NBS

Technical Note No. 334

AN INTRODUCTION TO SAMPLED DATA AND SWITCHING LOGIC

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NATIONAL BUREAU OF STANDARDS

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ISSUED April 1, 1966

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For sale by the Superintendent of Documents, U.S. Government Printing Office
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Price: 40¢

CONTENTS

AN INTRODUCTION TO SAMPLED DATA AND SWITCHING LOGIC

Thomas L. Davis

This note presents introductory material on the subjects of Fourier series and integral representations of data, operations on data (i.e. correlation, convolution, and sampling), Boolean algebra, and combinatorial logic circuits. The presentation is tutorial and stresses the operations and their effects rather than mathematical rigor.

1. INTRODUCTION

This note contains the material for a series of lectures which were given by the author in November 1962. These lectures were given to prepare the personnel of the Radio Systems Division and the Central Radio Propagation Laboratory for the training program on the use and operation of the Radio Systems Division's analog-to-digital conversion (ADC) system. The training program for the ADC was to be given by the contractor supplying the equipment.

The notes have been augmented to some extent to put them in a form more suitable for publication. They form a source of material for members of the CRPL technical staff and others who are not required to design digital systems, but may wish to know something about digital and sampled data techniques.

The report covers two main subjects. The first includes Fourier series and integral expressions of signals, correlations, description of the sampling process, the sampling theorem, and some precautions which must be exercised when using sampled data. The second major

Key words: switching logic, sampled data, Fourier series, Fourier integrals, correlation, computers

subject includes Boolean algebra, combinational circuits, and minimization methods.

2. FOUNDATIONS OF DATA ANALYSIS

We are generally concerned with three types of functions. These are periodic or recurrent signals, aperiodic or transient signals, and random signals. The periodic and aperiodic functions may be completely described for all time, both past and present, in terms of sin x, cos x, e^x, and other simple mathematical expressions. Random functions do not share this predictability for future time. They are completely determined for all past time, but their future nature can only be predicted within certain limits. We wish to study these types of functions and some fundamental operations with them so that we may see the operations which may be performed by a digital computer using the sampled data and see some of the shortcomings and precautions of data sampling techniques.

2.1. Fourier Series and Integral Expressions

a. Periodic Functions

A recurrent function, f(t), with period T_1, may be represented by the Fourier series expression,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_1 t + b_n \sin n\omega_1 t), \tag{2-1}$$

where

$$\omega_1 = \frac{2\pi}{T_1}, \text{ angular frequency,} \tag{2-2}$$

$$a_n = \frac{2}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f(t) \cos n\omega_1 t \, dt, \qquad n = 0, 1, 2, 3, \ldots, \tag{2-3}$$

$$b_n = \frac{2}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f(t) \sin n\omega_1 t \, dt, \qquad n = 1, 2, 3, \ldots \tag{2-4}$$

There also exists an exponential form,

$$f(t) = \sum_{n=-\infty}^{\infty} F(n) \exp(jn\omega_1 t) \tag{2-5}$$

where $F(n) = \frac{1}{2}(a_n - jb_n), \qquad n = 0, \pm 1, \pm 2, \ldots,$ (2-6)

and the inverse of (2-5) is

$$F(n) = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f(t) \exp(-jn\omega_1 t) \, dt, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{2-7}$$

Equations (2-5) and (2-7) are Fourier Transforms of each other and are known as a Fourier Transform Pair (FTP). F(n) may also be expressed in complex form:

$$F(n) = \frac{1}{2} \sqrt{a_n^2 + b_n^2} \exp\left[j \tan^{-1} \left(- \frac{b_n}{a_n} \right) \right] \tag{2-8}$$

where $|F(n)| = \frac{1}{2} \sqrt{a_n^2 + b_n^2}$ = amplitude spectrum, (2-9)

$$\theta(n) = \tan^{-1} \left(- \frac{b_n}{a_n} \right) = \text{phase spectrum.} \tag{2-10}$$

Example: 2-1

Consider the train of rectangular pulses of width b, amplitude E_m, and period T_1, shown in figure 2-1. Then from (2-7)

$$F(n) = \frac{1}{T_1}\int_0^b E_m \exp\left(-jn\omega_1 t\right) dt$$

$$= \frac{E_m b}{T_1}\left(\frac{\sin n\pi \frac{b}{T_1}}{n\pi \frac{b}{T_1}}\right)\exp\left[-j\omega_1 n\left(\frac{b}{2}\right)\right]. \tag{2-11}$$

If $b = \frac{T_1}{2}$,

$$F(n) = \frac{E_m}{2}\left(\frac{\sin n\frac{\pi}{2}}{n\frac{\pi}{2}}\right)\exp\left[-jn\omega_1\left(\frac{T_1}{4}\right)\right], \tag{2-12}$$

and by (2-5)

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{E_m}{2}\left(\frac{\sin n\frac{\pi}{2}}{n\frac{\pi}{2}}\right)\exp\left[jn\omega_1\left(t - \frac{T_1}{4}\right)\right] \tag{2-13}$$

$$= \frac{E_m}{2} + \frac{2E_m}{\pi}\left[\cos \omega_1\left(t - \frac{T_1}{4}\right) - \frac{1}{3}\cos 3\omega_1\left(t - \frac{T_1}{4}\right)\ldots\right]. \tag{2-14}$$

The spectrum, $F(n)$, is shown in figure 2-2.

b. Aperiodic Functions

We may write a single expression for $f(t)$ from (2-5) and (2-7),

$$f(t) = \sum_{n=-\infty}^{\infty} \exp\left(jn\omega_1 t\right)\frac{1}{T_1}\int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f(\sigma)\exp\left(-jn\omega_1\sigma\right)d\sigma . \tag{2-15}$$

Now let the period of f(t) increase without limit, which is the case for an aperiodic function, and we see that

$$T_1 \rightarrow \infty$$

$$\omega_1 \rightarrow d\omega$$

$$n\omega_1 \rightarrow \omega$$

so (2-15) becomes

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(j\omega t)\, d\omega \int_{-\infty}^{\infty} f(\sigma) \exp(-j\omega\sigma)\, d\sigma \qquad (2\text{-}16)$$

which may be broken down into two expressions:

$$f(t) = \int_{-\infty}^{\infty} F(\omega) \exp(j\omega t)\, d\omega \qquad (2\text{-}17)$$

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \exp(-j\omega t)\, dt. \qquad (2\text{-}18)$$

Note that in (2-17) , F(ω), is the <u>complex</u> <u>continuous</u> <u>spectrum</u> of f(t) and that (2-17) and (2-18) are FTP.

We may also write:

$$F(\omega) = P(\omega) + jQ(\omega) \qquad (2\text{-}19)$$

$$|F(\omega)| = \sqrt{P^2(\omega) + Q^2(\omega)} \qquad (2\text{-}20)$$

$$\theta(\omega) = \tan^{-1} \frac{Q(\omega)}{P(\omega)} \qquad (2\text{-}21)$$

Equation (2-20) is the continuous amplitude spectrum of f(t) and (2-21) is the continuous phase spectrum of f(t).

Consider a rectangular pulse of width b and amplitude E_m (figure 2-3). Then

$$F(\omega) = \frac{1}{2\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} E_m \exp(-j\omega t)\, dt \tag{2-22}$$

$$= \frac{E_m b}{2\pi} \left(\frac{\sin \omega \frac{b}{2}}{\omega \frac{b}{2}} \right). \tag{2-23}$$

The amplitude spectrum, $F(\omega)$, is shown in figure 2-4.

c. Random Functions

One may, under certain conditions, derive a complex spectrum for random noise voltages. The rigorous basis for such transforms is found in the theory of Lebesque measure and integration, a subject far beyond the scope of this paper. A heuristic approach is given in a paper by Bello [1964] to which the interested reader is referred.

2.2. Auto-correlation and Power Spectra

a. Periodic Functions

The auto-correlation function for periodic functions is defined as

$$\phi_{11}(\tau) = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f_1(t)\, f_1(t + \tau)\, dt. \tag{2-24}$$

Define the power spectrum,

$$\Phi_{11}(n) = |F_1(n)|^2 \tag{2-25}$$

where $F_1(n)$ is the spectrum of some periodic function $f_1(t)$.

Then, from (2-24) and (2-25), we may write *

$$\phi_{11}(\tau) = \sum_{n=-\infty}^{\infty} \Phi_{11}(n) \exp(-jn\omega_1 \tau) \tag{2-26}$$

and

$$\Phi_{11}(n) = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} \phi_{11}(\tau) \exp[-jn\omega_1 \tau] \, d\tau \tag{2-27}$$

Note that $\phi_{11}(\tau)$ is even and that (2-26) and (2-27) are FTP.

Example 2-3:

Consider the triangle $\frac{E_m}{b} t$, figure 2-5.

Then

$$f_1(t) = \frac{E_m}{b} t, \quad f_1(t+\tau) = \frac{E_m}{b}(t+\tau), \quad \tau = [0, b] \tag{2-28}$$

and

$$\phi_{11}(\tau) = \frac{1}{T_1} \int_0^{b-\tau} \frac{E_m^2}{b^2} t(t+\tau) \, dt$$

$$= \frac{E_m^2}{6b^2 T_1} \left(\tau^3 - 3b^2 \tau + 2b^3 \right), \quad \tau = [0, b]. \tag{2-29}$$

The autocorrelation function is plotted in figure 2-6 for $b = \frac{T_1}{2}$.

b. Aperiodic Functions

The autocorrelation function for some aperiodic function $f_1(t)$ is defined as

$$\phi_{11}(\tau) = \int_{-\infty}^{\infty} f_1(t) \, f_1(t+\tau) \, dt \tag{2-30}$$

*For a derivation of this and other equations in this section, refer to Chapter 2 of Lee [1960].

and the energy density spectrum as

$$\Phi_{11}(\omega) = 2\pi \left| F_1(\omega) \right|^2 . \tag{2-31}$$

We may now write

$$\phi_{11}(\tau) = \int_{-\infty}^{\infty} \Phi_{11}(\omega) \exp(j\omega\tau)\, d\omega \tag{2-32}$$

and

$$\Phi_{11}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{11}(\tau) \exp(-j\omega\tau)\, d\tau . \tag{2-33}$$

Note that (2-32) and (2-33) are FTP, and, since $\phi_{11}(\tau)$ is an even function, they become

$$\phi_{11}(\tau) = \int_{-\infty}^{\infty} \Phi_{11}(\omega) \cos \omega\tau \, d\omega \tag{2-34}$$

and

$$\Phi_{11}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{11}(\tau) \cos \omega\tau \, d\tau . \tag{2-35}$$

Example 2-4:

Consider the single rectangular pulse of width b, displaced by t_o, as shown in figure 2-7.

$$\phi_{11}(\tau) = \int_{-t_o}^{b-t_o-\tau} E^2_m \, dt = E^2_m (b - |\tau|), \quad \tau = [0, b] . \tag{2-36}$$

Figure 2-8 is a plot of $\phi_{11}(\tau)$.

We speak of power and energy when discussing periodic and aperiodic phenomena because of the ways in which they are defined. The energy in a function is given by $\int_{-\infty}^{\infty} |f(t)|^2 \, dt$ which may be infinite for some functions.

The power is given by $\frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 \, dt$ which will be finite for well behaved periodic functions. The condition that a function be square integrable (i.e. the power or energy is finite) is required for the existence of a Fourier transform.

We may also find the energy density spectrum

$$\Phi_{11}(\omega) = \frac{1}{2\pi} \int_{-b}^{b} E_m^2 (b - |\tau|) \cos \omega\tau \, d\tau \qquad (2\text{-}37)$$

$$= \frac{E_m^2 \, b^2}{2\pi} \left(\frac{\sin \omega \frac{b}{2}}{\omega \frac{b}{2}} \right)^2 \qquad (2\text{-}38)$$

Figure 2-9 demonstrates the shape of $\Phi_{11}(\omega)$.
Note that the initial displacement of the pulse, t_0, does not appear in the autocorrelation function or in the energy density spectrum.

c. Random Functions

We define the basic autocorrelation function for a random function, $f_1(t)$, as

$$\phi_{11}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_1(t) \, f_1(t+\tau) \, dt. \qquad (2\text{-}39)$$

This form is convenient for the analytical or experimental determination of the autocorrelation function of a random variable [Lee, 1960, p.66]. A procedure for this determination could be programmed on a digital computer [Blackman and Tukey, 1959; Ralston and Wilf, 1960]. This form is not very handy, however, for mathematical manipulation so a more convenient form will be introduced.
We state, without proof, the Wiener-Khintchine Theorem for Autocorrelation of Random Functions [Lee, 1960, p. 93].

$$\phi_{11}(\tau) = \int_{-\infty}^{\infty} \Phi_{11}(\omega) \cos\omega\tau \, d\omega \tag{2-40}$$

and

$$\Phi_{11}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{11}(\tau) \cos\omega\tau \, d\tau \tag{2-41}$$

where $\Phi_{11}(\omega)$ is the power density spectrum. Note that (2-40) and (2-41) are FTP.

Example 2-5:

Consider a random waveform of the type shown in figure 2-10 which has the Poisson distribution

$$P_{\xi}(n;\tau) = \frac{(k\tau)^n}{n!} \exp(-k\tau) \tag{2-42}$$

where k = average number of zero crossings per second,

τ = some time interval in seconds,

n = number of crossing in τ.

The autocorrelation function may then be shown to be [Lee, 1960, pp 221-224]

$$\phi_{11}(\tau) = E_m^2 \exp(-2k|\tau|) \tag{2-43}$$

and the power density spectrum will be

$$\Phi_{11}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_m^2 \exp(-2k|\tau|) \cos\omega\tau \, d\tau \tag{2-44}$$

$$= \frac{E_m^2}{\pi} \frac{2k}{(2k)^2 + \omega^2} . \tag{2-45}$$

The autocorrelation function is shown in figure 2-11 and the power density spectrum in figure 2-12. We see from this example how we may determine the power density spectrum of a random signal if we know its probability distribution or if we are able to determine the autocorrelation function by analytical means.

While we are discussing the autocorrelation function and power density spectrum, we should look at the concept of an integrated power spectrum.

Consider an ideal low-pass filter which has the response characteristic shown in figure 2-13. Such a filter is not physically realizable, (see appendix 10.1), but we may consider it theoretically. If we have such a filter whose cut-off frequency is variable, and connect it as shown in figure 2-14, we may obtain a curve of the power consumed in the resistor as a function of the cut-off frequency of the filter. When the filter input is a random function, this curve will have the appearance of figure 2-15. We may now plot the slope of the curve of figure 2-15 and arrive at the dotted curve of figure 2-16. This may be recognized as the power density spectrum defined only for positive frequency. If we divide every ordinate value by two, and plot the negative and positive frequencies, we have $\Phi_{11}(\omega)$, the solid curve of figure 2-16.

We are now in a position to define an integrated power spectrum as the power contained in any power density spectrum up to some frequency, ω

$$S_{11}(\omega) = \int_{-\infty}^{\omega} \Phi_{11}(\Omega)\, d\Omega. \tag{2-46}$$

If we wish to know the power between $\omega = 0$ (d.c.) and some frequency, ω_a,

$$S(\omega_a) = \int_{-\omega_a}^{\omega_a} \Phi_{11}(\omega)\, d\omega. \tag{2-47}$$

Note that the integration must include both positive and negative frequency. The power between two frequencies, ω_a and ω_b, would be

$$S(\omega_{a,b}) = \int_{-\omega_b}^{-\omega_a} \Phi_{11}(\omega)d\omega + \int_{\omega_a}^{\omega_b} \Phi_{11}(\omega)\, d\omega \tag{2-48}$$

$$= 2\int_{\omega_a}^{\omega_b} \Phi_{11}(\omega)\, d\omega. \tag{2-49}$$

Equation (2-49) can be written from (2-48) because $\Phi_{11}(\omega)$ has symmetry about $\omega = 0$. A typical curve for $S_{11}(\omega)$ is shown in Figure 2-17.

Example 2-6:

Consider the power spectrum $\Phi_{22}(n)$ of a periodic wave which is mixed with the power density spectrum, $\Phi_{11}(\omega)$ of a random wave. Such a mixture would have the appearance of figure 2-18. If we now integrate this mixed spectrum, we have the curve of figure 2-19 in which the jumps are caused by the presence of the periodic components in the spectrum and the magnitude of the jump determines the amplitude of that particular component. In this way, we could reconstruct the power spectrum of the original signal but no phase information would be available.

2.3. Crosscorrelation and Convolution

We shall now discuss crosscorrelation and convolution, but in less detail than the Fourier series and the autocorrelation function. This is not to detract from the importance of these functions in data analysis, but they are made up of two functions and are not in general used to represent a single function. An exception to this statement is the convolution between a time function and the unit-impulse. The unit-impulse acts as a scanning function of the original time function and reproduces it if its ordinates are stated only at the time of the unit-impulse. This type of operation will be discussed in more detail in a later section.

a. Crosscorrelation

(1) Periodic Functions. The crosscorrelation function of periodic functions $f_1(t)$ and $f_2(t)$ is defined as

$$\phi_{12}(\tau) = \frac{1}{T_1} \int_{\frac{-T_1}{2}}^{\frac{T_1}{2}} f_1(t)\, f_2(t+\tau)\, dt \tag{2-50}$$

and the cross-power spectrum as

$$\Phi_{12}(n) = \overline{F_1(n)}\, F_2(n) \tag{2-51}$$

where $\overline{F_1(n)}$ denotes the complex conjugate of $F_1(n)$.

The autocorrelation function may be considered as a special case of the crosscorrelation function where $f_1(t)$ and $f_2(t)$ are equal.

From (2-50) and (2-51) we may also write

$$\phi_{12}(\tau) = \sum_{n=-\infty}^{\infty} \Phi_{12}(n) \exp\left(jn\omega_1 \tau\right) \tag{2-52}$$

and

$$\Phi_{12}(n) = \frac{1}{T_1} \int_{\frac{-T_1}{2}}^{\frac{T_1}{2}} \phi_{12}(\tau) \exp\left(-jn\omega_1 \tau\right) d\tau. \tag{2-53}$$

Note that $\phi_{12}(\tau)$ and $\Phi_{12}(n)$ are FTP.

(2) Aperiodic Functions. We define the crosscorrelation function for aperiodic functions $f_1(t)$ and $f_2(t)$ as

$$\phi_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_2\,(t+\tau)\, dt \tag{2-54}$$

and the cross-energy density spectrum as

$$\Phi_{12}\,(\omega) = 2\pi\, \overline{F_1\,(\omega)}\, F_2\,(\omega). \tag{2-55}$$

We may now write

$$\phi_{12}(\tau) = \int_{-\infty}^{\infty} \Phi_{12}\,(\omega)\, \exp\left(j\omega\tau\right) d\omega \tag{2-56}$$

and

$$\Phi_{12}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{12}(\tau)\, \exp\left(-j\omega\tau\right) d\tau \tag{2-57}$$

where $\phi_{12}(\tau)$ and $\Phi_{12}\,(\omega)$ are FTP.

(3) Random Functions. The crosscorrelation function for random functions $f_1\,(t)$ and $f_2\,(t)$ is defined as

$$\phi_{12}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1\,(t)\, f_2\,(t+\tau)\, dt\ . \tag{2-58}$$

We may then define a cross-power density spectrum such that

$$\Phi_{12}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{12}(\tau) \exp\left(-j\omega\tau\right) d\tau \qquad (2\text{-}59)$$

and

$$\phi_{12}(\tau) = \int_{-\infty}^{\infty} \Phi_{12}(\omega) \exp\left(j\omega\tau\right) d\omega \qquad (2\text{-}60)$$

where $\phi_{12}(\tau)$ and $\Phi_{12}(\omega)$ are FTP.

b. Convolution

Convolution is similar to crosscorrelation in that two functions are used, one being scanned by the other. However, in convolution, the scanning function is first folded about the origin, then shifted to perform the scanning operation.

(1) Periodic Functions. We define the convolution of two periodic functions, $f_1(t)$ and $f_2(t)$, as

$$\rho_{12}(\tau) = f_1(t) * f_2(t) = \frac{1}{T_1} \int_{\frac{-T_1}{2}}^{\frac{T_1}{2}} f_1(t)\, f_2(\tau - t)\, dt. \qquad (2\text{-}61)$$

We may also write

$$\rho_{12}(\tau) = \sum_{n=-\infty}^{\infty} F_1(n)\, F_2(n) \exp\left(jn\omega_1 \tau\right) \qquad (2\text{-}62)$$

and

$$F_1(n)\, F_2(n) = \frac{1}{T_1} \int_{\frac{-T_1}{2}}^{\frac{T_1}{2}} \rho_{12}(\tau) \exp\left(-jn\omega_1 \tau\right) d\tau \qquad (2\text{-}63)$$

where $\rho_{12}(\tau)$ and $F_1(n)\, F_2(n)$ are FTP.

(2) Aperiodic Functions. We define the convolution of two aperiodic functions, $f_1(t)$ and $f_2(t)$ as

$$\rho_{12}(\tau) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(t)\, f_2(\tau - t)\, dt. \qquad (2\text{-}64)$$

Again we may write

$$\rho_{12}(\tau) = \int_{-\infty}^{\infty} 2\pi\, F_1(\omega)\, F_2(\omega) \exp\left(j\omega\tau\right) d\omega \qquad (2\text{-}65)$$

and

$$2\pi F_1(\omega)\, F_2(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho_{12}(\tau) \exp\left(-j\omega\tau\right) d\tau \qquad (2\text{-}65)$$

where $\rho_{12}(\tau)$ and $2\pi F_1(\omega)\, F_2(\omega)$ are FTP.

Note the similarity between the convolved power spectra and the cross-power spectra. The only difference between the two is that the complex conjugate is not used in the convoluted power spectrum.

(3) Random Functions. Random functions are not usually convoluted with themselves. However, random functions are used in the convolution integral in two important ways. The output of a linear device (or network) is given by the convolution between the input, $x_i(t)$, and the impulse response of the device, $h(t)$ [Lee, 1960, Chap. 13]

$$x_0(t) = \int_{-\infty}^{\infty} h(\tau)\, x_i(t-\tau)\, d\tau . \qquad (2\text{-}65)$$

In the sampling process (described below in Sect. 3.2) convolution occurs in the frequency domain so that the output of the sampler is given by

$$X_o(\omega) = \int_{-\infty}^{\infty} H(\nu)\, X_i(\omega - \nu)\, d\nu. \qquad (2\text{-}66)$$

These relationships hold equally well for periodic, aperiodic, or random functions.

2.4. Remarks

The purpose of this section has been to demonstrate some of the mathematical tools of the data analyst. Specifically, we have attempted to show how random functions may be treated in terms of power density spectra and how periodic, aperiodic, and random functions may be related. We have not delved too deeply into the physical significance of the various operations for two reasons. First, to do so would require much more time and space than are available and second, such accounts are available in the literature in very lucid form. The reader is specifically referred to three such accounts [Lee, 1960, pp. 4-96; Schwartz, 1959, pp. 18-72; Blackman and Tukey, 1959] which cover the matter in some detail.

Note also that the formulas given here require time records which are infinite in extent. Such records do not exist and could not be used if they did. Therefore, errors due to records of finite length must be considered. Blackman and Tukey [1959] demonstrate techniques to compensate for finite length records and a discussion of errors which might be expected.

3. SAMPLED DATA THEORY

3.1. Introduction

We wish now to investigate some of the properties of data sampling. The requirements for data sampling come about in two ways. Often, in communications work, only a single channel is available for many various data. Some method of sharing the channel by the various data must be devised if they are all to be transmitted. One method, frequency multiplexing, associates each data signal with a carrier, then all the carriers are mixed together in the channel and separated at the receiver. A second method which would allow the data to be transmitted over the common channel is time-division multiplexing. In this method, each datum is sampled for a short time and its value transmitted over the channel. At the receiver, each datum is reconstructed from these samples with the aid of a synchronizing signal to identify the various data. The theory of data sampling is important in this kind of system to determine the number of samples per datum cycle which are required to completely specify the signal [Stiltz, 1961] .

The second type of system which requires information about data sampling involves the use of a high-speed digital computer to process and analyze experimental data which are normally analog in form. The data must be translated from analog to digital form since the computers will only accept digital inputs. An analog-to-digital converter (ADC) which performs this translation has a finite interval between samples as some time is required for the conversion. We shall investigate how this sampling interval affects the maximum frequency of the data under study and some problems which will be encountered in attempting to reconstruct the data from its samples.

3.2. Data Sampling

Consider the single-ended chopper modulator shown in figure 3-1. The chopping signal, $f_c(t)$, has frequency $\omega_c = \frac{2\pi}{T}$, where T is the period, and is shown in figure 3-2. If the spectra of $f_c(t)$ and f(t) are $F_c(n)$ and $F(\omega)$ respectively, and if f(t) is ideally band-limited to some maximum frequency, ω_m, we may write

$$f(t) = \int_{-\omega_m}^{\omega_m} F(\omega) \exp\left(j\omega t\right) d\omega \qquad (3\text{-}1)$$

and

$$f_c(t) = \sum_{n=-\infty}^{\infty} F_c(n) \exp\left(jn\omega_c t\right). \qquad (3\text{-}2)$$

From example 2-1, we may write (3-2) as

$$f_c(t) = \frac{1}{2}\left[1 + 2\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}} \cos n\omega_c t\right]. \qquad (3\text{-}3)$$

Figure 3-3 shows a typical spectrum for f(t) and figure 2-2 shows a spectrum which is typical of $f_c(t)$. Note that we are considering f(t) as aperiodic. A random function will have a power spectrum $|F(\omega)|^2$, of similar shape. A periodic function will have a line spectrum over the same range and the arguments for $F(\omega)$ will apply equally well to any F(n) or $|F(\omega)|^2$ which is limited to ω_m [Linden, 1959].

If we now investigate the output conditions for the circuit of figure 3-1, we see that $f_s(t)$ will be present only when both f(t) and $f_c(t)$ are present. We may, therefore, write

$$f_s(t) = f(t)\, f_c(t). \qquad (3\text{-}4)$$

We also know that multiplication in the time domain corresponds to convolution in the frequency domain [Blackman & Tukey, 1959, pp. 72-73] so that we may write

$$f_s(t) = \frac{1}{2} f(t) \left[1 + 2 \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)} \cos n\omega_c t \right] \tag{3-5}$$

and represent the spectrum of $f_s(t)$, $F_s(\omega)$, obtained by convolution of $F(\omega)$ and $F_c(n)$, as shown in figure 3-4.

Let us now ask: What, if any, is the relationship between ω_c and ω_m? If we examine figure 3-4, we see that a lower limit for ω_c (for a particular ω_m) will be reached at the point where

$$\omega_c - \omega_m = \omega_m . \tag{3-6}$$

If ω_c were to decrease beyond this point, or ω_m to increase, the spectra would overlap and distortion would result. We may, therefore, establish a relationship for ω_m and ω_c such that

$$\omega_c \geq 2\omega_m \tag{3-7}$$

or

$$\omega_m \leq \frac{\omega_c}{2} \tag{3-8}$$

for the single-ended chopper modulator.

Let us now consider, for our chopping signal, a rectangular pulse whose width is small compared to the period, instead of the square wave we used above. Such a pulse train is shown in figure 3-5. We shall describe the functions for this discussion with the additional subscript 1 in order to distinguish them from the preceding discussion. We shall continue to use the circuit of figure 3-1 to illustrate the operation. The input signal, $f_1(t)$, has spectrum $F_1(\omega)$, and the chopping, or sampling signal, $f_{c_1}(t)$ has spectrum $F_{c_1}(n)$. Then, using similar arguments as above, we may write

$$f_1(t) = \int_{-\omega_m}^{\omega_m} F_1(\omega) \exp\left(j\omega t\right) d\omega \tag{3-9}$$

$$f_{c_1}(t) = \sum_{n=-\infty}^{\infty} F_{c_1}(n) \exp\left(jn\omega_c t\right)$$

$$= \frac{\tau}{T}\left[1 + 2\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi\tau}{T}\right)}{\frac{n\pi\tau}{T}} \cos n\omega_c t\right]$$

$$= d\left[1 + 2\sum_{n=1}^{\infty} \frac{\sin(n\pi d)}{n\pi d} \cos n\omega_c t\right] \tag{3-10}$$

where $d = \tau/T$, the duty cycle.

If $f_{s_1}(t)$ has spectrum $F_{s_1}(\omega)$, we may write

$$F_{s1}(\omega) = F_1(\omega) * F_{c_1}(n)$$

and

$$f_{s_1}(t) = f_1(t)\, f_{c1}(t)$$

$$= f_1(t)\, d\left[1 + 2\sum_{n=1}^{\infty} \frac{\sin(n\pi d)}{(n\pi d)} \cos n\omega_c t\right] \tag{3-11}$$

by the argument above concerning convolution.

The spectrum, $F_{s1}(\omega)$, obtained by convolution of $F_1(\omega)$ and $F_{c_1}(n)$ of the sample signal, $f_{c_1}(t)$, is shown in figure 3-6. Note the similarity between this spectrum and the one of figure 3-4.

If we now make the same comparison of ω_m and ω_c which was made for the square-wave chopping signal, we again have the relationship

$$\omega_c \geq 2\omega_m . \tag{3-12}$$

3.3. The Sampling Theorem

We state here, without proof (see appendix 10.2), the sampling theorem in the form given by Shannon [1949] :

"If a function f(t) contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced $\frac{1}{2W}$ seconds apart."

Stiltz [1961, p 86] gives the theorem in essentially the same form, except that the phrase"..., the series extending throughout the time domain" is added to the end of Shannon's statement.

Equation (3-12) provides a heuristic proof of the sampling theorem. However, there are some practical shortcomings of the theorem which make it difficult to apply exactly to a physical problem. The first of these shortcomings is the requirement for an ideal band-limited spectrum. We have mentioned in a previous discussion that such a spectrum cannot be obtained with physically realizable circuits (see appendix 10.1 for proof). Secondly, as indicated in the statement of the theorem by Stiltz, in order to completely recover all phase and amplitude information of the original signal, the samples must extend throughout all time, from negative to positive infinity. This is obviously not practical.

Since we cannot apply the sampling theorem exactly to the analysis of data, we should study the errors which are introduced, principally by the non-zero spectrum above $\frac{\omega_c}{2}$.

3.4. Recovery of Sampled Data

The recovery of data from the sampled form is often required where a digital communication link has been used, either for communication of data between two separated points, i.e., telemetry, or in a closed-loop control system. We shall describe a method, which while not physically realizable, does give some insight into the problem of data reconstruction.

Consider the signal whose spectrum is shown in figure 3-6. We wish to recover only that portion of the spectrum between $-\frac{\omega_c}{2}$ and $+\frac{\omega_c}{2}$. If any frequency components outside of this range are included, distortion (error) will result. We must, therefore, use an ideal low-pass filter to select only this portion of the spectrum.

Let us next consider the Fourier transform of $F_{s1}(\omega)$, $f_{s1}(t)$. If we assume that d is very small, we may say that $f_{s1}(t)$ is a train of amplitude modulated unit impulses. It is shown in appendix 10.1 that the response of an ideal low-pass filter to a unit impulse is given by

$$h(t) = \frac{\omega_m}{2\pi}\left(\frac{\sin \omega_m t}{\omega_m t}\right) \tag{3-13}$$

where ω_m is the cutoff frequency of the filter. If, instead of $\delta(t)$, the unit impulses, we use $f_{s1}(t)$, our sampled time function, it may be shown that

$$f_1(t) = \sum_{n=-\infty}^{\infty} f_{s1}(nT)\, h(t-nT) \tag{3-14}$$

where T is the sampling period [Susskind, 1957]. Each sample of the original function, occurring at a multiple of the sampling period, excites a $\frac{\sin \omega_c t}{\omega_c t}$ response from the ideal low-pass filter. The sum of all these individual outputs is the original function, $f_1(t)$.

3.5. Aliasing

a. Description of Aliasing

We see from figure 3-6 that when ω_m exceeds $\frac{\omega_c}{2}$, the two portions of the spectrum centered at zero and ω_c will overlap to some extent depending on the ratio of ω_c to ω_m. If a line is drawn at $\omega_c/2$, we see that, in effect, the spectrum of f(t) is folded about $\omega_c/2$, sometimes called the <u>Nyquist frequency</u> after H. Nyquist [1928]. This folding of the spectrum about $\omega_c/2$ "aliases" a frequency above $\frac{\omega_c}{2}$ to a frequency below $\frac{\omega_c}{2}$. For example, consider a sampling frequency of $\omega_c = 10$ k rad/sec. Then $\frac{\omega_c}{2} = 5$ k rad/sec. If we now sample a signal which contains a frequency component at say 6k rad/sec, then, because of the folding of the spectrum about $\frac{\omega_c}{2}$, this component is "aliased" to 4k rad/sec. If our input signal had a component at 4k rad/sec. already, the original and aliased signals would be added together and there would be no way to separate them.

b. Errors due to Aliasing

Since we cannot construct an ideal low-pass filter to limit the input spectrum to ω_m, let us investigate the errors which are introduced when a realizable filter is used and aliasing occurs.

Consider a filter with maximally flat amplitude response [Martin, 1955, pp. 184-191] such that

$$|H(\omega)|^2 = \frac{A}{1 + \left(\frac{\omega}{\omega_0}\right)^{2m}} \tag{3-15}$$

where

$H(\omega)$ = filter response

A = low frequency power gain (0 dB)

ω_0 = 3 dB frequency

m = roll-off factor

(i.e. for m = 1, roll-off = 6 dB/octave

= 2, roll-off = 12 dB/octave

etc.).

The response of such a filter is shown in figure 3-7. The shaded area to the right of $\frac{\omega_c}{2}$ represents the aliasing error power, that is, the power which will be aliased to some frequency below $\frac{\omega_c}{2}$ and will be indistinguishable from the spectral components already present there.

If $\frac{\omega_c}{2}$, the Nyquist frequency, is sufficiently in excess of ω_0, the cut-off frequency, we may, with negligible error, represent the filter, as for $\omega > \frac{\omega_c}{2}$, as

$$|H(\omega)|^2 \cong A\left(\frac{\omega_o}{\omega}\right)^{2m} . \qquad (3\text{-}16)$$

We may now find the amount of power above $\frac{\omega_c}{2}$, that is, the aliasing error power, which is given by (see 2-46)

$$S_a(\omega) = \int_{\frac{\omega_c}{2}}^{\infty} |H(\omega)|^2 \, d\omega = A \int_{\frac{\omega_c}{2}}^{\infty} \left(\frac{\omega_0}{\omega}\right)^{2m} d\omega$$

$$= A \frac{2^{2m-1}}{2m-1} \omega_0 \left(\frac{\omega_0}{\omega_c}\right)^{2m-1} . \qquad (3\text{-}17)$$

We may also find the total signal power

$$S_s(\omega) = \int_0^\infty |H(\omega)|^2 d\omega = A \int_0^\infty \frac{d\omega}{1+\left(\frac{\omega}{\omega_0}\right)^{2m}}$$

$$= \frac{\pi A\omega_0}{2m} \csc\left(\frac{\pi}{2m}\right). \qquad (3\text{-}18)$$

The relative rms error will now be given by the square root of the ratio of (3-15) to (3-16),

$$V_\varepsilon = \sqrt{\frac{S_a(\omega)}{S_s(\omega)}} = 2^m \left[\frac{m}{\pi(2m-1)}\left(\frac{\omega_0}{\omega_c}\right)^{2m-1} \sin\left(\frac{\pi}{2m}\right)\right]^{\frac{1}{2}}. \qquad (3\text{-}19)$$

This relative error is plotted for several values of m in figure 3-8 [Stiltz, 1961].

4. BINARY NUMBERS

4.1. Representation of Decimal and Octal Numbers

If we consider r as the radix of some numbering system, then any number $(x)_r$ may be represented

$$(x)_r = a_n a_{n-1} \ldots a_2 a_1 a_0$$

$$= a_n r^n + a_{n-1} r^{n-1} + \ldots + a_2 r^2 + a_1 r^1 + a_0 r^0 \qquad (4\text{-}1)$$

where: a_n = multiplier or weighting factor.

For example, the decimal number $(8271)_{10}$ may be represented as

$$8271 = 8 \times 10^3 + 2 \times 10^2 + 7 \times 10^1 + 1 \times 10^0 \tag{4-2}$$

where $r = 10$,

or similarly: $8000 + 200 + 70 + 1 = (8271)_{10}$. (4-3)

This is the familiar decimal number system. Each weighting factor may have as many values, or levels, as the radix. The radix exponent is determined by its position in the number.

If we now let $r = 8$, the octal base, we see that the weighting factors may only have the values 0 through 7. If we count in this scale, the successive numbers will be 0, 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, 22, etc. and any number $(x)_8$ will be represented as

$$(x)_8 = a_n 8^n + a_{n-1} 8^{n-1} + \ldots + a_2 8^2 + a_1 8^1 + a_0 8^0 \tag{4-4}$$

For example:

$$(100)_8 = 1 \times 8^2 + 0 \times 8^1 + 0 \times 8^0 = 64 + 0 + 0 = (64)_{10}. \tag{4-5}$$

If we let $r = 2$, as in the binary number system, then each weighting function, a_n, may take on only one of two values, 0 and 1. Again we may count in this system

0, 1, 10, 11, 100, 101, 110, 111, 1000, etc.,

and any number $(x)_2$ will be represented as

$$(x)_2 = a_n 2^n + a_{n-1} 2^{n-1} + \ldots + a_2 2^2 + a_1 2^1 + a_0 2^0. \tag{4-6}$$

For example:

$$(101000)_2 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$$

$$= 32 + 0 + 8 + 0 + 0 + 0 = (40)_{10} . \tag{4-7}$$

Note that if we consider

$$[(101)_2 (000)_2]_8 = [50]_8 = 5 \times 8^1 + 0 \times 8^0 = (40)_{10} . \tag{4-8}$$

We may convert $(40)_{10}$ into binary notation by a series of repeated subtractions.

So far we have considered only the direct conversion of a decimal number to a binary number. There is, however, an intermediate relationship where each decimal digit may be represented by a combination of 4 binary digits (bits). This is called the Binary-Coded-Decimal (BCD) form.

TABLE 4-1

Binary-Coded Decimal Representation of Decimal Numbers

	8421
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

The number $(8271)_{10}$ might be represented as

$$(8271)_{10} = (1000)\ (0010)\ (0111)\ (0001). \tag{4-9}$$

Many other combinations of BCD codes are possible such as 2421, 4221, etc. [see Phister, p. 244].

4.2. Binary Decision Elements

Let us now consider, instead of numbers, statements which are either true or false. For instance, the statement A may be either true or false, but not both. We may build other statements upon basic statements. The outcome would depend upon the conditions of the basic statement. For convenience, we shall represent a true statement by the number one (1) and a false statement by the number zero (0).

$$\begin{aligned} &A \text{ is true}, \quad A = 1 \\ &A \text{ is false}, \quad A = 0 \end{aligned} \tag{4-10}$$

We now introduce three combination statements:

(1) F_1 will be true when either A is true or B is true or if both are true, and false otherwise. We represent this as

$$F_1 = A \text{ or } B = A + B. \tag{4-11}$$

F_1 is called the "or" function (also called "inclusive or"). The circuit which performs this function is called the "or circuit or the "or" gate. A logic symbol for such a circuit is shown in figure 4-1.

We may make a "table of combinations" or a "truth table" for the "or" gate.

TABLE 4-2

Truth Table for "or" Circuit

A	B	F_1
0	0	0
0	1	1
1	0	1
1	1	1

(2) F_2 will be true only if both A and B are true, and false otherwise,

$$F_2 = A \text{ and } B = A \times B = AB. \qquad (4\text{-}12)$$

F_2 is called the "and" function and is performed by an "and" gate, shown symbolically in figure 4-2.

TABLE 4-3

Truth Table for "and" Circuit

A	B	F_2
0	0	0
0	1	0
1	0	0
1	1	1

(3) F_3 will be true only when A is false and false when A is true,

$$F_3 = A \text{ false} = \text{not } A = \overline{A} = A'. \qquad (4\text{-}13)$$

F_3 is the "not" function and is performed by the "not" or "inverter" gate (figure 4-3).

TABLE 4-4

Truth Table for "not" Circuit

A	F_3
0	1
1	0

5. SWITCHING LOGIC I

We shall now study an algebra which will lend more rigor and elegance to the statements introduced above. This is the Boolean algebra, named after the 19th century Irish mathematician, George Boole.

5.1. Boolean Algebra

The postulates given here follow those suggested by Huntington [1904]. Note that the symbol ϵ signifies "member of" or "belongs to."

Consider a class of elements, K, and two rules of combination " + " and " x." Also, assume the equality " = " so that a statement may be replaced by another which is equal to it. The postulates, which shall form a basis for our algebra, are:

P. 1: If $A \epsilon K$ and $B \epsilon K$, then $(A+B) \epsilon K$.

P. 2: If $A \epsilon K$ and $B \epsilon K$, then $(AxB) = (AB) \epsilon K$.

P. 3: There is an element 0 such that $A+0 = A$ for $A \epsilon K$.

P. 4: There is an element 1 such that $Ax1 = A$ for $A \epsilon K$.

P. 5: Whenever A and B belong to K, $(A+B) = (B+A)$.

P. 6: Whenever A and B belong to K, $AB = BA$.

P. 7: Whenever A, B, and C belong to K, $A+BC = (A+B)(A+C)$.

P. 8: Whenever A, B, and C belong to K, $A(B+C) = AB + AC$.

P. 9: If the 0 and 1 of postulates 3 and 4 are unique, then there is an element $\overline{A}$ such that $A\overline{A} = 0$ and $A+\overline{A} = 1$.

P. 10: There are at least two elements, X and Y, which belong to K such that $X \neq Y$.

Note that the postulate 7 gives rise to the idempotent relations

$$AA = A$$

$$A + A = A. \tag{5-1}$$

Using these 10 postulates, we could proceed to develop a series of identities for our algebra which would allow us to specify functions and reduce them to their simplest form. Before doing so, however, let us introduce an additional aid, geometric in nature, which will make the job a little simpler.

Consider the class of mammals. Human beings are a sub-group of this class. If we represent the class of mammals by an area in a diagram, the sub-class of humans may be represented as a smaller area contained in the larger. This is demonstrated in figure 5-1. Such a geometric interpretation of classes and sub-classes is called a Venn diagram. Note that the area is not necessarily an absolute representation of the size of the class.

We may now consider any universal class K, and two sub-classes, or sub-sets, A and B. They may be represented as shown in the Venn diagram of figure 5-2. The four areas which are labeled with two literals (i. e., letters) represent the intersections of A, $\bar{A}$, B, and $\bar{B}$. For instance, the area $A\bar{B}$ is that area of A which is not in B.

We shall now introduce and prove a few theorems which will complete our algebra.

Th. 1. DeMorgan's Theorem:

$$\overline{(XY)} = \bar{X}+\bar{Y}, \ \overline{(X+Y)} = \bar{X}\bar{Y} \tag{5-2}$$

This may be easily proven by use of a Venn diagram as shown in figure 5-3a. The complement ("not") of $\overline{XY}$ is shaded in figure 5-3a. The shaded areas in figure 5-3b are $\bar{X}$ and $\bar{Y}$. In each case, the area with no shading is XY and the area with any shading is $\overline{XY}$. Therefore, $\overline{XY} = \bar{X}+\bar{Y}$.

Th. 2. Absorption Theorem:

$$X+XY = X, \ X(X+Y) = X . \tag{5-3}$$

This theorem may be proven using the idempotent relationship, (5-1), and postulate 9.

$$X+XY = X \times 1 + X \times Y = X(1+Y) = X \times 1 = X$$

and $$X(X+Y) = XX+XY = X+XY.$$

Th. 3. Redundant Literal Theorem:

$$(X+\overline{Y})Y = XY, \; X\overline{Y}+Y = X+Y. \qquad (5\text{-}4)$$

In each of these expressions, it may be shown that the $\overline{Y}$ literal is not required.

$$(X+\overline{Y})Y = XY + Y\overline{Y} = XY \text{ (note: } Y\overline{Y} = 0)$$

and $$X\overline{Y}+Y = X\overline{Y}+Y + Y\overline{Y} + XY = (X+Y)(Y+\overline{Y}) = X+Y. \qquad (5\text{-}5)$$

Th. 4. Redundant Term Theorem:

$$XY + YZ + \overline{X}Z = XY + \overline{X}Z \qquad (5\text{-}6)$$

A truth table will be used to prove this theorem.

TABLE 5-1

Truth Table for Proof of Theorem 4

X	Y	Z	F_1	F_2
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	0	0
1	1	0	1	1
1	1	1	1	1

where $$F_1 = XY + YZ + \overline{X}Z, \; F_2 = XY + \overline{X}Z. \qquad (5\text{-}7)$$

Since F_1 and F_2 are equal, the theorem is proved.

5.2. NOR and NAND LOGIC

In section 4.2 above, we introduced three types of binary decision elements, the "and" gate, the "or" gate and the "not" gate. We shall now demonstrate two logic schemes which employ only two of these gates. These are the NOR logic which uses the "or" and "not" gates and the NAND logic which uses the "and" and "not" gates. In order to make this demonstration, we must use de Morgan's theorem

$$\overline{XY} = \overline{X} + \overline{Y} \text{ and } \overline{X + Y} = \overline{X}\,\overline{Y}.$$

The second form is the basis for the NOR logic system. To show this we consider some function F_1, take its complement, replace product (logical "and") terms according to de Morgan's theorem, and recomplement to obtain the original function using only "or" and "not" gates.

$$F_1 = (a+b+c)\ (x+y+z)$$

$$\overline{F}_1 = \overline{a}\ \overline{b}\ \overline{c} + \overline{x}\ \overline{y}\ \overline{z}. \tag{5-8}$$

By de Morgan's theorem,

$$\overline{a}\ \overline{b}\ \overline{c} = \overline{a+b+c}$$

$$\overline{x}\ \overline{y}\ \overline{z} = \overline{x+y+z}. \tag{5-9}$$

Substituting (5-9) into (5-8) we get

$$\overline{F}_1 = \overline{a+b+c} + \overline{x+y=z} \tag{5-10}$$

and

$$F_1 = \overline{\overline{F}}_1 = \overline{\overline{a+b+c} + \overline{x+y+z}}. \tag{5-11}$$

We have thus implemented a Boolean expression using only "or" and "not" gates. No "and" gates are required.

The first form of (5-2), $\overline{xy} = \overline{x} + \overline{y}$, is the basis for NAND logic.

We shall demonstrate the NAND logic in a manner similar to that used for NOR logic.

$$F_2 = abc + xyz$$

$$\overline{F}_2 = (\bar{a}+\bar{b}+\bar{c})\,(\bar{x}+\bar{y}+\bar{z})\ . \qquad (5\text{-}12)$$

By de Morgan's theorem,

$$\bar{a} + \bar{b} + \bar{c} = \overline{abc}$$

$$\bar{x} + \bar{y} + \bar{z} = \overline{xyz}. \qquad (5\text{-}13)$$

Substituting (5-13) into (5-12) we get

$$\overline{F}_2 = \overline{(abc)}\,\overline{(xyz)} \qquad (5\text{-}14)$$

and

$$F_2 = \overline{\overline{F}}_2 = \overline{\overline{(abc)}\,\overline{(xyz)}}. \qquad (5\text{-}15)$$

No "or" gates are required.

6. SWITCHING LOGIC II

We shall now introduce a standard, or canonical, form for Boolean functions of several variables. The standard form is required for some of the minimization methods to be discussed later.

We may express a function of n variables in one of two standard forms, as a sum of products, or minterms, or as a product of sums, or maxterms. The minterms and maxterms derive their names from the fact that they are respectively the minimum and maximum areas which may be defined on a Venn diagram. For instance, referring to the Venn diagram of figure 5-2, the four areas labeled there with two literals represent the four minterms of two variables, A and B. The maxterms on the same diagram would be $(A+B)$, $(A+\overline{B})$, and $(\overline{A}+\overline{B})$. Let us demonstrate the use of minterms and maxterms by defining a function $f = A$.

In general, a minterm, m_i, is represented by

$$m_i = (x_1 x_2 \ldots x_n), \qquad (6\text{-}1)$$

where n is the number of variables and the x_i may or may not be complemented. A function represented by minterms would have the form

$$f = \sum m_i . \qquad (6\text{-}2)$$

It can be shown that

$$\sum_{i=0}^{2^n} m_i = 1. \qquad (6\text{-}3)$$

If we wish now to find A in minterm form, from figure 5-2 we see that

$$AB + A\overline{B} = A(B+\overline{B}) = A \times 1 = A. \qquad (6\text{-}4)$$

Similarly, a maxterm is generally represented by

$$M_i = (x_1 + x_2 + \ldots + x_n), \qquad (6\text{-}5)$$

where the x_i may or may not be complemented. A function represented by maxterms would have the form

$$f = \pi M_i . \qquad (6\text{-}6)$$

As with the minterms above, it can be shown that

$$\prod_{i=0}^{2^n} M_i = 0. \qquad (6\text{-}7)$$

We now represent A in maxterm form

$$(A+B)(A+\overline{B}) = A+AB+A\overline{B}+B\overline{B} = A(1+B+\overline{B}) + 0 = A \qquad (6\text{-}8)$$

A convenient method of representing minterms and maxterms is with decimal notation. For instance, a minterm of four variables may be written

$$A\overline{B}\overline{C}D = 1001 = 9 \qquad (6\text{-}9)$$

and a maxterm

$$(\overline{A}+\overline{B}+C+\overline{D}) = 2. \qquad (6\text{-}10)$$

Using this notation, we may represent one function, A, from (6-4) and (6-8) above:

$$A = \sum(2,3) = \Pi\ (2,3). \qquad (6\text{-}11)$$

Example 6-1:

We shall now demonstrate how Boolean expressions may be expanded into their minterm and maxterm form. Consider

$$f_1 = AB+C(A+B) = AB+AC+BC \cdot \qquad (6\text{-}12)$$

This is a function of three variables. Therefore, for a minterm expression, each term must be a product which includes all three variables. We may multiply each term of (6-12) by 1 and not change its value. In this way, we can get a minterm expression for each term, and then add them all together.

$$AB = AB(C+\overline{C}) = ABC + AB\overline{C}$$
$$AC = AC(B+\overline{B}) = ABC + A\overline{B}C$$
$$BC = BC(A+\overline{A}) = ABC + \overline{A}BC$$

then

$$f_1 = ABC + AB\overline{C} + ABC + A\overline{B}C + ABC + \overline{A}BC$$
$$= ABC + AB\overline{C} + A\overline{B}C + \overline{A}BC \qquad (6\text{-}13a)$$
$$= \sum(7,6,5,3) = \sum(3,5,6,7). \qquad (6\text{-}13b)$$

Consider another function of three variables

$$f_2 = AB(A+C)+BC \tag{6-14}$$

which we desire in maxterm form. We shall express the complement of f_2, $\overline{f}_2$, in minterm form, and then take the complement again to get a maxterm expression for f_2. We may restate De Morgan's Theorem (5-2) as: "In order to find the complement of a Boolean expression, it is necessary to interchange the addition and multiplication operations and to replace each literal with its complement." Therefore,

$$f_2 = AB(A+C) + BC$$

$$\overline{f}_2 = (\overline{A}+\overline{B} + \overline{A}\,\overline{C})\,(\overline{B}+\overline{C}) = (\overline{A}+\overline{B})\,(\overline{B}+\overline{C})$$

$$= \overline{A}\,\overline{B}+\overline{B}+\overline{A}\,\overline{C}+\overline{B}\,\overline{C} = \overline{B}+\overline{A}\,\overline{C}$$

We now expand $\overline{f}_2$ in minterm form:

$$\overline{B} = \overline{B}(A+\overline{A})(C+\overline{C}) = A\overline{B}C + \overline{A}\,\overline{B}C + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C}$$

$$\overline{A}\,\overline{C} = \overline{A}\,\overline{C}(B+\overline{B}) = \overline{A}B\overline{C} + \overline{A}\,\overline{B}\,\overline{C}$$

$$\overline{f}_2 = A\overline{B}C+\overline{A}\,\overline{B}C+\overline{A}\,\overline{B}\,\overline{C}+A\overline{B}\,\overline{C}+\overline{A}B\overline{C}$$

$$f_2 = \overline{\overline{f}}_2 = (\overline{A}+B+\overline{C})(A+B+\overline{C})(A+B+C)(\overline{A}+B+C)(A+\overline{B}+C) \tag{6-15a}$$

$$= \Pi(2,6,7,3,5) = \Pi(2,3,5,6,7)\,. \tag{6-15b}$$

It should be noted that for n variables there are 2^{2^n} possible functions. For example, for two variables, there are 16 possible functions; for 3 variables, 256 possible functions; and for 6 variables, the number is greater than 18×10^{18} [Chu, 1962, p.108] .

We could, using only the algebra developed so far, reduce any given Boolean function to its simplest form. However, this becomes a very time-consuming process for more than three variables and quite impractical when five or six variables are included. Therefore, we shall now describe some reduction, or minimization, aids which will allow much easier simplification of Boolean expressions.

The most popular aids, for up to five or six variables, are the Veitch and Karnaugh maps. Both are slightly different forms of the same thing. We shall discuss the Veitch map first, then show how it is related to the Karnaugh map.

Consider the Venn diagram of figure 5-2. We wish to redraw this diagram so that the four areas representing the minterms are equal. Such an equal-area map is shown in figure 6-1(a). Note that A is always true on the left-hand half of the map and false on the right-hand side. B is true on the upper half and false on the lower half [figure 6-1(b)] . This situation will hold no matter how many variables are on the map. Each variable will be true on one-half of the squares and false on the other half.

Figure 6-1(c) demonstrates how the minterms for two variables are located on the map. We shall now give some examples, using the four-variable map of figure 6-2, of simplifications possible using the Veitch map. Consider any two adjacent squares (minterms) on the map, say 7 and 15. Using the algebraic techniques described above, we may simplify such an expression:

$$f = \sum(7,15) = \overline{A}BCD + ABCD = (\overline{A}+A)BCD = BCD. \qquad (6\text{-}15)$$

Note then that two adjacent squares on the map give rise to a three-literal term. Also note that the squares 0 and 4, 0 and 8, 4 and 12, and 8 and 12 are adjacent in that these minterms combine to give a reduced term. We see that there are three types of adjacencies of four squares, a row of four, as 0-1-5-4, a block of four, as 3-7-15-11, and the four corners, 0-4-12-8. For instance, the four corners simplify to the expression $\bar{C}\bar{D}$. Adjacencies of eight squares will reduce to a single literal, as A, $\bar{C}$, etc.

Example 6-2:

We shall now simplify a Boolean expression using the four-variable Veitch map.

Consider

$$f = \bar{B}C+\bar{B}\bar{C}\bar{D}+\bar{A}B\bar{C}+BCD. \tag{6-16}$$

Assume that we have derived this equation in one manner or another and wish to check to see that it is the simplest expression which will satisfy our conditions. Our technique is as follows: we first map the function (6-16) onto a Veitch map similar to figure 6-2. We shall designate a square for which the function is true with a 1 in that square. All other squares will be left blank, but we shall understand that they have a zero in them (figure 6-3). We may consider that the 1 or 0 is the multiplier of the minterm represented by that square.

In order to perform the mapping, we may expand (6-16) into minterm form and then place the appropriate multipliers on the map. It is much simpler, however, to map the function directly. Note that the first term is made up of those four squares for which B is false and C is true. We see from figure 6-2 that this corresponds to squares 2, 3, 10, and 11. We map these terms on figure 6-3. In a similar fashion, we may map the rest of the function and arrive with the complete map of figure 6-3. We shall now proceed to simplify the function. We wish first to select those terms which are essential, that is, have only one minimum combination with adjacent squares.

Let us begin with the square 8. This term will combine with those squares adjacent to it (10, or 10-11-9, or 9-13-12-14-15-11-10) or with the corners (0, or 12, or 0-4-12) or the row 10-2-0. The minimum combination here is 0-2-10-8, since all the other combinations for which all the terms are present give a larger number of literals. That is the combinations 8-0 or 8-10 give a three-literal term while 0-2-10-8 gives a two-literal term. The term given by the combination 0-2-10-8 is $\overline{B}\overline{D}$. We now place a small dot in the square indicating that we have used it as part of an essential term. Note that we must use every minterm at least once, but we may use it more than once if it allows greater simplification. Similarly, we note that the combination 3-7-15-11 gives the term CD. We now have left the squares 4 and 5 which have more than one minimum combination. The simplest expression will be obtained when we select the combination 4-5, since at least two terms will be required for any other. The combination 4-5 gives the term $\overline{A}B\overline{C}$. The total expression, simplified now from (6-16) is

$$f_s = \overline{B}\overline{D} + CD + \overline{A}B\overline{C} . \tag{6-17}$$

We have simplified an expression containing eleven literals to one that contains only seven.

We shall now describe the Karnaugh map and show how it is related to the Veitch map. Consider the Kanaugh map shown in figure 6-4, on which the squares have been labeled with their minterm designations. The labeling on the side and top of the map shows the condition of the variable in that row or column. For instance, the 01 on the top of the second column indicates that A is false and B is true. It may be easily verified that each variable is true on one half of the squares of the map and false on the other half. Adjacencies on the Veitch map are also adjacencies on the Karnaugh map, although their form may be different. For example, the four corners of the Veitch map appear as the top row of the Karnaugh map.

We shall conclude this section with a brief discussion of optional, or "don't care" terms. These are terms, or states, of a system which do not occur during operation of the system. Therefore, they may be used in the simplification process. As an example, consider the 8421 BCD code of Table 4-1. Assume we have a circuit which uses this particular code. We wish to convert this information from binary to decimal for a display. We note from Table 4-1 that the states 10 through 15 cannot occur. We may, therefore, use these states in our simplification of the terms for the binary-to-decimal converter. Let us consider the term 5. We could, with complete confidence, specify that the output 5 is given for the input condition $\bar{8}4\bar{2}1$. However, referring to the Veitch map shown in figure 6-5, we see that minterm 5 will combine with minterm 13 to give $(5)_{10} = 4\bar{2}1$. It may be demonstrated, using the map of figure 6-5, that two 4-input, six 3-input, and two 2-input circuits ("and" gates) are required to completely implement the BCD-to-decimal converter.

7. SWITCHING LOGIC III

We shall now discuss one of the exhaustive tabular methods such as might be used for a large number of variables or might be programmed on a computer for the automatic reduction of a Boolean expression [Bartee, 1959] . This method, known as the Quine-McCluskey method, was originally developed by Quine but its use was systemized and simplified by McClusky [1956] .

We know from a study of Boolean algebra that two terms differ only in one literal may be combined with the differing literal removed. In the Quine method, an exhaustive comparison of the minterms is made and then a selection is made from the remaining terms which results in the simplest expression. The literal expression for the minterm is used. In the McCluskey method, a numerical description, either binary or decimal, is used to describe the minterm. An order, or rank, is assigned to each minterm based on the number of ones in the binary expression.

For example, 0001, 0010, 0100, 1000 all have rank 1
0011, 0110, 0101, 1001 all have rank 2
etc.

McCluskey observed that a particular minterm need only be compared with the minterms of next higher rank since these were the only positions where a difference of one literal could occur.

We are required to use minterms for this method. If our function is not specified in minterms, we must operate upon it, as shown above, to get it in this form. We shall work an example which will demonstrate this technique. We shall include some optional terms to show how they are used in this method.

Example 7-1.

Consider

$$f = \sum (2,3,7,9.11,13) + \sum_{x} (1,10,15) \tag{7-1}$$

where $\sum_{x}$ indicates the optional minterms. Roughly, the procedure is as follows:

a) The minterms, including optional terms, are separated according to rank and placed in a column to the left of the page.

b) Comparisons are made between the terms in each rank and the one above it (rank 1 compared with rank 2, etc.).

c) When a simplification is made, a check mark is placed by the two terms and the simplified combination is placed in a second column. Terms may be used more than once but only one check mark need be placed by any term.

d) This process continues until all possible combinations are made in the first column. As each simplification is made, it is placed in the second column according to rank, except that a blank spot occurs where the literal was removed. For example: 1, 0001, and 3, 0011, combine to give a term in the second column, 1, 3 00-1.

e) Steps 4(a) through (d) are repeated for each column until no further simplifications are possible. This terminates the reduction part of the method.

f) All unchecked terms are now collected together as "prime implicants" that is, terms which contain all required information about the function.

g) A selection is made from the prime implicants which gives the minimum number of terms to specify the function.

We shall now demonstrate this process. From step (a), we set up the first column of Table 7-1. The terms are grouped according to rank (1, 2, 3, and 4). Note that the optional terms of (7-1) are included. The simplified terms from the first column are listed in the second, along with an indication of where they came from. Again, a separation is made according to rank. When making comparisons in the second column, only those terms of adjacent rank with the blank in the same space need be compared.

TABLE 7-1

Example of the Simplification of a Boolean Function Using the Quine-McCluskey Method

1	0001✓	1,3	00-1✓	1, 3, 9,11	-0-1	A
2	0010✓	1,9	-001✓	2, 3,10,11	-01-	B
		2,3	001-✓			
3	0011✓	2,10	-010✓	3, 7,11,15	--11	C
9	1001✓			9,11,13,15	1--1	D
10	1010✓	3,7	0-11✓			
		3,11	-011✓			
7	0111✓	9,11	10-1✓			
11	1011✓	9,13	1-01✓			
13	1101✓	10,11	101-✓			
15	1111✓	7,15	-111✓			
		11,15	1-11✓			
		13,15	11-1✓			

When all the possible comparisons have been made, all the unchecked terms, which may appear in any column, are labeled in the same manner as shown in Table 7-1. These unchecked terms are the prime implicants, and we wish to select a minimum number of them to specify our original function. In order to do this, we construct a prime implicant table of the prime implicants and the original minterms of the function. At this point we omit the optional terms, since they have already contributed to the simplification [Caldwell, 1958, pp. 145-156] .

TABLE 7-2

Prime Implicant Table

	2	3	7	9	11	13
A		x		x	x	
B	x	x			x	
C		x	x		x	
D				x	x	x

Each minterm from the original expression must be included at least once in our selection of prime implicants. Note that minterms 2,7, and 13 are only included once in a prime implicant. These prime implicants then are essential terms. We must include them in order to describe our function. We indicate this by circling the mark common to minterm and prime implicant and placing a check mark next to the minterm and prime implicant. Since B, C, and D must be included in the final expression, let us see what other minterms are covered by them. Note that B also includes 3 and 11, and D includes 9. Therefore, all our minterms are covered and the combination of B, C, and D will give us our simplest expression for (7-1). From table 7-1, we see that B = $\overline{X}Y$, C = YZ, D = WZ. Therefore:

$$f_s = \overline{X}Y + YZ + WZ. \tag{7-2}$$

Often there will be several choices available in the selection of prime implicants. In such cases, there are several, non-unique simplifications of a Boolean function. Several examples of these are given in Chu [1962] and Caldwell [1958] .

8. CONCLUSIONS

This report is not intended as an exhaustive dissertation on the subjects mentioned. Rather, we have sought to cover a few fundamentals so that the interested reader may proceed through the literature with a minimum of effort.

Most of the information in the literature on sampled-data systems is with regard to control and telemetry problems [Stiltz, 1961] [Susskind, 1957] , [Ragazzini & Franklin, 1958] . However, a very good, general description of sampling theorems is given by Linden [1959] . Monroe [1962] , while concerned mostly with control systems, gives a good account of computer tie-ins with sampled-data systems.

There is a wealth of texts on the subject of logic and Boolean algebra; Humphrey [1958] , and Maley and Earle [1962] have written good descriptions of sequential circuits, that is, circuits whose output depends on the history of the inputs, as well as their present states. Maley and Earle also give a good description of NOR and NAND logic.

It is hoped that the reader who is interested in system design using sampled data techniques or Boolean algebra will bypass the descriptions given here and refer to the more detailed and rigorous descriptions in the literature.

> "A little learning is a dangerous thing;
> Drink deep, or taste not the Pierian spring:
> There shallow draughts intoxicate the brain,
> And drinking largely sobers us again."
>
> Alexander Pope
> Essay on Criticism [1711]

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10. APPENDICES

10.1. The Ideal Low-Pass Filter

Consider the filter whose response is shown in figure 10-1, where $G(\omega)$ is a normalized power spectrum and $H(\omega)$ is the complex spectrum. Such a filter has unity gain up to ω_c and zero gain (infinite attenuation) for all frequencies above ω_c. We wish to demonstrate that such a filter is not physically realizable.

We know* that the frequency response of such a filter is the Fourier transform of its response to a unit impulse, $\delta(t)$. Therefore, we may write for the filter of figure 10-1:

$$h(t) = \int_{-\infty}^{\infty} H(\omega)\, \exp\left(j\omega t\right) d\omega \qquad (10\text{-}1)$$

where

$$H(\omega) = \begin{cases} \frac{1}{2}, & -\omega_c < \omega < \omega_c \\ 0, & \text{elsewhere} \end{cases}$$

so that

$$h(t) = \int_{-\omega_c}^{\omega_c} \frac{1}{2}\, \exp\left(j\omega t\right) d\omega$$

$$= \frac{1}{2\pi}\, \frac{\exp\left(j\omega t\right)}{jt}\Bigg|_{-\omega_c}^{\omega_c}$$

$$h(t) = \frac{\omega_c}{\pi}\, \frac{\exp\left(j\omega_c t\right) - \exp\left(-j\omega_c t\right)}{2j\omega_c t}$$

$$= \frac{\omega_c}{\pi}\left(\frac{\sin \omega_c t}{\omega_c t}\right). \qquad (10\text{-}2)$$

---

* cf. Lee [1960], pp. 328-330.

We see at once from (10-2) that a unit impulse, $\delta(t)$, applied at $t = 0$ excites an output from the filter for negative as well as positive times (from the symmetry of the $\frac{\sin \omega_c t}{\omega_c t}$ function). Therefore, we may conclude, since the filter produces an output before it receives an input, that the ideal low-pass filter is not physically realizable.

10.2. Proof of Sampling Theorem

We shall state the sampling theorem as:

"If a function f(t) contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced $\frac{1}{2W}$ seconds apart, the series extending throughout the time domain".

The points spaced $\frac{1}{2W}$ seconds apart will be located at

$$\frac{-2\pi}{\omega_o}, \; -\frac{\pi}{\omega_o}, \; 0, \; \frac{\pi}{\omega_o}, \; \frac{2\pi}{\omega_o} \ldots, \; \frac{n\pi}{\omega_o}, \qquad n = \ldots, -2, -1, 0, 1, 2, 3 \ldots.$$

$$\omega_o = 2\pi W \qquad (10\text{-}3)$$

Consider the Fourier integrals:

$$f(t) = \int_{-\omega_o}^{\omega_o} F(\omega) \exp\left(j\omega t\right) d\omega \qquad (10\text{-}4a)$$

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \exp\left(-j\omega t\right) dt \,. \qquad (10\text{-}4b)$$

Since $F(\omega)$ is limited to W cps, we may consider that it is periodic with period $2\omega_0$. Therefore, we may write in a dual form of (2-5)

$$F(\omega) = \sum_{n=-\infty}^{\infty} c_n \exp\left[j\left(\frac{n\pi}{\omega_0}\right)\omega\right] \text{for } |\omega| < |\omega_0| \tag{10-5}$$

where

$$c_n = \frac{1}{2\omega_0} \int_{-\omega_0}^{\omega_0} F(\omega) \exp\left[-j\left(\frac{n\pi}{\omega_0}\right)\omega\right] d\omega \tag{10-6}$$

is a dual of (2-7).

Now, from (10-4a), we may write

$$f\left(-\frac{n\pi}{\omega_0}\right) = \int_{-\omega_0}^{\omega_0} F(\omega) \exp\left[-j\left(\frac{n\pi}{\omega_0}\right)\omega\right] d\omega \tag{10-7}$$

so that

$$c_n = \frac{\pi}{\omega_0} f\left(-\frac{n\pi}{\omega_0}\right). \tag{10-8}$$

We may now substitute the value for c_n in (10-5) which determines the complex spectrum from only the ordinate values at times $\frac{n\pi}{W}$. Once the spectrum is uniquely determined, the original function is given by (10-4a) [Linden, 1959; Monroe, 1962].

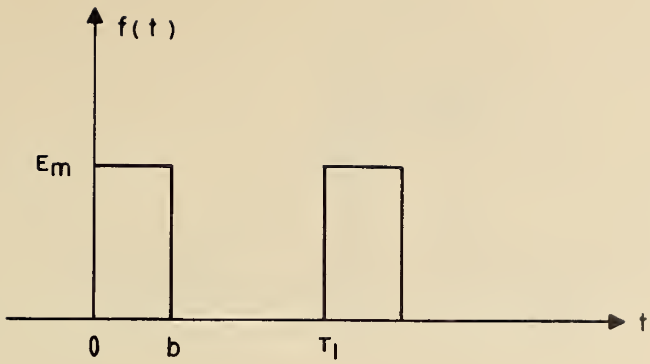


Figure 2-1. Recurrent pulse train.

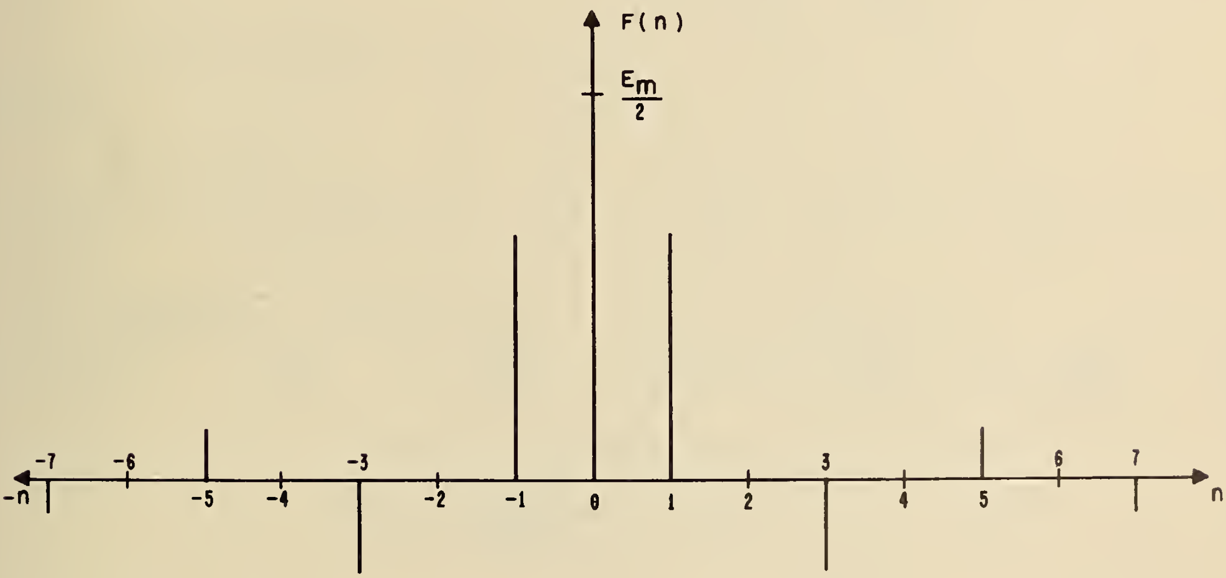


Figure 2-2. Spectrum of recurrent pulse train.

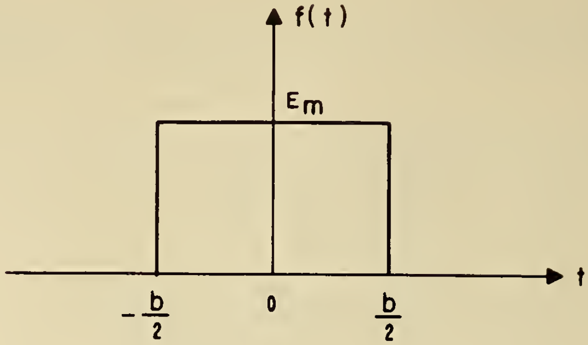


Figure 2-3. Rectangular pulse.

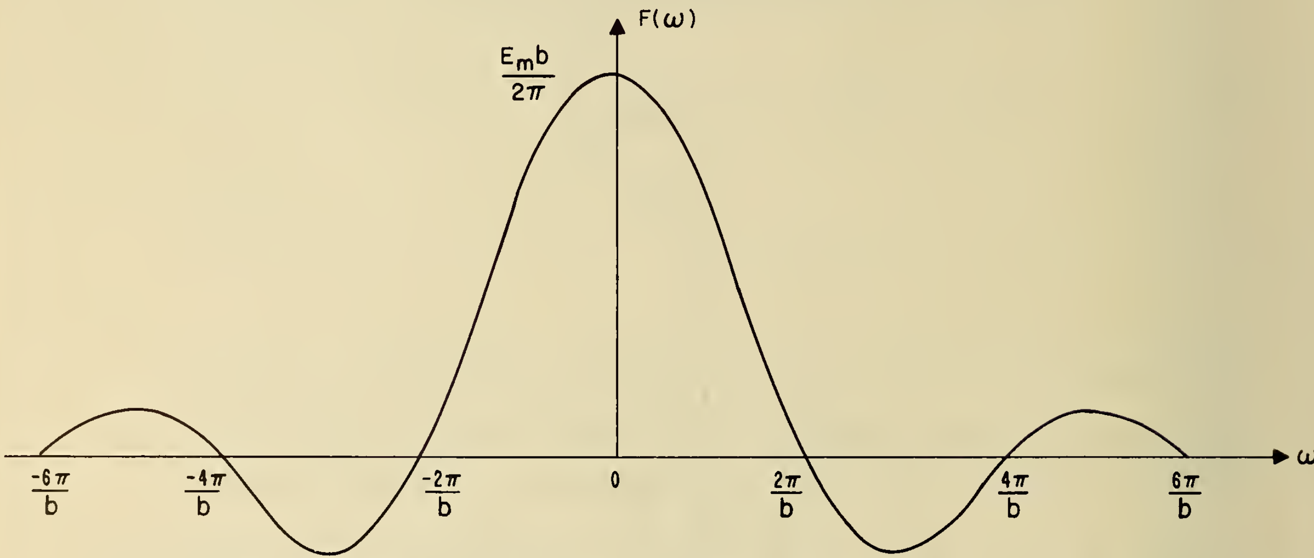


Figure 2-4. Spectrum of rectangular pulse.

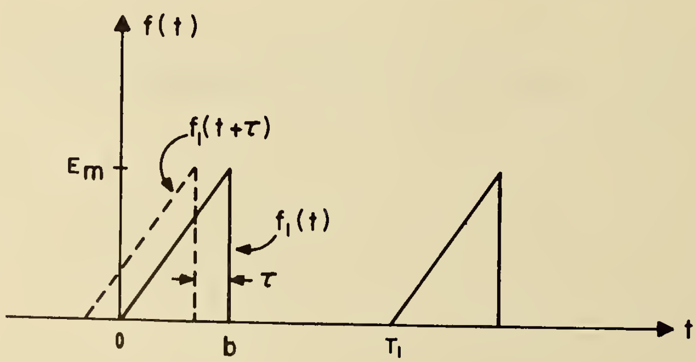


Figure 2-5. Periodic triangle for autocorrelation.

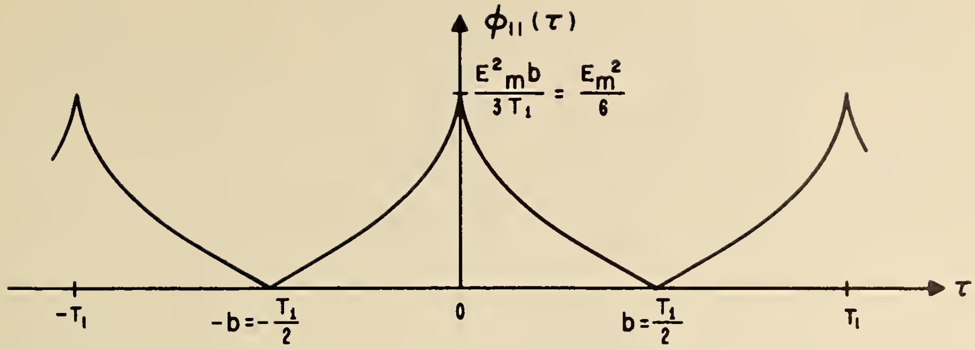


Figure 2-6. Autocorrelation function of a triangle.

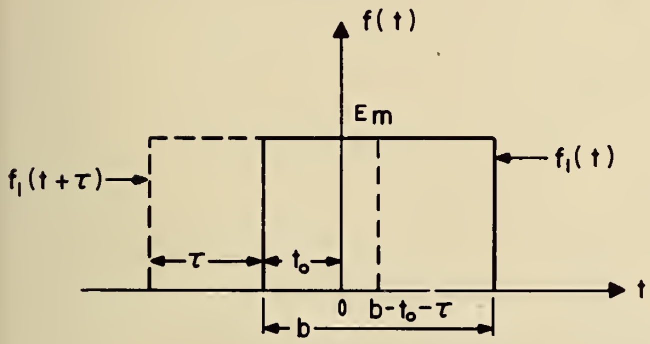


Figure 2-7. Single pulse

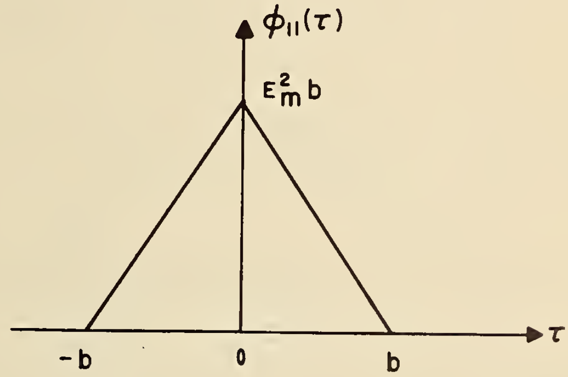


Figure 2-8.

Autocorrelation of single pulse.

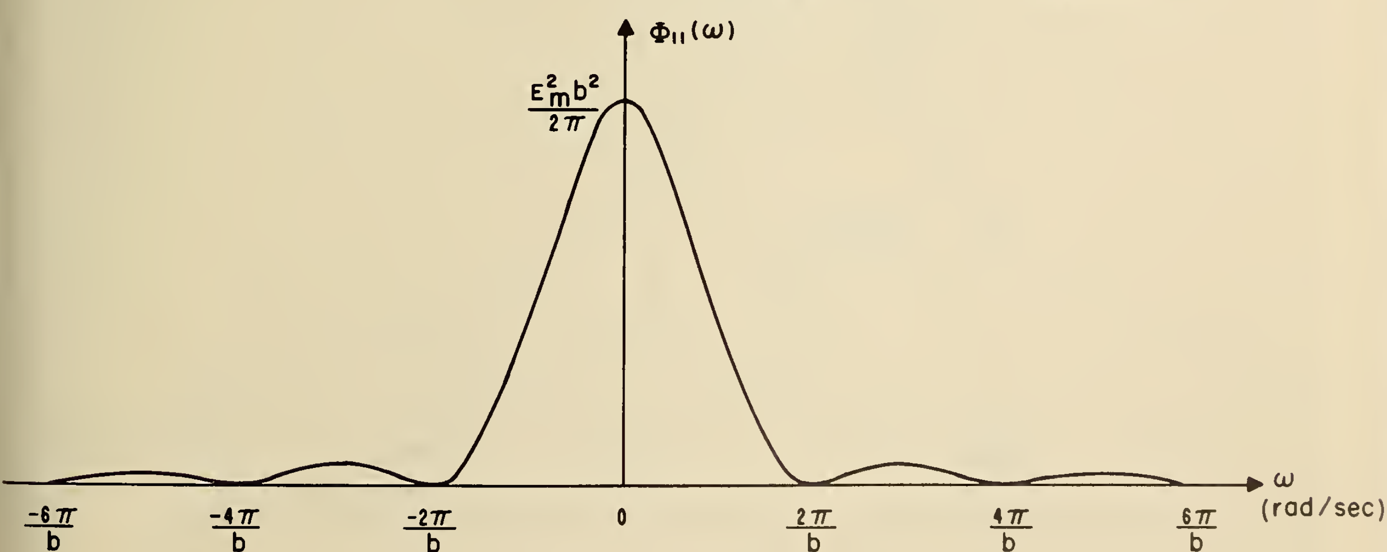


Figure 2-9. Energy density spectrum of a single pulse.

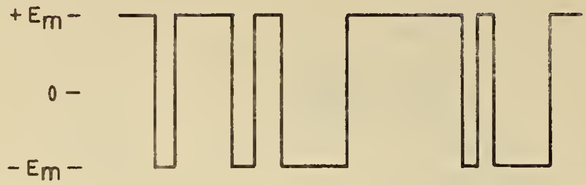


Figure 2-10. Random waveform.

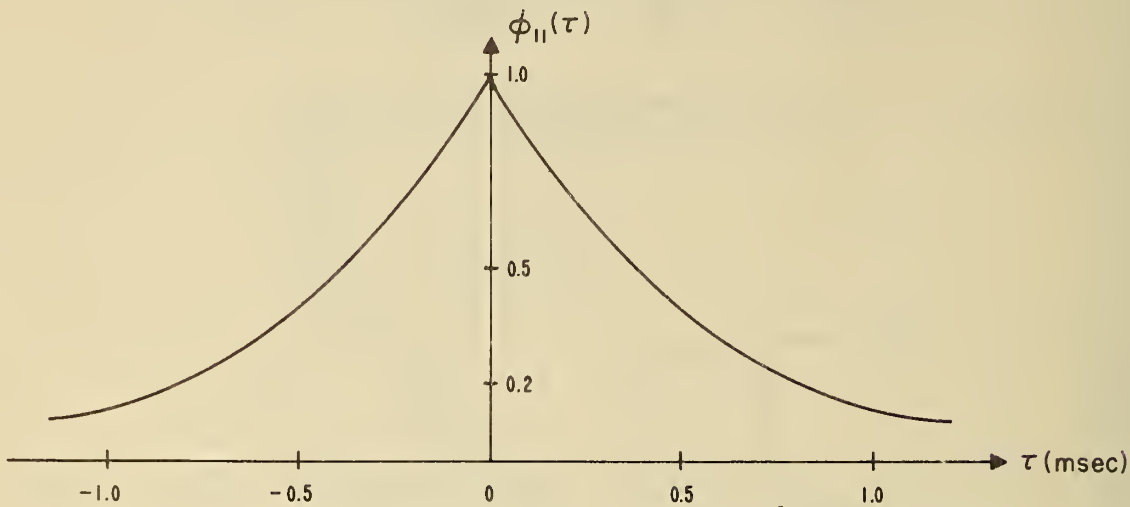


Figure 2-11. Autocorrelation function for waveform of figure 2-10 with $E_m = 1$ volt, $k = 1000/\text{sec}$.

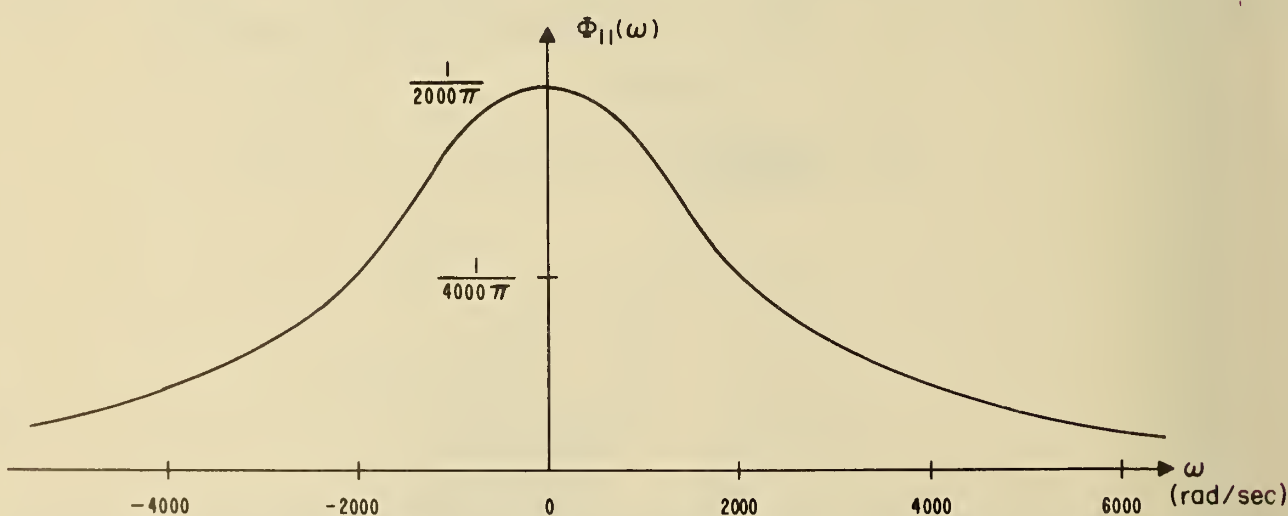


Figure 2-12. Power density spectrum for waveform of figure 2-10.

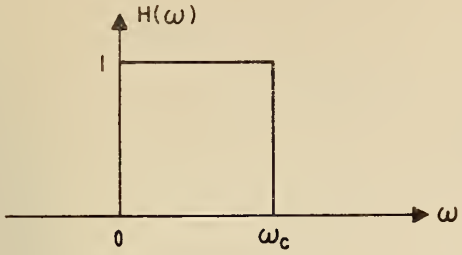


Figure 2-13.

Band-pass characteristic

of an ideal low-pass filter.

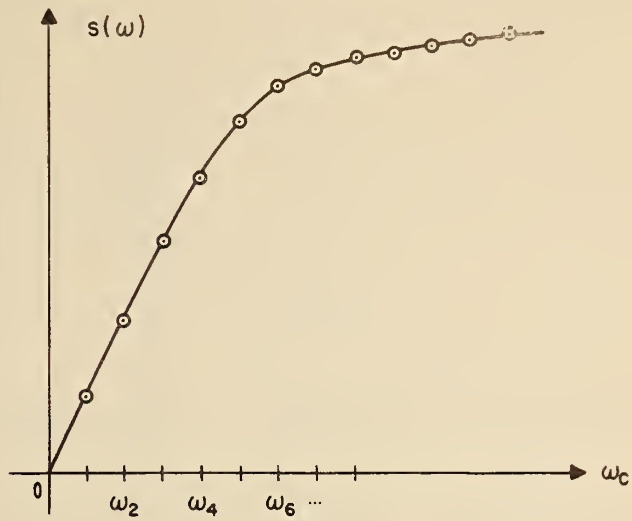


Figure 2-15. Power consumed in resistor as a function of cut-off frequency.

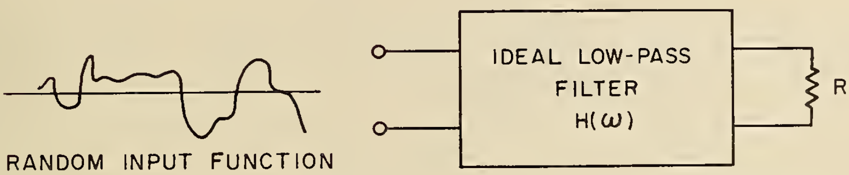


Figure 2-14. Measurement of the power spectrum of a random function.

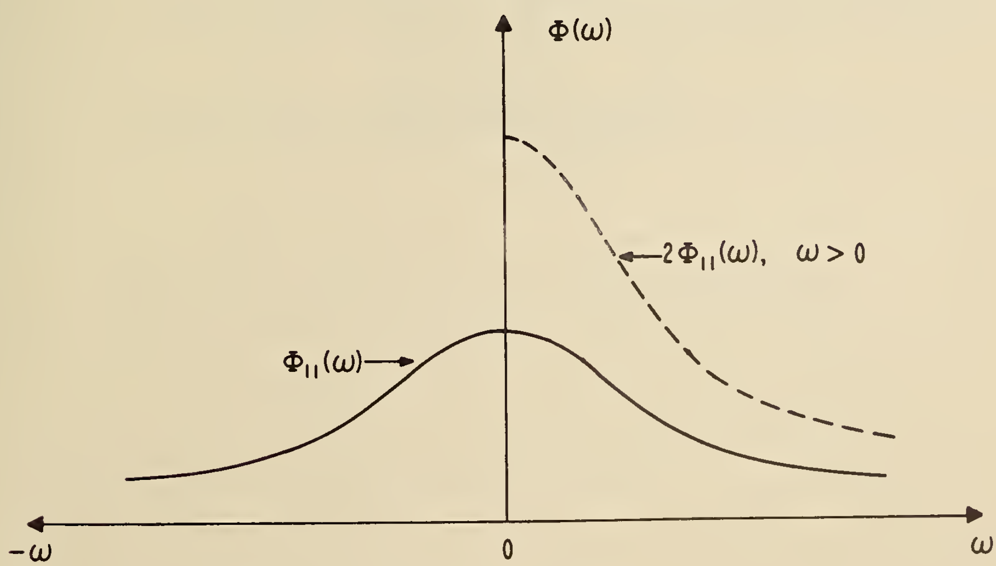


Figure 2-16. Power density spectrum of a random function.

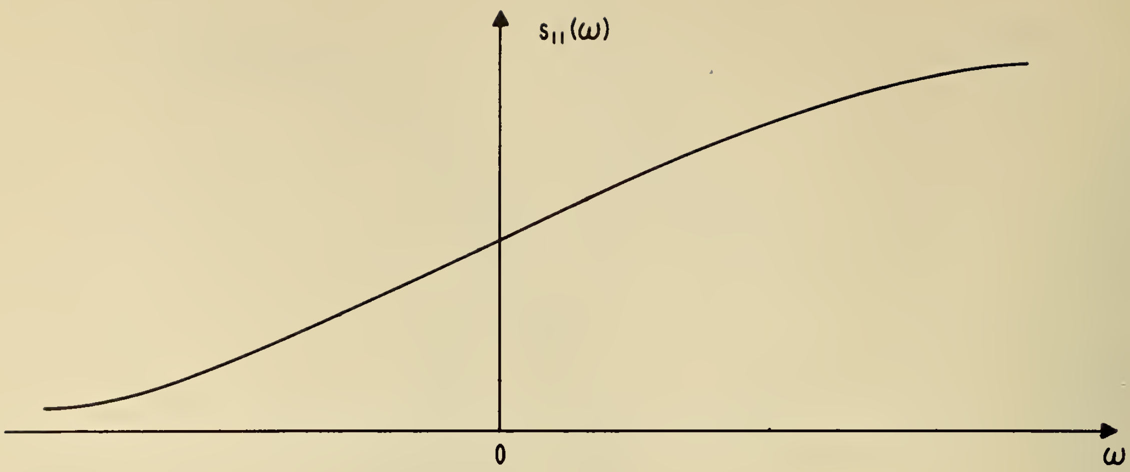


Figure 2-17. Integrated power spectrum of a random function.

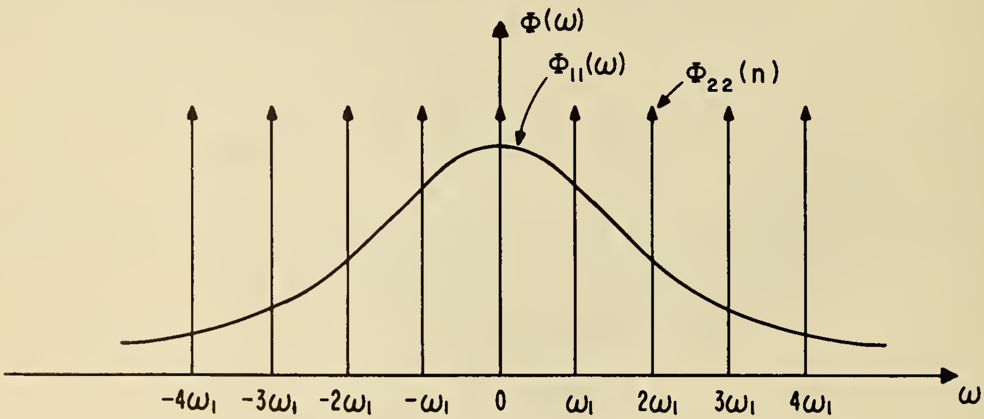


Figure 2-18. Spectrum of mixed random and periodic waves.

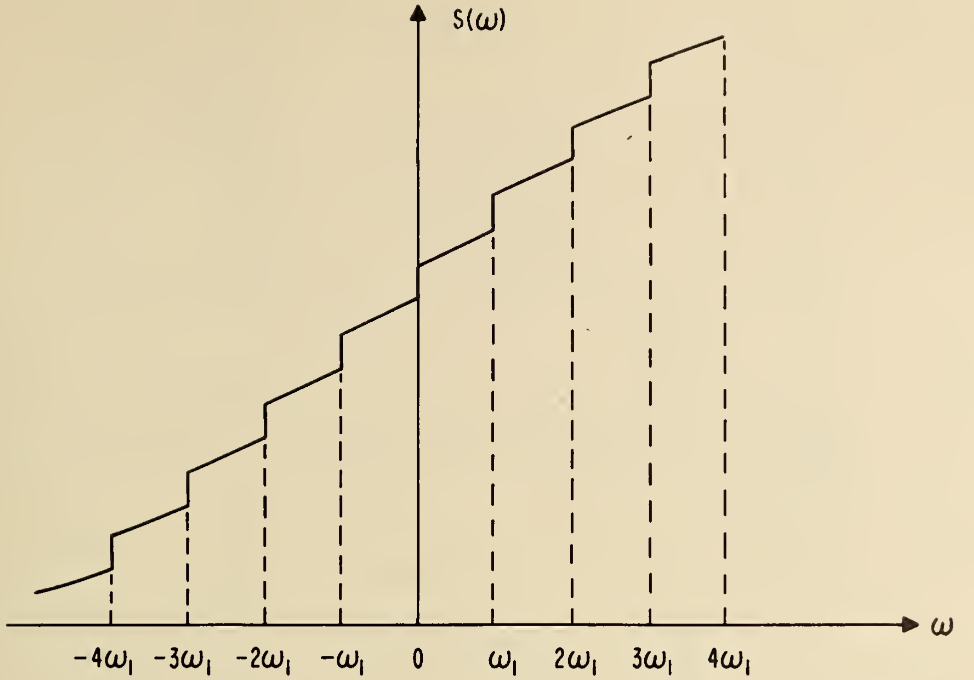


Figure 2-19. Integrated power spectrum for mixed random and periodic waves.

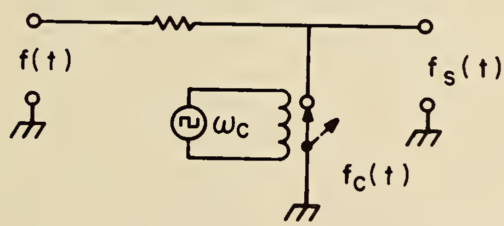


Figure 3-1. Single-ended chopper modulator.

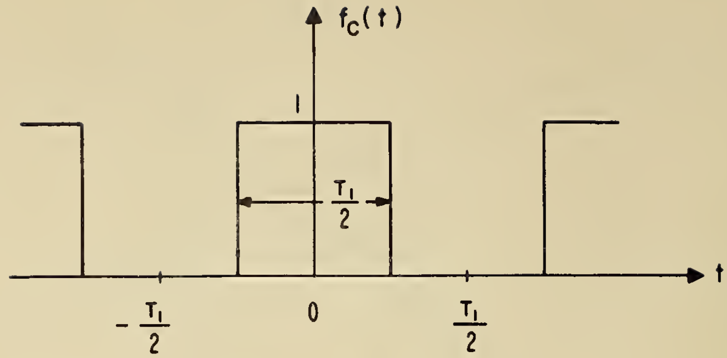


Figure 3-2. Chopping signal, $f_c(t)$, for figure 3-1.

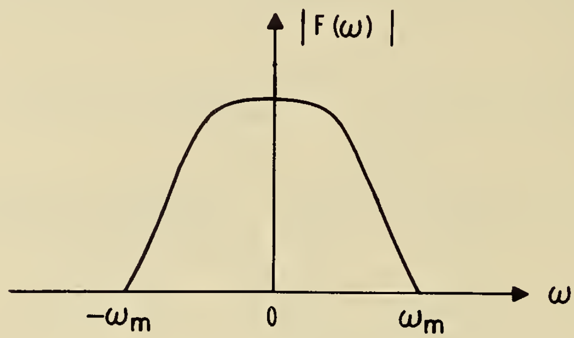


Figure 3-3. Band-limited spectrum of f(t).

Figure 3-5. Very narrow chopping (sampling) pulse.

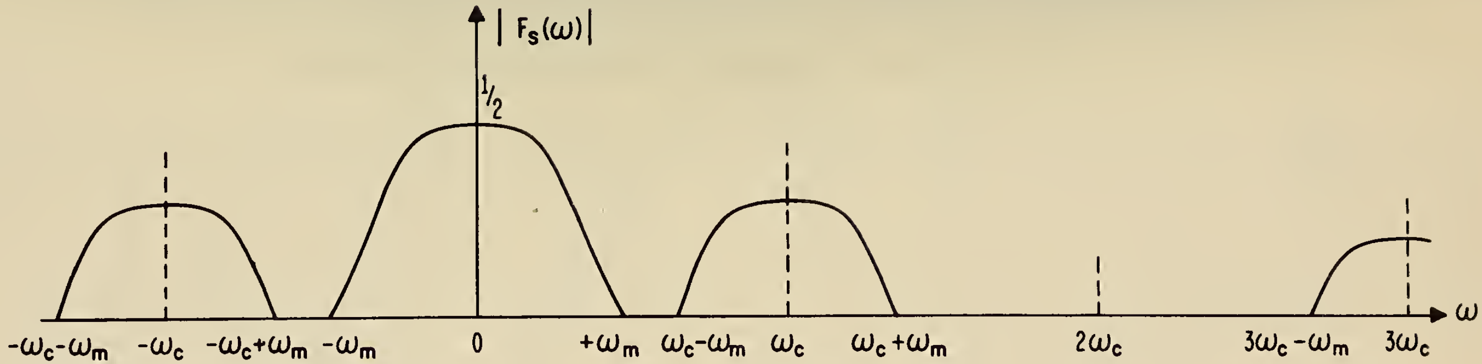


Figure 3-4. Spectrum of $F_s(t)$ for single-ended chopper.

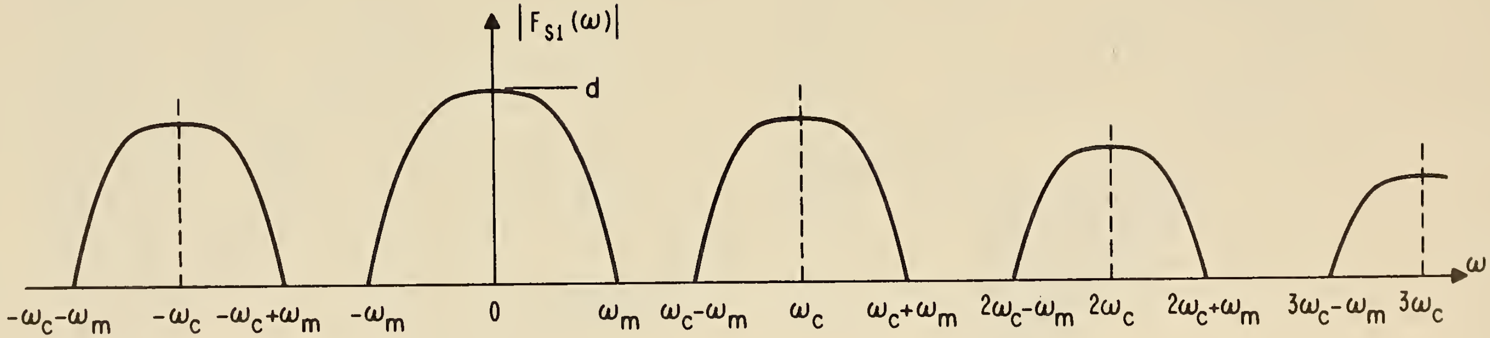


Figure 3-6. Spectrum of $f_{s_1}(t)$ for sampling with narrow pulse.

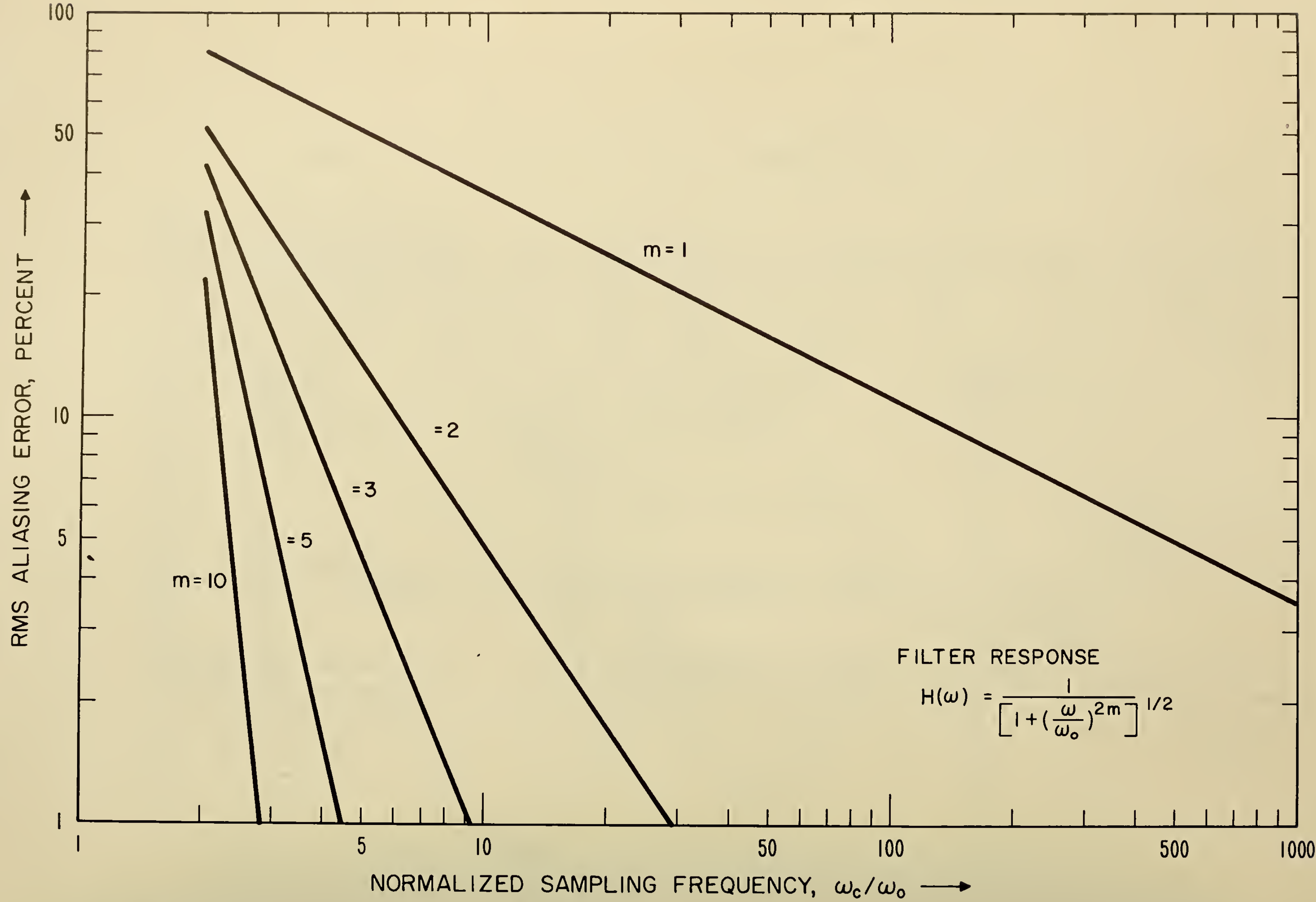


Figure 3-8. Aliasing error as a function of sampling frequency for a maximally flat filter [Stiltz, 1961].

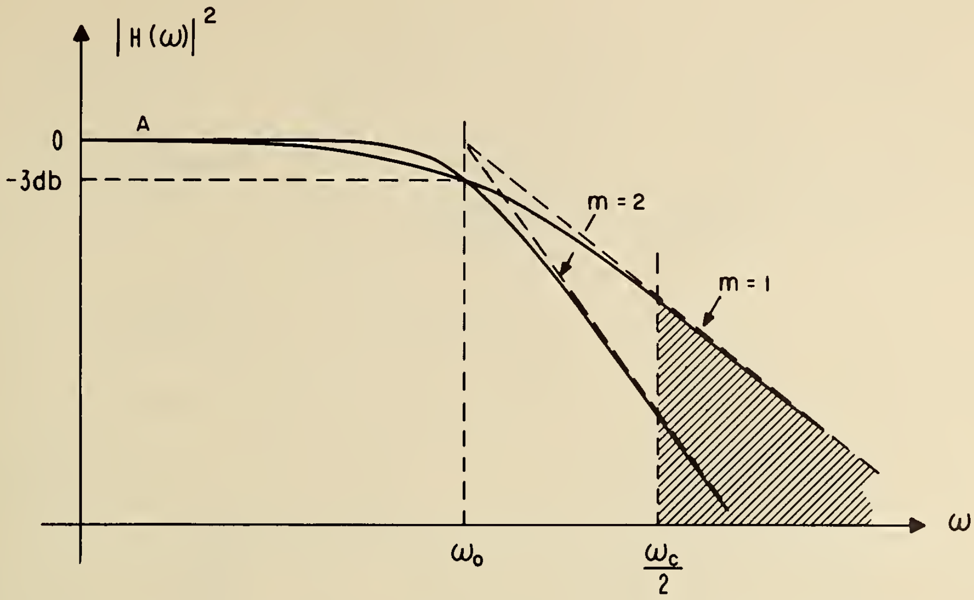


Figure 3-7. Response of maximally flat filter showing aliasing error power [Stiltz, 1961].

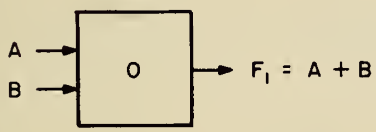


Figure 4-1. Logical OR function.

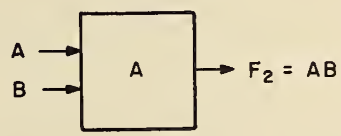


Figure 4-2. Logical AND function.

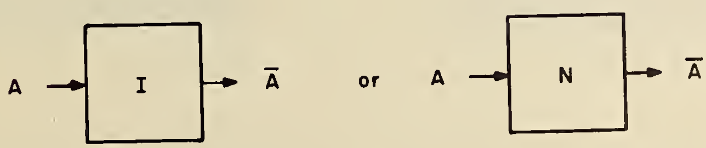


Figure 4-3. Logical NOT function.

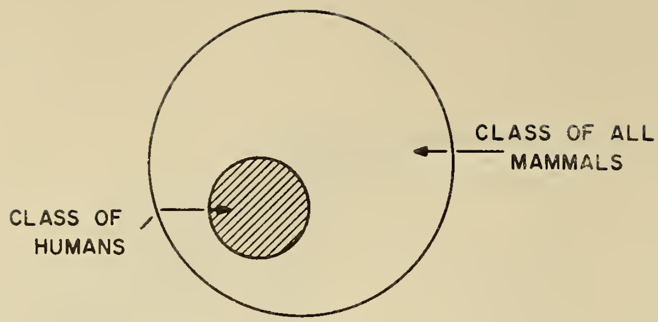


Figure 5-1. Example of a Venn diagram.

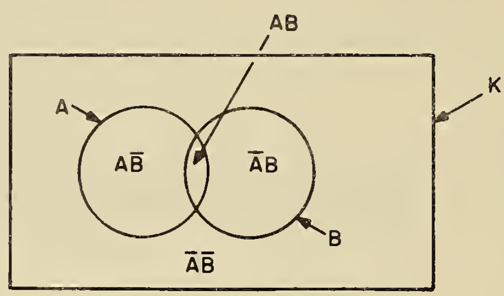
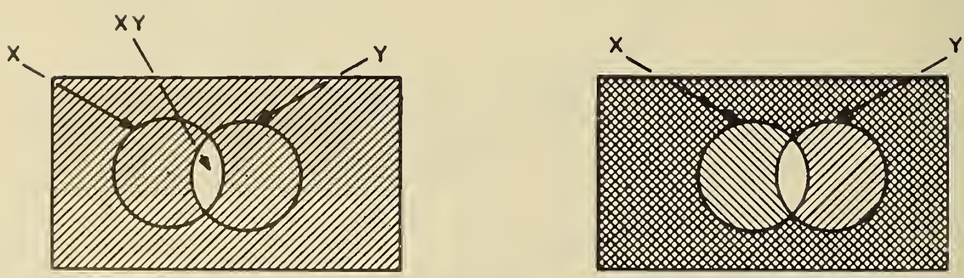


Figure 5-2. Universal class and two subsets on a Venn diagram.



(a) $\overline{XY}$ ////

(b) $\bar{Y}$ \\\\ , $\bar{X}$ ////

Figure 5-3. Proof of De Morgan's Theorem by Venn diagrams.

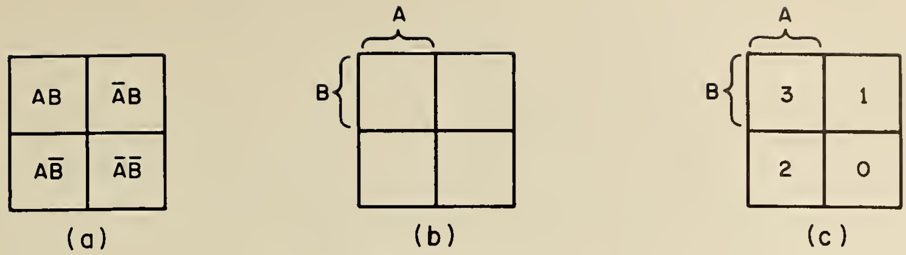


Figure 6-1. Development of Veitch map.

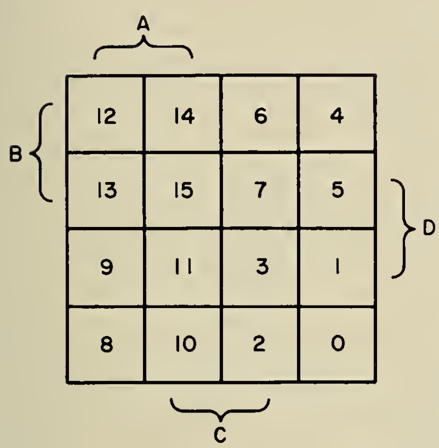


Figure 6-2.

A four-variable Veitch map.

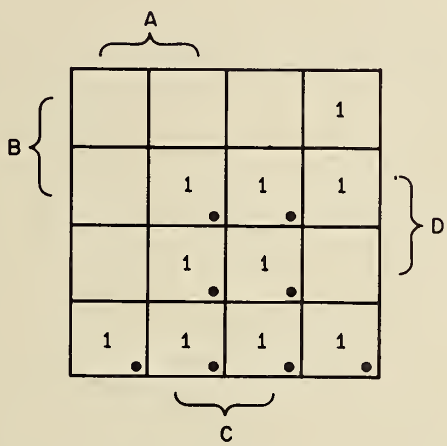


Figure 6-3.

Map for solution of equa. (6-16).

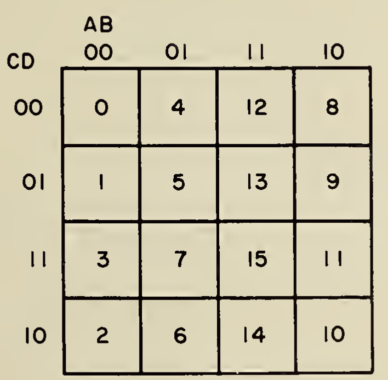


Figure 6-4. Example of a Karnaugh map.

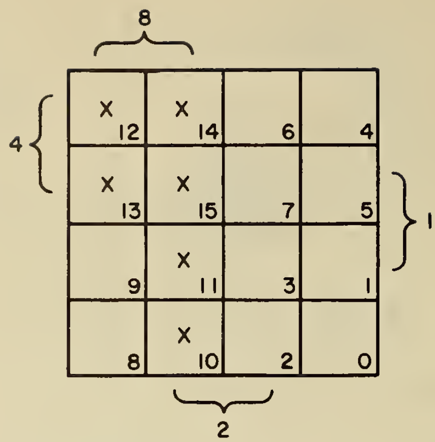


Figure 6-5. Veitch map for BCD-to-decimal converter.

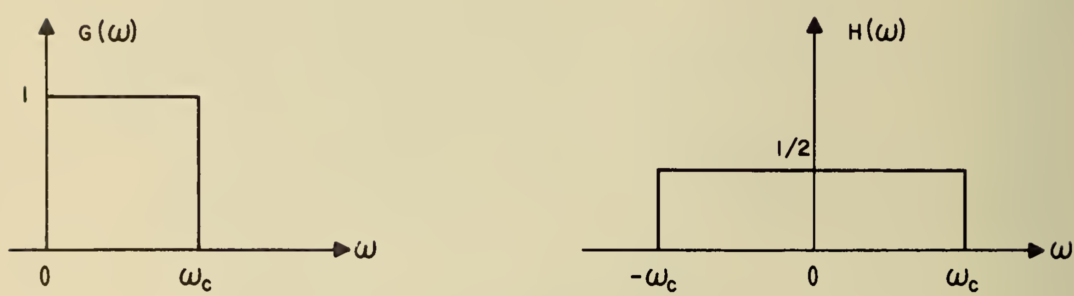


Figure 10-1. Frequency response of an ideal low-pass filter.

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