NBS

Technical Note No. 335

ZONAL HARMONICS IN LOW FREQUENCY TERRESTRIAL RADIO WAVE PROPAGATION

J. RALPH JOHLER

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

THE NATIONAL BUREAU OF STANDARDS

The National Bureau of Standards is a principal focal point in the Federal Government for assuring maximum application of the physical and engineering sciences to the advancement of technology in industry and commerce. Its responsibilities include development and maintenance of the national standards of measurement, and the provisions of means for making measurements consistent with those standards; determination of physical constants and properties of materials; development of methods for testing materials, mechanisms, and structures, and making such tests as may be necessary, particularly for government agencies; cooperation in the establishment of standard practices for incorporation in codes and specifications; advisory service to government agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; assistance to industry, business, and consumers in the development and acceptance of commercial standards and simplified trade practice recommendations; administration of programs in cooperation with United States business groups and standards organizations for the development of international standards of practice; and maintenance of a clearinghouse for the collection and dissemination of scientific, technical, and engineering information. The scope of the Bureau's activities is suggested in the following listing of its four Institutes and their organizational units.

Institute for Basic Standards. Electricity. Metrology. Heat. Radiation Physics. Mechanics. Applied Mathematics. Atomic Physics. Physical Chemistry. Laboratory Astrophysics.* Radio Standards Laboratory: Radio Standards Physics; Radio Standards Engineering.** Office of Standard Reference Data.

Institute for Materials Research. Analytical Chemistry. Polymers. Metallurgy. Inorganic Materials. Reactor Radiations. Cryogenics.** Office of Standard Reference Materials.

Central Radio Propagation Laboratory.** Ionosphere Research and Propagation. Troposphere and Space Telecommunications. Radio Systems. Upper Atmosphere and Space Physics.

Institute for Applied Technology. Textiles and Apparel Technology Center. Building Research. Industrial Equipment. Information Technology. Performance Test Development. Instrumentation. Transport Systems. Office of Technical Services. Office of Weights and Measures. Office of Engineering Standards. Office of Industrial Services.

---

* NBS Group, Joint Institute for Laboratory Astrophysics at the University of Colorado.

** Located at Boulder, Colorado.

NATIONAL BUREAU OF STANDARDS

Technical Note 335

ISSUED April 13, 1966

ZONAL HARMONICS IN LOW FREQUENCY TERRESTRIAL RADIO WAVE PROPAGATION

J. Ralph Johler
Institute for Telecommunication Sciences and Aeronomy*
Environmental Science Services Administration
Boulder, Colorado

NBS Technical Notes are designed to supplement the Bureau's regular publications program. They provide a means for making available scientific data that are of transient or limited interest. Technical Notes may be listed or referred to in the open literature.

* *Formerly the Central Radio Propagation Laboratory of the National Bureau of Standards; CRPL was transferred to ESSA in October 1965, but will temporarily use various NBS publication series pending inauguration of their ESSA counterparts.*

For sale by the Superintendent of Documents, U. S. Government Printing Office
Washington, D.C. 20402
Price: 25¢

Contents

Zonal Harmonics in Low Frequency Terrestrial Radio Wave Propagation

J. Ralph Johler

If R_n and T_n are ground and ionosphere reflection coefficient matrices and I is the unit matrix, then for the terrestrial waveguide boundaries, a geometric series $\left|I + \sum_{j=0}^{\infty} R_n^j T_n^j\right|$ for each zonal harmonic mode, n, results in a rapidly converging zonal harmonic series. Thus, by an interchange of the order of summation of the two series, the zonal harmonic series can be employed to calculate each term of the geometric series.

This method of analysis provides a practical solution to the propagated LF, VLF, ELF terrestrial waveguide field. Thus, geometric series terms (often called wavehops) can be calculated quickly and accurately with the zonal harmonic series. The summation of the geometric series then provides an efficient method for calculating the total field (or waveguide mode sum). The simplicity of the analysis is an attractive feature when compared with the complex integral method. Computation speed is obtained on the large scale computer by use of recursion formulas for the Legendre and spherical wave functions. It is noted that the terms of the geometric series in the high frequency limit give the geometric-optical rays.

An improved definition of the reflection coefficient matrix is presented which accounts for the reflection process in a manner which can be justified both mathematically and physically. In fact, the entire reflection coefficient matrix for the anisotropic ionosphere can be retained as a variable of the zonal harmonic summation process with resultant improvement in computation accuracy. Demonstrative computations indicate considerable advantage in the method as an alternate approach to the propagation problem.

Key Words: Extra low frequencies, geometric-optics, geometric-series, LF, VLF, ELF mode theory, low frequencies, terrestrial radio wave propagation, very low frequencies, zonal harmonics.

1. Introduction

In a previous paper [Johler, 1964] the notion of a zonal harmonic series for each term of the geometric series representation of the terrestrial radio wave field was introduced. The upper boundary of the terrestrial radio waveguide for low frequency radio waves (Johler and Berry, 1964) can be characterized by the reflection coefficient matrix T,

$$T = \begin{bmatrix} T_{ee}^{(s)} & T_{em}^{(s)} \\ T_{me}^{(s)} & T_{mm}^{(s)} \end{bmatrix}, \tag{1.1}$$

where[1] $T_{ee}^{(s)}$ is the reflection coefficient for vertical polarization (TM-mode) of the incident and reflected waves on the boundary $g = a + h$, figure 1, where a is the radius of the terrestrial sphere, and h is the height of the guide. Similarly, $T_{mm}^{(s)}$ refers to the vertical magnetic incident and reflected waves (TE-mode). Also, $T_{em}^{(s)}$ and $T_{me}^{(s)}$ are abnormal components resulting from the anisotropic nature of the ionosphere. For simplicity, it is instructive to assume the ionosphere to be isotropic, and the excitation TM. Then, $T = T_{ee}^{(s)}$. Also, for the ground,

$$R = \begin{bmatrix} R_{e}^{(s)} & 0 \\ 0 & R_{m}^{(s)} \end{bmatrix}, \tag{1.2}$$

where $R_e^{(s)}$ refers to vertical polarization of electric vector and $R_m^{(s)}$ refers to vertical polarization of the magnetic vector. For the TM excitation of the terrestrial waveguide with isotropic reflection at the ionosphere,

$$R = R_e^{(s)}, \tag{1.3}$$

or dropping the subscript, e,

$$R = R^{(s)}, \tag{1.4}$$

[1] Superscript (s) is here employed to remind the reader that the spherical convergence-divergence or focusing factor has not as yet been removed from the reflection coefficient. See (1.10).

The classical zonal harmonic solution [Johler, 1964], can be written for terminals S and O in figure 1, on the ground, $r = a$, for the particular case of the vertical electric polarization, E_r,

$$E_r = \frac{I_0 \ell}{k_1^2 a^4} \frac{\mu_0 c}{4\pi} \sum_{n=0}^{\infty} n(n+1)(2n+1) P_n(\cos\theta)\, \zeta_{1a}^{(2)} \psi_{1a}$$

$$\times \left\{1 + R_n^{(s)} \frac{-\zeta_{1a}}{\psi_{1a}}\right\}\left\{1 + T_n^{(s)} \frac{-\psi_{1a}}{\zeta_{1a}^{(2)}}\right\}\left\{1 - R_n^{(s)} T_n^{(s)}\right\}^{-1}. \qquad (1.5)$$

$$(n = 0, 1, 2, 3\ldots)$$

Here $I_0 \ell$ is the intensity of the electrical point source dipole, S, figure 1, in ampere-meters which can be for convenience taken as unit, $I_0 \ell = 1$. The permeability of space is μ_0, $\mu_0 = 4\pi(10^{-7})$ henry/meter and c is the speed of light ($c \sim 2.997925\,(10^8)$ m/s). k_1 is the wave number, $k_1 = \frac{\omega}{c}\eta_1$ in air where $\eta_1 \sim 1.0001$ to 1.0003 and the frequency $f = \omega/2\pi$. In vacuum, $\eta_1 = 1$, $k_0 = \frac{\omega}{c}$. The subscript n on T_n and R_n reminds the reader that these quantities are calculated for each n. $P_n(z)$ is the solution of Legendre's differential equation,

$$(1 - z^2)\frac{d^2 P_n(z)}{dz^2} - 2z \frac{dP_n(z)}{dz} + n(n+1) P_n(z) = 0. \qquad (1.6)$$

The abbreviations $\zeta_{1a}^{(1,2)}$ and ψ_{1a} are, letting $k_1 a = z$,

$$\zeta_{1a}^{(1,2)} = \zeta_n^{(1,2)}(z) = \sqrt{\frac{\pi z}{2}}\, H_{n+\frac{1}{2}}^{(1,2)}(z) \qquad (1.7)$$

$$\psi_{1a} = \psi_n(z) = \sqrt{\frac{\pi z}{2}}\, J_{n+\frac{1}{2}}(z) \qquad (1.8)$$

where $J_{n+\frac{1}{2}}(z)$ and $H_{n+\frac{1}{2}}^{(1,2)}(z)$ are Bessel and Hankel functions of order $n+\frac{1}{2}$ and argument z and $H_n^{(1,2)}$ may be of the first or second kind.

The reflection coefficients $R_n^{(s)}$ and $T_n^{(s)}$ contain focusing or convergence-divergence factors. Thus, the spherical reflection coefficients will be designated by R_n and T_n, dropping the superscript (s), or, after substitution in (1. 5) of the exact relationship,

$$\psi_n(z) = \tfrac{1}{2}\left[\zeta_n^{(1)}(z) + \zeta_n^{(2)}(z)\right] \tag{1.9}$$

$$R_n^{(s)} = \frac{-\zeta_{1a}^{(1)}}{\zeta_{1a}^{(2)}} R_n \tag{1.10}$$

$$T_n^{(s)} = \frac{-\zeta_{1g}^{(2)}}{\zeta_{1g}^{(1)}} T_n . \tag{1.11}$$

Here, the spherical reflection coefficients are defined,

$$R_n = \frac{\dfrac{\zeta_{1a}^{(1)\prime}}{\zeta_{1a}^{(1)}} - \dfrac{k_1}{k_2}\dfrac{\zeta_{2a}^{(1)\prime}}{\zeta_{2a}^{(1)}}}{\dfrac{-\zeta_{1a}^{(2)\prime}}{\zeta_{1a}^{(2)}} + \dfrac{k_1}{k_2}\dfrac{\zeta_{2a}^{(1)\prime}}{\zeta_{2a}^{(1)}}}, \tag{1.12}$$

and,

$$T_n = \frac{\dfrac{\zeta_{1g}^{(2)\prime}}{\zeta_{1g}^{(2)}} - \dfrac{k_1}{k_3}\dfrac{\zeta_{3g}^{(2)\prime}}{\zeta_{3g}^{(2)}}}{\dfrac{-\zeta_{1g}^{(1)\prime}}{\zeta_{1g}^{(1)}} + \dfrac{k_1}{k_3}\dfrac{\zeta_{3g}^{(2)}}{\zeta_{3g}^{(2)}}} \tag{1.13}$$

where $\zeta^{(1,2)\prime}(z) = \dfrac{d}{dz}\,\zeta^{(1,2)}(z)$.

The wave number, k_3, is,

$$k_3 = \frac{\omega}{c}\sqrt{1 - i\,\frac{\omega_N^2}{\omega(\nu + i\omega)}} \tag{1.14}$$

for a plasma frequency squared $\omega_N^2 \sim \frac{Ne^2}{\epsilon_0 m}$ where N is the number density of the plasma ($N = N_e$ el/m^3 for an electron plasma and $m = m_e$ the electronic mass) and $\epsilon_0 = \frac{1}{c^2 \mu_0}$. The electron-neutral collision frequency is ν, sec^{-1}. The wave number of the ground, k_2, is given by,

$$k_2 = \frac{\omega}{c} \sqrt{\epsilon_2 - i \frac{\sigma \mu_0 c^2}{\omega}} \tag{1.15}$$

where the permittivity, $\epsilon = \epsilon_0 \epsilon_2$. The conductivity is given by σ, mhos/m.

The geometric series expansion was obtained from the denominator in (1.5),

$$\left[1 - R_n^{(s)} T_n^{(s)}\right]^{-1} = 1 + \sum_{j=1}^{\infty} \left[R_n^{(s)} T_n^{(s)}\right]^j, \tag{1.16}$$

which converges absolutely, if $\left|R_n^{(s)} T_n^{(s)}\right| < 1$.

This basic idea has been employed in integral form in the complex n-plane by Bremmer [1949] and Wait [1961]. Indeed, Wait [1961] suggested a rigorous type of geometric-optics applicable to low frequency propagation. Berry [1964a, b] and Berry and Chrisman [1965] developed elegant computation methods based on this idea using asymptotic formulas for the spherical wave functions (1.6), (1.7), and (1.8) in the complex n-plane. In effect, the method presented in this paper extends the classical geometric-optics [Johler, 1961; 1962] to great distance (> 3000 km). Such an approach is an alternate to the waveguide mode solution described in detail by Wait [1962]. The asymptotic formulas for the spherical wave functions [Berry, 1964a, b] are intricate to program for a large scale computer. On the other hand, along the real axis of the complex n-plane, the spherical wave functions (1.6), (1.7), and (1.8), for integer values of n, can be calculated with great speed and simplicity on a computer with standard recursion formulas. Furthermore, the recursion formulas give an exact rather than an asymptotic solution. This paper, therefore, presents an alternate approach to the calculation of the low frequency fields propagated in the terrestrial waveguide. Finally the solution obtained leads to an improved method for introducing the anisotropic ionosphere reflection coefficient matrix.

2. Theory of Zonal Harmonic and Geometric Series

In the previous paper [Johler, 1964] the propagated field (vertical polarization of the source) was written as the series,

$$E_r = E_{r,0} + \sum_{j=1}^{\infty} E_{r,j}, \tag{2.1}$$

where the ground wave, $E_{r,0}$, can be regarded as the zero order (j = 0) term of the series,

$$E_{r,0} = \frac{\mu_0 c}{8\pi} \frac{I_0 \ell}{k_1^2 a^4} \sum_{n=0}^{\infty} F(n)\, \zeta_{1a}^{(2)} \zeta_{1a}^{(1)} (1 + R_n), \tag{2.2}$$

and a particular ionosphere wave, j,

$$E_{r,j} = \frac{\mu_0 c}{8\pi} \frac{I_0 \ell}{k_1^2 a^4} \sum_{n=0}^{\infty} F(n)\, \zeta_{1a}^{(2)} \zeta_{1a}^{(1)} (1 + R_n)^2$$

$$\times \left[\frac{\zeta_{1a}^{(1)}}{\zeta_{1a}^{(2)}} \cdot \frac{\zeta_{1g}^{(2)}}{\zeta_{1g}^{(1)}}\right]^j \left[R_n\right]^{j-1} \left[T_n\right]^j, \tag{2.3}$$

where $F_n = n(n+1)(2n+1)\, P_n(\cos\theta)$. Here the order of summation of the zonal harmonics and geometric series has been interchanged. This does not impose any serious mathematical restrictions, since the j-series and the n-series are absolutely convergent.

The Legendre function, $P_n(z)$, can be calculated exactly for integral n from the recursion formula,

$$P_{n+1}(z) = \frac{(2n+1)z}{n} P_n(z) - \frac{n-1}{n} P_{n-1}(z). \tag{2.4}$$

where,

$$P_0(z) = 1 \tag{2.5}$$

and,

$$P_1(z) = z. \tag{2.6}$$

Both integer and non-integer values of n can be used in a formula [Bremmer, 1949] based on the saddle point approximation of the integral of Sommerfeld (letter $z = \cos\,\theta$).

$$P_n(\cos\,\theta) \sim \frac{2\cos\left[(n+\frac{1}{2})\,\theta - \frac{\pi}{4}\right]}{\sqrt{2\pi(n+1)\,\sin\,\theta}}\,. \tag{2.7}$$

This formula becomes inaccurate for $\sin\,\theta \sim 0$ and n small. This formula (2.7) was checked with formulas (2.4), (2.5) and (2.6) in a computer subroutine. The agreement was excellent. The recursion formulas were used, however, for the computations illustrated in this paper.

The spherical wave functions can also be calculated from the recursion formulas,

$$\zeta_{n+1}^{(1,2)}(z) = \frac{2n+1}{z}\,\zeta_n^{(1,2)}(z) - \zeta_{n-1}^{(1,2)}(z), \tag{2.8}$$

starting with the values,

$$\zeta_0^{(1,2)}(z) = \mp\, i\,\exp\,[\pm iz] \tag{2.9}$$

$$\zeta_{-1}^{(1,2)}(z) = \exp\,[\pm\, iz]. \tag{2.10}$$

Using a computation technique of Goldsteing and Thayler [1959], a computer program was written to calculate the spherical wave functions (2.8), (2.9), and (2.10). An extensive comparison was made with a program developed by Berry [1964a] using the asymptotic formulas (designed for the complex n-plane) along the real axis of the n-plane.

Excellent agreement was found. The recursion formulas (2. 8), (2. 9), and (2. 10) are of course calculated with much greater speed and accuracy on the computer.

The ratio of the derivative of a spherical wave function to the function can also be determined by recursion [Johler and Berry, 1962],

$$\frac{\zeta_n^{(1,2)'}(z)}{\zeta_n^{(1,2)}(z)} = \frac{1}{\frac{n}{z} - \frac{\zeta_{n-1}^{(1,2)'}(z)}{\zeta_{n-1}^{(1,2)}(z)}} - \frac{n}{z} \tag{2.11}$$

where,

$$\frac{\zeta_0^{(1,2)'}(z)}{\zeta_0^{(1,2)}(z)} = \pm\, i.$$

For complex argument, $z = k_2 a$, the Debye approximation can be employed,

$$\frac{\zeta_n^{(1)'}(k_2 a)}{\zeta_n^{(1)}(k_2 a)} \sim \sqrt{\frac{n(n+1)}{(k_2 a)^2} - 1}. \tag{2.12}$$

This very simple approximation becomes quite excellent since $\sqrt{n(n+1)}$ does not grow as large as $|k_2 a|$, since the series converges at $n \sim k_1 a$. Alternately, the function $\frac{\psi_n'(k_2 a)}{\psi_n(k_2 a)}$ can be readily used in (2. 11) by replacing the $\zeta_n^{(2)}$ and $\zeta_n^{(2)'}$ function with ψ_n and ψ_n' functions. Thus,

$$\frac{\zeta_n^{(1)'}(k_2 a)}{\zeta_n^{(1)}(k_2 a)} \sim \frac{\psi_n'(k_2 a)}{\psi_n(k_2 a)}. \tag{2.13}$$

Equations (2. 2) and (2. 3) can be readily generalized for elevated terminals, with the source S situated at $b = a + h_s$ above the ground and the observer O situated at $r = a + h_0$, instead of the surface $r = b = a$ illustrated in figure 1. Thus, the ground wave can be written, $r > b$,

$$E_{r,0} = \frac{\mu_0 c}{8\pi} \frac{I_0 \ell}{k_1^2 r^2 b^2} \sum_{n=0}^{\infty} F(n) \left[\zeta_{1b}^{(1)} + \zeta_{1b}^{(2)} \frac{\zeta_{1a}^{(1)}}{\zeta_{1a}^{(2)}} R_n \right] \zeta_{1r}^{(2)}, \qquad (2.14)$$

and the ionosphere waves can be written,

$$E_{r,j} = \frac{\mu_0 c}{8\pi} \frac{I_0 \ell}{k_1^2 r^2 b^2} \sum_{n=0}^{\infty} F(n) \left[\zeta_{1b}^{(1)} + \frac{\zeta_{1a}^{(1)}}{\zeta_{1a}^{(2)}} \zeta_{1b}^{(2)} R_n \right]$$

$$\times \left[\zeta_{1r}^{(2)} R_n + \frac{\zeta_{1a}^{(2)}}{\zeta_{1a}^{(1)}} \zeta_{1r}^{(1)} \right] p^j R_n^{j-1} T_n^j. \qquad (2.15)$$

Here the abbreviations, $\zeta_{1r}^{(1,2)} = \zeta_n^{(1,2)}(k_1 r)$; $\zeta_{1b}^{(1,2)} = \zeta_n^{(1,2)}(k_1 b), \ldots,$ have been employed.

The series (2.2) is the zonal harmonics representation of the ground wave and converges quite slowly. Thus, approximately 15 $k_1 a$ terms are required to get a reasonably accurate answer. At 10 kc/s this would be 20,000 terms. Hence, the zonal harmonic series (2.2) should be replaced by the classical ground wave theory. An efficient computer program has been developed for this purpose Johler [1962]. Once the ground wave (2.2) has been removed as a problem, the second term of (2.1) and the higher order terms can be readily evaluated with zonal harmonics. This can be made evident by introducing the Wronskian (Johler and Berry, 1962; page 768).

$$\begin{vmatrix} \zeta_r^{(1)\prime} & \zeta_r^{(2)\prime} \\ \zeta_r^{(1)} & \zeta_r^{(2)} \end{vmatrix} = 2i, \qquad (2.16)$$

into the factor $1 + R_n$ in equation (2.3). Thus,

$$1 + R_n = \frac{2i}{\zeta_{1a}^{(1)} \zeta_{1a}^{(2)} \left[\frac{-\zeta_{1a}^{(2)\prime}}{\zeta_{1a}^{(2)}} + \frac{k_1}{k_2} \frac{\zeta_{2a}^{(1)\prime}}{\zeta_{2a}^{(1)}} \right]} \qquad (2.17)$$

Thus, the $\zeta_{1a}^{(1)}\zeta_{1a}^{(2)}$ in (2. 17) cancels the $\zeta_{1a}^{(1)}\zeta_{1a}^{(2)}$ in series (2. 2), and as n grows large, advantage cannot be taken of the fact that $\zeta_{1a}^{(1)}\zeta_{1a}^{(2)}$ grows large. This is the reason for the notoriously slow convergence of the harmonic series [Johler and Berry, 1962]. On the other hand, the ionospheric waves converge rapidly as $n \to k_1 a$ since the factor $(1+R_n)^2$ replaces $(1+R_n)$ in (2. 3). It is noted that $R_n \to -1$ as n grows large. This would lead to computation problems if (2. 17) were not employed. In any case approximately 1400 terms at 10 kc/s or approximately $k_1 a$ terms required. In fact, 1500 terms gives very high accuracy (> 6 significant figures).

Although this is still a large number of terms, a simple recursion method for calculating each term allows the electronic computer to sum the series as quickly as it could calculate the smaller number of complex terms used in the evaluation of an integral in the complex n-plane. The complete field can then be calculated by a summation of the geometric series (2. 1) which requires only a few terms if the upper boundary is lossy. However, 100 terms or more may be required if the boundaries are not very lossy, but this adds little to the computation time. In (2. 3) the factor,

$$F_c(n) = \frac{\zeta_{1a}^{(1)}}{\zeta_{1a}^{(2)}} \cdot \frac{\zeta_{1g}^{(2)}}{\zeta_{1g}^{(1)}} = \exp(i\varphi) \tag{2. 18}$$

is of unity magnitude since $\zeta_{1a}^{(1)} = \zeta_{1a}^{(2)*}$ for real arguments $k_1 a$ and $k_1 g$. Also, as n becomes large, φ is a phase angle which approaches zero,

$$\lim_{n \to \infty} F_c(n) = 1. \tag{2. 19}$$

The analysis procedure described above which in essence removes the ground wave from the problem, permits the practical use of zonal harmonics techniques to frequencies as great as 300 kc/s. Of course, the linear dependence on $k_1 a$ makes the technique more efficient at 10 kc/s and lower. Below 2 kc/s the zonal harmonics can also be used for the ground wave without any significant increase in computation time on a computer. In the case of elevated terminals (2. 13) and (2. 14) the convergence is improved by the factor $\left(\frac{a}{r}\right)^n$.

Thus, the factor (see (2.14) and (2.15)),

$$\left[\zeta_{1b}^{(1)} + \zeta_{1b}^{(2)} \frac{\zeta_{1a}^{(1)}}{\zeta_{1a}^{(2)}} R_n\right] = [1 + R_n]\, \zeta_{1a}^{(1)} \tag{2.20}$$

if b = a (receiver only elevated). For $n \gg k_1 a$, Johler and Berry [1962] used a more quickly convergent series based on an approximation of Watson [1919], or using (2.17),

$$(1 + R_n)\left(\zeta_{1r}^{(2)} R_n + \frac{\zeta_{1a}^{(2)}}{\zeta_{1a}^{(1)}} \zeta_{1r}^{(1)}\right) \sim \frac{4ik_1 a}{2n+1} \left[\exp(i\varphi) + R_n\right]\left(\frac{a}{r}\right)^n \tag{2.21}$$

where,

$$\exp(i\varphi) = \frac{\zeta_{1r}^{(1)}}{\zeta_{1r}^{(2)}} \frac{\zeta_{1a}^{(2)}}{\zeta_{1a}^{(1)}} .$$

Hence, the terminals S and O on the ground require the greatest number of terms. It is also noted that a large number of cases, say 100 distances, can be calculated simultaneously since the distance factor or Legendre function is merely multiplied by the frequency dependent factor, term-by-term to introduce the distance dependence.

3. Plane Reflection Coefficients

The ionosphere spherical reflection coefficient (1.13) can be replaced by the Fresnel reflection coefficient, which, for vertical polarization can be written,

$$T_{ee} = \frac{\cos\varphi_i - \frac{k_1}{k_3}\sqrt{1 - \left(\frac{k_1}{k_3}\sin\varphi_i\right)^2}}{\cos\varphi_i + \frac{k_1}{k_3}\sqrt{1 - \left(\frac{k_1}{k_3}\sin\varphi_i\right)^2}} \tag{3.1}$$

where [Johler and Berry, 1964]

$$\cos\varphi_i = i\,\frac{\zeta_{1g}^{(2)\prime}}{\zeta_{1g}^{(2)}} \sim \sqrt{1 - \frac{n(n+1)}{(k_1 g)^2}} \tag{3.2}$$

or,

$$\sin \varphi_i \sim \frac{\sqrt{n(n+1)}}{k_1 g} \tag{3.3}$$

with the stipulation that $\sin \varphi_i = 1$ if $\frac{\sqrt{n(n+1)}}{k_1 g} > 1$. This latter stipulation causes only a small loss in computation precision since near $n = k_1 a$ the series converges abruptly (see (4.10) below), and the spherical reflection coefficient reaches a value of -1. The plane reflection coefficient, T_{ee}, exhibits the same asymptotic behavior of $n \to \infty$. The introduction of plane Fresnel reflection coefficients permits use of available reflection coefficients for an anisotropic plane plasma developed by Johler and Harper [1962], Johler [1963], and Wait and Walters [1963]. Fresnel reflection coefficients are not necessary to the isotropic case (3.1) except to demonstrate numerically that the procedure causes only small loss in computation precision. The capability of using Fresnel reflection coefficients is an important consideration in the anisotropic ionosphere, since available reflection coefficients are restricted to the Fresnel type.

4. Inhomogeneous, Anisotropic Ionosphere

The generalization of equation (2.3) to the inhomogeneous, anisotropic ionosphere is similar to methods introduced in previously published work [Johler and Berry, 1964]. A consequence of the presence of the anisotropic ionosphere is the existence of TE (transverse electric) propagation in addition to the TM (transverse magnetic) propagation, notwithstanding the fact that the source is TM. Equation (2.3) can be re-written,

$$E_{r,j} = \frac{\mu_0 c}{8\pi} \frac{I_0 \ell}{k_1^2 a^4} \sum_{n=0}^{\infty} F(n)\, \zeta_{1a}^{(2)} \zeta_{1a}^{(1)} (1 + \mathcal{R}_{e,n})^2 \left\{ \frac{\zeta_{1a}^{(1)}}{\zeta_{1a}^{(2)}} \cdot \frac{\zeta_{1g}^{(2)}}{\zeta_{1g}^{(1)}} \right\} C_j \tag{4.1}$$

Let

$$p = \left\{ \frac{\zeta_{1a}^{(1)}}{\zeta_{1a}^{(2)}} \; \frac{\zeta_{1g}^{(2)}}{\zeta_{1g}^{(1)}} \right\},$$

$$\mathcal{R} = R_{e,n}\, \rho_n$$

$$\mathcal{R}_{e,n} = \begin{bmatrix} R_e & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{T}_n = \begin{bmatrix} T_{ee} & T_{em} \\ T_{me} & T_{mm} \end{bmatrix}$$

$$\rho_n = p \begin{bmatrix} 1 & 0 \\ 0 & -R_m \end{bmatrix}, \quad \mathcal{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then,

$$\left(\rho_n \mathcal{T}_n \mathcal{R}_{e,n}\right)^{j-1} \rho_n \mathcal{T}_n = p^j \begin{bmatrix} C_j & x_j \\ y_j & z_j \end{bmatrix}. \qquad (4.2)$$

Then it can be readily shown that,

$$C_1 = T_{ee}$$

$$C_2 = R_e T_{ee}^2 + R_m T_{em} T_{me}$$

$$C_3 = 2 R_e R_m T_{ee} T_{em} T_{me} + R_e^2 T_{ee}^2 + R_m^2 T_{mm} T_{em} T_{me} \qquad (4.3)$$

. . . .

These results (4.3) were given by Johler [1961]. These results (4.2) and (4.3) can be deduced from equations given by Johler and Berry [1964]--see equations on page 11; (43) and (52) on pages 8 and 9.

The solution for the vertical electric field, E_r, can be written, [Berry, 1964b],

$$E_r \sim \frac{\mu_0 c}{8\pi} \frac{I_0 \ell}{k_1^2 a^4} \sum_{n=0}^{\infty} F(n)\, \zeta_{1a}^{(2)} \zeta_{1a}^{(1)} (1 + R_e) \frac{|\mathcal{I} + \rho_n \mathcal{T}_n|}{|\mathcal{I} - \rho_n \mathcal{R}_{e,n} \mathcal{T}_n|}. \qquad (4.4)$$

The ratio of the determinants in (4.4) can be expanded in a geometric type series by Berry [1964b]; Johler and Berry [1964]--(43) and (52), pages 8 and 9,

$$\frac{|1 + \rho_n T_n|}{|1 - \rho_n R_{e,n} T_n|} = |1 - \rho_n R_{e,n} T_n|^{-1} \, |1 + \rho_n T_n| \tag{4.5}$$

or,

$$\frac{|1 + \rho_n T_n|}{|1 - \rho_n R_{e,n} T_n|} = \left|1 + (1 + R_{e,n}) \sum_{j=1}^{\infty} (\rho_n R_{e,n} T_n)\, \rho_n^{j-1} T_n \right|. \tag{4.6}$$

This, of course, leads to (4. 2). Berry [1964] pre-multiplied $(\rho_n T_n)^j$ by $R_{e,n}^{j-1}$ which would cause the second term of C_2 to read $- R_e R_m T_{em} T_{me}$. Similarly, the higher order C_j's would be altered. Whereas this permits removal of a factor γ_j from the integral (or in this paper the summation), such that $\gamma_j = R_e^{j-1} C_j$, the procedure, although a good approximation, does not seem to be physically real, since the second term is a TE wave and does not require a TM reflection coefficient, R_e, (see figure 2),

The expansion (4. 6) converges absolutely for all n. Thus the expansion of the denominator (1. 16) of the left side of (4. 6) is the geometric series which converges if,

$$|\rho_n R_{e,n} T_n| < 1,$$

which is certainly true for all physical values of $R_{e,n}$, ρ_n, and T_n, if n is not too large. However,

$$\lim_{n \to \infty} \rho_n T_n = -1,$$

$$\lim_{n \to \infty} R_{e,n} = -1,$$

hence (4. 5),

$$\lim_{n \to \infty} |1 + \rho_n T_n| = 0,$$

which multiplies into the geometric series (4. 6). Hence (4. 6) converges absolutely for all values of interest, i. e.

$$|\rho_n \mathcal{R}_{e,n} \mathcal{T}_n| \leq 1.$$

The geometric series representation permits the introduction of local reflection coefficients. This is illustrated by analogy to geometric optics in figure 1. Thus, the first ionosphere wave, j = 1, has a reflection point [1, 1], located in the middle of the propagation path. In the region of this point, it can be conjectured, most of the reflection occurs. This becomes evident from an examination of the angle of incidence, φ_i, on the ionosphere of the spherical wave [Johler and Berry, 1964], determined by

$$\cos\varphi_i = \operatorname{Re} i \frac{\zeta_{1g}^{(2)\prime}}{\zeta_{1g}^{(2}} \sim \operatorname{Re}\sqrt{1 - \frac{n(n+1)}{(k_1 g)^2}} \tag{4. 7}$$

or

$$\sin\varphi_i \sim \frac{\sqrt{n(n+1)}}{k_1 g} .$$

For $\varphi_i \sim 0$, or n small, the contribution of the terms of the harmonic series is small. As $\sin\varphi_i$ approaches unity, the field is determined. Thereafter, the series converges rapidly as a consequence of the behavior of (2. 17) previously discussed. Therefore, within a small range of values near $k_1 g$ the field is primarily determined. Similarly, the local reflection points [2, 1], [2, 2], [3, 1], [3, 2], [3, 3], [4, 1], [4, 2], [4, 3], and [4, 4] can be defined. The anisotropy and non-homogeneity, of course, causes the calculation of C_j to be quite complicated. Thus, instead of (4. 2), use,

$$\prod_{\kappa=2}^{j} (\rho_{n,\kappa} \mathcal{T}_{n,\kappa} \mathcal{R}_{e,n,\kappa})(\rho_{n,1} \mathcal{T}_{n,1}) = \prod_{\kappa=1}^{j} p_\kappa \begin{bmatrix} C_j & x_j \\ y_j & z_j \end{bmatrix} \tag{4. 8}$$

where for j = 1,

$$(\rho_{n,1} \mathcal{T}_{n,1}) = p_1 \begin{bmatrix} C_1 & x_1 \\ y_1 & z_1 \end{bmatrix} .$$

Explicit expression has been given by Johler [1961] which can be written

$$C_1 = T_{ee}(1,1)$$

$$C_2 = T_{ee}(2,1)\,T_{ee}(2,2)\,R_e(2,1) + R_n(2,1)\,T_{em}(2,1)\,T_{me}(2,2)$$

$$\begin{aligned} C_3 = {} & R_e(3,2)\,R_m(3,1)\,T_{ee}(3,3)\,T_{em}(3,1)\,T_{me}(3,2) \\ & + R_e(3,1)\,R_m(3,2)\,T_{ee}(3,1)\,T_{em}(3,2)\,T_{me}(3,3) \\ & + R_e(3,1)\,R_e(3,2)\,T_{ee}(3,1)\,T_{ee}(3,2)\,T_{ee}(3,3) \\ & + R_m(3,1)\,R_m(3,2)\,T_{mm}(3,2)\,T_{em}(3,1)\,T_{me}(3,3) \qquad (4.9) \\ & \cdots \end{aligned}$$

Since higher order C_j's are quite complicated, it is perhaps simpler to calculate C_j from the general matrix formula (4.8). The detailed reflection, figure 1, can also be altered by the introduction of a local reflection height at each point [1,1], [2,1], and [2,2] $\cdots$. This permits an extension of the theory to take account of irregularities of the ionosphere. The limitations of this procedure are not treated in this paper, but the procedure is intuitively evident from the rays depicted in figure 1. It should be emphasized here that the analysis presented here does not depend upon the geometric-optics depicted for tutorial purposes in figure 1. The solution is exact since the waves fill all of the space shown. The significance of the geometric-optical ray limit has been discussed elsewhere, for example, Johler [1964] and Wait [1961].

The error introduced in (4.9) by using available plane (Fresnel) reflection coefficients for real angles of incidence, φ_1, is small compared with errors introduced by lack of knowledge of the ionosphere model, for example, but the error is not mathematically negligible. Some improvement can be obtained by multiplication of (4.8) by a spherical correction factor, CF, based on the spherical reflection coefficient for infinitely conducting ionosphere,

$$CF = -\left(\frac{\zeta_{1g}^{(2)\prime}}{\zeta_{1g}^{(2)}}\right)\left(\frac{\zeta_{1g}^{(1)\prime}}{\zeta_{1g}^{(1)}}\right)^{-1} . \qquad (4.10)$$

These functions are, of course, available from the computer code written for (2.2), and (2.3). However, the best procedure would be the introduction of the complex angle of incidence, φ_1, (see (3.2)) into the calculation of the Fresnel ionosphere reflection coefficients directly. Thus, $\cos\varphi_1$ in the analysis procedure of Johler and Harper [1962] is replaced by the complex number after equality sign in (3.2).

5. Discussion of a Sample Calculation

Figures 2 and 3 illustrate a sample calculation using the zonal harmonic--geometric series double summation at 10 kc/s and 20 kc/s. An entire calculation of this type requires approximately two minutes at 10 kc/s, (or perhaps less with optimization of the computer code) on the electronic computer. The ground wave, j = 0, was calculated with the aid of a standard computer code using the classical ground wave theory. At frequencies less than 2 kc/s, (2.2) becomes efficient, and should be employed. The final answer, $\sum_j$, employed 20 terms. However, graphical accuracy can be obtained with fewer terms. Since $k_1 g \sim 1340$, 1400 terms of the harmonic series were employed in figure 2 and 2800 in figure 3. In practice, it is usually only necessary to form $\sum_{n=1000}^{1400}$ at 10 kc/s and $\sum_{n=2000}^{2800}$ at 20 kc/s. . . , (2.3) since the small n terms ordinarily contribute a negligible amount to the total field. This results in approximately 1/5 reduction in computer time. It is, however, necessary to recurse from n = 0 in (2.8) and (2.11) but (2.7) can in this case be used to replace (2.4). The $\sum_j$ curve and curves j = 0, 1, 2, 3 agree with previous results of Berry [1964b] using the integral form. The number of distances calculated by any summation can be increased to 100 or more without appreciably increasing the computation time, since $P_n(\cos\theta)$ is a distance multiplier for each frequency dependent term which can be calculated with great speed using (2.4) or (2.7). The latter is quite accurate, except at extremely low frequencies where small n is important or at distances, $\sin\theta \sim 0$.

It is of interest to note the characteristic behavior of each term of the geometric series at great distance. Thus, the attenuation with distance approaches a ground wave slope. However, since full wave type solution is employed, this attenuation with distance does differ slightly from the ground wave curve.

The complicated behavior of the particular case, f = 10 kc/s, N = 56 el/cc, j = 4, 5, 6, in the region immediately preceding the diffraction region is of particular interest. The apparent standing wave appears to be a consequence of the fact that a full wave solution has been employed. Thus, the geometric optics solution would yield a simple pseudo-Brewster angle type cusp in the curve. Here, however, a multiplicity of waves exist and would be expected under some circumstance to yield a more complicated behavior near the pseudo-Brewster angle.

6. Conclusions

A new approach to the calculation of LF, VLF, ELF fields has been demonstrated. The solution permits introduction of existing plane, anisotropic reflection coefficients locally as a function of a real or complex angle of incidence. This permits retention of the more exact method of calculation in which the anisotropic ionosphere reflection coefficient matrix is left inside the summation or integration process without an excessive increase in computer time. The method appears to be tractable up to frequencies of approximately 300 kc/s although the efficiency is reduced at LF and MF. The method is very efficient at VLF and ELF. The method also can be developed to introduce local irregularities and local electrical constant changes at both the ground and ionosphere.

The author is indebted to his colleague, R. L. Lewis, for the design of the computer subroutines based on the recursion formulas. The author is also indebted to L. A. Berry for the design of the asymptotic subroutines for the Hankel functions against which the recursion formulas were checked. Finally, the author wishes to note the able assistance of Carlene M. Lilley with many of the details of the zonal harmonic program design.

7. References

Berry, L. A. (June, 1964a), Computation of Hankel functions, NBS Technical Note 216.

Berry, L. A. (Dec. 1964b), Wave hop theory of long distance propagation of LF radio waves, Radio Sci. NBS/USNC-URSI 68D, No. 12, 1275-1282.

Berry, L. A., and M. E. Chrisman (Nov. 1965), The path integrals of LF/VLF wave hop theory, Radio Sci. J. Res. NBS 64D, 1469-1480.

Bremmer, H. (1949), Terrestrial radio waves--theory of propagation (Elsevier Publ. Co., Inc., New York, N. Y.).

Goldstein, M., and R. M. Thayler (1959), Recurrence techniques for the calculation of Bessel functions, MTAC XIII (13), 102-108.

Johler, J. R. (Oct. 1961), On the analysis of LF ionospheric radio propagation phenomena, J. Res. NBS 65D (Radio Prop.), No. 5, 507-529.

Johler, J. R. (Apr. 1962), Propagation of the low-frequency radio signal, Proc. IRE 50, No. 4, 404-427.

Johler, J. R. (1963), Radio wave reflections at a continuously stratified plasma with collisions proportional to energy and arbitrary magnetic induction, Proc. International Conference on the Ionosphere, 1962 (The Institute of Physics and the Physical Society, London), 436-443.

Johler, J. R. (Jan. 1964), Concerning limitations and further corrections to geometric-optical theory for LF, VLF propagation between the ionosphere and the ground, Radio Sci. J. Res. NBS/USNC-URSI 68D, No. 1, 67-78.

Johler, J. R., and L. A. Berry (Sept.-Oct. 1962), Propagation of terrestrial radio waves of long wavelength--theory of zonal harmonics with improved summation techniques, J. Res. NBS 66D (Radio Prop.), No. 6, 737-773.

Johler, J. R., and J. D. Harper, Jr. (Jan.-Feb. 1962), Reflection and transmission of radio waves at a continuously stratified plasma with arbitrary magnetic induction, J. Res. NBS 66D (Radio Prop.), No. 1, 81-99.

Johler, J. R., and L. A. Berry (Oct. 1964), A complete mode sum for LF, VLF, ELF terrestrial radio wave fields, NBS Monograph 78.

Wait, J. R. (June 1961), A diffraction theory of LF sky wave propagation, J. Geophys. Res. 66, No. 6, 1713-1724.

Wait, J. R. (1962), Electromagnetic wave in stratified media (Pergamon Press, Oxford).

Wait, J. R., and L. C. Walters (Sept.-Oct. 1963), Reflection of VLF radio waves from an inhomogeneous ionosphere, J. Res. NBS 62D (Radio Prop.), No. 5, 519-523.

Watson, G. N. (1919), The transmission of electric waves--round the earth, Proc. Roy. Soc. London ACXV (95), p. 546.

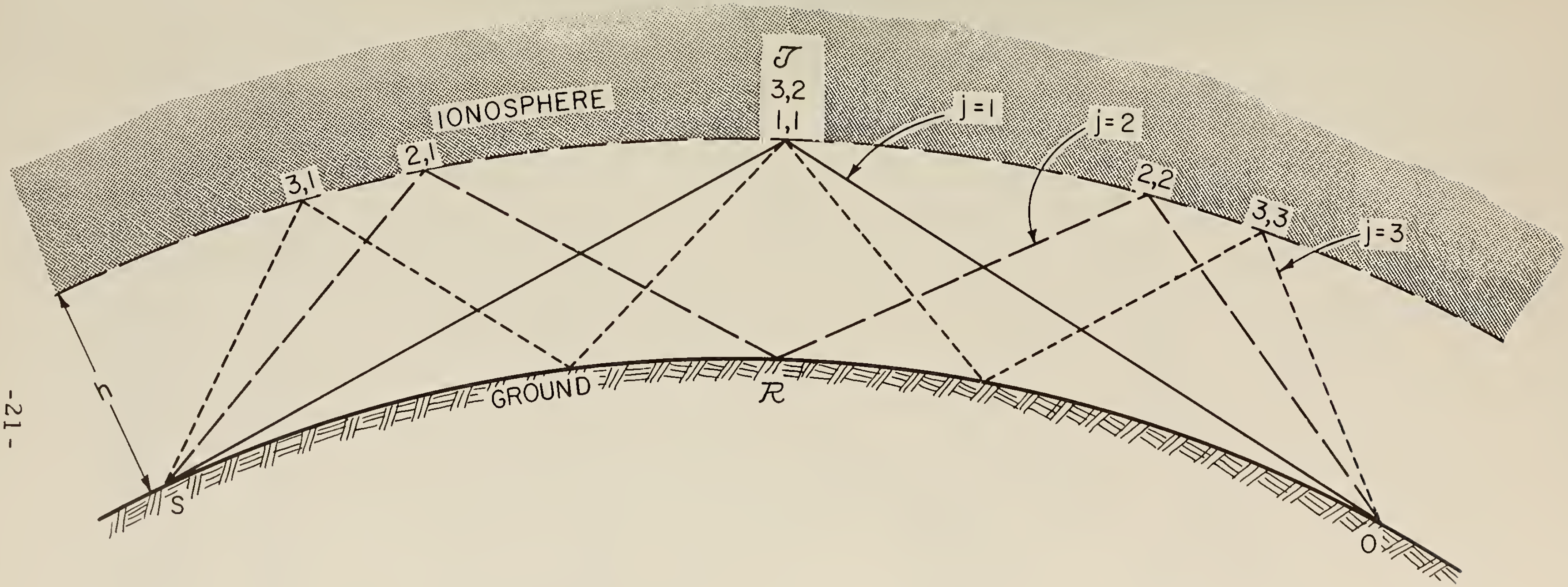

Figure 1. Illustrating geometric-optical rays between the ionosphere and the ground with corresponding regions of reflection for each ray (j, k). j = order of term of geometric series, j = 1, 2, 3 . . . and k is the order of reflection region.

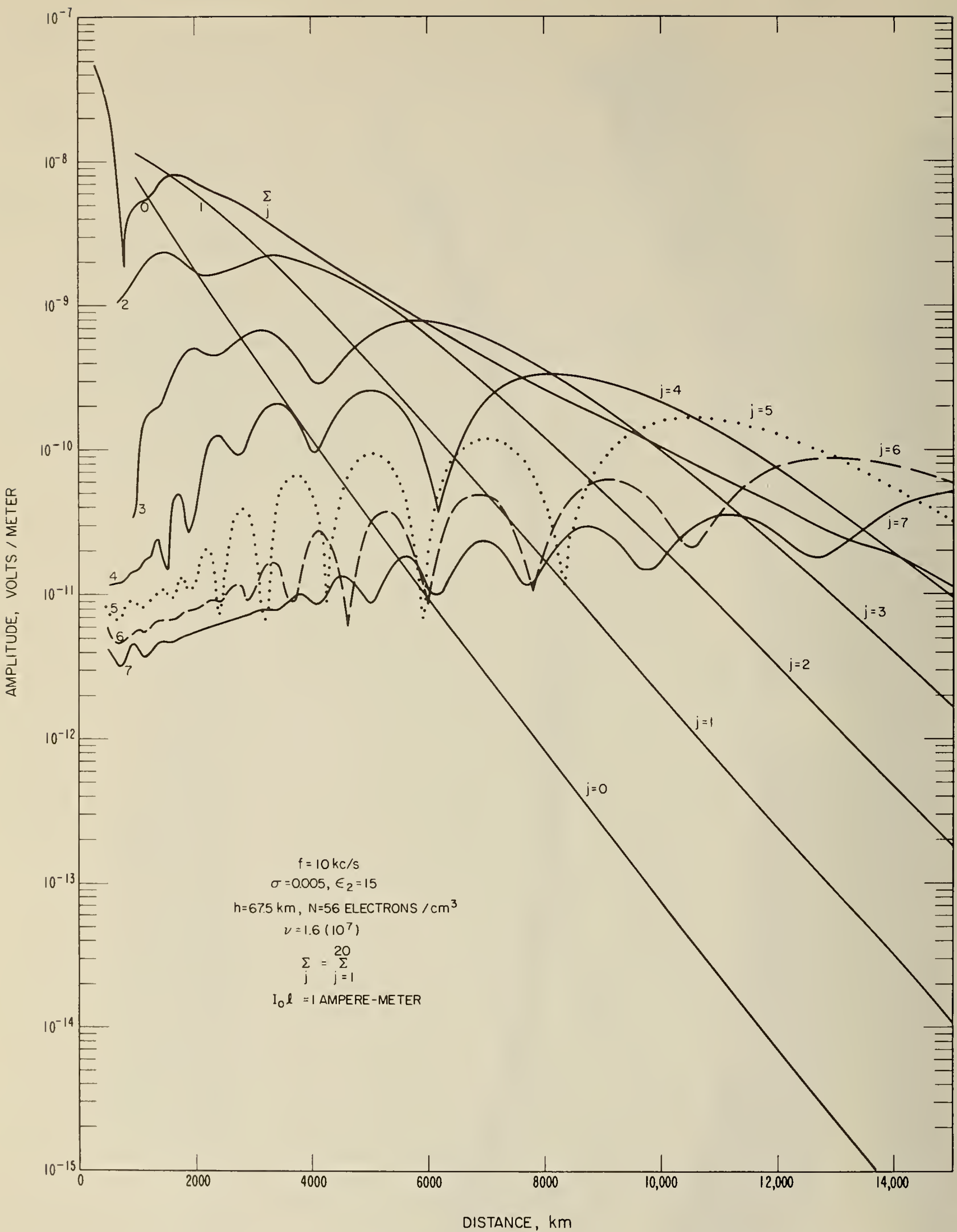

Figure 2. Illustrating components of the rigorous geometric series j = 0, 1, 2, 3... together with complete field, Σ_j, for terrestrial waveguide propagation at 10 kc/s.

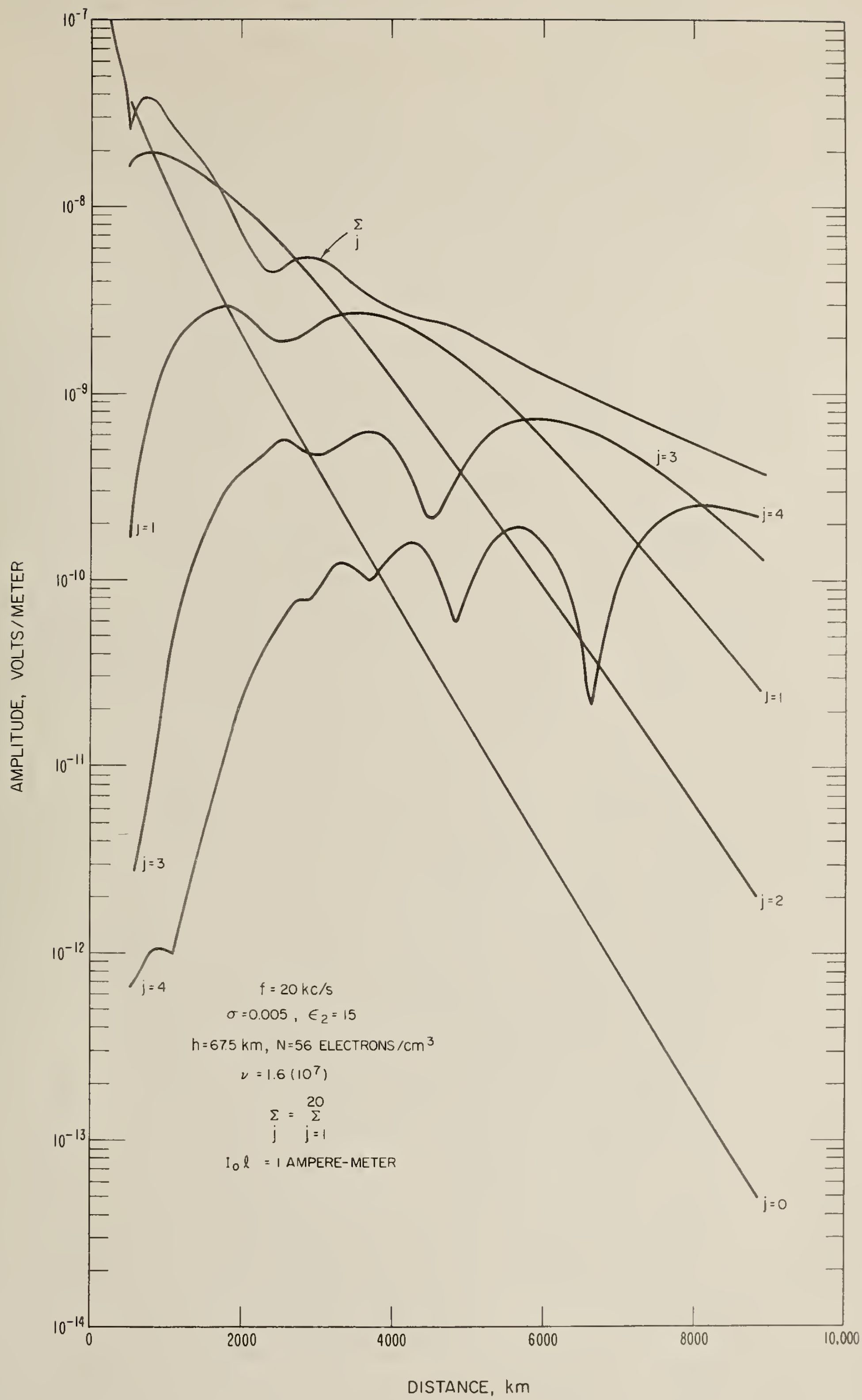

Figure 3. Illustrating components of the rigorous geometric series $j = 0, 1, 2, 3...$ together with complete field, $\sum_j$, for terrestrial waveguide propagation at 20 kc/s.

GPO 857-896

U.S. DEPARTMENT OF COMMERCE
WASHINGTON, D.C. 20230

OFFICIAL BUSINESS

POSTAGE AND FEES PAID
U.S. DEPARTMENT OF COMMERCE