_2) - \bar{C}_n^b K_0(\bar{\lambda}_n r_2) = -\frac{\lambda_n}{\bar{\lambda}_n} K_0(\lambda_n r_2) \\
&S_n^a I_1(\lambda_n r_2) + S_n^b K_1(\lambda_n r_2) = \bar{S}_n^a I_1(\bar{\lambda}_n r_2) + \bar{S}_n^b K_1(\bar{\lambda}_n r_2) \\
&S_n^a I_0(\lambda_n r_2) - S_n^b K_0(\lambda_n r_2) = \frac{\bar{\lambda}_n}{\lambda_n}\left[\bar{S}_n^a I_0(\bar{\lambda}_n r_2) - \bar{S}_n^b K_0(\bar{\lambda}_n r_2)\right] \\
&\bar{S}_n^a I_1(\bar{\lambda}_n r_1) - \bar{S}_n^b K_1(\bar{\lambda}_n r_1) = I_1(\lambda_n r_1) / C_n^b \\
&\bar{S}_n^a I_0(\bar{\lambda}_n r_1) - \bar{S}_n^b K_0(\bar{\lambda}_n r_1) = \frac{\lambda_n}{\bar{\lambda}_n} I_0(\lambda_n r_1) / C_n^b .
\end{aligned}
$$

Upon solving these equations, we find that:

$$
\begin{pmatrix} \bar{C}_n^a \\ \bar{C}_n^b \end{pmatrix} = \begin{pmatrix} K_0(\bar{\lambda}_n r_2) & K_1(\bar{\lambda}_n r_2) \\ I_0(\bar{\lambda}_n r_2) & -I_1(\bar{\lambda}_n r_2) \end{pmatrix} \begin{pmatrix} K_1(\lambda_n r_2) \\ -\frac{\lambda_n}{\bar{\lambda}_n} K_0(\lambda_n r_2) \end{pmatrix} / D_1 ,
$$

$$
D_1 = I_1(\bar{\lambda}_n r_2) K_0(\bar{\lambda}_n r_2) + K_1(\bar{\lambda}_n r_2) I_0(\lambda_n r_2) = \frac{1}{\lambda}\left[I_0' K_0 - K_0' I_0\right]
$$
$$
= \frac{1}{r_2 \bar{\lambda}_n} ,
$$

[See (10) for this reduction.]

$$
\begin{pmatrix} \bar{S}_n^a \\ \bar{S}_n^b \end{pmatrix} = \begin{pmatrix} K_0(\bar{\lambda}_n r_1) & K_1(\bar{\lambda}_n r_1) \\ I_0(\bar{\lambda}_n r_1) & -I_1(\bar{\lambda}_n r_1) \end{pmatrix} \begin{pmatrix} I_1(\lambda_n r_1) \\ \frac{\lambda_n}{\bar{\lambda}_n} I_0(\lambda_n r_1) \end{pmatrix} \left(\frac{r_1 \bar{\lambda}_n}{C_n^b}\right) ,
$$

$$\begin{pmatrix} C_n^a \\ C_n^b \end{pmatrix} = \begin{pmatrix} K_0(\lambda_n r_1) & K_1(\lambda_n r_1) \\ I_0(\lambda_n r_1) & -I_1(\lambda_n r_1) \end{pmatrix} \begin{pmatrix} \bar{C}_n^a I_1(\bar{\lambda}_n r_1) + \bar{C}_n^b K_1(\bar{\lambda}_n r_1) \\ \frac{\bar{\lambda}_n}{\lambda_n}\left[\bar{C}_n^a I_0(\bar{\lambda}_n r_1) - \bar{C}_n^b K_0(\bar{\lambda}_n r_1)\right] \end{pmatrix} / D_2)$$

$D_2 = K_0 I_1 + I_0 K_1 = \frac{1}{\lambda_n}[K_0 I_0' - I_0 K_0'] = \frac{1}{\lambda_n r_1}$) and finally,

$$\begin{pmatrix} S_n^a \\ S_n^b \end{pmatrix} = \begin{pmatrix} K_0(\lambda_n r_2) & K_1(\lambda_n r_2) \\ I_0(\lambda_n r_2) & -I_1(\lambda_n r_2) \end{pmatrix} \begin{pmatrix} \bar{S}_n^a I_1(\bar{\lambda}_n r_2) + \bar{S}_n^b K_1(\bar{\lambda}_n r_2) \\ \frac{\bar{\lambda}_n}{\lambda_n}\left[\bar{S}_n^a I_0(\bar{\lambda}_n r_2) - \bar{S}_n^b K_0(\bar{\lambda}_n r_2)\right] \end{pmatrix} (\lambda_n r_2) .$$

We next construct the integral equation for R_n. That is

$$\begin{aligned} R_n - \int G_n(r,\bar{r})\, f_0(\bar{r})\, d\bar{r}\, B_n - G_n(r,r_1)\, D_n^a - G_n(r,r_2)\, D_n^b \\ = \omega^2 \sum_{n'\neq n} \int_{r_1}^{r_2} G_n(r,\bar{r})\, d\bar{r}\, R_{n'}(\bar{r})\, A_{nn'} \\ + \left[G_n \overrightarrow{\partial_{\bar{r}}} R_n - G_n \bar{r}\, \overleftarrow{\partial_{\bar{r}}} \frac{1}{\bar{r}} R_n\right]_{\bar{r}=r_3} . \end{aligned} \quad (23)$$

Now we assume R_n has only outgoing waves; therefore, we have the following integral equation:

$$\begin{aligned} R_n(r) = \int_{r_1}^{r_2} G_n(r,\bar{r})\, f_0(\bar{r})\, d\bar{r}\, B_n + D_n^a G_n(r,r_1) + D_n^b G_n(r,r_2) \\ + \omega^2 \sum_{n'\neq n} A_{nn'} \int_{r_1}^{r_2} G_n(r,\bar{r})\, R_{n'}(\bar{r})\, d\bar{r} . \end{aligned} \quad (24)$$

The solution of this integral equation by standard iteration is for $n \neq 0$:

$$R_n(r) = \int_{r_1}^{r_2} G_n(r,\bar{r})\, f_0(\bar{r})\, d\bar{r}\, B_n + D_n^a G_n(r,r_1) + D_n^b G_n(r,r_2); \quad (25)$$

and for $n = 0$:

$$\begin{aligned} R_0(r) = \int_{r_1}^{r_2} G_0(r,\bar{r})\, f_0(\bar{r})\, d\bar{r}\, B_0 + D_0^a G_0(r,r_1) + D_0^b G_0(r,r_2) \\ + \omega^2 \sum_{n'>0} A_{0n'} \int_{r_1}^{r_2} G_0(r,\bar{r})\, R_{n'}(\bar{r})\, d\bar{r} . \end{aligned}$$

The final form for f, at $r = r_3$, is

$$f = f_1 + R_0 = B^a \left[B^b K_1(i\omega r_3) + I_1(i\omega r_3)\right] r_3 . \quad (26)$$

The admittance is given with $r = r_3$ as follows:

$$Y = \frac{2\pi r_3}{z_4 - z_1}\left[\frac{B^b K_1(i\omega r_3) + I_1(i\omega r_3)}{-B^b K_0(i\omega r_3) + I_0(i\omega r_3)}\right]. \tag{27}$$

The basic computational problem is to determine B^b and B^a. We make the expansion of the f_o and R_o function explicit so that we can determine B^b:

$$f_o = r\left[A I_1 + B K_1\right], \qquad r > r_2, \tag{28}$$

$$\begin{aligned} R_n(r) = {} & r\, C_n(r)\left\{\int_{r_1}^{r} T_n(\bar{r})\bar{r}\left(\bar{A} I_1(\bar{x}) + \bar{B} K_1(\bar{x})\right)B_n\, d\bar{r} + D_n^a T_n(r_1)\right\} \\ & + r\, T_n(r)\left\{\int_{r}^{r_2} C_n(\bar{r})\bar{r}\left(\bar{A} I_1(\bar{x}) + \bar{B} K_1(\bar{x})\right)B_n\, d\bar{r} + D_n^b C_n(r_2)\right\}, \end{aligned} \tag{29}$$

$$\begin{aligned} R_0(r_3) = {} & r_3 C_0(r_3)\left\{B_0 \int_{r_1}^{r_2} T_0(\bar{r})\, \bar{r}\, f_0(\bar{r})\, d\bar{r} + D_0^a T_0(r_1) + D_0^b T_0(r_2)\right\} \\ & + \omega^2 \sum_{n'>0} A_{on'}\, r_3\, C_0(r_3) \int_{r_1}^{r_2} T_0(\bar{r})\, R_{n'}(\bar{r})\, d\bar{r} \end{aligned} \tag{30}$$

Here $\bar{x} = \bar{\lambda} r$, $n \neq 0$, and $\bar{\lambda} = i\omega\sqrt{\mu_2 \epsilon_2}$.

Thus we define

$R_o = r_3 K_1(i\omega r_3)\left[H^{(1)} + \omega^2 H^{(2)}\right]$ and note that B^b is now

$$B^b = \left(B + H^{(1)} + \omega^2 H^{(2)}\right) / A \quad .$$

The $H^{(1)}$, $H^{(2)}$ are defined as follows:

$$\begin{aligned} H^{(1)} = {} & B_0 \int_{r_1}^{r_2} T_0(\bar{r})\, d\bar{r}\;\; f_0(\bar{r})\,\bar{r} \\ & + D_0^a T_0(r_1) + D_0^b T_0(r_2), \end{aligned} \tag{31}$$

$$H^{(2)} = \sum_{n'>0} A_{on'} \int_{r_1}^{r_2} T_o(\bar{r}) R_{n'}(\bar{r})\, d\bar{r}. \tag{32}$$

We define:

$$\begin{aligned}
P_1^n(r) &\equiv \int_{r_1}^{r} I_1(\bar{x}_n)\, I_1(\bar{x})\, r\, dr, \qquad (\bar{x}_n \equiv \bar{\lambda}_n r), \\
P_2^n(r) &\equiv \int_{r_1}^{r} I_1(\bar{x}_n)\, K_1(\bar{x})\, r\, dr, \\
P_3^n(r) &\equiv \int_{r_1}^{r} K_1(\bar{x}_n)\, I_1(\bar{x})\, r\, dr, \\
P_4^n(r) &\equiv \int_{r_1}^{r} K_1(\bar{x}_n)\, K_1(\bar{x})\, r\, dr.
\end{aligned} \tag{33}$$

The integrals are:

$$\begin{aligned}
P_1^n(r) &= \left[r \left(I_1(\bar{x}_n) I_0(\bar{x}) \bar{\lambda} - \bar{\lambda}_n I_0(\bar{x}_n) I_1(\bar{x}) \right) \right]_{r_1}^{r} / z_n, \\
P_2^n(r) &= \left[-r \left(I_1(\bar{x}_n) K_0(\bar{x}) \bar{\lambda} + \bar{\lambda}_n I_0(\bar{x}_n) K_1(\bar{x}) \right) \right]_{r_1}^{r} / z_n, \\
P_2^n(r) &= \left[r \left(K_1(\bar{x}_n) I_0(\bar{x}) \bar{\lambda} + \bar{\lambda}_n K_0(\bar{x}_n) I_1(\bar{x}) \right) \right]_{r_1}^{r} / z_n, \\
P_4^n(r) &= \left[-r \left(K_1(\bar{x}_n) K_0(\bar{x}) \bar{\lambda} - \bar{\lambda}_n K_0(\bar{x}_n) K_1(\bar{x}) \right) \right]_{r_1}^{r} / z_n.
\end{aligned} \tag{34}$$

$(z_n \equiv \bar{\lambda}_n^2 - \bar{\lambda}^2)$.

We have defined:

$$\begin{aligned}
H_n^{(5)}(r) &\equiv \bar{A}\, P_1^n(r) + \bar{B}\, P_2^n(r), \\
H_n^{(6)}(r) &\equiv \bar{A}\, P_3^n(r) + \bar{B}\, P_4^n(r), \\
H_n^{(3)}(r) &\equiv D_n^a\, T_n(r_1) + B_n \left(\bar{S}_n^a H_n^{(5)}(r) + \bar{S}_n^b H_n^{(6)}(r) \right), \\
H_n^{(4)}(r) &\equiv D_n^b\, C_n(r_2) + B_n \left\{ \begin{array}{l} \bar{C}_n^a \left(H_n^{(5)}(r_2) - H_n^{(5)}(r) \right) + \\ \bar{C}_n^b \left(H_n^{(6)}(r_2) - H_n^{6}(r) \right) \end{array} \right\},
\end{aligned} \tag{35}$$

$$\begin{aligned}
R_n(r) &\equiv r \left[C_n(r)\, H_n^{(3)}(r) + T_n(r)\, H_n^{(4)}(r) \right], \\
H_n^{(2)} &\equiv \int_{r_1}^{r_2} T_o(r)\, R_n(r)\, dr,
\end{aligned} \tag{36}$$

hence,

$$H^{(2)} = \sum_{n>0} A_{on} H_n^{(2)}. \tag{37}$$

Now, in addition we have:

$$H^{(1)} \equiv B_o H^{(7)} + D_o^a T_o(r_1) + D_o^b T_o(r_2) \tag{38}$$

$$H^{(7)} \equiv \bar{S}_o^a H_o^{(5)}(r_2) + \bar{S}_o^b H_o^{(6)}(r_2). \tag{39}$$

4. Discussion

The analysis has gone as far as possible without numerical integration of (36). We summarize the basic necessary results in Appendix A. In Appendix B we have a FORTRAN 63 listing of the machine program. In Appendix C is a sample calculation. (Here μ_M = UM +I UM* and ε_M = EM + I EM* in the computer program. UM, EM are the real parts and UM*, EM* are the imaginary parts . They must be negative to describe a lossy system. Appendix D gives a flow chart of the input data necessary to run the machine program.

The quasistatic formula used is $Y_{qs} = i\omega\pi \dfrac{(r_2^2 - r_1^2)}{z_4 - z_1} \varepsilon_2$

The computation has explicitly separated out the highest order correction, $H^{(2)}$, for the determination of the admittance of the sample. This correction can be used to get an estimate of the accuracy of the admittance calculation. This correction is calculated by Simpson intervals to an accuracy of about 3%. Thus if this $H^{(2)}$ term is less than 10%, and subject to discussion of the introduction we can conclude that the admittance is known to at least 0.5%. In addition, one has an option of varying r_3 by 10%. Under this option one can get an estimate of how sensitive the experimental system is to the inexact knowledge of r_3. As a consequence we have a simple measure of the degree the radiation part of the electromagnetic field is present.

The program prints out the results in mho's for the admittance.

5. Acknowledgements

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APPENDIX A

SUMMARY OF FORMULAE USED IN MACHINE PROGRAM

$$\begin{pmatrix} A \\ B \end{pmatrix} = x_2 \begin{pmatrix} K_1(x_2) & K_0(x_2) \\ -I_1(x_2) & I_0(x_2) \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\mu_2}{\epsilon_2}} \left(\bar{A} I_0(\bar{x}_2) - \bar{B} K_0(\bar{x}_2) \right) \\ \bar{A} I_1(\bar{x}_2) + \bar{B} K_1(\bar{x}_2) \end{pmatrix} \quad \text{(A1)}$$

$$\begin{pmatrix} \bar{A} \\ \bar{B} \end{pmatrix} = \bar{x}_1 \begin{pmatrix} K_0(\bar{x}_1) & K_1(\bar{x}_1) \\ I_0(\bar{x}_1) & -I_1(\bar{x}_1) \end{pmatrix} \begin{pmatrix} I_1(x_1) \\ \sqrt{\frac{\epsilon_2}{\mu_2}} I_0(x_1) \end{pmatrix} \quad \text{(A2)}$$

$$f_0 = \bar{A} I_1(\bar{x}) + \bar{B} K_1(\bar{x}) \qquad \bar{x} \equiv i\omega\sqrt{\mu_2 \epsilon_2}\, r \qquad r_1 \le r \le r_2 \quad \text{(A3)}$$

$$A_{0n} = (\mu_m \epsilon_m - \mu_1 \epsilon_1) \frac{\sqrt{2}}{z_4 - z_1} \left. \frac{\sin k_n (z - z_1)}{k_n} \right|_{z_2}^{z_3} \quad \text{(A4)}$$

$$A_{nn} = \mu_1 \epsilon_1 - 1 + (\mu_m \epsilon_m - \mu_1 \epsilon_1) \left(\frac{1 - \frac{1}{2}\delta_{n0}}{z_4 - z_1} \right) \left. \left(z + \frac{\sin 2k_n (z - z_1)}{2k_n} \right) \right|_{z_2}^{z_3} \quad \text{(A5)}$$

$$D_n^a = r_1 i\omega\, I_0(i\omega r_1) E_n \quad \text{(A6)}$$

$$D_n^b = -r_2 i\omega \left[A I_0(i\omega r_2) - B K_0(i\omega r_2) \right] E_n \quad \text{(A7)}$$

$$E_n = \left[\epsilon_2 \left(\frac{1}{\epsilon_m} - \frac{1}{\epsilon_1} \right) F_n + \left(\frac{\epsilon_2}{\epsilon_1} - 1 \right) \frac{\delta_{n0}}{N_0} \right], \qquad N_0 = \frac{1}{\sqrt{z_4 - z_1}}$$

$$F_n = \frac{1}{k_n} \sqrt{\frac{2 - \delta_{n0}}{z_4 - z_1}} \left. \sin k_n (z - z_1) \right|_{z_2}^{z_3} \quad \text{(A8)}$$

$$\lambda_n^2 = k_n^2 - \omega^2, \qquad \lambda_n = +\sqrt{\lambda_n^2} \quad \text{(A9)}$$

$$\bar{\lambda}_n^2 = \lambda_n^2 - \omega^2 A_{nn}$$

$$T_n(r) = \bar{S}_n^a I_1(\bar{\lambda}_n r) + \bar{S}_n^b K_1(\bar{\lambda}_n r) \qquad r_1 \le r \le r_2 \quad \text{(A10)}$$

$$C_n(r) = \bar{C}_n^a I_1(\bar{\lambda}_n r) + \bar{C}_n^b K_1(\bar{\lambda}_n r)$$

$$B_n = \omega^2 \left[\begin{array}{l} \left\{ (\mu_m - \mu_1) - \epsilon_2 \mu_2 \left(\frac{1}{\epsilon_m} - \frac{1}{\epsilon_1} \right) \right\} F_n \\ + \left(\mu_1 - \frac{\epsilon_2 \mu_2}{\epsilon_1} \right) \frac{\delta_{n0}}{N_0} \end{array} \right. \quad \text{(A11)}$$

$$\begin{pmatrix} \bar{C}_n^a \\ \bar{C}_n^b \end{pmatrix} = r_2 \bar{\lambda}_n \begin{pmatrix} K_0(\bar{\lambda}_n r_2) & K_1(\bar{\lambda}_n r_2) \\ I_0(\bar{\lambda}_n r_2) & -I_1(\bar{\lambda}_n r_2) \end{pmatrix} \begin{pmatrix} K_1(\lambda_n r_2) \\ \frac{\lambda_n}{\bar{\lambda}_n} K_0(\lambda_n r_2) \end{pmatrix} \tag{A12}$$

$$\begin{pmatrix} \bar{S}_n^a \\ \bar{S}_n^b \end{pmatrix} = \frac{r_1 \bar{\lambda}_n}{C_n^b} \begin{pmatrix} K_0(\bar{\lambda}_n r_1) & K_1(\bar{\lambda}_n r_1) \\ I_0(\bar{\lambda}_n r_1) & -I_1(\bar{\lambda}_n r_1) \end{pmatrix} \begin{pmatrix} I_1(\lambda_n r_1) \\ \frac{\lambda_n}{\bar{\lambda}_n} I_0(\lambda_n r_1) \end{pmatrix} \tag{A13}$$

$$\begin{pmatrix} C_n^a \\ C_n^b \end{pmatrix} = \lambda_n r_1 \begin{pmatrix} K_0(\lambda_n r_1) & K_1(\lambda_n r_1) \\ I_0(\lambda_n r_1) & -I_1(\lambda_n r_1) \end{pmatrix} \begin{pmatrix} \bar{C}_n^a I_1(\bar{\lambda}_n r_1) + \bar{C}_n^b K_1(\bar{\lambda}_n r_1) \\ \frac{\bar{\lambda}_n}{\lambda_n}\left[\bar{C}_n^a I_0(\bar{\lambda}_n r_1) - \bar{C}_n^b K_0(\bar{\lambda}_n r_1)\right] \end{pmatrix} \tag{A14}$$

$$\begin{pmatrix} S_n^a \\ S_n^b \end{pmatrix} = \lambda_n r_2 \begin{pmatrix} K_0(\lambda_n r_2) & K_1(\lambda_n r_2) \\ I_0(\lambda_n r_2) & -I_1(\lambda_n r_2) \end{pmatrix} \begin{pmatrix} \bar{S}_n^a I_1(\bar{\lambda}_n r_2) + \bar{S}_n^b K_1(\bar{\lambda}_n r_2) \\ \frac{\bar{\lambda}_n}{\lambda_n}\left[\bar{S}_n^a I_0(\bar{\lambda}_n r_2) - \bar{S}_n^b K_0(\bar{\lambda}_n r_2)\right] \end{pmatrix} \tag{A15}$$

$$\Delta Y = \frac{2\pi r_3}{z_4 - z_1}\left[\frac{B^b K_1(i\omega r_3) + I_1(i\omega r_3)}{-B^b K_0(i\omega r_3) + I_0(i\omega r_3)} - \frac{I_1(i\omega r_3)}{I_0(i\omega r_3)} \right] \tag{A16}$$

$$B^b = \frac{B + H^{(1)} + \omega^2 H^{(2)}}{A} \tag{A17}$$

$$H^{(1)} = B_0 H^{(7)} + D_0^a T_0(r_1) + D_0^b T_0(r_2) \tag{A18}$$

$$H^{(7)} = \bar{S}_0^a \left(H_0^{(5)}(r_2) - H_0^{(5)}(r_1) \right) + \bar{S}_0^b \left(H_0^{(6)}(r_2) - H_0^{(6)}(r_1) \right) \tag{A19}$$

$$H^{(2)} = \sum_{n>0} A_{0n} H_n^{(2)} \tag{A20}$$

$$H_n^{(2)} = \int_{r_1}^{r_2} T_0(r)\, R_n(r)\, dr \tag{A21}$$

$$R_n(r) = r\left[C_n(r)\, H_n^{(3)}(r) + T_n(r)\, H_n^{(4)}(r) \right] \tag{A22}$$

$$H_n^{3}(r) = D_n^a T_n(r_1) + B_n \left\{ \begin{array}{l} \bar{S}_n^a \left[H_n^{(5)}(r) - H_n^{(5)}(r_1) \right] \\ + \bar{S}_n^b \left[H_n^{(6)}(r) - H_n^{(6)}(r_1) \right] \end{array} \right\} \tag{A23}$$

$$H_n^{(4)}(r) = D_n^b C_n(r_2) + B_n \left\{ \begin{array}{l} \bar{C}_n^a \left[H_n^{(5)}(r_2) - H_n^{(5)}(r) \right] \\ + \bar{C}_n^b \left[H_n^{(6)}(r_2) - H_n^{(6)}(r) \right] \end{array} \right\} \tag{A24}$$

$$H_n^{(5)}(r) = \bar{A} P_1^n(r) + \bar{B} P_2^n(r) \tag{A25}$$

$$H_n^{(6)}(r) = \bar{A} P_3^n(r) + \bar{B} P_4^n(r) \tag{A26}$$

$$P_1^n(r) = \left[r \left(I_1(\bar{x}_n) I_0(\bar{x}) \bar{\lambda} - \bar{\lambda}_n I_0(\bar{x}_n) I_1(\bar{x}) \right) \right] / Z_n \tag{A27}$$

$$P_2^n(r) = \left[-r \left(I_1(\bar{x}_n) K_0(\bar{x}) \bar{\lambda} + \bar{\lambda}_n I_0(\bar{x}_n) K_1(\bar{x}) \right) \right] / Z_n \tag{A28}$$

$$P_3^n(r) = \left[r \left(K_1(\bar{x}_n) I_0(\bar{x}) \bar{\lambda} + \bar{\lambda}_n K_0(\bar{x}_n) I_1(\bar{x}) \right) \right] / Z_n \tag{A29}$$

$$P_4^n(r) = \left[-r \left(K_1(\bar{x}_n) K_0(\bar{x}) \bar{\lambda} - \bar{\lambda}_n K_0(\bar{x}_n) K_1(\bar{x}) \right) \right] / Z_n \tag{A30}$$

$$\bar{Z}_n = \bar{\lambda}_n^2 - \bar{\lambda}^2, \quad \bar{x}_n = \bar{\lambda}_n r, \quad \bar{x} = \bar{\lambda} r$$

APPENDIX B

FORTRAN LISTING

```
      PROGRAM CAPACTOR                                                   CAP
      COMMON  R,Z,F,U1, SK,ALM,BB,BA,  AL,AA,AB,AD,ADB,BK0,BK1,BI0,BI1, CAP
     1CA,CB,SA,SB,X1,X2,      Z21,Z31,Z41,UM,E1,EM,RATIO,Y1,Z32,IB,ONA,  CAP
     2SN1,SN2,DU,DE,AK,CFT,SUM,F2,ICYLE, TEMP,ST,C,B,    SUB     ,I1     CAP
     3 ,STEP2
      DIMENSION R(4),Z(4),         SK(100,2),ALM(100,2),AL(100),AA(100), CAP
     1AB(100),AD(100),ADB(100),BK0(500),BK1(500),BI0(500),BI1(500),     CAP
     2CA(100,2),CB(100,2),SA(100,2),SB(100,2),X1(2),X2(2),RATIO(2),Y2(3)CAP
     3,SN1(100,2),SN2(100,2),CFT(6),SUM(7),SUB(2),ST(6)                  AP
     4 ,C(3,2)
      TYPE COMPLEX U1, ALM,BB,BA,  AL,AA,AB,AD,ADB,BK0,BK1,BI0,BI1,CA,  CAP
     1CB,SA,SB,X1,X2,                X,   UM,E1,EM,    RATIO,Y1,AREA,Y2, CAP
     2Y3,   ONA,DU,DE,    SUM,           TEMP,ST          ,SUB          CAP
     3,C,CFT      ,E2,U2
      DATA ( CON=2.0944E-10),(PI=1.668E-02 )
C     THIS PROGRAM COMPUTES THE ADMITTANCE IN MHOS OF A TOROID SAMPLE   CAP
C     IN PARALLEL PLATE CAPACITOR. THE INPUT DATA IS THE THREE RAD1(CM),CAP
C     THE FOUR Z AXIS VALUES, THE FREQUENCY (HZ),AND THE TWO SETS OF    CAP
C     VALUES FOR  THE PERMITTIVITY AND PERMIBILITY. THE OUTPUT DATA     CAP
C     IS  THE ADMITTANCES--      SECOND ORDER CORRECTION,QUASISTATIC,   CAP
C     AND  TOTAL.                                                       CAP
      SENSE LIGHT 1
      ONA=(1.,0.)
2     READ 1,(R(I),I=1,3),Z,IA,IB,E1,U1                                 CAP
1     FORMAT(3E18.5/4E18.5/      2I4/ 2C(E18.5,E18.5 ))                  CAP
C   R IS RADIUS VALUES,Z IS HEIGHT VALUES IA=-1 IS ADMITTANCE TEST
C      IA=0 SKIPS TEST OF SENSITIVITY IB=1 SKIPS HIGHER MODE =0 DOES IT
      IT=-1                                                             CAP
101   READ 102,E,IC,IS,ITES                                             CAP
C     E IS FREQUENCY IN HERTZ IC= NUMBER OF DIFFERENT EM AND UM
C     IS IS PAGE NUMBER ITES=-1 TO EXIT AFTER NEXT BATCH OF DATA
C     ITES=0 NEW SPACIAL DIMENSION   =+1 NEW FREQUENCY
      F= CON*E   $ F2=F**2                                              CAP
102   FORMAT(E18.5,3I4)                                                 CAP
6     PRINT 7,IS,E,(R(I),I=1,3),Z,E1,U1                                 CAP
7     FORMAT( 51H1ADMITTANCE FOR CAPACITOR WITH TOROID GEOMETRY PAGE I4/CAP
     118H THE FREQUENCY IS E18.4, 4HHZ.  /                              CAP
     227H THE RADIAL DIMENSIONS ARE        3E18.4 /                     CAP
     327H THE Z AXIS DIMENSIONS ARE        4E18.4 /14H SPACER E1 IS     CAP
     4C(E18.4,E18.4),7H U1 IS   C(E18.4,E18.4)/                         CAP
     591H THE LENGTH MEASUREMENTS ARE IN CM. ADMITTANCES ARE IN MHOS.  TCAP
     6HE REST ARE DIMENSIONLESS.          //)                           CAP
      PRINT 701                                                         CAP
701   FORMAT(115H            MU                         EPSILON       QUACAP
     1SI STATIC ADM        CORRECT ADM              DBL SUM ADM       / CAP
     2125H UM          +I UM*           EM          +I EM*          Y1          +ICAP
     3 Y2          YC1          +I YC2          DY1          +I DY2          CNT    )CAP
      ITT=1                                                             CAP
9     IS=IS+1                                                           CAP
      IF(IT)71,72,13                                                    CAP
 71   Z41= Z(4)-Z(1)  $ B= 1./Z41                                       CAP
      STEP2=(R(2)-R(1))/32.
      FIX= PI  /Z41                                                     CAP
 212  Z21= Z(2)-Z(1)$ Z32=Z(3)-Z(2)                                     CAP
```

```
      Z31= Z(3)-Z(1)    $ AK=(3.14159265)/Z41                          CAP
 72   AREA= FIX *(R(2)**2-R(1)**2)*(0.,0.5)*F                          CAP
12    DO  13 I1=1,IC                                                   CAP
 211  FORMAT(2C(E18.5,E18.5))                                          CAP
225   READ 211, EM,UM                                                  CAP
      Z5=Z32/Z41
 .    E2=E1/(ONA+Z5*(E1/EM-ONA))
      U2=U1+Z5*(UM-U1)
      Y1=AREA*(E2-ONA)
15    CALL ADMIT(E2,U2)
 216  X=R(3) *F*(0.,1.) $ CALL  BESSL2(X,500) $ F3=FIX*R(3)
      DO 215 J=1,2                                                     CAP
      Y2(J)=F3*((RATIO(J)*BK1(500)+BI1(500))/(RATIO(J)*(-BK0(500))+    CAP
     1BI0(500))-BI1(500)/BI0(500))
      IF(IB)215,215,213
 215   CONTINUE
      IF(IA)217,218,218                                                CAP
 217  X= 0.9*X $ F3=0.9*F3 $CALL  BESSL2(X,500) $ ITT=ITT+1
      Y2(3)=F3*((RATIO(2)*BK1(500)+BI1(500))/(RATIO(2)*(-BK0(500))+    CAP
     1BI0(500))-BI1(500)/BI0(500))
      PER= CABS( (Y2(2)-Y2(3))/Y2(2))                                  CAP
 219  FORMAT(60H NEXT ADMITTANCE FRACTIONAL CHANGE WITH R(3)= 0.9*R(3)  CAP
     1IS  E12.4)                                                       CAP
      PRINT 219,PER                                                    CAP
      GO TO 218
 213   Y2(2)=Y2(1)
 218  Y3=Y2(2)-Y2(1)                                                   CAP
302   PRINT 8,UM,EM,Y1,Y2(2),Y3,ICYLE                                  CAP
 8    FORMAT(C(E13.4,E12.4),4C(E12.4,E12.4),I4)                        CAP
      ITT=ITT+1                                                        CAP
      IF(ITT-46)13,6,6                                                 CAP
 13   CONTINUE                                                         CAP
      IF(ITES)240.2,103                                                CAP
 103  IT=0                                                             CAP
      GO TO 101                                                        CAP
 240  CALL EXIT                                                        CAP
      END                                                              CAP
      SUBROUTINE SAVE(J)                                               SAVE
      COMMON  R,Z,F,U1, SK,ALM,BB,BA,  AL,AA,AB,AD,ADB,BK0,BK1,BI0,BI1, CAP
     1CA,CB,SA,SB,X1,X2,      Z21,Z31,Z41,UM,E1,EM,RATIO,Y1,Z32,IB,ONA,  CAP
     2SN1,SN2,DU,DE,AK,CFT,SUM,F2,ICYLE, TEMP,ST,C,B,   SUB  ,IT        CAP
      DIMENSION R(4),Z(4),      SK(100,2),ALM(100,2),AL(100),AA(100),  CAP
     1AB(100),AD(100),ADB(100),BK0(500),BK1(500),BI0(500),BI1(500),    CAP
     2CA(100,2),CB(100,2),SA(100,2),SB(100,2),X1(2),X2(2),RATIO(2),Y2(3)CAP
     3,SN1(100,2),SN2(100,2),CFT(6),SUM(7),SUB(2),ST(6)                AP
     4 ,C(3,2)
      TYPE COMPLEX U1, ALM,BB,BA,  AL,AA,AB,AD,ADB,BK0,BK1,BI0,BI1,CA,  CAP
     1CB,SA,SB,X1,X2,                X,   UM,E1,EM,    RATIO,Y1,AREA,Y2, CAP
     2Y3,   ONA,DU,DE,    SUM,        TEMP,ST          ,SUB            CAP
     3.,C,CFT
      IF(IT)50,101,101
 50    J1=0
      IT=1
      B1=B*1.414213562 $B2=SQRTF(B) $B3=B2*1.414213562  $B4=B/2.0
```

```
      X6=Z32*B $SN2(1,1)=X6    $SN2(1,2)=B2*( Z32)
2      GO TO 9                                                            SAVE
101   IF( J-J1) 10,10,3
 3    SK(J,1)=SK(J-1,1)+AK                                                SAVE
      SK(J,2)=SK(J,1)**2                                                  SAVE
      X4=SK(J,1)*Z21                                                      SAVE
      X5=SK(J,1)*Z31                                                      SAVE
      SN1(J,2)=(SINF(X5)-SINF(X4))/SK(J,1)  $ SN2(J,2)=  SN1(J,2)*B3      SAVE
      SN1(J,1)=((SINF( 2.*X5)-SINF(2.*X4))/(   SK(J,1)))*B4               SAVE
      SN2(J,1)=      (SN1(J,1)+X6                                         SAVE
      SN1(J,2)=   SN1(J,2)*B1                                             SAVE
 9    J1=J1+1                                                             SAVE
 10   RETURN                                                              SAVE
      END                                                                 SAVE
      SUBROUTINE INTGR(J)                                                 INTGR
      COMMON  R,Z,F,U1, SK,ALM,BB,BA,  AL,AA,AB,AD,ADB,BK0,BK1,BI0,BI1,   CAP
     1CA,CB,SA,SB,X1,X2,     Z21,Z31,Z41,UM,E1,EM,RATIO,Y1,Z32,IB,ONA,    CAP
     2SN1,SN2,DU,DE,AK,CFT,SUM,F2,ICYLE, TEMP,ST,C,B,   SUB               CAP
     3,IT,STEP2,T,HC,AF,P
      DIMENSION R(4),Z(4),       SK(100,2),ALM(100,2),AL(100),AA(100),    CAP
     1AB(100),AD(100),ADB(100),BK0(500),BK1(500),BI0(500),BI1(500),       CAP
     2CA(100,2),CB(100,2),SA(100,2),SB(100,2),X1(2),X2(2),RATIO(2),Y2(3)CAP
     3,SN1(100,2),SN2(100,2),CFT(6),SUM(7),SUB(2),ST(6)
     4,H(8,2),T(6,2),HC(5),      P(4),C(3,2),AF(33)
      TYPE COMPLEX U1, ALM,BB,BA,  AL,AA,AB,AD,ADB,BK0,BK1,BI0,BI1,CA,    CAP
     1CB,SA,SB,X1,X2,T0,P1,P2,P3,P4,X,X3,UM,E1,EM,    RATIO,Y1,AREA,Y2,   CAP
     2Y3,   ONA,DU,DE,CFT,SUM,H,T,CR,HC,TEMP,ST  ,P,C,AF,SUB              CAP
      TYPE COMPLEX HF1,HF2,HF3,HF4,HF5,    HFB3,HFB4,HFB5,HFB6,           INTGR
     1    RF,TF,CF,RFB,TFB,CFB,X0,X5,RT,RTB,DM1,DM2 ,RA1,RA2,RC,AL4       INTGR
      TF(K,K1)= SA(K,2)*BI1(K1)+SB(K,2)*BK1(K1)                           INTGR
      CF(K,K1)= CA(K,2)*BI1(K1)+CB(K,2)*BK1(K1)                           INTGR
      P1(K1,K2)= BI1(K1)*BI0(K2)*X0- X*BI0(K1)*BI1(K2)                    INTGR
      P2(K1,K2)=-(BI1(K1)*BK0(K2)*X0+X*BI0(K1)*BK1(K2) )
      P3(K1,K2)=  BK1(K1)*BI0(K2)*X0+X*BK0(K1)*BI1(K2)                    INTGR
      P4(K1,K2)=-(BK1(K1)*BK0(K2)*X0-X*BK0(K1)*BK1(K2)   )
      HF5(I)=(SA(100,2)*P(I)+SB(100,2)*P(I+1))
      HF3(K) =(SA(K,2)*H(5,1)+SB(K,2)*H(6,1) )*ADB(K)                     INTGR
      HF4(K) =(CA(K,2)*H(5,1)+CB(K,2)*H(6,1)   )*ADB(K)                   INTGR
      IF(J-1)1,1,3
 84   SUM(4)=(AF(1)+AF(33))/(2.,0.)                                       INTGR
      DO 85 L=5,30,4                                                      INTGR
 85   SUM(4)=SUM(4)+AF(L)                                                 INTGR
      SUM(5)= 2.*(SUM(4)+2.*SUM(3))                                       INTGR
      SUM(6)= SUM(4)+SUM(3)+2.*SUM(2)                                     INTGR
      SUM(1)= (SUM(6)-SUM(5))/(SUM(6)*15.)                                INTGR
       TOO = CABS(SUM(1))  $ J3=J-1                                       INTGR
      IF( TOO -0.02) 87,87,86
 86   PRINT 90,TOO,J3                                                     INTGR
 90   FORMAT(* THE INTEGRATION ERROR IS* E10.2 *THE MODE NUMBER IS* I4)   INTGR
 87   ST(4)=  (STEP*0.666666667)*SUM(6)                                   INTGR
      RETURN                                                              INTGR
1                            I8=1  $ GO TO 2
3     I8=2
 2    JB1=J+100                                                           INTGR
```

```
      JB2=J+300                                                          INTGR
      R(4)=R(1)                                                          INTGR
      JR=JB1      $ JS=200
      I6=1   $ I7=2                                                      INTGR
      GO TO 10                                                           INTGR
 20   R(4)=R(2)                                                          INTGR
      H(1,1)= H(3,1)                  $ H(7,1)=H(4,1)                    INTGR
      T(1,1)= T(3,1)                                                     INTGR
      I6=2                                                               INTGR
      HC(2)=AL(J)*T(1,1)-H(1,1)
      JR=JB2     $JS=400                                                 INTGR
      GO TO 10                                                           INTGR
 30   H(2,1)=H(3,1)                   $H(8,1)=H(4,1)                     INTGR
      T(2,1)=T(3,1)                                                      INTGR
      GO TO (90,91),I8
91    C(2,1)=C(3,1)                                                      INTGR
      HC(3)= AB(J)*C(2,1)+H(8,1)                                         INTGR
      I7=1                                                               INTGR
      R(4)=R(2)                                                          INTGR
      RA1   = (C(3,1)*(H(3,1)+HC(2))+T(3,1)*(HC(3)-H(4,1)))   *R(4)
       JR= J+400      $JS=500                                            INTGR
      AF(1)= RA1*HC(5)
      STEP=0.15/ CABS(ALM(J,2))                                          INTGR
      IF(STEP-STEP2)32,32,31
 31    STEP=STEP2
32    DO 80 L=2,33                                                       INTGR
      R(4)=R(4)- STEP                                                    INTGR
      I6=3                                                               INTGR
      GO TO 10                                                           INTGR
40    X=ALM(1,2)*R(4)
      CALL BESSL2(X,401) $ T(6,1)=TF(1,401)
      RA1 =   (C(3,1)*(H(3,1)+HC(2))+T(3,1)*(HC(3)-H(4,1)))   *R(4)
80    AF(L)=RA1*T(6,1)
      DO 81 L=2,3                                                        INTGR
 81   SUM(L)=(0.,0.)                                                     INTGR
      DO 82 L=2,33,2                                                     INTGR
 82   SUM(2)=SUM(2)+AF(L)                                                INTGR
      DO 83 L=3,33,4                                                     INTGR
 83   SUM(3)=SUM(3)+AF(L)                                                INTGR
      GO TO 84                                                           INTGR
 10   X0= ALM(100,2)*R(4)
      X = ALM(J,2)*R(4)                                                  INTGR
                                 GO TO (11,12),I7
 11   CALL BESSL2(X0,500)
      CALL BESSL2(X ,JR )                                                INTGR
 12   P(1)= P1(JR,JS )
      P(2)= P2(JR,JS )
      P(3)= P3(JR,JS )
      P(4)= P4(JR,JS )
      H(5,1)=HF5(1)                                                      INTGR
      H(6,1)=HF5(3)                                                      INTGR
      H(3,1)=HF3(J)    $ T(3,1)=TF(J,JR)                                 INTGR
      GO TO (851,861),I8
861   H(4,1)=HF4(J)    $ C(3,1)=CF(J,JR)                                 INTGR
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851   GO TO (20,30,40),I6                                                   INTGR
90    HC(3)=AB(1)*T(2,1)
      ST(5)=HC(3)+HC(2)+H(2,1)
      HC(5)=T(2,1)
      RETURN
      END                                                                   INTGR
      SUBROUTINE COEF(J)                                                    COEF
      COMMON  R,Z,F,U1, SK,ALM,BB,BA,  AL,AA,AB,AD,ADB,BK0,BK1,BI0,BI1, CAP
     1CA,CB,SA,SB,X1,X2,      Z21,Z31,Z41.UM,E1.EM,RATIO,Y1,Z32,IB,ONA,   CAP
     2SN1,SN2,DU,DE,AK,CFT,SUM,F2,ICYLE, TEMP,ST,C,B,    SUB              CAP
      DIMENSION R(4),Z(4),        SK(100,2),ALM(100,2),AL(100),AA(100), CAP
     1AB(100),AD(100),ADB(100),BK0(500),BK1(500),BI0(500),BI1(500),       CAP
     2CA(100,2),CB(100,2),SA(100,2),SB(100,2),X1(2),X2(2),RATIO(2),Y2(3)CAP
     3,SN1(100,2),SN2(100,2),CFT(6),SUM(7),SUB(2),ST(6)                   AP
     4 ,C(3,2)
      TYPE COMPLEX U1, ALM,BB,BA,  AL,AA,AB,AD,ADB,BK0,BK1,BI0,BI1,CA,   CAP
     1CB,SA,SB,X1,X2,                X,   UM,E1,EM,    RATIO,Y1,AREA,Y2,  CAP
     2Y3,   ONA,DU,DE,    SUM,         TEMP,ST          ,SUB              CAP
     3,C,CFT
        J1= J                                                               COEF
        JB1=J+100                                                           COEF
        J2 =J+200                                                           COEF
        JB2=J+300                                                           COEF
        CALL BESSL2(X1(1),J1)                                               COEF
        CALL BESSL2(X1(2),JB1)                                              COEF
        CALL BESSL2(X2(1),J2 )                                              COEF
        CALL BESSL2(X2(2),JB2)                                              COEF
        TEMP=          ALM(J,1)/ALM(J,2)                                    COEF
        IF(J-100)5,7,7
7     TEMP=EM*TEMP
      CB(100,1)=(1.,0.)
      GO TO 6
5     CA(J,2)=(BK0(JB2)*BK1(J2)-BK1(JB2)*TEMP*BK0(J2))*X2(2)              COEF
      CB(J,2)=(BI0(JB2)*BK1(J2)+BI1(JB2)*TEMP*BK0(J2))*X2(2)              COEF
      CB(J,1)=         ( BI0(J1)*(CA(J,2)*BI1(JB1)+CB(J,2)*BK1(JB1))      COEF
     1   -BI1(J1)*(CA(J,2)*BI0(JB1)-CB(J,2)*BK0(JB1))/TEMP)*X1(1)         COEF
6     SA(J,2)=(BK0(JB1)*BI1(J1)+BK1(JB1)*TEMP*BI0(J1))*X1(2)/CB(J,1)      COEF
      SB(J,2)=(BI0(JB1)*BI1(J1)-BI1(JB1)*TEMP*BI0(J1))*X1(2)/CB(J,1)      COEF
      IF(J-1) 1,1,2                                                         COEF
 1    ST(1)=(SA(J,2)*BI1(JB2)+SB(J,2)*BK1(JB2)) *X2(1)                    COEF
      ST(2)=(SA(J,2)*BI0(JB2)-SB(J,2)*BK0(JB2))*X2(1)/TEMP                COEF
      SA(J,1)= BK0(J2)*ST(1)+BK1(J2)*ST(2)                                  COEF
      SB(J,1)= BI0(J2)*ST(1)-BI1(J2)*ST(2)                                  COEF
3     RETURN                                                                COEF
2     IF(J-100)3,1,3
      END                                                                   COEF
      SUBROUTINE ADMIT(E2,U2)
      COMMON  R,Z,F,U1, SK,ALM,BB,BA,  AL,AA,AB,AD,ADB,BK0,BK1,BI0,BI1, CAP
     1CA,CB,SA,SB,X1,X2,      Z21,Z31,Z41,UM,E1,EM,RATIO,Y1,Z32,IB,ONA,   CAP
     2SN1,SN2,DU,DE,AK,CFT,SUM,F2,ICYLE, TEMP,ST,C,B,    SUB   ,IT        CAP
      DIMENSION R(4),Z(4),        SK(100,2),ALM(100,2),AL(100),AA(100), CAP
     1AB(100),AD(100),ADB(100),BK0(500),BK1(500),BI0(500),BI1(500),       CAP
     2CA(100,2),CB(100,2),SA(100,2),SB(100,2),X1(2),X2(2),RATIO(2),Y2(3)CAP
     3,SN1(100,2),SN2(100,2),CFT(6),SUM(7),SUB(2),ST(6)                   AP
```

```
      4 ,C(3,2)
       TYPE COMPLEX U1, ALM,BB,BA,  AL,AA,AB,AD,ADB,BK0,BK1,BI0,BI1,CA,  CAP
      1CB,SA,SB,X1,X2,                  X,   UM,E1,EM,    RATIO,Y1,AREA,Y2, CAP
      2Y3,   ONA,DU,DE,     SUM,          TEMP,ST           ,SUB              CAP
      3,C,CFT    ,DE1   ,DE2,AAA,DE3,AAB
      4,DE4,DE5,DE6,E2,U2,E3,AAC,AAD ,AL4,AL5       ,AL8
       TYPE REAL MS1,MS2,MIC1,MIC2
       IF(IT)58,63,63
58      Z6=SQRTF(Z41)
63     DU=UM*EM-U1*E1
       DE=E1*U1-1. $DE6=U2*E2 $DE5=E2/E1-1.$DE3=CSQRT(DE6)
       DE1=E2*(1./EM-1./E1) $DE2=-F2*(DE1*U2-UM+U1)  $DE4=(U1-DE6/E1)*F2
       DE4=DE4*Z6  $DE5=DE5*Z6
C      FO CASE HAS I=100
C      BAR STUFF IN EQUATIONS IS LABLED BY TWO                          ADMIT
       SK(1,1)=0.                                                       ADMIT
       IF DIVIDE CHECK 101,101
101    CALL SAVE(1)
       AD(1)= DU*SN2(1,1)    +DE                                        ADMIT
       ALM(1,1)= (0.,1.)*F        $ALM(100,1)=ALM(1,1)                  ADMIT
       E3=EM  $EM=E2
       ALM(100,2)=ALM(100,1)*DE3
11      AL4=ALM(100,2)**2
       ALM(1,2)=  CSQRT( F2*(1.+AD(1)))*(0.,1.)
       AL8=ALM(1,2)**2-AL4
       IF(CABS(AL8/AL4)-1.0E-6)600,600,604
600     PRINT 602
602    FORMAT(* CONDITIONS OF SYSTEM ARE SUCH THAT THE ACCURACY IS */
      1*  TOTALLY UNRELIABLE *)
       CALL EXIT
604    ADB(1)=DE2*SN2(1,2)      +DE4
        ADB(1)=ADB(1)/AL8
        DO 150,J=1,100,99
       X1(1)=  ALM(J,1)*R(1)                                            ADMIT
       X1(2)=  ALM(J,2)*R(1)                                            ADMIT
       X2(1)=  ALM(J,1)*R(2)                                            ADMIT
       X2(2)=  ALM(J,2)*R(2)                                            ADMIT
150    CALL  COEF (J)                                                   ADMIT
       EM=E3
       AAC= X1(1)    *BI0(100)
       AAD=-X2(1)   *(SA(100,1)*BI0(300)-SB(100,1)*BK0(300))
       AAA=DE1*AAC $AAB=DE1*AAD
       AL(1)=AAA*SN2(1,2)        +AAC*DE5
       AB(1)=AAB*SN2(1,2)      +AAD*DE5
       IF DIVIDE CHECK 40,200
 200   BA= SA(1,1)  -1.                                                 ADMIT
       IF(CABS(BA)-1.E-7)201,201,220
 220    PRINT 42
 42     FORMAT(*NORMALIZATION ERROR HAS OCCURRED*)
201    BB= SB(100,1)
       ICYLE =0
       IA=-1
        IF EXPONENT FAULT 499,499
499    CALL INTGR(1)
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```
      IF EXPONENT FAULT 500,501
500   RATIO(1)=RATIO(2)=(0.,0.)     $ GO TO 40
501   BB=BB+ST(5)
      SUB(2)=BB
      RATIO(1)=BB/SA(100,1)
      IF(IB)1,1,2
1     DO 100,I=2,99      $ I=I
      CALL SAVE(I)                                                      ADMIT
      ICYLE =ICYLE+1
      AD(I)=DU*SN2(I,1)+DE
                          AL(I)=AAA*SN2(I,2) $AB(I)=AAB*SN2(I,2)
      ALM(I,1)=       ( SK(I,2) -F2)                                    ADMIT
      ALM(I,2)=       ( ALM(I,1)   -F2*AD(I))                           ADMIT
      AL5=ALM(I,2)-AL4 $ALM(I,2)=CSQRT(ALM(I,2))
      IF(CABS(AL5/AL4)-1.0E-6)600,600,601
601   ADB(I)=DE2*SN2(I,2)/AL5
      ALM(I,1)= CSQRT(ALM(I,1))  $AA(I)=SN1(I,2)*DU
      X1(1)=ALM(I,1)*R(1)                                               ADMIT
      X1(2)=ALM(I,2)*R(1)                                               ADMIT
      X2(1)=ALM(I,1)*R(2)                                               ADMIT
      X2(2)=ALM(I,2)*R(2)                                               ADMIT
      CALL COEF(I)                                                      ADMIT
      IF DIVIDE CHECK 40,300
300   IF EXPONENT FAULT 301,301
301   CALL INTGR(I)
      IF EXPONENT FAULT 40,400
400    ST(6)       = ST(4)*AA(I)    *F2                                 ADMIT
       MS2= CABS(SUB(2))*(0.001)
      MIC2=CABS(ST(6))                                                  ADMIT
      SUB(2)=SUB(2)+ST(6)
 20   IF( MS2-MIC2) 100,100,99
99    IF(IA)102,103,2
102    I4=I+1 $IA=   0 $ GO TO 100
103   IF(I-I4)2,2,97
97    I4=I+1
 100  CONTINUE                                                          ADMIT
 2    RATIO(2)=SUB(2)/SA(100,1)
      RETURN                                                            ADMIT
 40   PRINT 41
 41    FORMAT(*NEXT LINE HAS HAD AN OVERFLOW OR DIVIDE CHECK*)
      RETURN
      END                                                               ADMIT
      SUBROUTINE BESSL2(X7,I)                                           BESSL2
      COMMON  R,Z,F,U1, SK,ALM,BB,BA,  AL,AA,AB,AD,ADB,BK0,BK1,BI0,BI1, CAP
     1CA,CB,SA,SB,X1,X2,      Z21,Z31,Z41,UM,E1,EM,RATIO,Y1,Z32,IB,ONA,  CAP
     2SN1,SN2,DU,DE,AK,CFT,SUM,F2,ICYLE, TEMP,ST,C,B,    SUB             CAP
      DIMENSION R(4),Z(4),         SK(100,2),ALM(100,2),AL(100),AA(100), CAP
     1AB(100),AD(100),ADB(100),BK0(500),BK1(500),BI0(500),BI1(500),      CAP
     2CA(100,2),CB(100,2),SA(100,2),SB(100,2),X1(2),X2(2),RATIO(2),Y2(3)CAP
     3,SN1(100,2),SN2(100,2),CFT(6),SUM(7),SUB(2),ST(6)                  AP
     4 ,C(3,2)
      TYPE COMPLEX U1, ALM,BB,BA,  AL,AA,AB,AD,ADB,BK0,BK1,BI0,BI1,CA,  CAP
     1CB,SA,SB,X1,X2,                X,   UM,E1,EM,    RATIO,Y1,AREA,Y2, CAP
     2Y3,   ONA,DU,DE,    SUM,          TEMP,ST          ,SUB            CAP
```

```
     3,C,CFT
      TYPE DOUBLE X4,X5,H01A,H01B,H02A,H02B,H11A,H11B,H12A,H12B
      TYPE COMPLEX X3,PH,X7
      DATA (PH=(0.,1.))
      IF DIVIDE CHECK 2,1
1     X3=X7*PH
      X4=X3 $ X5=X7
      CALL ALLH(X4,X5,H01A,H01B,H02A,H02B,H11A,H11B,H12A,H12B)
      BI1(I)=(0.,-0.5)*((H11A+H12A)+PH*(H11B+H12B))
      BK1(I)=(-1.57079633,0.)*(H11A+PH*H11B)
      IF(I-401)100,200,100
100   BI0(I)=0.5*((H01A+H02A)+PH*(H01B+H02B))
      BK0(I)=(0.,1.57079633)*(H01A+PH*H01B)
200   IF DIVIDE CHECK 22,3
2     J=1  $ GO TO 21
22    J=2
21    PRINT 4,I,J
4     FORMAT(* NEXT LINE HAS BESSL2 DIVIDE CHECK I= *,2I4)
3     RETURN                                                            BESSL2
      END                                                               BESSL2
      SUBROUTINE ALLH(ZA,ZB,H01A,H01B,H02A,H02B,H11A,H11B,H12A,H12B)
C      THE REMAINING SUBROUTINES ARE A PACKAGE FOR HANKEL FUNCTION
C      GENERATION .       THIS PACKAGE WAS DEVELOPED BY GARNEY HARDY
      TYPE DOUBLE ZA,ZB,H01B,H01A,H02A,H02B,H11A,H11B,H12A
     2,H12B,X,Y,ABSZ
      X=ZA
      Y=ZB
      ABSZ=DSQRT(X*X+Y*Y)
      IF(ABSZ-7.7D)1,1,3
    1 CALL HSMALL(X,Y,H01A,H01B,H02A,H02B,H11A,H11B,H12A,H12B)
      RETURN
    3 CALL HLARGE(X,Y,H01A,H01B,H02A,H02B,H11A,H11B,H12A,H12B)
      RETURN
      END
      SUBROUTINE CHIS(CHI)
      DIMENSION CHI(20)
         TYPE DOUBLE CHI
      CHI(1)=1.57D
      CHI(2)=2.D
      CHI(3)=2.36D
      CHI(4)=2.67D
      CHI(5)=2.95D
      CHI(6)=3.2D
      CHI(7)=3.44D
      CHI(8)=3.66D
      CHI(9)=3.87D
      CHI(10)=4.06D
      CHI(11)=4.26D
      CHI(12)=4.43D
      CHI(13)=4.61D
      CHI(14)=4.77D
      CHI(15)=4.94D
      CHI(16)=5.09D
      CHI(17)=5.25D
```

```
      CHI(18)=5.39D
      CHI(19)=5.54D
      CHI(20)=5.68D
      RETURN
      END
      SUBROUTINE AS(A0,A1)
      DIMENSION A0(20),A1(20)
      TYPE DOUBLE A0,A1
         TYPE DOUBLE AN
      A0(1)=-.125D
      A1(1)=.375D
      DO 1 N=2,20
      AN=N
      A0(N)=-(((2.D*AN-1.D)**2)*A0(N-1))/(AN*8.D)
    1 A1(N)=((4.D0-((2.D0*AN-1.D0)**2))*A1(N-1))/(AN*8.D0)
      RETURN
      END
      SUBROUTINE JSMALL(X,Y,AJ0A,AJ0B,AJ1A,AJ1B)
      TYPE DOUBLE X,Y,AJ0A,AJ0B,AJ1A,AJ1B,ZA,ZB,AM,ZMA,
     1ZMB,SIGN   ,SUMA,SUMB,FACTM,TERMA,TERMB
      ZA=X
      ZB=Y
      ZA=ZA/2.D
      ZB=ZB/2.D
       CALL ZMPY(ZA,ZB,ZA,ZB,ZA,ZB)
      AM=0.D
      FACTM=1.D
      ZMA=1.D
      ZMB=0.D
      SUMA=1.D
      SUMB=0.D
      SIGN=1.D
    1 CALL ZMPY(ZA,ZB,ZMA,ZMB,ZMA,ZMB)
      SIGN=-SIGN
      AM=AM+1.D
      FACTM=FACTM*AM
      TERMA=ZMA/(FACTM*FACTM)
      TERMB=ZMB/(FACTM*FACTM)
      SUMA=SUMA+SIGN*TERMA
      SUMB=SUMB+SIGN*TERMB
    2 IF(DABS(TERMA)+DABS(TERMB)-1.D-13)3,1,1
    3 AJ0A=SUMA
      AJ0B=SUMB
      AM=1.D
      FACTM=1.D
      ZMA=1.D
      ZMB=0.D
      SUMA=1.D
      SUMB=0.D
      SIGN=1.D
    5 CALL ZMPY(ZA,ZB,ZMA,ZMB,ZMA,ZMB)
      SIGN=-SIGN
      FACTM=FACTM*AM
      AM=AM+1.D
```

```
      TERMA=ZMA/(FACTM*FACTM*AM)
      TERMB=ZMB/(FACTM*FACTM*AM)
      SUMA=SUMA+SIGN*TERMA
      SUMB=SUMB+SIGN*TERMB
    4 IF(DABS(TERMA)+DABS(TERMB)-1.D-13)7,5,5
    7 AJ1A=SUMA/2.D
       AJ1B=SUMB/2.D
      CALL ZMPY(X,Y,AJ1A,AJ1B,AJ1A,AJ1B)
      RETURN
      END
      SUBROUTINE HSMALL(X,Y,H01A,H01B,H02A,H02B ,H11A,H11B,H12A,H12B)
      TYPE DOUBLE X,Y,H01A,H01B,H02A,H02B,H11A,H11B,H12A,
     1H12B,ZA,ZB,AJ0A,AJ0B,AJ1A,AJ1B,PSI,SIX,SUMA,
     2SUMB,S,FACTS,TERMA,TERMB,ALOG,BLOG,FUNNYA,
     3FUNNYB,ZMA,ZMB,SIGN
      ZA=X
      ZB=Y
      CALL JSMALL(ZA,ZB,AJ0A,AJ0B,AJ1A,AJ1B)
      PSI=-.57721566490153286060651 2D
      SIX=.63661977236758134307553 50D
      ZA=ZA/2.D
      ZB=ZB/2.D
       CALL ZMPY(ZA,ZB,ZA,ZB,ZA,ZB)
      ZMA=1.D
      ZMB=0.D
      SUMA=PSI
      SUMB=0.D
      S=0.D
      FACTS=1.D
      SIGN=1.D
    1 CALL ZMPY(ZA,ZB,ZMA,ZMB,ZMA,ZMB)
      SIGN=-SIGN
      S=S+1.D
      FACTS=FACTS*S*S
      PSI=PSI+1.D/S
      TERMA=PSI*ZMA/FACTS
      TERMB=PSI*ZMB/FACTS
      SUMA=SUMA+SIGN*TERMA
      SUMB=SUMB+SIGN*TERMB
      IF(DABS(TERMA)+DABS(TERMB)-1.D-13)3,1,1
    3 H01A=AJ0A+SUMB*SIX
      H01B=AJ0B-SUMA*SIX
      H02A=AJ0A-SUMB*SIX
      H02B=AJ0B+SUMA*SIX
      PSI=-.57721566490153286060651 2D
      ZMA=1.D
      ZMB=0.D
      SUMA=PSI+PSI+1.D
      PSI=SUMA
      SUMB=0.D
      S=0.D
      FACTS=1.D
      SIGN=1.D
    5 CALL ZMPY(ZA,ZB,ZMA,ZMB,ZMA,ZMB)
```

```
      SIGN=-SIGN
      S=S+1.D
      FACTS=FACTS*S*(S+1.D)
      PSI=PSI+1.D/S+1.D/(S+1.D)
      TERMA=PSI*ZMA/FACTS
      TERMB=PSI*ZMB/FACTS
      SUMA=SUMA+SIGN*TERMA
      SUMB=SUMB+SIGN*TERMB
      IF(DABS(TERMA)+DABS(TERMB)-1.D-13)7,5,5
    7 CALL ZMPY(ZA,ZB,SUMA,SUMB,SUMA,SUMB)
      SUMA=SUMA+1.D
      CALL ZMPY(X,Y,AJ1A,AJ1B,AJ1A,AJ1B)
      H11A=AJ1A+SIX*SUMB
      H11B=AJ1B-SIX*SUMA
      H12A=AJ1A-SIX*SUMB
      H12B=AJ1B+SIX*SUMA
      ZA=X/2.D
      ZB=Y/2.D
         ALOG=DSQRT(ZA*ZA+ZB*ZB)
          ALOG=DLOG(ALOG)
      BLOG=DATAN2(Y,X)
      FUNNYA=-SIX*BLOG
      FUNNYB=SIX*ALOG
      CALL ZDIV(AJ1A,AJ1B,X,Y,AJ1A,AJ1B)
      CALL ZMPY(FUNNYA,FUNNYB,AJ0A,AJ0B,AJ0A,AJ0B)
      CALL ZMPY(FUNNYA,FUNNYB,AJ1A,AJ1B,AJ1A,AJ1B)
      H01A=H01A+AJ0A
      H01B=H01B+AJ0B
      H02A=H02A-AJ0A
      H02B=H02B-AJ0B
      CALL ZDIV(H11A,H11B,X,Y,H11A,H11B)
      CALL ZDIV(H12A,H12B,X,Y,H12A,H12B)
      H11A=H11A+AJ1A
      H11B=H11B+AJ1B
      H12A=H12A-AJ1A
      H12B=H12B-AJ1B
      RETURN
      END
      SUBROUTINE HLARGE(X,Y,H01A,H01B,H02A,H02B,H11A,H11B,H12A,H12B)
      TYPE DOUBLE X,Y,H01A,H01B,H02A,H02B,H11A,H11B,H12A,
     1H12B,ZA,ZB,RIPI,PI,ABSZ,ARGZ,ABZM,FACT0,FACT1,SUM01A,
     2SUM02A,SUM01B,SUM02B,SUM11A,SUM11B,SUM12A,SUM12B,
     3CXA,CXB,CXNEGA,CXNEGB,ZSA,ZSB,EPS,EPA,PA,PB,QA,QB,
     4ZHALA,ZHALB,EY
      DIMENSION CHI(20),A0(20).A1(20)
         TYPE DOUBLE A0,A1,CHI
      ZA=X
      ZB=Y
      IF(SENSE LIGHT 1)1,3
    1 RIPI=.56418958354775628694807950
      PI=3.14159265358979323846264330
      CALL CHIS(CHI)
      CALL AS(A0,A1)
3        ABSZ=DSQRT(ZA*ZA+ZB*ZB)
```

```
      ARGZ=DATAN2(ZB,ZA)
      IF(DABS(ARGZ)-PI/2.D)29,29,31
   31 IF(ARGZ-PI/2.D)33,29,35
   35 ZA=-ZA
      ARGZ=PI-ARGZ
      M=1
      GO TO 37
   33 ZA=-ZA
      ZB=-ZB
      ARGZ=ARGZ+PI
      M=2
      GO TO 37
   29 M=3
   37 ABZM=1.D
      FACT0=2.D*DEXP(PI/(8.D*ABSZ))
      FACT1=2.D*DEXP((3.D*PI)/(8.D*ABSZ))
      SUM01A=1.D
      SUM01B=0.D
      SUM02A=1.D
      SUM02B=0.D
      SUM11A=1.D
      SUM11B=0.D
      SUM12A=1.D
      SUM12B=0.D
      CXA=1.D
      CXB=0.D
      CXNEGA=1.D
      CXNEGB=0.D
      ZSA=1.D
   13 ZSB=0.D
      EPS=1.D
      DO 5 K=1,20
      EPA=EPS
      ABZM=ABZM*ABSZ
      EPS=FACT0*CHI(K)*DABS(A0(K))/ABZM
      IF(EPS-EPA)19,7,7
   19 CALL ZMPY(ZSA,ZSB,ZA,ZB,ZSA,ZSB)
      CALL ZMPY(CXA,CXB,0.D,1.D,CXA,CXB)
      CALL ZMPY(CXNEGA,CXNEGB,0.D,-1.D,CXNEGA,CXNEGB)
      PA=CXA*A0(K)
   15 PB=CXB*A0(K)
      CALL ZDIV(PA,PB,ZSA,ZSB,QA,QB)
      SUM01A=SUM01A+QA
      SUM01B=SUM01B+QB
      PA=CXNEGA*A0(K)
   17 PB=CXNEGB*A0(K)
      CALL ZDIV(PA,PB,ZSA,ZSB,QA,QB)
      SUM02A=SUM02A+QA
      SUM02B=SUM02B+QB
   18 IF(EPS-1.D-10)7,7,5
    5 CONTINUE
    7 CALL ZMPY(RIPI,-RIPI,SUM01A,SUM01B,H01A,H01B)
      CALL ZMPY(RIPI,RIPI,SUM02A,SUM02B,H02A,H02B)
      CXA=1.D
```

```
      CXB=0.D
      CXNEGA=1.D
      CXNEGB=0.D
      ZSA=1.D
      ZSB=0.D
      ABZM=1.D
      EPS=1.D
      DO 9 K=1,20
      EPA=EPS
      ABZM=ABZM*ABSZ
      EPS=FACT1*CHI(K)*DABS(A1(K))/ABZM
      IF(EPS-EPA)25,11,11
   25 CALL ZMPY(CXA,CXB,0.D,1.D,CXA,CXB)
      CALL ZMPY(CXNEGA,CXNEGB,0.D,-1.D,CXNEGA,CXNEGB)
      CALL ZMPY(ZSA,ZSB,ZA,ZB,ZSA,ZSB)
       PA=CXA*A1(K)
   21 PB=CXB*A1(K)
      CALL ZDIV(PA,PB,ZSA,ZSB,QA,QB)
      SUM11A=SUM11A+QA
      SUM11B=SUM11B+QB
      PA=CXNEGA*A1(K)
   23 PB=CXNEGB*A1(K)
      CALL ZDIV(PA,PB,ZSA,ZSB,QA,QB)
      SUM12A=SUM12A+QA
      SUM12B=SUM12B+QB
   24 IF(EPS-1.D-10)11,11,9
    9 CONTINUE
   11 CALL ZMPY(-RIPI,-RIPI,SUM11A,SUM11B,H11A,H11B)
      CALL ZMPY(-RIPI,RIPI,SUM12A,SUM12B,H12A,H12B)
      ZHALA=DSQRT(ABSZ)*DCOS(.5D*ARGZ)
      ZHALB=DSQRT(ABSZ)*DSIN(.5D*ARGZ)
         EY=DEXP(ZB)
      PA=DCOS(ZA)/EY
         PB=DSIN (ZA)/EY
      CALL ZDIV(PA,PB,ZHALA,ZHALB,QA,QB)
      CALL ZMPY(H01A,H01B,QA,QB,H01A,H01B)
      CALL ZMPY(H11A,H11B,QA,QB,H11A,H11B)
      CALL ZMPY(PA,PB,ZHALA,ZHALB,QA,QB)
      CALL ZDIV(H02A,H02B,QA,QB,H02A,H02B)
      CALL ZDIV(H12A,H12B,QA,QB,H12A,H12B)
      GO TO (39,41,43),M
   39 H02A=2.D*H01A+H02A
      H02B=-2.D*H01B-H02B
      H12A=-2.D*H11A-H12A
      H12B=2.D*H11B+H12B
      H01A=-H01A
      H11B=-H11B
      RETURN
   41 PA=H01A
      PB=H01B
      QA=H11A
      QB=H11B
      H01A=2.D*H01A+H02A
      H01B=2.D*H01B+H02B
```

```
      H11A=-2.D*H11A-H12A
      H11B=-2.D*H11B-H12B
      H02A=-PA
      H02B=-PB
      H12A=QA
      H12B=QB
   43 RETURN
      END
```

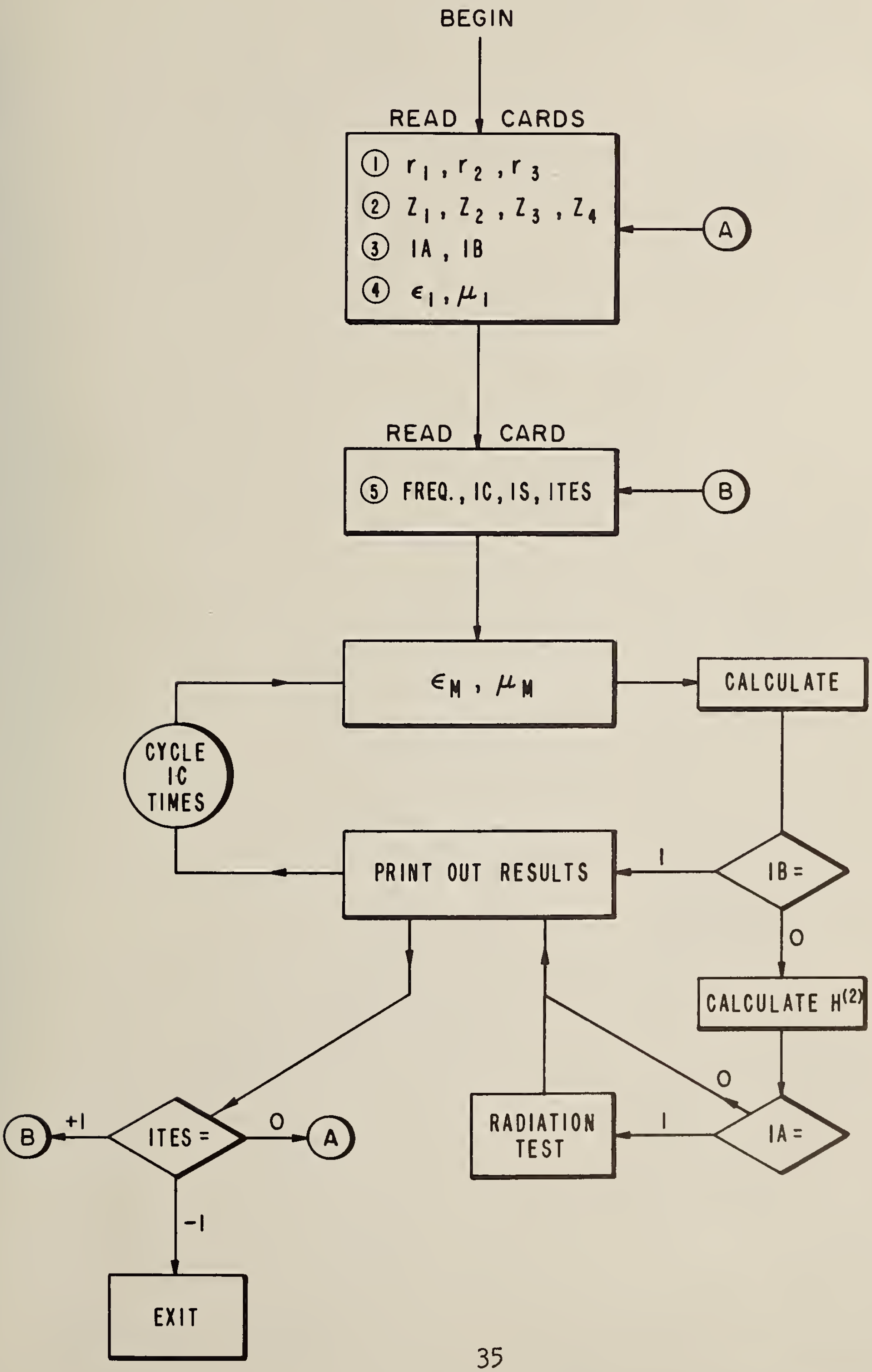
APPENDIX C

SAMPLE CALCULATION

```
ADMITTANCE FOR CAPACITOR WITH TOROID GEOMETRY PAGE   1
THE FREQUENCY IS          1.0000+006HZ.
THE RADIAL DIMENSIONS ARE         1.1176+000        1.2446+000         2.5400+000
THE Z AXIS DIMENSIONS ARE        -6.5550-002       -6.2750-002         6.2750-002        6.5550-002
SPACER E1 IS           2.0000+000       -0.0000+000 U1 IS         1.0000+000        -0.0000+000
THE LENGTH MEASUREMENTS ARE IN CM. ADMITTANCES ARE IN MHOS.  THE REST ARE DIMENSIONLESS.


           MU                     EPSILON              QUASI STATIC ADM        CORRECT ADM            DBL SUM ADM
UM         +I UM*         EM          +I EM*        Y1          +I Y2        YC1         +I YC2        DY1         +I DY2        CNT
NEXT ADMITTANCE FRACTIONAL CHANGE WITH R(3)= 0.9*R(3)  IS    5.8630-008
  2.7050+003 -5.8000+001  1.2060+005 -1.0900+005  3.4586-008  1.8311-004  3.6301-008  1.8312-004  1.5731-009 -1.2497-009   2
```

APPENDIX D
FLOW CHART OF MACHINE PROGRAM



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