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STABILITY OF TWO-PHASE ANNULAR FLOW IN A VERTICAL PIPE

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U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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CONTENTS

Page

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The theory of hydrodynamic stability for infinitesimal disturbances is applied to the steady symmetric annular flow of two incompressible fluids in a vertical pipe. The resulting 12 x 12 complex determinant for the determination of curves of neutral stability is reduced, by suitable approximations, to an 8 x 8 one, and numerical methods are used to determine some neutral curves for air-water mixtures.

1. Introduction

In this report, an analysis is made of the stability of two-component annular flow in vertical tubes. The approach employs the well-known stability theory of infinitesimal disturbances[1].

Because of its importance in many engineering applications (e.g., fluid transport, heat exchangers), two-phase flow in tubes and orifices has received a great deal of attention in the last decade. But because the flow is very complex, even in straight vertical pipes, most of the effort has been devoted to accumulating a large quantity of experimental information, and to attempting to correlate data in particular cases with theories based on empirical relations. Furthermore, most of the work deals with the apparently simple (though important) case of annular flow without heat transfer or phase exchange; yet the results to date for this case are theoretically unsatisfactory and of modest practical utility. It is not the aim of this paper to correct this situation, but merely to apply the theory of hydrodynamic stability to this simple case in a more general way than has been used heretofore in this area. The stability theory has a record of significant (though not uniform) success.

Restricting ourselves to two-component incompressible flow in vertical pipes (e.g., air and water), the flows resulting experimentally are readily described in terms of "modes" dependent essentially on the ratio ν, of gas flow rate Q_g, to liquid flow rate Q_ℓ. As ν increases from zero, we have several modes manifesting themselves in the following order:

A. Aerated flow: gas bubbles entrained in an otherwise normal liquid flow.

B. Piston flow: the gas collects into a train of regularly-shaped slugs which nearly span the pipe.

C. Churn flow: a chaotic flow of large gas and liquid masses of irregular form.

D. Wave entrainment: the liquid lies against the pipe wall, while the gas fills the central core, the interface being marked by large waves which nearly span the pipe.

E. Annular flow: the large waves of D, damp to form a barely rippled surface on the liquid film.

F. Entrained flow: liquid droplets are torn from the film surface and carried along with the gas.

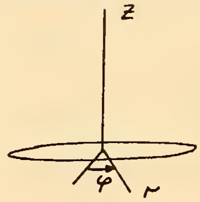
One is tempted to attribute to annular flow, E, a low-λ instability leading to wave entrainment, D, and a high-λ instability leading to droplet entrainment, F. There is, however, a conceptual difficulty in studying the stability of this annular flow via infinitesimal disturbances on an otherwise steady flow. The gas flow and liquid flows are usually turbulent in cases of interest, while the small interfacial ripples are known to be responsible (probably thru profile drag) for a significant part of the pressure drop. To what extent the small wavy disturbances admitted in the stability analysis can qualitatively or quantitatively approximate the experimentally observed flow can only be determined a posteriori. Some encouragement is given by the results of the only other stability study of two-phase flow known to the author, that of Feldman[2]. He considered the oversimplified model comprising two-dimensional film flow on a flat wall, with a linear velocity profile in both the film and the semi-infinite gas flow above it. While the critical Reynolds number determined was very poor, the destabilizing effects of surface tension and gravity (normal to the wall) were demonstrated. The most important result, however, was the existence of instability, although for a homogeneous flow with linear velocity profile (Couette flow), the analysis is known to predict stability for all Reynolds numbers (contrary to the physical facts however!). The instability must therefore be associated with the interface[3] rather than with the inception of turbulence in either fluid. However, Feldman appears to have erred in his formulation of the interface conditions, which may cause the failure of his result to match film cooling Reynolds numbers.

The results obtained here, when reduced to Feldman's case, give very much lower critical Reynolds numbers, and agree in an acceptable manner with film data of Charvonia[4].

Squire's result [8] for homogeneous two-dimensional flow, which states that three-dimensional infinitesimal disturbances are more stable than two-dimensional ones, is not clearly applicable to Feldman's model, and is not applicable to the model considered here. (Squire's result appears to fail for finite disturbances.)

2. The Model and Basic Equations

The model chosen for this analysis is that of vertical annular flow of two incompressible fluids in a circular pipe of radius L. The core fluid is referred to as the gas, the film fluid is referred to as the liquid. Gravity is assumed to act downward, in the (- z) - direction. An undisturbed flow



is assumed to exist with cylindrical symmetry having only a vertical velocity component W (r). We assume W is continuous, W (L) = 0, and $W'(r) < 0$ for $0 < r \leq L$. This latter assumption simplifies the analysis by permitting, at most, one critical layer while placing a limitation on the influence of gravity which is realized in most flows of practical interest. The pressure drop $\gamma = -\frac{dP}{dz}$ is assumed constant. In the gas and liquid, respectively, we have the pressures:

$$P_g = -\gamma z + P_{go}$$

$$P_\ell = -r z + P_{\ell o}$$

The constant pressure difference is associated with surface tension at the interface

$$\Delta P \equiv P_{og} - P_{o\ell} = \frac{\sigma^*}{x_*}$$

where x_* is the undisturbed interface radius and σ^* the coefficient of surface tension. While the base flow W (r) is assumed to satisfy the Navier-Stokes equations in each medium, more general flows will be considered; they are discussed in Section 7. The remainder of the analysis requires no further specification of the base flow. On this base flow are superimposed small disturbances to the velocities (radial, azimuthal and axial, respectively) and the pressure of the form:

$$\begin{pmatrix} u(r) \\ v(r) \\ w(r) \\ \pi(r) \end{pmatrix} e^{i\alpha_*(z^* - c_* t^*) + im\varphi} \tag{1}$$

whose amplitudes are taken to be sufficiently small that the linearized equations of flow suffice for their determination. Only real parts are taken to be physically significant. The longitudinal wave number is α_*, m gives the rotational mode, and c_{*R} is the wave speed: thus $c_* = c_{*R} + i c_{*I}$. The sign of c_{*I} determines whether the disturbance will grow ($c_{*I} > 0$) or decay ($c_{*I} < 0$). When (α_*, m) are given, c_* is to be determined from the analysis.

From the Navier-Stokes equations in cylindrical coordinates,[5] we obtain the following ordinary differential equations by linearization:

$$A u + \pi' = \mu \left\{ u'' + \frac{1}{r} u' - \frac{m^2+1}{r^2} u - \frac{2im}{r^2} v - \alpha_*^2 u \right\} \tag{2}$$

$$A v + \frac{im}{r} \pi = \mu \left\{ v'' + \frac{1}{r} v' - \frac{m^2+1}{r^2} v + \frac{2im}{r^2} u - \alpha_*^2 v \right\} \tag{3}$$

$$A w + \rho W' u + i\alpha_* \pi = \mu \left\{ w'' + \frac{1}{r} w' - \frac{m^2}{r^2} w - \alpha_*^2 w \right\} \tag{4}$$

$$u' + \frac{1}{r} u + \frac{im}{r} v + i\alpha_* w = 0 \tag{5}$$

$$A(r) \equiv i\alpha_* \rho \left(W(r) - c_* \right)$$

where (ρ, μ) must be given their respective values in the film and in the core.

At the center of the pipe, r = 0, we require the velocity components to be bounded. At the wall, r = L, the disturbance components must vanish:

$$u(0),\ v(0),\ w(0) \qquad \text{bounded} \tag{6}$$

$$u(L) = v(L) = w(L) = 0\,. \tag{7}$$

We assume the interface to be disturbed in the same way as the base flow, and put for the interface

$$f(z, r, \varphi, t) \equiv r - x_* + F_0\, e^{i\alpha_*(z^* - c_* t^*) + im\varphi} = 0\,, \tag{8}$$

where again $|F_0| << 1$ is assumed. Since on the interface, with $\mathbb{V}$ the complete velocity vector,

$$\frac{\partial f}{\partial t} + \mathbb{V} \cdot \nabla f = 0$$

we find, using $W_x = W(x_*)$,

$$F_0 = + \frac{u(x_*)}{i\alpha_*(W_x - c_*)}\,. \tag{9}$$

Now across the interface, u, v, W + w must be continuous for this viscous flow. The remaining physical conditions are that the total shearing stress be continuous across the interface, while the total normal stress jump, $\Delta\sigma_n$, be given in terms of the mean curvature of the interface

$$\Delta\sigma_n = \sigma^* K_m\,,$$

where σ^* is again the surface tension coefficient. The base flow has the normal stress

$$\sigma^0_{rr} = -P_0 + \gamma z$$

and the shear stress $\sigma^0_{rz} = \tau_0 = \mu W'(r)$ for laminar flow. The complete stress tensor is then [5]

$$\sigma_{rr} = \sigma_{rr}^{0} + (-\pi + 2\mu u')E_0$$

$$\sigma_{r\varphi} = \sigma_{\varphi r} = \mu\left(r\left(\frac{v}{r}\right)' + \frac{im}{r}u\right)E_0$$

$$\sigma_{rz} = \sigma_{zr} = \tau_0 + \mu(i\alpha_* u + w')E_0$$

$$\sigma_{\varphi z} = \sigma_{z\varphi} = \mu\left(i\alpha_* v + \frac{im}{r}w\right)E_0 \tag{10}$$

$$\sigma_{\varphi\varphi} = \sigma_{rr}^{0} + \left(-\pi + 2\mu\left(\frac{im}{r}v + \frac{1}{r}u\right)\right)E_0$$

$$\sigma_{zz} = \sigma_{rr}^{0} + (-\pi + 2\mu i\alpha_* w)E_0$$

where $E_0 = e^{i\alpha_*(z^* - c_* t^*) + im\varphi}$

The unit outward normal to the interface has the components

$$n_j = (1, n_\varphi, n_z)$$

where $|n_\varphi|, |n_z| << 1$, where:

$$n_\varphi = -\frac{im}{r}F_0 E_0$$

$$n_z = -i\alpha_* F_0 E_0 . \tag{11}$$

Thus the force on the interface has the components (neglecting quadratic terms in the disturbances):

$$\begin{pmatrix} \sigma_{rr}^{0} + (-\pi + 2\mu u' - i\alpha_* F_0 \tau_0)E_0 \\ \left(\mu r\left(\frac{v}{r}\right)' + \mu\frac{im}{r}u - \frac{im}{r}F_0\sigma_{rr}^{0}\right)E_0 \\ \tau_0 + (\mu i\alpha_* u + \mu w' - i\alpha_* F_0 \sigma_{rr}^{0})E_0 \end{pmatrix}. \tag{12}$$

Now a tedious calculation of the mean curvature K_m of the interface gives

$$K_m = -\frac{1}{X_*} - \left(\alpha_*^2 + \frac{m^2-1}{X_*^2}\right)E_0 F_0 \tag{13}$$

to the first order in small quantities. (In this calculation, only the real part of f is used, leading to K_m which is the real part of (13).)

Noting that on the undisturbed interface:

$$
\left.\sigma^{o}_{rr}\right|_{\text{liquid}}^{\text{gas}} = -\frac{\sigma^{*}}{x_{*}}
\qquad\qquad (14)
$$

$$
\left.\tau_{o}\right|_{\text{liquid}}^{\text{gas}} = 0
$$

for the base flow, and putting

$$
\left[\tau_{o}(r)\right]_{f=0} = \tau_{o}(x_{*}) + \tau_{o}'(x_{*})\,F_{o}\,E_{o}
$$

to first order, we determine the jump conditions for the disturbances. In the following $\delta A = A\Big|_{\text{liquid}}^{\text{gas}}$ and $r = x_{*}$:

$$
\delta u = \delta v = 0, \qquad \delta w + \frac{u(x_{*})}{i\alpha_{*} W_{x} - c_{*}}\,\delta W_{x}' = 0 \qquad (15)
$$

$$
\left\{
\begin{aligned}
&2\,\delta(\mu u') - \delta\pi = \frac{-iu\sigma^{*}}{\alpha_{*}(W_{x} - c_{*})}\left(\alpha_{*}^{2} + \frac{m^{2}-1}{x_{*}^{2}}\right)\\
&\delta(\mu v') + \frac{imu - v}{x_{*}}\,\delta\mu = \frac{\sigma^{*} m u}{\alpha_{*} x_{*}^{2}(W_{x} - c_{*})}\\
&\delta(\mu w') + i\alpha_{*} u\,\delta\mu + \frac{u}{i\alpha_{*}(W_{x} - c_{*})} = \frac{\sigma^{*} u}{x_{*}(W_{x} - c_{*})}
\end{aligned}
\right. \qquad (16)
$$

The twelve conditions (6, 7, 15 and 16) are homogeneous in (u, v, w, π) as are the pair of 6th order systems (2–5), associated with the film and core flows. This will be seen in Section 6 to lead to an eigenvalue problem for the determination of c_{*} as a function of the other parameters of the problem. In fact, we shall be led to require that a complex determinant $\Delta = \Delta_{R} + i\,\Delta_{I}$ vanish:

$$
\left\{
\begin{aligned}
\Delta_{R} &= 0\\
\Delta_{I} &= 0
\end{aligned}
\right.
$$

We shall, in effect, set $c_{*I} = 0$, and eliminate c_{*R} between these real equations, obtaining a single relation among all the parameters α_{*}, μ, etc., of the problem which when satisfied, assures that the solutions so specified will be neutrally stable, that is, they will neither grow nor decay. This hypersurface in the space of the parameters will, in general, separate a region of stable solutions from a region of unstable solutions.

We shall now proceed to the details of determining solutions to the system (2–5), bearing in mind that c_* will be taken to be real.

3. Fundamental Solution System for Small Viscosity

The customary approach to solving the complicated set of (2–5) is to assume that the viscosity $\mu \to 0$. The resulting solutions can be proved to be rigorously asymptotic in domains devoid of the singularities of the solutions, i.e., $r = 0$ and $r = r_c$, where $W(r_c) = c_*$ (if such exists), the so-called "critical layer". We shall require that when r_c exists for real c_*,

$$r_c \neq (0, x_*, L)$$

Setting $\mu = 0$, the system (2–5) may be reduced to the system:

$$\begin{aligned} &\pi'' + \left(\frac{1}{r} - \frac{2W'}{W - c_*}\right)\pi' - \left(\frac{m^2}{r^2} + \alpha_*^2\right)\pi = 0 \\ &u = -\frac{\pi'}{A} \\ &v = -\frac{im}{rA}\pi \\ &w = -\frac{i\alpha_*}{A}\pi + \frac{\rho W'}{A^2}\pi' \end{aligned} \qquad (17)$$

This system is evidently of 2nd order, and are the so-called "inviscid equations". They provide only two of the independent set of 6 solutions we must obtain for (2–5) in order to be able to satisfy general boundary conditions. Before proceeding to the development of a complementary set of four solutions, we shall establish certain facts about the solutions of (17). The points $r = 0$ and $r = r_c$ are regular singular points of (17).

It does not appear possible to integrate these equations in closed form for general values of the parameters m and α_* for the profiles $W(r)$ of interest. In studying the stability of plane boundary layer flow, the simpler equation $\pi'' - \frac{2W'}{W-c}\pi' - \alpha_*^2 \pi = 0$ is obtained which can be integrated in terms of hypergeometric functions for a useful profile. On the other hand, it turns out that the neutral stability curve is associated with $\alpha_* L^* << 1$ (where L^* is the boundary layer thickness), so that the solutions may be determined adequately by two terms of an expansion in α_*^2. In the present case, we could solve the equation in terms of (untabulated) hypergeometric functions for $\alpha_* L \to 0$, but Feldman's results indicate the neutral stability curve involves values α_* between the values $0.01 \leqslant \alpha_*(L - x_*) \leqslant 1$, so that we must be prepared to have wave numbers of the order of the inverse film thickness, too large for this approximation.

When a critical layer occurs in the flow at $r = r_c$, it will be apparent from Section 5 that certain phases must be adopted to continue properly asymptotic solutions across the singularity:

$$
\begin{aligned}
0 &\leqslant \arg(r - r_c) \leqslant \pi \\
\pi &\leqslant \arg(W - c) \leqslant 2\pi \\
\frac{3\pi}{2} &\leqslant \arg A \leqslant \frac{5\pi}{2} \\
\arg t &= \frac{3\pi}{2} \\
\frac{\pi}{2} &\leqslant \arg \zeta \leqslant \frac{3\pi}{2},
\end{aligned}
\qquad (18)
$$

where

$$t = \frac{dA}{dr}, \qquad \zeta = \mu^{-1/3} t^{-2/3} A. \qquad (19)$$

It is convenient to define here specific solutions of the inviscid equation which will be used in the numerical work. For this we shall introduce the dimensionless quantities:

$$\xi = r/L, \qquad \alpha = \alpha_* L, \qquad \mathcal{W}(\xi) = \frac{W(r)}{W_x}, \tag{20}$$
$$x = x_*/L, \qquad c = c_*/W_x,$$

and put

$$\pi(r) = \Pi(\xi), \qquad Q(\xi) = \Pi'(\xi). \tag{21}$$

The inviscid equations for (Π, Q) are then

$$\begin{cases} \Pi' = Q \\ Q' = \left(\frac{m^2}{\xi^2} + \alpha^2\right)\Pi + \left(\frac{2\mathcal{W}'}{\mathcal{W}-c} - \frac{1}{\xi}\right)Q. \end{cases} \tag{22}$$

We define a pair of independent solutions of (22) in the liquid film by their values at the (assumed regular) point $\xi = x$:

$$\begin{cases} \left(\Pi_5(\xi),\ Q_5(\xi)\right)_{\xi=x} = (1,\ 0) \\ \left(\Pi_6(\xi),\ Q_6(\xi)\right)_{\xi=x} = (0,\ 1). \end{cases} \tag{23}$$

Their values at the wall $\xi = 1$ (r = L) will be denoted by (Π_{jL}, Q_{jL}; $j = 5, 6$) which may be complex.

In the gas core, we shall define a unique H $\equiv \xi \dfrac{Q_2(\xi)}{\Pi_2(\xi)}$ by the requirement (since $r_c \neq 0$):

$$\Pi_2(\xi) \xrightarrow[\xi\to 0]{} \xi^m \tag{24}$$

(Any linearly independent solution $\Pi_1(\xi)$ is singular at $\xi = 0$.) In Section 8, we shall assume a core flow of the form:

$$\mathcal{W}(\xi) = K - f\xi^2.$$

Then with

$$\bar{\sigma} = \alpha\xi, \qquad \bar{\beta} = \alpha\left(\frac{K-c}{f}\right)^{1/2},$$

$H(\bar{\sigma}; \bar{\beta}; m)$ satisfies the equations

$$H(\bar{\sigma}) \equiv H_R(\bar{\sigma}) + i\, H_I(\bar{\sigma})$$

$$\begin{cases} \bar{\sigma} H_R' + H_R^2 - H_I^2 + \dfrac{4\bar{\sigma}^2}{\bar{\beta}^2 - \bar{\sigma}^2} H_R - (m^2 + \bar{\sigma}^2) = 0 \\ \bar{\sigma} H_I' + 2 H_R H_I + \dfrac{4\bar{\sigma}^2}{\bar{\beta}^2 - \bar{\sigma}^2} H_I = 0 \end{cases} \tag{25}$$

for real and imaginary parts. The initial condition for H is:

$$\begin{cases} H_I(0) = 0 \\ H_R(\bar{\sigma}) \xrightarrow[\bar{\sigma} \to 0]{} m + \dfrac{\bar{\sigma}^2}{2(1+m)}\left(1 - \dfrac{4m}{\bar{\beta}^2}\right) \\ \qquad + O(\bar{\sigma}^4). \end{cases} \tag{26}$$

The critical layer is at $\bar{\sigma} = \bar{\sigma}_c = \bar{\beta}$. Of course, $H_I(\bar{\sigma}) \equiv 0$ for $\bar{\sigma} < \bar{\beta}$.

We now turn to the derivation of a complementary set of solutions for (2-5). The approach is based on the results of characteristic theory for inviscid flows, in which jumps in tangential velocity across streamlines are permitted. Then for small viscosity, we expect derivatives across the base flow streamlines (y = constant) to be large. We set:

$$\begin{pmatrix} u \\ v \\ w \\ \pi \end{pmatrix} = \begin{pmatrix} \mu^{1/2} U_0 + \mu U_1 + \cdots \\ V_0 + \mu^{1/2} V_1 + \cdots \\ W_0 + \mu^{1/2} W_1 + \cdots \\ \mu^{3/2} \Pi_0 + \mu^2 \Pi_1 + \cdots \end{pmatrix} e^{\mu^{-1/2} \int_{x_*}^{r} dr\, g(r)}. \tag{27}$$

Substituting these in (2-5), and ordering in powers of $\mu^{1/2}$, the leading terms in each equation are homogeneous algebraic equations in $(\mathcal{U}_0, \mathcal{V}_0, \mathcal{W}_0, \Pi_0)$. For solutions to exist, the determinant of the coefficients must vanish. This leads to two determinations of $g(r)$:

$$\begin{cases} g_1(r) = A^{1/2} \\ g_2(r) = A^{1/2} e^{-i\pi}. \end{cases} \tag{28}$$

The next highest order set of equations leads, for each g, to two independent 1st order differential systems in $(\mathcal{U}_0, \mathcal{V}_0, \mathcal{W}_0, \Pi_0)$:

$$\begin{cases} 2r^2 g\, \mathcal{V}_0' + r^2 \mathcal{V}_0 g' + r g\, \mathcal{V}_0 = 0 \\ 2r^2 g\, \mathcal{W}_0' + r^2 \mathcal{W}_0 g' + r g\, \mathcal{W}_0 - \rho r^2 W' \mathcal{U}_0 = 0 \end{cases}$$

with the auxiliary specifications:

$$\begin{cases} \mathcal{U}_0 = -\dfrac{1}{rg}\left(im\, \mathcal{V}_0 + i\alpha_* r\, \mathcal{W}_0\right) \\ \Pi_0 = \dfrac{2}{(rg)^{1/2}}\left((rg)^{1/2}\,\mathcal{U}_0\right)' - \dfrac{2im}{r^2 g}\,\mathcal{V}_0. \end{cases}$$

Equations 25 and 26 together with the solutions of (17) enable us to write the general solution in the gas core:

$$v = \frac{C_1 e_1}{(r^2 A)^{1/4}} + \frac{i C_2 e_2}{(r^2 A)^{1/4}} + C_5 v_1 + C_6 v_2$$

$$w = -\frac{m \displaystyle\int_{x_*}^{r} \frac{dr}{r} \frac{W'}{(W-c_*)^{1/2}}}{2\alpha_* (W-c_*)^{1/2} (r^2 A)^{1/4}} \left(C_1 e_1 + i C_2 e_2\right)$$

$$+ \frac{1}{(r^2 A)^{1/4} (W-c_*)^{1/2}} \left(C_3 e_1 + i C_4 e_2\right)$$

$$+ C_5 w_1 + C_6 w_2$$

$$u = -\frac{i\alpha_* r \mu^{1/2}}{(W-c_*)^{1/2} (r^2 A)^{3/4}} \left(C_3 e_1 - i C_4 e_2\right)$$

$$- \frac{im\mu^{1/2}}{(r^2 A)^{3/4}} \left(1 - \frac{\frac{1}{2} r}{(W-c_*)^{1/2}} \int_{x_*}^{r} \frac{dr}{r} \frac{W'}{(W-c_*)^{1/2}}\right) \left(C_1 e_1 - i C_2 e_2\right)$$

$$+ C_5 u_1 + C_6 u_2$$

$$\pi = C_5 \pi_1 + C_6 \pi_2 \qquad (29)$$
$$+ \mu^{3/2} L(C_1, C_2, C_3, C_4).$$

Here $$e_{1,2} = \exp\left\{ \pm \mu^{-1/2} \int_{X_*}^{r} dr \, A^{1/2} \right\}.$$

The latter term in π need not be explicitly given. A similar solution can be written for the liquid film.

Clearly, the solutions (29) are singular at $r = (0, r_c)$, but the conditions (18) determine their correct continuations.

However, since boundary conditions on the general solution must be applied at $r = (0, X_*, L)$, it is desirable to derive solutions which are regular at these points without restricting a priori the location of the critical layer. Furthermore, this analysis will determine the continuation conditions (18).

For convenience later in joining solutions, we shall show how the solutions (29) behave as $r \to 0$ and $r \to X_*$.

$\underline{r \to 0}$:

Writing $A(0) = A_0$, and

$$E_{1,2} = e^{\pm \mu^{-1/2} A_0^{1/2} r}$$

$$P_0 = e^{-\mu^{-1/2} \int_0^{X_*} dr \, A^{1/2}}$$

$$P_1 = - \int_0^{X_*} dr \, \frac{1}{r} \frac{W'}{(W - c_*)^{1/2}}$$

we find as $r \to 0$:

$$v \rightarrow \frac{C_1 p_0}{A_0^{1/4}} \frac{E_1}{r^{1/2}} + \frac{i\, C_2}{p_0 A_0^{1/4}} \frac{E_2}{r^{1/2}} + C_5 v_1 + C_6 v_2$$

$$w \longrightarrow \frac{-m p_1}{2 \alpha_* (W_0 - c_*)^{1/2} A_0^{1/4}} \left\{ C_1 p_0 \frac{E_1}{r^{1/2}} + i\, C_2 \frac{E_2}{p_0 r^{1/2}} \right\}$$

$$+ \frac{1}{A_0^{1/4} (W_0 - c_*)^{1/2}} \left\{ C_3 p_0 \frac{E_1}{r^{1/2}} + i\, C_4 \frac{E_2}{p_0 r^{1/2}} \right\}$$

$$+ C_5 w_1 + C_6 w_2 \tag{30}$$

$$u \longrightarrow \frac{-i \alpha_* \mu^{1/2}}{A_0^{3/4} (W_0 - c_*)^{1/2}} \left\{ C_3 p_0 \frac{E_1}{r^{1/2}} - i\, C_4 \frac{E_2}{p_0 r^{1/2}} \right\}$$

$$- \frac{i m \mu^{1/2}}{A_0^{3/4}} \left\{ C_1 p_0 \frac{E_1}{r^{3/2}} - i\, C_2 \frac{E_2}{p_0 r^{3/2}} \right\}$$

$$+ C_5 u_1 + C_6 u_2$$

$$\pi \longrightarrow C_5 \pi_1 + C_6 \pi_2 + L \mu^{3/2}$$

$\underline{r \longrightarrow x_*}$:

Writing

$$A \rightarrow p + t(r - x_*), \qquad \left\{ \begin{array}{l} t \equiv i \alpha_* \rho W_x' \\ p = A(x_*) \neq 0 \end{array} \right.$$

$$\zeta = A \mu^{-1/3} t^{-2/3},$$

$$E_{3,4} = e^{\pm \frac{2}{3} \zeta^{3/2}}$$

$$P_2 = e^{-\frac{2}{3} \mu^{-1/2} t^{-1} p^{3/2}}$$

Then as $r \to X_*$:

$$v \to (\mu^{1/3} t^{2/3} X_*^2)^{-1/4} \left(\frac{C_1 E_3 P_2}{\varsigma^{1/4}} + \frac{i C_2 E_4}{P_2 \varsigma^{1/4}} \right)$$

$$+ C_5 v_1 + C_6 v_2$$

$$w \to (X_* W_x' \mu^{1/2})^{-1/4} \left(\frac{C_3 E_3 P_2}{\varsigma^{3/4}} + \frac{i C_4 E_4}{P_2 \varsigma^{3/4}} \right)$$

$$+ C_5 w_1 + C_6 w_2$$

$$u \to \frac{-i \alpha_* \mu^{1/12}}{(X_* W_x' t^{2/3})^{1/2}} \left(\frac{C_3 E_3 P_2}{\varsigma^{5/4}} - \frac{i C_4 E_4}{P_2 \varsigma^{5/4}} \right) \tag{31}$$

$$- \frac{i m \mu^{1/2}}{(X_*^6 \mu t^2)^{1/4}} \left(\frac{C_1 E_3 P_2}{\varsigma^{3/4}} - \frac{i C_2 E_4}{P_2 \varsigma^{3/4}} \right)$$

$$+ C_5 u_1 + C_6 u_2$$

$$\pi \to C_5 \pi_1 + C_6 \pi_2 + \mu^{3/2} L.$$

4. Singular Point $\underline{r = 0}$.

Regular solutions to the system (2–5) may be obtained by introducing the singular - layer coordinate:

$$z = r \mu^{-1/2}$$

and expanding the dependent variables and coefficients in the equations in powers of $r^{1/2}$:

$$u = \mathcal{U}_0(z) + \cdots$$

$$v = \mathcal{V}_0(z) + \cdots$$

$$w = r^{1/2}\, \mathcal{W}_0(z) + \cdots$$

$$\pi = r^{1/2}\, \Pi_0(z) + \cdots$$

$$W = W_0 + \mathcal{O}(r^2).$$

Substituting in (2–5), one can derive a single factorable 4th order equation in $\mathcal{U}_0$.

Putting $D \equiv \frac{d}{dz}$:

$$\left[D^2 + \frac{3}{z} D + \left(\frac{1-m^2}{z^2} - A_0\right)\right] Q = 0$$

$$\left[D^2 + \frac{3}{z} D + \frac{1-m^2}{z^2}\right] \mathcal{U}_0 = Q.$$

The remaining variables are determined by the 2nd order auxiliary system

$$\mathcal{V}_0 = -\frac{1}{im}\left(z\, \mathcal{U}_0' + \mathcal{U}_0\right)$$

$$\Pi_0 = \frac{z}{im}\left[\mathcal{V}_0'' + \frac{1}{z}\mathcal{V}_0' - \left(\frac{1+m^2}{z^2} + A_0\right)\mathcal{V}_0 + \frac{2im}{z^2}\mathcal{U}_0\right]$$

$$\left[D^2 + \frac{1}{z} D - \left(\frac{m^2}{z^2} + A_0\right)\right] \mathcal{W}_0 = i\alpha\, \Pi_0 + \rho\, W_0''\, z\, \mathcal{U}_0 .$$

These equations are readily solved in terms of Bessel functions. In applying the regularity conditions at r = 0, we must distinguish the cases m = 0 and m ≠ 0. The required solutions are as follows:

$$\mathcal{U}_0 = B_2\, z^{-1+m} + B_3 \left\{ z^{-1-m} \int_0^z d\varsigma\, \varsigma^{1+m} I_m(\varsigma A_0^{1/2}) - z^{-1+m} \int_0^z d\varsigma\, \varsigma^{1-m} I_m(\varsigma A_0^{1/2}) \right\} \qquad (32)$$

(m ≠ 0)

$$\mathcal{W}_0 = B_5\, I_m(z A_0^{1/2}) + \hat{L}(B_2, B_3)$$

the latter term depending on $\mathcal{U}_o$.

$$\mathcal{V}_o = B_1 I_1 (\mathfrak{z} A_o^{1/2})$$

$$\mathcal{W}_o = B_3 I_o (\mathfrak{z} A_o^{1/2})$$

$$\mathcal{U}_o = -i \alpha_* \mathfrak{z} \int_0^{\mathfrak{z}} d\varsigma \; \mathcal{W}_o(\varsigma A_o^{1/2}) \qquad (m = 0) \quad (33)$$

$$\Pi_o = \bar{B}_5 + \int_0^{\mathfrak{z}} d\varsigma \left[\mathcal{U}_o'' + \frac{1}{\varsigma} \mathcal{U}_o' - \left(A_o + \frac{1}{\varsigma^2}\right) \mathcal{U}_o \right].$$

From the known asymptotic behavior of the Bessel functions, one can deduce forms for these solutions as $|\mathfrak{z} A_o^{1/2}| \longrightarrow \infty$. These are matched with the corresponding asymptotic solutions evaluated as $r \to 0$ (30), and relations between the constants B_j, C_k are found. Elimination of the B_j gives three relations among the C_k required for regularity of the solutions on the axis. Details of this procedure are given in Appendix 3 where the notation is defined. We obtain the following:

$$\begin{cases} C_2 - C_1 P_o^2 = 0 \\ C_4 + C_3 P_o^2 = 0 \\ C_5 - \dfrac{i \alpha_* \sqrt{2\pi}\, \mu^{3/4} P_o i_{01}}{(W_o - c_*)^{1/2}} C_3 = 0 \end{cases} \qquad (m = 0) \quad (34)$$

$$\begin{cases} C_2 - \dfrac{\gamma^{**}}{\gamma^*} i P_o^2 C_1 = 0 \\ C_4 + P_o^2 e^{-im\pi} C_3 = 0 \\ C_5 + \dfrac{i_{m,1+m}}{\gamma^*} i P_o \mu^{\frac{1+2m}{4}} A_o^{\frac{1-m}{2}} C_1 = 0. \end{cases} \qquad (m \neq 0) \quad (35)$$

5. Critical Layer Solutions

We shall now derive a set of solutions to the system (2–5) which are regular in a neighborhood of the singular point, yet are of the same character as the asymptotic solutions outside such a neighborhood.

We shall assume that in a neighborhood of a fixed (though arbitrary) point r_s,

$$A = p + t(r - r_s) + \mathcal{O}((r-r_s)^2)$$

$$W' = W_s' + \mathcal{O}(r - r_s) = \frac{t}{i\alpha_* \rho} + \mathcal{O}(r - r_s)$$

and confine ourselves to a neighborhood of r_s in which the higher order terms are negligible. If $r_s = r_c$, $p = 0$; no restriction will be placed on the magnitude of p.

Let
$$\rho = (\mu t^2)^{-1/3} A .$$

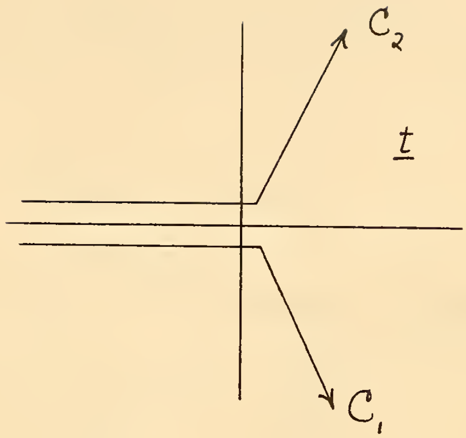
Let
$$\begin{pmatrix} v \\ w \\ u \\ \pi \end{pmatrix} = \begin{pmatrix} \mathcal{V}(\rho) \\ \mathcal{W}(\rho) \\ \mu^{1/3}\,\mathcal{U}(\rho) \\ \mu\,\Pi(\rho) \end{pmatrix} .$$

To the highest order in $\mu^{1/3}$, we obtain the 6th order system

$$\begin{cases} \mathcal{V}'' - \rho\,\mathcal{V} = 0 \\ \mathcal{U}''' - \rho\,\mathcal{U}' + \mathcal{U} = 0 \\ \mathcal{W} = -\frac{1}{i\alpha_*}\left(t^{1/3}\,\mathcal{U}' + \frac{im}{x_*}\,\mathcal{U}\right) \\ \Pi' = t^{1/3}(\mathcal{U}'' - \rho\,\mathcal{U}). \end{cases} \qquad (36)$$

The first two do not depend on the geometry of the problem. They occur in stability analyses in general, and lead to the well-known Tietjen's function and related functions.

Let



$$\begin{cases} Z_\kappa = \displaystyle\int_{C_1} \frac{dt}{t^\kappa}\, e^{\frac{1}{3}t^3 - \rho t} \\[2ex] \Upsilon_\kappa = \displaystyle\int_{C_2} \frac{dt}{t}\, e^{\frac{1}{3}t^3 - \rho t} \end{cases} \tag{37}$$

where the contour $C_2(\infty e^{i\pi}$ to $\infty e^{i\frac{\pi}{3}})$ passes over $t = 0$ and $C_1(\infty e^{-i\pi}$ to $\infty e^{-i\frac{\pi}{3}})$ passes under $t = 0$. We write the expressions:

$$\begin{aligned} v &= A_1 Z_0 + A_2 \Upsilon_0 + A_5 v_1 + A_6 v_2 \\ u &= A_3 Z_2 + A_4 \Upsilon_2 + A_5 u_1 + A_6 u_2 \\ w &= \frac{1}{i\alpha_*}\left(\frac{t}{\mu}\right)^{1/3}\left(A_3 Z_1 + A_4 \Upsilon_1\right) \\ &\quad - \frac{m}{\alpha_* x_*}\left(A_1 Z_0 + A_2 \Upsilon_0\right) + A_5 w_1 + A_6 w_2 \\ \pi &= (\mu^2 t)^{1/3}\left\{A_3(Z_4 + \rho Z_3 - Z_1) + A_4(\Upsilon_4 + \rho \Upsilon_3 - \Upsilon_1)\right\} \\ &\quad + A_5 \pi_1 + A_6 \pi_2 . \end{aligned} \tag{38}$$

When $A_5 = A_6 = 0$, this set satisfies (36). The inviscid solutions (π_1, π_2) are appended as extensions to the approximations to the inviscid solutions which are obtained from the pole contributions of (37).

It is readily shown by the method of steepest descents that with the specification (18):

$$\frac{\pi}{2} \leq \arg \rho \leq \frac{3\pi}{2}$$

we have the asymptotic evaluations:

$$Z_\kappa(\varsigma) \sim \sqrt{\pi}\,(-)^\kappa e^{\frac{2}{3}\varsigma^{3/2}} \varsigma^{-\frac{1+2\kappa}{4}} \cdot$$

$$\cdot\left[1 + \varsigma^{-3/2}\left(\frac{5}{48} + \frac{\kappa(\kappa+2)}{4}\right) + \mathcal{O}(\varsigma^{-3})\right]$$

$$Y_\kappa(\varsigma) \sim i\sqrt{\pi}\, e^{-\frac{2}{3}\varsigma^{3/2}} \varsigma^{-\frac{1+2\kappa}{4}} \cdot \tag{39}$$

$$\cdot\left[1 - \varsigma^{-3/2}\left(\frac{5}{48} + \frac{\kappa(\kappa+2)}{4}\right) + \mathcal{O}(\varsigma^{-3})\right]$$

Specifying $r_s = x_*$, ($r \leq x_*$) in (38), and using the asymptotic forms (39), we readily identify the coefficients A_j with the C_j of (31):

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \frac{1}{\sqrt{\pi}\,(\mu^{1/3} t^{2/3} x_*^2)^{1/4}} \begin{pmatrix} p_2 C_1 \\ C_2/p_2 \end{pmatrix}$$

$$\begin{pmatrix} A_3 \\ A_4 \end{pmatrix} = -\frac{i\alpha_* \mu^{\frac{1}{12}}}{\sqrt{\pi}\,(x_* W_x' t^{2/3})^{1/2}} \begin{pmatrix} -p_2 C_3 \\ C_4/p_2 \end{pmatrix} \tag{40}$$

$$\begin{pmatrix} A_5 \\ A_6 \end{pmatrix} = \begin{pmatrix} C_5 \\ C_6 \end{pmatrix}$$

Thus the solution is determined in the core near the core-film interface in terms of the C_j .

An entirely analogous procedure can be applied in the liquid film. That is, asymptotic forms analogous to (29) can be joined to critical layer forms analogous to (38) near the interface ($r \to x_*$, $r \geq x_*$) and near the wall ($r \to L$, $r \leq L$). It turns out, however, that with the approximation to be made in Section 6 on exponential orders of magnitude, the same result is obtained when it is assumed that a critical layer form is valid throughout the film and the evaluations of ς and t are made locally:

$$\hat{\varsigma} = (\hat{u}\,\hat{t}^{2})^{-1/3}\,\hat{A}$$
$$\hat{t} = \frac{d\hat{A}}{dr} \tag{41}$$

The $\hat{(\ )}$ refers to evaluations in the film. Thus throughout the film, we take:

$$v = \hat{A}_1 \hat{Z}_0 + \hat{A}_2 \hat{\varphi}_0 + \hat{A}_5 v_5 + \hat{A}_6 v_6$$
$$u = \hat{A}_3 \hat{Z}_2 + \hat{A}_4 \hat{\varphi}_2 + \hat{A}_5 u_5 + \hat{A}_6 u_6$$
$$w = \frac{1}{i\alpha_*}\left(\frac{\hat{t}}{\hat{u}}\right)^{1/3}\left(\hat{A}_3 \hat{Z}_1 + \hat{A}_4 \hat{\varphi}_1\right)$$
$$- \frac{m}{\alpha_* x_*}\left(\hat{A}_1 \hat{Z}_0 + \hat{A}_2 \hat{\varphi}_0\right) + \hat{A}_5 w_5 + \hat{A}_6 w_6 \tag{42}$$
$$\pi = (\hat{u}^{2}\hat{t})^{1/3}\left[\hat{A}_3\left(\hat{Z}_4 + \hat{\varsigma}\hat{Z}_3 - \hat{Z}_1\right) + \hat{A}_4\left(\hat{\varphi}_4 + \hat{\varsigma}\hat{\varphi}_3 - \hat{\varphi}_1\right)\right]$$
$$+ \hat{A}_5 \pi_5 + \hat{A}_6 \pi_6$$

where $(\hat{Z}_K, \hat{\varphi}_K) \equiv (Z_K(\hat{\varsigma}), \varphi_K(\hat{\varsigma}))$.

6. The Eigenvalue Problem

The regularity conditions (6) have given three linear homogeneous conditions (34, 35) on the C_j. Applying the wall conditions (7) to the film solution (38) leads to three linear homogeneous conditions in the A_j:

$$\begin{cases} \hat{A}_1 \hat{Z}_{0L} + \hat{A}_2 \hat{\varphi}_{0L} + \hat{A}_5 v_{5L} + \hat{A}_6 v_{6L} = 0 \\ \hat{A}_3 \hat{Z}_{2L} + \hat{A}_4 \hat{\varphi}_{2L} + \hat{A}_5 u_{5L} + \hat{A}_6 u_{6L} = 0 \\ \frac{1}{i\alpha_*}\left(\frac{\hat{t}}{\hat{u}}\right)^{1/3}\left(\hat{A}_3 \hat{Z}_{1L} + \hat{A}_4 \hat{\varphi}_{1L}\right) \\ \quad - \frac{m}{\alpha_* x_*}\left(\hat{A}_1 \hat{Z}_{0L} + \hat{A}_2 \hat{\varphi}_{0L}\right) + \hat{A}_5 w_{5L} + \hat{A}_6 w_{6L} = 0 \end{cases} \tag{43}$$

where $\hat{Z}_{jL} \equiv Z_j(\hat{\xi}(L))$.

The six interface conditions (15, 16) lead to six linear, homogeneous relations among the $(\hat{A}_j, C_j)$. We shall not write them down explicitly because of their length, but shall refer to them as (44) and (45) respectively:

$$\left\{ \begin{array}{l} \text{6 linear homogeneous relations in } (\hat{A}_j, C_j) \\ \text{derived from interface conditions (15, 16)} \end{array} \right\}. \qquad \begin{array}{r} (44) \\ (45) \end{array}$$

The twelve linear homogeneous relations (34 or 35, 43, 44, 45) suffice to determine non-trivial $(C_j, \hat{A}_j)$ to within a single scaling factor if the twelfth order determinant of the coefficient vanishes. This determinant is given explicitly in Appendix 2 in the following form:

	$\hat{A}_1, \ldots, \hat{A}_6$	$C_1, \ldots, C_6$
(44)	Interface Velocity Conditions	
(45)	Interface Shear Conditions	
(43)	Wall Conditions	0
(34) or (35)	0	Axis Conditions

$$= |A_{ij}| \qquad (i, j = 1, \ldots, 12).$$

In the determinant, we have used:

$$\hat{\mu} = m_0 \mu$$

$$\hat{\rho} = m_1 \rho$$

$$q_* = i\alpha_*(m_0 - 1) + \frac{\sigma^*}{\mu x_*(c_* - W_x)}$$

$$\beta_* = \frac{+1}{i\alpha_* \mu (W_x - c_*)} \left[(\tau_0')_{liquid} - (\tau_0')_{gas} \right]_{r = x_*}$$

The prime reduction to be made in this determinant is the elimination of exponentially small terms, i.e., terms down $O(e^{-\mu^{-1/2}\ell})$ for some ℓ from the leading terms. To do this, we first replace row 9, R_9, by (row 9)$'$; $R_9' = R_9 + \frac{m}{\alpha_* x_*} R_7$. This makes $A_{91} = 0$, $A_{92} = 0$,

$$A_{95} = w_{5L} + \frac{m}{\alpha_* x_*} v_{5L}, \qquad A_{96} = w_{6L} + \frac{m}{\alpha_* x_*} v_{6L},$$

and does not effect the other elements. Now we divide columns 1 through 4 by, respectively, ($\hat{Z}_{0x}$, $\hat{\varphi}_{0x}$, $\hat{Z}_{2x}$, $\hat{\varphi}_{2x}$) and columns 7 through 10 by, respectively, (A_{17}, A_{18}, A_{29}, $A_{2\,10}$), row 10 by A_{18}, row 12 by $A_{2,10}$. Then the first 6 rows are purely algebraic in μ (i.e., algebraic in ς as $\varsigma \to \infty$). On the other hand (39):

$$\frac{\hat{Z}_{KL}}{\hat{Z}_{nx}} \underset{|\varsigma|}{\sim} f_{Kn}(\hat{\varsigma}_L, \hat{\varsigma}_x)\, e^{\frac{2}{3}(\hat{\varsigma}_L^{\,3/2} - \hat{\varsigma}_x^{\,3/2})}$$

for integral (k,n), where f_{Kn} is algebraic in its arguments. Now the real part of $\varsigma^{3/2}$:

$$|\varsigma|^{3/2}\cos\left(\frac{3}{2}\arg\varsigma\right) = \left(\frac{\rho\,\alpha_*|W - c_*|}{\mu^{1/3}|t|^{2/3}}\right)^{3/2} \cdot \begin{Bmatrix} \cos\frac{9\pi}{4} & (r < r_c) \\ \cos\frac{3\pi}{4} & (r > r_c) \end{Bmatrix}$$

is monotonically decreasing with increasing r in each medium. Hence

$$\left|\frac{\hat{Z}_{KL}}{\hat{Z}_{nx}}\right| \underset{|\varsigma|}{\sim} |f_{Kn}|\, e^{-\mu^{-1/2}\delta}$$

for some positive δ independent of μ associated with L - x and is exponentially small as $\mu \to 0$. On the other hand,

$$\left|\frac{\hat{\varphi}_{KL}}{\hat{\varphi}_{nx}}\right| \underset{|\varsigma|}{\sim} |f_{Kn}|\, e^{\mu^{1/2}\delta}$$

is exponentially large as $\mu \to 0$. .

Furthermore,

$$|p_0| = \left|e^{-\mu^{-1/2}\int_0^{x_*} dr\, A^{1/2}}\right| \underset{\mu \to 0}{\sim} e^{\mu^{-1/2}\bar{X}}$$

is exponentially large as $\mu \to 0$, with $\bar{X}$ related to the core radius x. Then

$$\frac{p_0^2 A_{18}}{A_{17}} \sim \frac{p_0^2 A_{2\,10}}{A_{29}} \sim e^{2\mu^{-1/2}\bar{X}}$$

$$\frac{p_0}{A_{17}} \sim \frac{p_0}{A_{29}} \sim e^{\mu^{-1/2}\bar{X}}$$

Clearly the largest terms in the determinant will contain the factors:

$$A_{10\,7} \cdot A_{12\,9} \cdot A_{11\,11} \cdot A_{72} \cdot \begin{pmatrix} A_{84} \\ A_{94} \end{pmatrix}$$

which is

$$\approx e^{\mu^{-1/2}(2\bar{X} + 2\delta)} .$$

Then dropping the exponentially small terms, the vanishing of the 12 x 12 determinant leads to the vanishing of the 8 x 8 determinant obtained from the cofactor of

$$A_{10\,7} \cdot A_{12\,9} \cdot A_{11\,11} \cdot A_{72}$$

in which we also set

$$(A_{83}, A_{93}) = 0$$

$$(A_{14}, A_{24}, \ldots, A_{64}) = 0.$$

Now we replace row (3) by $\frac{m}{\alpha_* X_*}\Big[$row (1) + row (3)$\Big]$, row (5) by $\Big[$row (5) + $\frac{\alpha_* X_*}{m}$ row (6) $-\left(\frac{m_0 - 1}{X_*}\right)$ row (1)$\Big]$, and row (6) by $\Big[$row (6)$-\frac{m}{\alpha_* X_*}$ row (5)$\Big]$, in that order, and use the row and column multipliers:

row	mult	col	mult
1	1	1	1
2	m	3	$\frac{1}{m}$
3	m	4	$\frac{1}{m}$
4	$\frac{m}{X_* A_x} = \frac{m\, m_1}{X_* \hat{A}_x}$	5	1
5	$m^2 X_*$	6	1
6	$\frac{X_*}{m}$	8	1
8	m	10	$\frac{1}{m}$
9	m	12	1

We obtain the 8 x 8 determinant as shown in Appendix 2.

One can now examine algebraic orders of magnitude in μ as $\zeta \to \infty$, and expand the determinant in the form

$$(\det) = (\det)_0\, \mu^{-5/2} + (\det)_1\, \mu^{-2} + O(\mu^{-3/2}).$$

One obtains in this way, retaining the top two orders, a bilinear form

$$det = \sum B_i \bar{B}_i$$

of 11 terms, where the B_i are the 3 x 3 minors of the "inviscid" terms, (cols. 5, 6, 12), while the $\bar{B}_i$ are dependent on the viscous elements, (cols. 1, 3, 4, 8, 10). The result is still sufficiently complicated to make further analysis fruitless and require numerical methods for further evaluation.

The 8 x 8 determinant will be put in a final dimensionless form by setting

$$\begin{aligned} \xi &= r/L & \qquad \Sigma &= \frac{\sigma^*}{\hat{\rho} L (W_x)^2} \\ \alpha &= \alpha_* L & \qquad \frac{1}{\mathcal{F}^2} &= \frac{g^* L}{(W_x)^2} \\ c &= c_*/W_x \\ \mathcal{W}(\xi) &= W(r)/W_x \\ x &= x_*/L \\ \mathcal{R}e &= \frac{\hat{\rho} W_x L}{\hat{\mu}} \end{aligned} \tag{46}$$

so that Re is a Reynold's number, $\mathcal{F}$ the Froude number (dependent on sign g^*), and Σ a dimensionless surface tension parameter. We divide column 5 by $v_{5x} = -\frac{im}{x_* \hat{A}_x}$, column 6 by $u_{6x} = \frac{1}{L\hat{A}_x}$, and column 12 by Π_{2x}. We have, then, finally:

$$0 = \Delta = |A_{ij}| \tag{47}$$

$$
\Delta = \begin{vmatrix}
-1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\
0 & -1 & 0 & 0 & A_{26} & 0 & 1 & A_{2\,12} \\
0 & A_{33} & 0 & A_{35} & A_{36} & 0 & A_{3\,10} & A_{3\,12} \\
0 & A_{43} & 0 & A_{45} & A_{46} & 0 & A_{4\,10} & A_{4\,12} \\
0 & A_{53} & 0 & A_{55} & A_{56} & A_{58} & A_{5\,10} & 0 \\
A_{61} & A_{63} & 0 & A_{65} & A_{66} & A_{68} & 0 & 0 \\
0 & 0 & 1 & A_{85} & A_{86} & 0 & 0 & 0 \\
0 & 0 & A_{94} & A_{95} & A_{96} & 0 & 0 & 0
\end{vmatrix} . \qquad (48)
$$

The values of the A_{ij} follow.

(The validity of the basic reduction to an 8 x 8 determinant is difficult to assess. At some points of the critical curves computed, the exponential factors dropped were calculated to be down by only $e^{-2.5}$ from the factors retained, which is certainly not as much as one would wish. The effect of this on the location of the critical curves is, nevertheless, probably not very great, although it would be desirable to have some results for the complete 12 x 12 case.)

$$A_{61} = \frac{m_0}{\alpha}\ \frac{\hat{W}_x'}{1-c}\left(\hat{\rho}_x\ \frac{\hat{Z}_{-1x}}{\hat{Z}_{0x}}\right)$$

$$A_{33} = -\frac{1}{i\alpha}\ \frac{\hat{W}_x'}{1-c}\left(\hat{\rho}_x\ \frac{\hat{Z}_{1x}}{\hat{Z}_{2x}}\right) + \frac{W_x' - \hat{W}_x'}{i\alpha(1-c)}$$

$$A_{43} = -\frac{m_1\,\hat{W}_x'}{i\alpha x\,R_e\,(1-c)^2}\left(\hat{\rho}_x\ \frac{\hat{Z}_{1x} + \hat{\rho}_x\hat{Z}_{3x} + \hat{Z}_{4x}}{\hat{Z}_{2x}}\right)$$

$$+ \frac{m_1 Z}{x(1-c)^2}\left(1 + \frac{m^2-1}{\alpha^2 x^2}\right)$$

$$A_{53} = \alpha x\left[b + q\left(1 + \frac{m^2}{\alpha^2 x^2}\right)\right] + i m_0 x^2\left(\frac{\hat{W}_x'}{1-c}\right)^2\left(\frac{\hat{\rho}_x^2\,\hat{Z}_{0x}}{\hat{Z}_{2x}}\right)$$

$$A_{63} = -\frac{1}{\alpha^2 x}\ q$$

$$A_{94} = -\frac{1}{i\alpha}\left(\frac{\hat{W}_1'}{c}\right)\left(\frac{\hat{\rho}_L\ \hat{\varphi}_{1L}}{\hat{\varphi}_{2L}}\right)$$

$$A_{58} = -(m_0 - 1)\,m^2$$

$$A_{68} = -\frac{1}{\alpha}\ \frac{W_x'}{1-c}\left(\frac{\rho_x\,\varphi_{-1x}}{\varphi_{0x}}\right) + \frac{m_0-1}{\alpha x}$$

$$A_{3\,10} = \frac{1}{i\alpha}\ \frac{W_x'}{1-c}\left(\frac{\rho_x\,\varphi_{1x}}{\varphi_{2x}}\right)$$

$$A_{4\,10} = \frac{m_1 \mathcal{W}_x'}{i\alpha m_0 x \mathcal{R} (1-c)^2} \left(\rho_x \, \frac{\varphi_{1x} + \Gamma_x \varphi_{3x} + \varphi_{4x}}{\varphi_{2x}} \right)$$

$$A_{5\,10} = -i x^2 \left(\frac{\mathcal{W}_x'}{1-c} \right)^2 \left(\frac{\rho_x^2 \varphi_{0x}}{\varphi_{2x}} \right)$$

$$A_{26} = i x$$

$$A_{2\,12} = -i H$$

$$A_{35} = -\alpha x \left(1 + \frac{m^2}{\alpha^2 x^2} \right)$$

$$A_{36} = - \frac{x \mathcal{W}_x'}{\alpha (1-c)}$$

$$A_{3\,12} = \frac{\mathcal{W}_x'}{\alpha (1-c)} H + \alpha x \left(1 + \frac{m^2}{\alpha^2 x^2} \right)$$

$$A_{45} = -i m_1 - \frac{2\alpha m_1}{\mathcal{R} (1-c)} \left(1 + \frac{m^2}{\alpha^2 c^2} \right)$$

$$A_{46} = - \frac{i m_1 \Sigma}{(1-c)^2} \left(1 + \frac{m^2 - 1}{\alpha^2 x^2} \right) + \frac{2 m_1}{\alpha \mathcal{R} (1-c) x} \left(1 - \frac{x \hat{\mathcal{W}}_x'}{1-c} \right)$$

$$A_{4\,12} = i - \frac{2 m_1 \alpha}{m_0 \mathcal{R} (1-c)} \left[\frac{H}{\alpha^2 x^2} \left(1 - \frac{x \mathcal{W}_x'}{1-c} \right) - \left(1 + \frac{m^2}{\alpha^2 x^2} \right) \right]$$

$$A_{55} = -m_0 m^2$$

$$A_{56} = m_0 \alpha^2 x^3 \left(1 + \frac{m^2}{\alpha^2 x^2}\right) - i\alpha x^2 \left[\sigma + q\left(1 + \frac{m^2}{\alpha^2 x^2}\right)\right]$$

$$- m_0 x^2 \left(\frac{\hat{W}_x' - x\hat{W}_x''}{1-c}\right)$$

$$A_{5\,12} = -m^2(m_0 - 2) - \alpha^2 x^2 H\left[\left(1 + \frac{m^2}{\alpha^2 x^2}\right) + \frac{x^2 W_x'' - x W_x'}{\alpha^2 x^2 (1-c)}\right]$$

$$A_{65} = \frac{m_0}{\alpha}\left(\frac{1}{x} + \frac{\hat{W}_x'}{1-c}\right)$$

$$A_{66} = -x\left(\frac{m_0}{\alpha x} - \frac{iq}{\alpha^2 x^2}\right)$$

$$A_{6\,12} = \frac{1}{\alpha x}\left(m_0 - 2 - \frac{x W_x'}{1-c} + H\right)$$

$$A_{85} = i x \frac{1-c}{c} Q_{5L}$$

$$A_{86} = i x \frac{1-c}{c} Q_{6L}$$

$$A_{95} = -\frac{\alpha x(1-c)}{c}\left(1 + \frac{m^2}{\alpha^2 x}\right)\Pi_{5L} + \frac{x W_1'(1-c)}{\alpha c^2} Q_{5L}$$

$$A_{96} = \frac{x(1-c)\hat{\mathcal{W}}_1'}{\alpha c^2} Q_{6L} - \frac{\alpha x(1-c)}{c}\left(1 + \frac{m^2}{\alpha^2 x^2}\right) \Pi_{6L}$$

$$q = i\alpha x(m_0 - 1) - \frac{m_0 \mathcal{R}_e \Sigma}{1-c}$$

$$b = \frac{-ix}{\alpha(1-c)} \left(\frac{L^2}{\mu W_x}\right) \Big/ \left(\frac{d}{dr} \tau_0(r)\right)_{r = x_*} \Bigg|_{\text{gas}}^{\text{liquid}}$$

$$= \frac{-ixm_0}{\alpha(1-c)} \frac{m_1 - 1}{m_1} \frac{\mathcal{R}_e}{\mathcal{F}}$$

$$\zeta_x = z_1 e^{i \arg \zeta_x},$$

$$z_1 = |1-c| \left(\frac{m_0 \alpha \mathcal{R}_e}{m_1 (\mathcal{W}_x')^2}\right)^{1/3}$$

$$\hat{\zeta}_x = z_2 e^{i \arg \hat{\zeta}_x},$$

$$z_2 = |1-c| \left(\frac{\alpha \mathcal{R}_e}{(\hat{\mathcal{W}}_x')^2}\right)^{1/3}$$

$$\hat{\zeta}_L = z_3 e^{i \arg \hat{\zeta}_L},$$

$$z_3 = |c| \left(\frac{\alpha \mathcal{R}_e}{(\hat{\mathcal{W}}_1')^2}\right)^{1/3}$$

7. Base Flow

The inviscid disturbance solutions depend on the entire base flow velocity profile $\mathcal{W}(\xi)$, while the viscous solutions depend on certain local properties of $\mathcal{W}$. Strictly speaking a stability analysis has meaning only when applied to an exact steady solution of the Navier-Stokes equations. (It has been applied with success, however, to boundary layer flow when the streamlines are assumed to be parallel to the wall.) The velocity profile for laminar pipe flow is exactly a function of the radius only. Feldman's analysis, however, shows instability of a two-phase flow to disturbances of wave length associated with film thickness while instability in homogeneous Poiseuille flow occurs only for wave lengths comparable to the radius. Then to some extent, we may be able to distinguish in the thin film flow two types of instability, the one associated with the gas core, the other with the film-core interface. It may be possible that the core be turbulent while in a neighborhood of the thin film the turbulent fluctuations could be of so small a magnitude as to be contained in a linearized stability theory. Near the film in this case the turbulent core makes itself felt through a radical change in mean velocity profile and shear stress. Then in a non-rigorous way one might extend the stability analysis to thin film flows with velocity profiles of a turbulent type. This extension is not included in this note.

We shall adopt velocity profiles in the gas core and liquid film as follows:

$$\mathcal{W}_g(\xi) = K - f\,\xi^2 \tag{49}$$

$$\mathcal{W}_\ell(\xi) = -\mathcal{W}_1'(1-\xi) + \lambda(1-\xi)^2 \tag{50}$$

$$\mathcal{W}_g(x) = \mathcal{W}_\ell(x) = 1. \tag{51}$$

The core flow profile is of the correct form for laminar flow. A determination of the constants (K, f, W_1', λ) must be made.

The requirements $\frac{dp}{dz} = -\tau$ = constant, $W = W(r)$ only, lead to the base flow equation

$$\mu\left(W'' + \frac{1}{r}W'\right) + (\tau - \rho g^*) = 0, \tag{52}$$

which has the general solution in non-dimensional form

$$\mathcal{W}(\xi) = A_0 \ln \xi + A_1 - \Gamma \xi^2$$
$$\Gamma \equiv \frac{\gamma - \rho g^*}{4\mu} \frac{L^2}{W_x^2} . \tag{53}$$

In the gas core then

$$\mathcal{W}_g(\xi) = B_1 - \Gamma_g \xi^2$$
$$\Gamma_g = \frac{\gamma - \rho g^*}{4\mu} \frac{L^2}{W_x^2} \tag{54}$$

while in the liquid film, to 2nd order in film thickness:

$$\mathcal{W}_\ell(\xi) = A_0 \ln(1 - (1 - \xi)) + A_1$$
$$- \Gamma_\ell (1 - (1 - \xi))^2$$
$$\doteq A_0\left[(1 - \xi) + \frac{1}{2}(1 - \xi)^2\right] \tag{55}$$
$$+ \Gamma_\ell\left[2(1 - \xi) - (1 - \xi)^2\right]$$
$$\Gamma_\ell \equiv \frac{\gamma - \hat{\rho} g^*}{4\hat{\mu}} \frac{L^2}{W_x^2} .$$

We have the wall condition

$$\mathcal{W}_\ell(1) = 0, \tag{56}$$

the interface velocity conditions

$$\mathcal{W}_\ell(x) = \mathcal{W}_g(x) = 1 \tag{57}$$

and the interface shear condition

$$m_0 \mathcal{W}_\ell'(x) = \mathcal{W}_g'(x) \tag{58}$$

to determine the unknown constants. Identifying the resulting forms with equations 49 through 51, we find:

$$\mathcal{W}_g = K - f\,\xi^2 \tag{59}$$

$$\mathcal{W}_\ell = -\mathcal{W}_1'(1-\xi) + \lambda(1-\xi)^2$$

$$\left\{\begin{aligned}
K &= 1 + f x^2 \\
f &= \frac{m_0}{1-x^2} + \frac{m_0(m_1-1)}{2m_1}\,\frac{\mathcal{R}e}{\mathcal{T}^2}\,\frac{1-x}{(1+x)(2-x)} \\
\mathcal{W}_1' &= -\frac{2}{1-x^2} + \frac{m_1-1}{m_1}\,\frac{\mathcal{R}e}{\mathcal{T}^2}\,\frac{x(1-x)}{(1+x)(2-x)} \\
\lambda &= -\frac{1}{1-x^2} + \frac{m_1-1}{m_1}\,\frac{\mathcal{R}e}{\mathcal{T}^2}\,\frac{x}{(1+x)(2-x)} \\
\gamma &= \frac{W_x^2\,\hat{\rho}}{L}\left[\frac{4}{\mathcal{R}e\,(1-x^2)}\right. \\
&\qquad\left.+\frac{1}{\mathcal{T}^2}\left(1-\frac{m_1-1}{m_1}\,\frac{x(3-x)}{(1+x)(2-x)}\right)\right] \\
\mathcal{W}_x' &= -2fx \\
\hat{\mathcal{W}}_x' &= \mathcal{W}_1' - 2\lambda(1-x)
\end{aligned}\right. \tag{60}$$

8. Numerical Computation

It did not appear that anything much could be done analytically with this 8 x 8 complex determinant, even in the thin film case with linear velocity profiles, corresponding to Feldman's study. He, indeed, used numerical methods also.

The $\mu = 0$ limit, the so-called "inviscid limit," may be obtained from the determinant, and corresponds to setting (See Appendix 6):

$$\Delta_{\mu=0} = \begin{vmatrix} 0 & A_{26} & A_{2\,12} \\ A_{45} & A_{46} & A_{4\,12} \\ A_{85} & A_{86} & 0 \end{vmatrix} = 0$$

The same result is readily found by solving the inviscid boundary value problem from scratch. (Feldman goes to same length to excuse the fact that his two inviscid limits differ, when in fact they do not, as a simple algebraic manipulation shows!) While the inviscid solution cannot give a "critical Reynolds number" for stability, it locates any non-stable domains in the (α, $\mathcal{R}e$) plane which extend to infinity in $\mathcal{R}e$, and provides the inviscid limit of one of the two functions defined by the vanishing of the 8 x 8 complex determinant Δ;

$$\Delta = E(\alpha, \mathcal{R}e, c) + i\,F(\alpha, \mathcal{R}e, c) = 0$$

Searching this limit curve, F (α, ∞, c) = 0 in the approach used, enabled us to find a zero Δ (α, $\mathcal{R}e$, c) = 0 quite easily, even though the resulting critical Reynolds number $(\mathcal{R}e)_{cr}$ was not very large. Varying the neutral wave speed c by small steps, one is able to trace out a neutral stability curve:

$$\begin{cases} E(\alpha, \mathcal{R}e, c) = 0 \\ F(\alpha, \mathcal{R}e, c) = 0 \end{cases}$$

$$\begin{cases} \alpha = \alpha(c) \\ \mathcal{R}e = \mathcal{R}e(c) \end{cases}$$

defined by the parameter c . These curves have a nose defining a minimum ($\mathcal{R}_{cr}$, α_{cr}) for which the flow is unstable. The position of this nose, the critical $\mathcal{R} = \mathcal{R}_{cr}$, is then sought for other flows by small variation of the other parameters:

$$\begin{cases} h = 1 - x \\ m \\ m_0 \\ m_1 \\ \dfrac{\alpha}{t}^2 \\ \Sigma \end{cases}$$

All this would require a great deal of computing, and attempts to simplify Δ further were made.

First, all the elements A_{ij} were expanded in the top two orders in $\mathcal{R}$, and then Δ itself was expanded, retaining only the top two orders again. It was found that the order of magnitude of $\mathcal{R}_{cr}$, $\mathcal{O}$ (100), was far too small to justify the asymptotic expansion used, values of ζ of $\mathcal{O}$ (10^{-1}) occurring in some cases, where $\zeta = \mathcal{O}$ (10) would be required to use asymptotic forms for the $Y_i(\zeta)$, $Z_i(\zeta)$. Since full range numerical tabulations of Y and Z [6] are sparse in ζ, and with too few significant figures in the mid-range $2 \leq |\zeta| \leq 8$, and since complex routines for Ai (ζ), Bi (ζ) had recently been written and checked out [7] , further routines were written (see Appendix 4) to permit at least four-place evaluation of the required functions on $0 \leq |\zeta| < \infty$. Also, the entire determinant is calculated using single-precision complex arithmetic with no restriction on the order of $\mathcal{R}_{cr}$.

A more satisfactory simplification was the approximations made for H and for (Π_{5L}, Q_{5L} , Π_{6L} , Q_{6L}). When the inviscid equations for Π (ξ) and H ($\bar{\sigma}$) are solved numerically, typical times for these alone were a minute. The approximating forms used (see Appendix

E) are very good for thin films $h \leq 0.1$, and probably adequate to $h \approx 0.5$. Numerical checks have shown that $\mathcal{R}_{cr}$ is not significantly affected by using the approximate forms for thin films, but no comparisons were made at the thicker films values because of the long computing times required. Our primary interest is in the thin film anyway, as being the normal form in annular flow.

In its final form, in evaluation of the elements A_{ij} and Δ took about 0.003 minutes. A single $\mathcal{R}_{cr}(h)$ curve for $(0.001 \leq h \leq 0.3)$ took 30 minutes at a minimum.

In actual fact, the variations of the nose of the critical curve with respect to the parameters computed were:

$$\frac{\partial \mathcal{R}_F}{\partial h}, \quad \frac{\partial \mathcal{R}_F}{\partial m_0}, \quad \frac{\partial \mathcal{R}_F}{\partial m_1}, \quad \frac{\partial \mathcal{R}_F}{\partial \sigma}, \quad \frac{\partial \mathcal{R}_F}{\partial g}, \quad \frac{\partial \mathcal{R}_F}{\partial \beta}$$

where

$$g \equiv \frac{\mathcal{R}_F}{h\,\tilde{t}} \operatorname{sgn} g^* = \frac{\hat{\rho}}{\hat{\mu}} \sqrt{|g^*| L^3} \operatorname{sgn} g^*$$

$$\sigma \equiv \frac{\Sigma\, \mathcal{R}_F}{h^2} = \frac{\sigma^* L \hat{\rho}}{\hat{\mu}^2}$$

$$\beta \equiv \frac{m\alpha}{X}$$

defining, respectively, gravity and surface tension parameters independent of flow rates, and an interface azimuthal wave number factor, or "wave pitch". Also, the derivative:

$$\frac{\partial c_I}{\partial \mathcal{R}_F}$$

was computed on the nose curve to measure the rate of growth of the disturbance moving off the neutral line in the direction of increasing $\mathcal{R}_F$. It is found from the requirement:

$$\delta \Delta = \frac{\partial \Delta}{\partial \mathcal{R}_F} \delta \mathcal{R}_F + \frac{\partial \Delta}{\partial c} \delta c = 0$$

where $\delta c = \delta c_R + i\,\delta c_I$

and $c_I = 0$ on neutral curves. This complex equation leads to:

$$\frac{\partial c_I}{\partial \mathcal{R}_F} = \frac{\dfrac{\partial E}{\partial \mathcal{R}_F}\dfrac{\partial F}{\partial c} - \dfrac{\partial F}{\partial \mathcal{R}_F}\dfrac{\partial E}{\partial c}}{\left(\dfrac{\partial F}{\partial c}\right)^2 + \left(\dfrac{\partial E}{\partial c}\right)^2}$$

the partial derivatives being estimated by taking small independent changes in $\mathcal{R}_F$ and <u>real</u> c. We must expect $\dfrac{\partial c_I}{\partial \mathcal{R}_F} > 0$ if the flow is unstable for values of $\mathcal{R}_F$ exceeding the values on the neutral curve.

Because of the extensive computing required, most computing was done for the air-water case, taking

$$m_0 = 47.0$$

$$m_1 = 814.4$$

The automated computation process, curve following by using a linear-variation predictor-corrector method, broke down for thick films, and could not be continued. In the cases shown, all values appeared numerically sound; values showing instability have not been included.

To be more precise, starting from some known point on an (α_F, $\mathcal{R}_F$) curve, α_F was stepped by a small increment, then ($\mathcal{R}_F$, c) readjusted to lie on curve, using the complex equation

$$\delta\Delta = 0 = \frac{\partial \Delta}{\partial \alpha_F}\delta\alpha_F + \frac{\partial \Delta}{\partial \mathcal{R}_F}\delta\mathcal{R}_F + \frac{\partial \Delta}{\partial c}\delta c$$

to determine (δc, $\delta \mathcal{R}_F$), and small variations in (α_F, c, $\mathcal{R}_F$) to compute the partial derivatives. We continue to follow this curve through decreasing values of $\mathcal{R}_F$ until $\mathcal{R}_F$ increases, and call the smallest value$(\mathcal{R}_F)_{cr}$. In a similar way, starting from a point on one (α_F, $\mathcal{R}_F$) curve, one finds a point on another curve (α_F, $\mathcal{R}_F$)

corresponding to changing some parameter p into $p + \delta p$ by solving

$$\delta \Delta = 0 = \frac{\partial \Delta}{\partial p} \delta p + \frac{\partial \Delta}{\partial c} \delta c + \frac{\partial \Delta}{\partial \mathcal{R}_F} \delta \mathcal{R}_F$$

and this curve is again followed to a nose $((\mathcal{R}_F)_{cr}, (\alpha_F)_{cr})$. Breakdown of this procedure appears to occur when the estimates of partial derivatives fail to give useful predictions. While one would like to take smaller variation then to compute partial derivatives, one is then limited by the four to five place accuracy of the computations.

$\frac{\partial \mathcal{R}_F}{\partial p}$ At each point near the nose of an $(\alpha_F, \mathcal{R}_F)$ - curve, variations in the position of the critical curve for fixed α_F are made for parameters $h, m_0, m_1, \sigma, g, \beta$ as an estimate of the change to be expected $\frac{(\delta \mathcal{R}_F)_{cr}}{\delta p}$. Much of the irregularity of these derivatives is doubtless due to the irregularity of the determination of $(\mathcal{R}_F)_{cr}$, due to the discrete stepping of α_F.

9. Results

Because of the large number of parameters involved in this stability analysis and the large computing times involved, it has not been practical to generate neutral stability curves $(\alpha_F, \mathcal{R}_F)$ and critical curves $((\mathcal{R}_F)_{cr}, h)$ for the wide variety of materials and flow conditions common in modern engineering. For this reason, the results given are all for fluids with density ratio $m_1 = 814.4$ and viscosity ratio $m_0 = 47.0$, reasonably good values for an air-water system. The derivatives $\left(\frac{\partial \mathcal{R}_F}{\partial m_0}, \frac{\partial \mathcal{R}_F}{\partial m_1}\right)$ can be used to estimate the critical curve changes for fluids with somewhat different ratios m_0 and m_1.

Tables I through VII contain the computed data on the critical curves as functions of the basic parameters σ, h, β, g. The values of $(\mathcal{R}_F)_{cr}, c$ and $(\alpha_F)_{cr}$ given are those nearest the nose of the particular neutral stability curve $(\alpha_F, \mathcal{R}_F)$ obtained by stepping through discrete increments, a change of from one to five percent as seemed desirable. This discretization in α_F is the major cause of the irregularity in the differences in derivatives shown with respect to h, as was remarked earlier. Data are plotted in figures 1–9.

The tables are arranged in order of increasing surface tension parameter, $0 \leq \sigma \leq .23(10)^7$. Within each table, the non-gravity flows are given first, all variables given as functions of film thickness h for increasing values of wave pitch β. In the last two tables, Tables VI and VII , for which a gravity flow is given in each case, the $\beta = 0$ curve develops complexities at larger values at h which are difficult to trace with assurance because of the rapid changes. There is most certainly more than one curve involved, and consequent intersections evidenced by changes in sign of $\frac{\partial c_I}{\partial R_F}$; the linear extrapolation method cannot cope with these complications. For values of h smaller than those at which these problems arise, however, there is a change in the sign of $\frac{\partial R_F}{\partial \beta}$, indicating that the axially symmetric wave is not the least stable for larger values of h . The scattered values of β included beyond this indicate the locus of $\frac{\partial R_F}{\partial \beta} = 0$, and should give then the <u>lowest</u> critical Reynolds number $(R_F)_{cr}$ in its dependence on both h and β.

Trends are not, in general, easily characterized, and the following generalizing remarks should be checked in individual cases:

(1) In the absence of gravity, surface tension is destabilizing for very thin films, stabilizing for thick films*. For gravity flows, a second destabilizing trend occurs at the thickest films studied.

*(For the very thick films, $h \geqslant .15$, an unaccounted for irregularity in $\frac{\partial R_F}{\partial \sigma}$ with h occurs.)

(2) For very thin films, $\frac{\partial R_F}{\partial m_0}$ is positive while $\frac{\partial R_F}{\partial m_1}$ is negative. These signs change at approximately the same small value of h , so that for all larger h , increased viscosity ratio is destabilizing, increased density ratio is stabilizing.

(3) For very thin films, increasing the downward $(-g)$ acceleration due to gravity destabilized the flow, while for non-thin films it stabilizes the flow. (Again, an unexplained irregularity in $\frac{\partial R_F}{\partial \sigma}$ appears for very thick films.)

(4) Thin films are more stable for $\beta > 0$ than for $\beta = 0$, while for non-thin films, there are many cases in which the reverse is true.

Figure 10 is extracted from Charvonia's paper, and shows a disturbance instability line based on apparent droplet entrainment occurring in experimental air-water flows downward in vertical pipes. Charvonia's dimensionless parameters are thickness parameter T_p and liquid Reynolds number N_{Re_1} :

$$T_p = h^* \left[\frac{1}{g^*} \left(\frac{\hat{\rho}}{\hat{\mu}} \right)^2 \right]^{1/3}$$

$$N_{Re_1} = \frac{4 G'}{g^* \hat{\mu}}$$

where G' is the peripheral liquid flow rate:

$$G' = \frac{1}{2} h^* \hat{\rho} W_x$$

for thin films. Using his pipe radius $L = 1.25''$ and $\hat{\mu} = 2.09 \; 10^{-5} \left(\frac{slugs}{ft\text{-}sec}\right)$

$$T_p = 678. \, h$$

$$N_{Re_1} = 2 R_F .$$

The critical curves of figures 8 and 9 are plotted as curves 8 and 9 on figure 10. Charvonia's experiments lead to:

$$g = \frac{(g^* L^3)^{1/2} \hat{\rho}}{\hat{\mu}} \doteq 0.360(10)^5$$

$$\sigma = \frac{\sigma^* L \hat{\rho}}{\hat{\mu}^2} \doteq 0.230(10)^7$$

corresponding to curve 9. Curve 8 represents intermediate values for g and σ .

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11. Glossary

Dimensions are indicated in brackets: [L] length, [T] time, [M] mass, and combinations of them.

Subscripts x and L indicate evaluations at $\xi = x$ and $\xi = L$, respectively.

Cap, $(\hat{\ })$, implies a liquid film quantity.

$A(r^*) = i\alpha_* \rho (W - c_*)$: coefficient in linearized equations (2-4). $[ML^{-3}T]$

A_{ij}: elements of the stability determinant, Δ.

$b = \frac{i\chi m_0}{\alpha c} \frac{m-1}{m_1} \frac{\mathcal{R}}{\mathcal{F}^2}$: quantity in Δ, see p. 30. [1]

$c = c_R + i c_I$: complex wave speed; $c_I = 0$ on neutral curve. [1]

$c_* \doteq W_x c$. $[LT^{-1}]$

E: real (Δ)

F: imaginary (Δ)

$\mathcal{F}$: Froude number, $\frac{W_x \operatorname{sgn} g^*}{\sqrt{|g^*|L}}$. [1]

$\mathcal{G} = \frac{\hat{\rho}}{\hat{\mu}} \sqrt{|g^*|L^3} \operatorname{sgn} g^*$: a gravity number independent of flow rates. [1]

g^*: gravitational constant. $[LT^{-2}]$

h : film thickness, $1 - x^*/L$. [1]

H: parameter based on inviscid disturbances in the gas core. [1]

$i_{m,n}$: numbers defined by asymptotic expansions of viscous flow near $r = 0$ (see Appendix 3).

L: pipe radius. [L]

m : azimuthal mode number in wave factor $e^{im\varphi}$.

m_0 : viscosity ratio, $\hat{\mu}/\mu$. [1]

m_1 : density ratio, $\hat{\rho}/\rho$. [1]

p_0, p_2 : exponential factors occurring in viscous flow in core (see pp. 13-14).

$q = i\alpha\chi (m_0 - 1) - \frac{m_0 \mathcal{R} \Sigma}{1-c}$: quantity in A_{ij}, see p. 30. [1]

$q_* = q/x^*$. [L]

$Q(\xi) = \Pi'(\xi)$.

r : radial coordinate. [L]

$\mathcal{R}e = \dfrac{\hat{\rho}\, W_x\, L}{\hat{\mu}}$, pipe Reynolds number. [1]

$\mathcal{R}e_F = h\, \mathcal{R}e$, film Reynolds number. [1]

$(\mathcal{R}e_F)_{cr}$: critical film Reynolds number, a minimum with respect to α , dependent on other parameters. [1]

t^*: time coordinate [T]

$u(r)$, $v(r)$, $w(r)$: radial, azimuthal and axial velocity coefficients for disturbance wave factor $e^{i\alpha_*(z^* - c_* t^*) + im\varphi}$. [LT^{-1}]

W (r): undisturbed axial velocity profile. [LT^{-1}]

$\mathcal{W}(\xi) = W/W_x$.

x: x_*/L. [1]

x_*: undisturbed interface radius. [L]

$\mathcal{Y}(\zeta)$, $\mathcal{Z}(\zeta)$: functions describing flow near viscous critical layer $\mathcal{W} = c$; see (37).

z^*: axial coordinate. [L]

α: $\alpha_* L$.

α_*: wave number for axial component of wave factor $e^{i\alpha_* z^*}$. [L^{-1}]

$\beta = \dfrac{m}{\alpha x}$; interface azimuthal wave number factor, or "wave pitch". [1]

$\beta_* = b/x_*$.

γ: axial base flow pressure gradient: [$ML^{-2}T^{-2}$]

γ^*, γ^{**}: numbers defined by asymptotic expansions of viscous flow near $r = 0$ (see Appendix 3).

$\Delta = E + iF$: complex determinant whose vanishing defines neutral stability curves.

ζ : coordinate basic to viscous flow near critical layer $\mathcal{W} = c$.

μ: dynamic viscosity. [$ML^{-1}T^{-1}$]

$\xi = r/L$. [1]

$\pi(r)$: pressure coefficient for disturbance wave factor

$$e^{i\alpha_*(z^*-c_*t^*)+im\varphi}. \quad [ML^{-1}T^{-2}]$$

$\Pi(\xi)$: inviscid pressure coefficient.

$\Pi_2(\xi)$: pressure coefficient satisfying $\Pi_2(\xi) \to \xi^m$ as $\xi \to 0$.

$\Pi_5(\xi)$: pressure coefficient satisfying $(\Pi_5(x), \; Q_5(x)) = (1,0)$.

$\Pi_6(\xi)$: pressure coefficient satisfying $(\Pi_6(x), \; Q_6(x)) = (0,1)$.

ρ : density. [ML^{-3}]

$\sigma = \dfrac{\sigma^* L \hat{\rho}}{\hat{\mu}^2}$: a surface tension number independent of flow rates. [1]

σ^* = surface tension. [MT^{-2}]

$\Sigma = \dfrac{\sigma^*}{\hat{\rho} L (W_x)^2}$: Weber's number. [1]

$\sigma_{rr}^{\circ}(r)$: $\widehat{rr}$-component of base flow stress tensor. [$ML^{-1}T^{-2}$]

$\tau_0(r)$: $\widehat{rz}$-component of base flow stress tensor. [$ML^{-1}T^{-2}$]

φ: azimuthal coordinate, in radians.

Appendix 1

12 x 12 DETERMINANT

Elements of the basic 12 x 12 determinant are given in the following charts.

<table>
<tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th></tr>
<tr><td>1</td><td>$-\hat{Z}_{0x}$</td><td rowspan="6">{Column 1 with $Z \to \varphi$}</td><td>0</td><td rowspan="6">{Column 3 with $Z \to \varphi$}</td><td>$-v_{5x}$</td><td rowspan="6">{Column 5 with $(5) \to (6)$}</td></tr>
<tr><td>2</td><td>0</td><td>$-\hat{Z}_{2x}$</td><td>$-u_{5x}$</td></tr>
<tr><td>3</td><td>$\frac{m}{\alpha_* x_*}\hat{Z}_{0x}$</td><td>$-\frac{1}{i\alpha_*}\left(\frac{\hat{t}}{\hat{u}}\right)^{1/3}\hat{Z}_{1x} + \frac{W_x' - \hat{W}_x'}{i\alpha_*(W_x - c_*)}\hat{Z}_{2x}$</td><td>$-w_{5x} + \frac{W_x' - \hat{W}_x'}{i\alpha_*(W_x - c_*)}u_{5x}$</td></tr>
<tr><td>4</td><td>0</td><td>$\frac{i\sigma\alpha_*}{W_x - c_*}\left(1 + \frac{m^2-1}{\alpha_*^2 x_*^2}\right)\hat{Z}_{2x} - \hat{u}\left(\frac{\hat{t}}{\hat{u}}\right)^{1/3}\left\{\hat{Z}_{4x} + \hat{\zeta}_x\hat{Z}_{3x} + \hat{Z}_{1x}\right\}$</td><td>$2\hat{u}u_{5x}' + \frac{i\sigma\alpha_*}{(W_x - c_*)}\left(1 + \frac{m^2-1}{\alpha_*^2 x_*^2}\right)u_{5x} - \pi_{5x}$</td></tr>
<tr><td>5</td><td>$-m_0\left(\frac{\hat{t}}{\hat{u}}\right)^{1/3}\hat{Z}_{-1x} - \frac{m_0 - 1}{x_*}\hat{Z}_{0x}$</td><td>$\hat{Z}_{2x}\cdot\left\{\frac{im(m_0-1)}{x_*} + \frac{m\sigma}{\mu x_*^2\alpha_*(c_* - W_x)}\right\}$</td><td>$m_0 v_{5x}' - \frac{m_0-1}{x_*}v_{5x} + \left(\frac{im(m_0-1)}{x_*} + \frac{\sigma m}{\mu x_*^2\alpha_*(c_* - W_x)}\right)u_{5x}$</td></tr>
<tr><td>6</td><td>$\frac{m m_0}{\alpha_* x_*}\left(\frac{\hat{t}}{\hat{u}}\right)^{1/3}\hat{Z}_{-1x}$</td><td>$-\frac{m_0}{i\alpha_*}\left(\frac{\hat{t}}{\hat{u}}\right)^{2/3}\hat{Z}_{0x} + (\beta_* + q_*)\hat{Z}_{2x}$</td><td>$m_0 w_{5x}' + (\beta_* + q_*)u_{5x}$</td></tr>
</table>

	1	2	3	4	5	6
7	$\hat{z}_{0L}$		0		υ_{5L}	
8	0	$\{$Column 1 with $Z \to \varphi\}$	$\hat{Z}_{2L}$	$\{$Column 3 with $Z \to \varphi\}$	u_{5L}	$\{$Column 5 with $(5) \to (6)\}$
9	0		$\frac{1}{i\alpha_*}\left(\frac{\hat{t}}{\hat{u}}\right)^{1/3} \hat{Z}_{1L}$		$w_{5L} + \frac{m}{\alpha_* x_*} \upsilon_{5L}$	
10	0		0		0	
11	0		0		0	
12	0		0		0	

	7	8	9	10	11	12
1	$\frac{(\mu^{1/3} t^{2/3} x_*^2)^{-1/4}}{\sqrt{\pi}} p_2 Z_{0x}$		0		v_{1x}	
2	0		$\frac{i\alpha_* \mu^{\frac{1}{12}} p_2 Z_{2x}}{\sqrt{\pi}(t^{2/3} x_* W_x')^{1/2}}$		u_{1x}	
3	$-\frac{m p_2 Z_{0x}}{\alpha_* \sqrt{\pi}(\mu^{1/3} t^{2/3} x_*^6)^{1/4}}$	{Column 7 with $Z \to \Psi$}	$\frac{\mu^{-\frac{1}{4}} p_2 Z_{1x}}{\sqrt{\pi}(x_* W_x')^{1/2}}$	{Column 9 with $Z \to \Psi$}	w_{1x}	{Column 11 with $(1) \to (2)$}
4	0		$\frac{i\alpha_* \mu^{\frac{3}{4}} p_2}{\sqrt{\pi}(x_* W_x')^{1/2}} \cdot (Z_{4x} + \varsigma_x Z_{3x} + Z_{1x})$		$-2uu_{1x} + \pi_{1x}$	
5	$\frac{\mu^{-\frac{5}{12}} p_2 Z_{-1x}}{\sqrt{\pi}(x_*^2 t^{-2/3})^{1/4}}$		0		$-v_{1x}'$	
6	$-\frac{m t^{1/6} \mu^{-5/12} p_2 Z_{-1x}}{\alpha_* \sqrt{\pi}\, x_*^{3/2}}$		$\frac{\mu^{-\frac{7}{12}} t^{\frac{1}{3}} p_2 Z_{0x}}{\sqrt{\pi}(x_* W_x')^{1/2}}$		$-w_{1x}'$	

	7	8	9	10	11	12
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	$-P_0^2$ $-i \frac{\gamma^{**}}{\gamma^*} P_0^2$	1	0	0	0	0
11	0 $\frac{i P_0 i_{m,m+1} u^{\frac{1+2m}{4}}}{\gamma^* A_0^{\frac{m-1}{2}}}$	0	$\frac{i \alpha_* \sqrt{2\pi} u^{\frac{3}{4}} P_0 i_{01}}{(W_0 - c_*)^{1/2}}$ 0	0	1	0
12	0	0	$-P_0^2 e^{-im\pi}$	1	0	0

Upper Value, $m = 0$
Lower Value, $m \neq 0$

Appendix 2

INTERMEDIATE 8 x 8 DETERMINANT

Elements of the 8 x 8 determinant after the exponential reduction and certain sealing operations have been performed are given in the following charts. Quantities are still dimensional.

	1	3	4	5
1	-1	0	0	$-v_{5x}$
2	0	-1	0	$-m u_{5x}$
3	0	$-\frac{1}{i\alpha_*}\left(\frac{\hat{t}}{\hat{u}}\right)^{1/3} \frac{\hat{Z}_{1x}}{\hat{\hat{Z}}_{2x}}$ $+\frac{W_x' - \hat{W}_x'}{i\alpha_*(W_x - c_*)}$	0	$-m\left(w_{5x} + \frac{m}{\alpha_* x_*} v_{5x}\right)$ $+m \frac{W_x' - \hat{W}_x'}{i\alpha_*(W_x - c_*)} u_{5x}$
4	0	$-\frac{m_1 \hat{u}}{x_* \hat{A}_x}\left(\frac{\hat{t}}{\hat{u}}\right)^{1/3} \cdot$ $\cdot\left\{\frac{\hat{\hat{Z}}_{1x} + \zeta_x \hat{Z}_{3x} + \hat{Z}_{4x}}{\hat{\hat{Z}}_{2x}}\right\}$ $+\frac{i m_1 \sigma \alpha_*}{x_* \hat{A}_x (W_x - c_*)}\left(1 + \frac{m^2 - 1}{\alpha_*^2 x_*^2}\right)$	0	$\frac{i\alpha_* \sigma m m_1}{x_* \hat{A}_x (W_x - c_*)} \cdot$ $\cdot\left(1 + \frac{m^2-1}{\alpha_*^2 x_*^2}\right) u_{5x}$ $-\frac{m m_1}{x_* \hat{A}_x} \pi_{5x} + 2\hat{u} \frac{m m_1}{x_* \hat{A}_x} u_{5x}'$

	1	3	4	5
5	0	$\alpha_* x_*^2 \left[\beta_* + q_*\left(1 + \frac{m^2}{\alpha_*^2 x_*^2}\right)\right] + i x_*^2 m_0 \left(\frac{\hat{t}}{\hat{u}}\right)^{2/3} \frac{\hat{Z}_{0x}}{\hat{Z}_{2x}}$	0	$m \alpha_* x_*^2 u_{5x} \cdot \left[\beta_* + q_*\left(1 + \frac{m^2}{\alpha_*^2 x_*^2}\right)\right] + m_0 m^2 x_* v'_{5x} + m_0 m \alpha_* x_*^2 w'_{5x}$
6	$\frac{m_0}{\alpha_*}\left(\frac{\hat{t}}{\hat{u}}\right)^{1/3} \frac{\hat{Z}_{-1x}}{\hat{Z}_{0x}}$	$-\frac{1}{\alpha_*^2 x_*} q_*$	0	$-\frac{1}{\alpha_*}\left(m_0 v'_{5x} + \frac{m}{\alpha_* x_*} q_* u_{5x}\right)$
8	0	0	1	$m u_{5L}$
9	0	0	$\frac{1}{i\alpha_*}\left(\frac{\hat{t}}{\hat{u}}\right)^{1/3} \frac{\hat{\varphi}_{1L}}{\hat{\varphi}_{2L}}$	$m\left(w_{5L} + \frac{m}{\alpha_* x_*} v_{5L}\right)$

	6	8	1.0	12
1	{Column 5 with (5) → (6)}	1	0	v_{2x}
2		0	1	$m u_{2x}$
3		0	$\frac{1}{i\alpha_*}\left(\frac{t}{\mu}\right)^{1/3}\frac{\varphi_{1x}}{\varphi_{2x}}$	$m\left(\omega_{2x}+\frac{m}{\alpha_* x_*}v_{2x}\right)$
4		0	$\frac{\mu}{x_* A_x}\left(\frac{t}{\mu}\right)^{1/3}\cdot\left\{\frac{\varphi_{1x}+\Gamma_x\varphi_{3x}+\varphi_{4x}}{\varphi_{2x}}\right\}$	$\frac{m}{x_* A_x}\pi_{2x} - \frac{2\mu m}{x_* A_x}u'_{2x}$

	6	8	10	12
5	{Column 5 with (5) → (6)}	$-m^2(m_0-1)$	$-i x_*^2 \left(\frac{t}{\mu}\right)^{2/3} \frac{\Upsilon_{0x}}{\Upsilon_{2x}}$	$-m^2 X_* \cdot \left\{ v_{2x}' + \frac{m_0-1}{X_*} v_{2x} + \frac{\alpha_* X_*}{m} w_{2x}' \right\}$
6		$-\frac{1}{\alpha_*}\left(\frac{t}{\mu}\right)^{1/3} \frac{\Upsilon_{-1x}}{\Upsilon_{0x}} + \frac{m_0-1}{\alpha_* X_*}$	0	$\frac{1}{\alpha_*}\left(v_{2x}' + \frac{m_0-1}{X_*} v_{2x}\right)$
8		0	0	0
9		0	0	0

Appendix 3

Letting $z = \zeta A_0^{1/2}$, we have the asymptotic expansion:

$\left(-\frac{3\pi}{2} \leq \arg z \leq \frac{\pi}{2}\right)$

$$I_m(z) \underset{|z|}{\sim} \frac{e^z}{(2\pi z)^{1/2}}\left[1 + S_{1m}\left(\frac{1}{z}\right)\right]$$

$$+ \frac{e^{-z}}{(2\pi z)^{1/2}} e^{-i(m+\frac{1}{2})\pi}\left[1 + S_{2m}\left(\frac{1}{z}\right)\right]$$

where the S_{jm} are asymptotic series in $\frac{1}{z}$ which vanish as $|z| \to \infty$. Then for some constant i_{01}:

$$\int_0^z d\zeta\, \zeta\, I_0(\zeta) = i_{01} + \frac{z^{1/2}}{\sqrt{2\pi}}\left\{ e^z\left[1 + S_{10}^{(1)}\left(\frac{1}{z}\right)\right] + i e^{-z}\left[1 + S_{20}^{(1)}\left(\frac{1}{z}\right)\right]\right\}$$

by integration by parts involving new series $S_{j0}^{(1)}$.

For $m = 0$, we obtain the asymptotic forms

$$v \sim B_1 \frac{\mu^{1/4} A_0^{-1/4}}{\sqrt{2\pi}}\left(\frac{e^z}{r^{1/2}} + i\frac{e^{-z}}{r^{1/2}}\right)$$

$$w \sim B_3 \frac{\mu^{-1/4} A_0^{-1/4}}{\sqrt{2\pi}}\left(\frac{e^z}{r^{1/2}} - i\frac{e^{-z}}{r^{1/2}}\right)$$

$$u \sim B_3\left\{-\frac{i\alpha\mu^{1/2} i_{01}}{A_0 r} - \frac{i\alpha\mu^{1/4}}{\sqrt{2\pi}\, A_0^{3/4}}\left(\frac{e^z}{r^{1/2}} + \frac{i e^{-z}}{r^{1/2}}\right)\right\}$$

$$\pi \sim B_5 + L(B_3)$$

Comparing these with (30) for $m = 0$, and noting $\left\{ \begin{array}{l} u_1 \to -\dfrac{1}{A_0 r} \\ u_2 \to \text{Constant} \end{array} \right\}$ we obtain the relations

$$\frac{P_0 C_1}{A_0^{1/4}} = B_1 \frac{\mu^{1/4} A_0^{-1/4}}{\sqrt{2\pi}} = \frac{C_2}{P_0 A_0^{1/4}}$$

$$\frac{P_0 C_3}{(W_0 - c_*)^{1/2} A_0^{1/4}} = \frac{B_3}{\sqrt{2\pi}\, \mu^{1/4} A_0^{1/4}} = -\frac{C_4}{P_0 (W_0 - c_*)^{1/2} A_0^{1/4}}$$

$$-\frac{C_5}{A_0} = -\frac{i \alpha \mu^{1/2} i_{01} B_3}{A_0}\ ; \qquad C_6 = B_5$$

These provide the three relations (34).

For $m \geq 1$, the matching is a little more difficult. The constants $i_{m,p}$, $i^{*}_{m,p}$, $i^{**}_{m,p}$ are defined by the $|z| \to \infty$ result: $\left(-\frac{3\pi}{2} \leq \arg z \leq \frac{\pi}{2}\right)$

$$\int_0^z d\varsigma\, \varsigma^p I_m(\varsigma) \underset{|z|}{\sim} i_{mp} + \left[\frac{1}{\sqrt{2\pi}} z^{p-\frac{1}{2}} e^{z} - \frac{1}{\sqrt{2\pi}} z^{p-\frac{1}{2}} e^{-z} e^{-i\pi(m+\frac{1}{2})}\right]$$

$$+ \left[i^{*}_{mp} z^{p-\frac{3}{2}} e^{z} + i^{**}_{mp} z^{p-\frac{3}{2}} e^{-z}\right]$$

$$+ \mathcal{O}\left(z^{p-\frac{5}{2}} e^{\pm z}\right)$$

Using this in the relations (32) and their auxiliary relations, we obtain the asymptotic forms

$$u \underset{|z|}{\sim} \left[B_2 z^{m-1} + B_3 A_0^{-1/2}\left(i_{m,1+m} z^{-1-m} - i_{m,1-m} z^{-1+m}\right)\right]$$

$$+ z^{-3/2} e^{z}\left[B_3 A_0^{-1/2}\left(i^{*}_{m,1+m} - i^{*}_{m,1-m}\right)\right]$$

$$+ z^{-3/2} e^{-z}\left[B_3 A_0^{-1/2}\left(i^{**}_{m,1+m} - i^{**}_{m,1-m}\right)\right]$$

$$w \underset{|z|}{\sim} \mu^{1/2} B_5 \left[\frac{e^{z}}{(2\pi z)^{1/2}} + \frac{e^{-z} e^{-i\pi(m+\frac{1}{2})}}{(2\pi z)^{1/2}}\right] + \mathcal{O}(u, \pi)$$

For the matching, it suffices to consider the above relations with respect to their counterparts in (30). Since the exponential terms in $u(B_2, B_3)$ are correct to a factor $(1 + O(\frac{1}{z}))$ while $u(C_1, \ldots, C_4)$ in (30) is correct to a factor $(1 + O(r))$, the best match possible to the level of approximation taken is in the leading terms only, $u(C_1, C_2)$. The $u(C_3, C_4)$ terms are due to coupling with other functions in the $u(C_1, \ldots, C_4)$ - approximation and should be suppressed for this match. Then in w, the $w(C_1, C_2)$ portion is induced by (ω, u) coupling which is the part of $w(B_2, B_3, B_5)$ not explicitly given. The remaining part, $w(C_3, C_4)$, is not coupled to u (the $u(C_3, C_4)$ having been suppressed) and should be matched with the free part $w(B_5)$. We obtain then

$$-\frac{i m \mu^{1/2} p_0}{A_0^{3/4}} C_1 = B_3 A_0^{-5/4} \mu^{3/4} \left(i^*_{m,1+m} - i^*_{m,1-m} \right)$$

$$-\frac{m \mu^{1/2}}{p_0 A_0^{3/4}} C_2 = B_3 A_0^{-5/4} \mu^{3/4} \left(i^{**}_{m,1+m} - i^{**}_{m,1-m} \right)$$

and

$$\frac{p_0}{(W_0 - c_*)^{1/2} A_0^{1/4}} C_3 = \frac{1}{\sqrt{2\pi}} \mu^{3/4} B_5$$

$$\frac{i}{(W_0 - c_*)^{1/2} A_0^{1/4} p_0} C_4 = \frac{1}{\sqrt{2\pi}} A_0^{-1/4} \mu^{3/4} e^{-i\pi(m + \frac{1}{2})} B_5 .$$

For the inviscid parts, $[u(B_j), u(C_j)]$ matching gives, since $\left\{ u_1 \to \frac{m}{A_0} r^{-m-1}, \; u_2 \to -\frac{m}{A_0} r^{m-1} \right\}$:

$$\frac{m}{A_0} C_6 = \mu^{\frac{1+m}{2}} A_0^{-(1+\frac{m}{2})} i_{m,1+m} B_3$$

$$-\frac{m}{A_0} C_5 = \mu^{\frac{1-m}{2}} B_2 - \mu^{\frac{1-m}{2}} A_0^{-(1-\frac{m}{2})} i_{m,1-m} B_3 .$$

Eliminating the B_j from the above lead to (33), where

$$\begin{pmatrix} \gamma^* \\ \gamma^{**} \end{pmatrix} = \begin{pmatrix} i^*_{m,1+m} - i^*_{m,1-m} \\ i^{**}_{m,1+m} - i^{**}_{m,1-m} \end{pmatrix} .$$

Appendix 4

TIETJEN'S AND RELATED FUNCTIONS.

In the elements of the determinant A_{ij} occur the Tietjen's function and some related functions incorporating the viscous effects in the so-called "critical layers". Asymptotic expansions for large $\mathcal{R}$ are inadequate approximations on the neutral stability curves, while the tabulations of some of them given by Holstein[6] do not give adequate accuracy in the crossover range between convergent (small $\mathcal{R}$) and asymptotic representations. A computing routine was developed to compute these functions taking advantage of a routine written by Strand[7] giving Airy's functions $\mathcal{A}i(z)$ and $\mathcal{B}i(z)$ to eight decimal digits.

We introduce the notation:

$$Q_p(z\,;\,B,\,C) \equiv \left\{ \begin{array}{ll} \dfrac{Z_{p-1}(\zeta)}{Z_p(\zeta)} & (C=1) \\[2ex] \dfrac{\Upsilon_{p-1}(\zeta)}{\Upsilon_p(\zeta)} & (C=-1) \end{array} \right\}$$

where

$$\zeta = z\, e^{i\pi\left(1-\frac{B}{2}\right)}, \quad z \geq 0.$$

Then

$$\arg \zeta = \left\{ \begin{array}{ll} \frac{\pi}{2} & (B=1) \\ \frac{3\pi}{2} & (B=-1) \end{array} \right\}.$$

Putting

$$\arg \zeta_x = \pi\left(1-\frac{B_1}{2}\right), \qquad z_1 \equiv |\zeta_x|,$$

$$\arg \hat{\zeta}_x = \pi\left(1-\frac{B_2}{2}\right), \qquad z_2 \equiv |\hat{\zeta}_x|,$$

$$\arg \hat{\zeta}_L = \pi\left(1-\frac{B_3}{2}\right), \qquad z_3 = |\hat{\zeta}_L|,$$

we have:

	B_1	B_2	B_3
$c < 0$	-1	-1	-1
$0 < c < 1$	-1	-1	1
$c > 1$	1	1	1

and

$$\zeta_x = i B_1 z_1$$

$$\hat{\zeta}_x = i B_2 z_2$$

$$\hat{\zeta}_L = i B_3 z_3 .$$

We also have the identity:

$$\begin{pmatrix} Z_1 \\ Y_1 \end{pmatrix} + \zeta \begin{pmatrix} Z_3 \\ Y_3 \end{pmatrix} + \begin{pmatrix} Z_4 \\ Y_4 \end{pmatrix} = \frac{4}{3} \begin{pmatrix} Z_1 \\ Y_1 \end{pmatrix} + \frac{2}{3} \zeta \begin{pmatrix} Z_3 \\ Y_3 \end{pmatrix}$$

eliminating the need for determining Z_4, Y_4.
Let also

$$\mathcal{Q}_j(z; B, c) \equiv i B z \, Q_j(z; B, c).$$

The elements A_{ij} containing (Z_j, Y_j) functions take the following forms:

$$A_{61} = \frac{i m_0}{\alpha} \frac{\hat{W}_x'}{1-c} B_2 z_2 \mathcal{Q}_0(z_2; B_2, 1)$$

$$A_{33} = -\frac{\hat{W}_x'}{(1-c)\alpha} B_2 z_2 \mathcal{Q}_2(z_2; B_2, 1) + \frac{W_x' - \hat{W}_x'}{i\alpha(1-c)}$$

$$A_{43} = -\frac{2m_1}{x} \frac{\hat{W}_x'}{(1-c)^2 \alpha R} B_2 z_2 \left[\frac{2}{3} \mathcal{Q}_2(z_2; B_2, 1) + \frac{i}{3} \frac{B_2 z_2}{\mathcal{Q}_3(z_2; B_2, 1)} \right]$$

$$+ \frac{m_1 \Sigma}{x(1-c)^2} \left(1 + \frac{m^2 - 1}{\alpha^2 x^2} \right)$$

$$A_{53} = \alpha x\left(b + q\left(1 + \frac{m^2}{\alpha^2 x^2}\right)\right) - i m_o x^2 \left(\frac{\hat{W}_x'}{1-c}\right)^2 z_2^2 \mathcal{Q}_1(z_2; B_2, 1)\, \mathcal{Q}_2(z_2; B_2, 1)$$

$$A_{94} = -\frac{\hat{W}_1'}{\alpha c} B_3 z_3 \mathcal{Q}_2(z_3; B_3, -1)$$

$$A_{68} = \frac{m_o - 1}{\alpha x} - \frac{W_x'}{\alpha(1-c)} i B_1 z_1 \mathcal{Q}_o(z_1; B_1, -1)$$

$$A_{310} = \frac{W_x'}{\alpha(1-c)} B_1 z_1 \mathcal{Q}_2(z_1; B_1, -1)$$

$$A_{410} = \frac{2m_1}{m_o} \frac{W_x'}{\alpha x(1-c)^2 R} B_1 z_1 \left[\frac{2}{3}\mathcal{Q}_2(z_1; B_1, -1) + \frac{i}{3}\frac{B_1 z_1}{\mathcal{Q}_3(z_1; B_1, -1)}\right]$$

$$A_{510} = i x^2 \left(\frac{W_x'}{1-c}\right)^2 z_1^2 \mathcal{Q}_2(z_1; B_1, -1)\, \mathcal{Q}_1(z_1; B_1, -1).$$

The following formulae are readily verified:

$$Q_o(z; B, C) \equiv R(z; BC) + i\, C\, I(z; BC)$$

$$Q_o(z; 1, 1) = \tilde{Q}_o(z; 1, -1)$$

$$= -e^{i\frac{4\pi}{3}} \frac{\mathcal{A}i'(z e^{-i\frac{\pi}{6}})}{\mathcal{A}i(z e^{-i\frac{\pi}{6}})}$$

$$Q_o(z; -1, 1) = \tilde{Q}_o(z; -1, -1)$$

$$= -\frac{\mathcal{A}i'(z e^{i\frac{3\pi}{2}}) + i\, \mathcal{B}i'(z e^{i\frac{3\pi}{2}})}{\mathcal{A}i(z e^{i\frac{3\pi}{2}}) + i\, \mathcal{B}i(z e^{i\frac{3\pi}{2}})}$$

when R and I are real. In addition, we compute the function

$$\chi_3(z; B, C) = S(z; BC) + i\,C\,T(z; BC)$$

where S, T are real. Then with

$$\chi_2 \equiv (1 - \chi_3)/iBz$$

$$\chi_1 \equiv (Q_0 - \chi_2)/iBz$$

we have may derive:

$$Q_1 = 1/\chi_1$$

$$Q_2 = \chi_1/\chi_2$$

$$Q_3 = 2\chi_2/\chi_3 .$$

These are found from the Airy function properties:

$$\begin{pmatrix} Z_{p+1} \\ \varphi_{p+1} \end{pmatrix} = -\begin{pmatrix} Z_p' \\ \varphi_p' \end{pmatrix}$$ (Recursion formula)

$$\zeta \begin{pmatrix} Z_0 \\ \varphi_0 \end{pmatrix} = \begin{pmatrix} Z_{-2} \\ \varphi_{-2} \end{pmatrix}$$ (Airy equation)

giving

$$Q_p = \frac{p-1}{Q_{p-1}Q_{p-2} - \zeta} \qquad (p \geq 2).$$

$\chi_3(z\,;\,B\,,C)$ itself is found from the formula:

$$\chi_3(z;B,C) = \begin{pmatrix} \dfrac{2\,Z_3}{Z_0} & (C=1) \\ \\ \dfrac{2\,Y_3}{Y_0} & (C=-1) \end{pmatrix}$$

calculating Z_3 and Y_3 as follows:

BC = 1	Convergent series:	$0 \le z \le 7.5$
	Integral evaluation:	$7.5 < z \le 13.0$
	Asymptotic series:	$z > 13.0$
BC = -1	Convergent series:	$0 \le z \le 10.6$
	Asymptotic series:	$z > 10.6$

Tests indicate a minimum of four significant figures near the joint for BC = -1, five significant figures minimum elsewhere. The integrals used, and the basis for the asymptotic expansions, arise from the saddle point method applied to the defining integrals for (Z_p , Y_p). They are:

$$Z_p(\tau) = e^{\frac{2}{3}\tau^{3/2}}\,\tau^{-\frac{1+2p}{4}}(-)^p\int_{-\infty}^{\infty} d\sigma\, e^{-\sigma^2}\,\frac{e^{\sigma^3/3\,\tau^{3/4}}}{\left(1-\frac{\sigma}{\tau^{3/4}}\right)^p}$$

$$Y_p(\tau) = i\,e^{-\frac{2}{3}\tau^{3/2}}\,\tau^{-\frac{1+2p}{4}}\int_{-\infty}^{\infty} d\sigma\, e^{-\sigma^2}\,\frac{e^{-i\sigma^3/3\,\tau^{3/4}}}{\left(1+\frac{i\sigma}{\tau^{3/4}}\right)^p}$$

$$\left(\frac{\pi}{2} \le \arg\tau \le \frac{3\pi}{2}\right)$$

Appendix 5

THIN FILM APPROXIMATIONS FOR $(H, \Pi_{5L}, Q_{5L}, \Pi_{6L}, Q_{6L})$.

For the determination of $H = x \dfrac{Q_2(x)}{\Pi_2(x)}$ from the inviscid pressure equation

$$\Pi'' + \left(\frac{1}{\rho} - \frac{2W'}{W-c}\right)\Pi' - \left(\frac{m^2}{\rho^2} + \alpha^2\right)\Pi = 0$$

in the core, satisfying $\Pi(\rho) \longrightarrow \rho^m$ as $\rho \to 0$, we shall break the core into an inner core with $W = W_0$ constant for $0 \leq \rho \leq \rho_I$ and $m = 0$ assumed, and an outer core with linear velocity profile $W = 1 + W_x'(\rho - x)$ for $\rho_I < \rho \leq x$ and in which we drop the cylindrical terms $\left(\frac{1}{\rho}, \frac{m^2}{\rho^2}\right)$ from the equation. The constants ρ_I and W_0 are determined to give the same gas flow rate as the true profile:

$$W = 1 + f(x^2 - \rho^2) \qquad (0 \leq \rho \leq x),$$

and $W_0 = 1 + W_x'(\rho_I - x)$ gives continuity in W at $\rho = \rho_I$.

In the inner core,

$$\Pi(\rho) = a I_0(\alpha\rho)$$

$$Q(\rho) = \Pi'(\rho) = \alpha a I_{-1}(\alpha\rho).$$

Assuming $\alpha\rho_I >> 1$ as occurs, we shall use leading asymptotic terms at $\rho = \rho_I$:

$$\Pi(\rho_I^-) \doteq \frac{a e^{\alpha\rho_I}}{\sqrt{2\pi\alpha\rho_I}}$$

$$Q(\rho_I^-) \doteq \frac{a\alpha e^{\alpha\rho_I}}{\sqrt{2\pi\alpha\rho_I}}.$$

In the outer core, we have:

$$\Pi'' - \frac{2}{\rho - \bar{\rho}_c}\Pi' - \alpha^2\Pi = 0$$

where $\bar{\rho}_c$ is such that

$$1 - c = -W_x'(\bar{\rho}_c - x).$$

Thus $\bar{\xi}_c$ is the critical layer location assuming the same linear profile is valid to $\mathcal{W} = c$ i.e., if $1 < c < \mathcal{W}_0$. If $\mathcal{W}(\xi) = c$ in the film, $\bar{\xi}_c$ is not the true critical point ξ_c. Note $\mathcal{W}_x' = -2fx$. It is assumed that $\bar{\xi}_c > \xi_I$. Then

$$\Pi(\xi) = A\left(1 - \alpha(\xi - \bar{\xi}_c)\right) e^{\alpha(\xi - \bar{\xi}_c)}$$
$$+ B\left(1 + \alpha(\xi - \bar{\xi}_c)\right) e^{-\alpha(\xi - \bar{\xi}_c)}.$$

Joining (Π, Π') continuously at $\xi = \xi_I$, determines the ratio B/A:

$$\frac{B}{A} = \frac{-e^{2\alpha(\xi_I - \bar{\xi}_c)}}{1 + 2\alpha(\xi_I - \bar{\xi}_c)}.$$

Then we obtain the approximation

$$H_1(x;\alpha) \doteq \frac{\alpha^2 x(\bar{\xi}_c - x)\left[1 + \frac{B}{A} e^{-2\alpha(x - \bar{\xi}_c)}\right]}{\left[1 + \alpha(\bar{\xi}_c - x)\right] + \frac{B}{A}\left[1 + \alpha(x - \bar{\xi}_c)\right] e^{-2\alpha(x - \bar{\xi}_c)}}.$$

The mass flow calculation leads to

$$\xi_I = \frac{x}{4^{1/3}} \doteq 0.630\,x.$$

For the thin film limit $h \to 0$, another approximation may be made with

$$m = \alpha\beta$$
$$a = \alpha\sqrt{1 + \beta^2}.$$

We can, to principal order in h, obtain:

$$\Pi'' - \frac{2}{\xi - \bar{\xi}_c}\Pi' - a^2\Pi = 0.$$

The solution bounded as $(\xi - \bar{\xi}_c)a \to -\infty$ yields the approximation:

$$H_2 \doteq \frac{x\,a^2}{a + \dfrac{1}{\bar{\xi}_c - x}}$$

agreeing with the earlier approximation H_1 when $B e^{-2\alpha(x - \bar{\xi}_c)} = 0$.
Since the terms with $\frac{B}{A}$ are at best approximate the form we have used is the first with α set to a,

$$H = H_1(x, a).$$

The approximation is weakest for non-thin films with $\beta > 0$. This might be remedied by retaining $m \neq 0$ in the inner core solution, and using the asymptotic expansion associated with $\alpha \to \infty$ in the solution:

$$\Pi(\xi_I) = a \cdot I_{\alpha\beta}(\alpha\,\xi_I).$$

These expansions are very complicated and the result would still be approximate in the outer core even if the broken velocity-profile were exact, so this approximation was rejected.

For the liquid film, the profile is again linear, and the thin film case again leads to

$$\Pi'' - \frac{2}{\xi - \xi_c}\Pi' - a^2\Pi = 0$$

as above, where now we have

$$\xi_c = 1 - c(1 - x)$$

is the critical layer when $0 < c < 1$.
Setting

$$\eta_x = x - \xi_c$$

$$\eta_1 = 1 - \xi_c$$

and again using

$$a = \alpha\sqrt{1 + \beta^2},$$

the solutions satisfying the requirements:

$$\left(\Pi_5(x) = 1, \quad Q_5(x) = 0\right)$$

$$\left(\Pi_6(x) = 0, \quad Q_6(x) = 1\right)$$

give on $\xi = L$:

$$\begin{cases} \Pi_{5L} = \dfrac{\eta_1}{\eta_x}\left[\cosh a(\eta_1 - \eta_x) - \dfrac{1}{a\eta_x}\sinh a(\eta_1 - \eta_x)\right]. \\ \\ Q_{5L} = a\dfrac{\eta_1}{\eta_x}\sinh a(\eta_1 - \eta_x) \end{cases}$$

$$\begin{cases} \Pi_{6L} = \dfrac{a^2\eta_1\eta_x - 1}{a^3\eta_x^2}\sinh a(\eta_1 - \eta_x) \\ \qquad\qquad + \dfrac{a(\eta_1 - \eta_x)}{a^3\eta_x^2}\cosh a(\eta_1 - \eta_x) \\ \\ Q_{6L} = \dfrac{\eta_1}{\eta_x}\cosh a(\eta_1 - \eta_x) + \dfrac{\eta_1}{a\eta_x^2}\sinh a(\eta_1 - \eta_x). \end{cases}$$

Special care must be taken in computing Π_{6L} when $a(\eta_1 - \eta_x)$ is small since much cancellation occurs.

These thin film approximations cannot be expected to give good results if gravity causes significant film profile curvature.

Appendix 6

INVISCID LIMIT

As mentioned earlier, the inviscid stability limit may be found by a completely inviscid analysis, or by an order analysis as $\mu \to 0$, giving the result:

$$\begin{vmatrix} 0 & A_{26} & A_{2\,12} \\ A_{45} & A_{46} & A_{4\,12} \\ A_{85} & A_{86} & 0 \end{vmatrix} = 0$$

The values of these elements for $\mu = 0$ are as follows:

$$A_{26} = i\,x$$

$$A_{2\,12} = -iH$$

$$A_{45} = -i\,m_1$$

$$A_{46} = -\frac{i\,m_1 \Sigma}{(1-c)^2}\left(1 + \frac{m^2-1}{\alpha^2 x^2}\right)$$

$$A_{4\,12} = i$$

$$A_{85} = i\,x\,\frac{1-c}{c}\,Q_{5L}$$

$$A_{86} = i\,x\,\frac{1-c}{c}\,Q_{6L}.$$

The result is written simply as :

$$x + m_1 H \frac{Q_{6L}}{Q_{5L}} - m_1 H \frac{1}{(1-c)^2} \Sigma \cdot \left(1 + \frac{m^2-1}{\alpha^2 x^2}\right) = 0.$$

We use the thin film approximations derived in Appendix 5,

$$H \doteq \frac{x a^2}{a + \dfrac{2 f x}{1-c}}$$

since $1 - c = -W_x'(\xi_c - x)$.

Noting also

$$\eta_1 = 1 - \xi_c = c h$$

$$\eta_x = -h(1-c)$$

we have

$$Q_{5L} \doteq -a \frac{c}{1-c} \sinh a_F$$

$$Q_{6L} \doteq -\frac{c}{1-c} \cosh a_F + \frac{c}{(1-c)^2 a_F} \sinh a_F$$

where $a_F = a h$.
Then

$$\frac{Q_{6L}}{h Q_{5L}} \doteq \frac{\operatorname{ctnh} a_F}{a_F} - \frac{1}{(1-c) a_F^2}$$

and putting $\lambda \to 1$,

$$h H \doteq \frac{a_F^2}{a_F + \frac{2}{1-c}(h f)} .$$

Introducing

$$h f \doteq \frac{m_0}{2}\left[1 + \frac{m_1 - 1}{2 m_1} \frac{q^2 h^5}{\mathcal{R}_F} \operatorname{sgn} q^*\right]$$

for small h,

$$\frac{\Sigma}{h} = \frac{h \sigma}{\mathcal{R}_F} ,$$

we obtain the inviscid stability curve for thin films:

$$\begin{aligned} & a_F (1-c)^2 (1 + m_1 \operatorname{ctnh} a_F) \\ & \quad - (1-c)\left[m_1 - m_0\left(1 + \frac{m_1 - m_0}{2 m_1} \frac{q^2 h^5}{\mathcal{R}_F} \operatorname{sgn} q^*\right)\right] \\ & \quad - m_1 a_F^2 \frac{h \sigma}{\mathcal{R}_F}\left(1 + \beta^2 - \frac{1}{\alpha^2}\right) = 0. \end{aligned}$$

(Note that only the disturbance flow is assumed inviscid. The base flow must be viscous.)

When $\sigma = 0$, we have the neutral curve

$$c = 1 - \frac{m_1 - m_0\left(1 + \frac{m_1 - 1}{2m_1}\frac{g^2h^5}{R_F}\operatorname{sgn} g^*\right)}{a_F\left(1 + m_1 \operatorname{ctnh} a_F\right)}$$

which agrees with Feldman's result when $g = 0$. When $\sigma \neq 0$, the equation is quadratic. (Feldman's erroneous formulation of the interface equation again leads to a linear equation.) Since concurrent flow implies for the film

$$\hat{w}'(1) < 0, \quad \hat{w}'(x) < 0$$

we have the restrictions on g:

$$|g| < \sqrt{\frac{2m_1}{m_1 - 1} R_F h^5}$$

for small h. With the restriction to concurrent flow then, we have

$$m_1 - 2m_0 < m_1 - m_0\left(1 + \frac{m_1 - 1}{2m_1}\frac{g^2h^5}{R_F}\operatorname{sgn} g^*\right) < m_1 .$$

Thus in general, since $m_1 > 2m_0$, the sums of the roots (c_1, c_2) satisfies

$$(1 - c_1) + (1 - c_2) > 0 .$$

Then for waves not too long such that

$$\alpha > \frac{1}{\sqrt{1 + \beta^2}}$$

the roots (1 - c) are of opposite sign: one travels faster than the interface, one slower. For <u>long</u> waves such that

$$\alpha < \frac{1}{\sqrt{1 + \beta^2}}$$

the roots (1 - c) are both positive, each wave travelling more slowly than the interface.

For non-long waves, we have for the second solution for small surface tension:

$$c = 1 + \frac{m_1 a_F^2 \dfrac{h\sigma}{\mathcal{R}_F}\left(1 + \beta^2 - \alpha^{-2}\right)}{m_1 - m_0\left(1 + \dfrac{m_1 - 1}{2m_1}\dfrac{g^2 h^5}{\mathcal{R}_F}\operatorname{sgn} g^*\right)}.$$

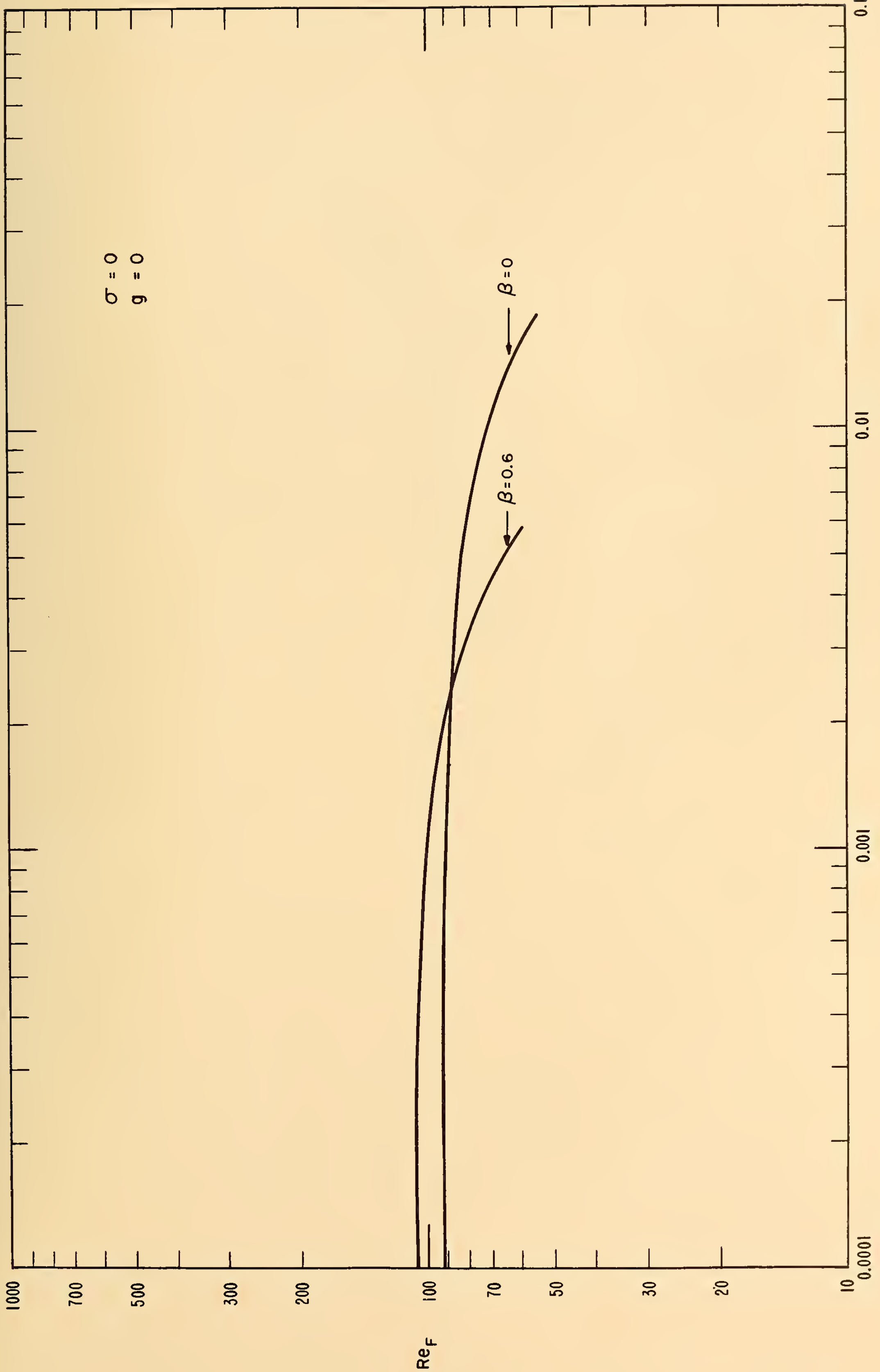

FIG. 1 CRITICAL CURVES

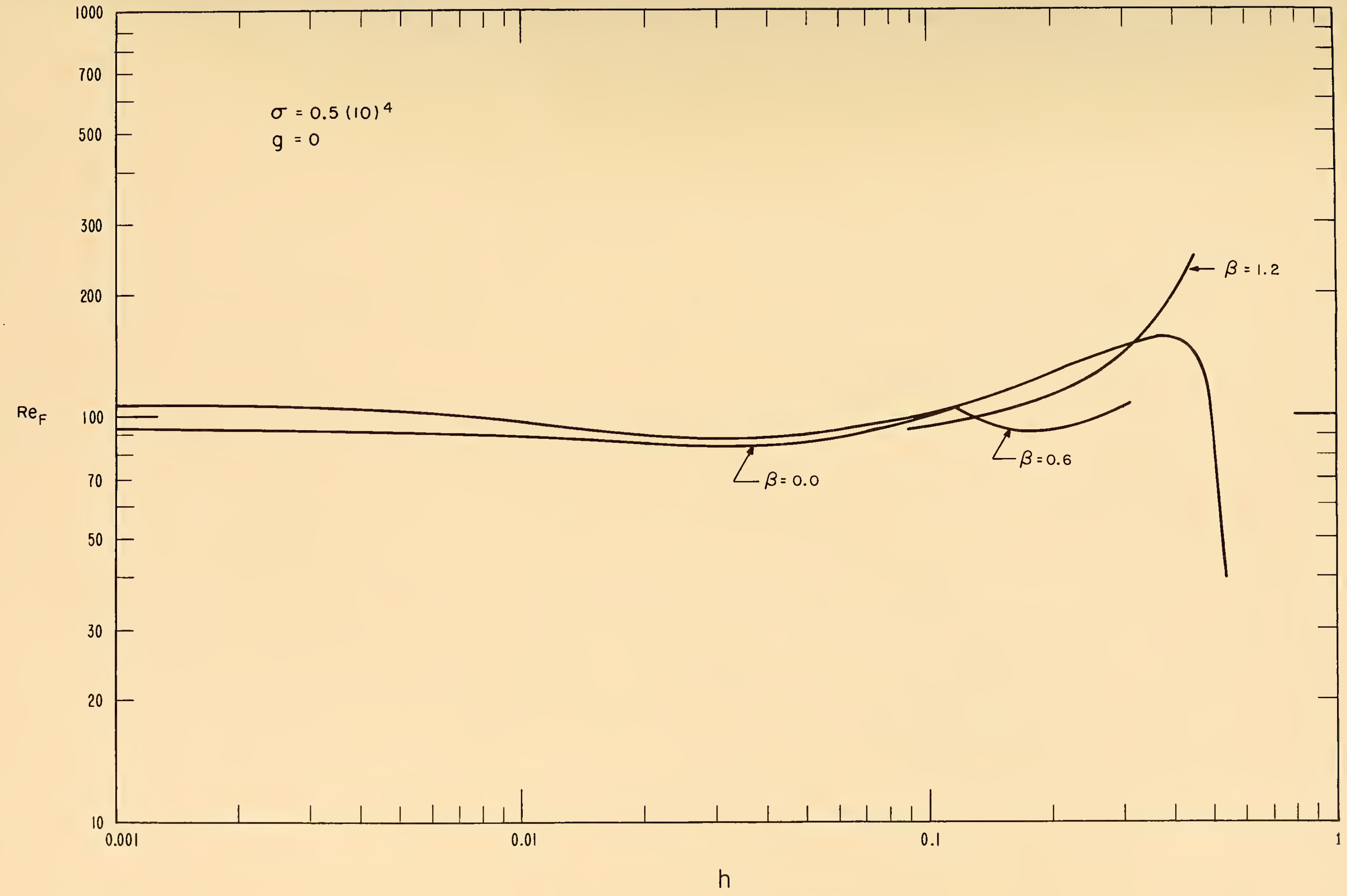

FIG.2 CRITICAL CURVES

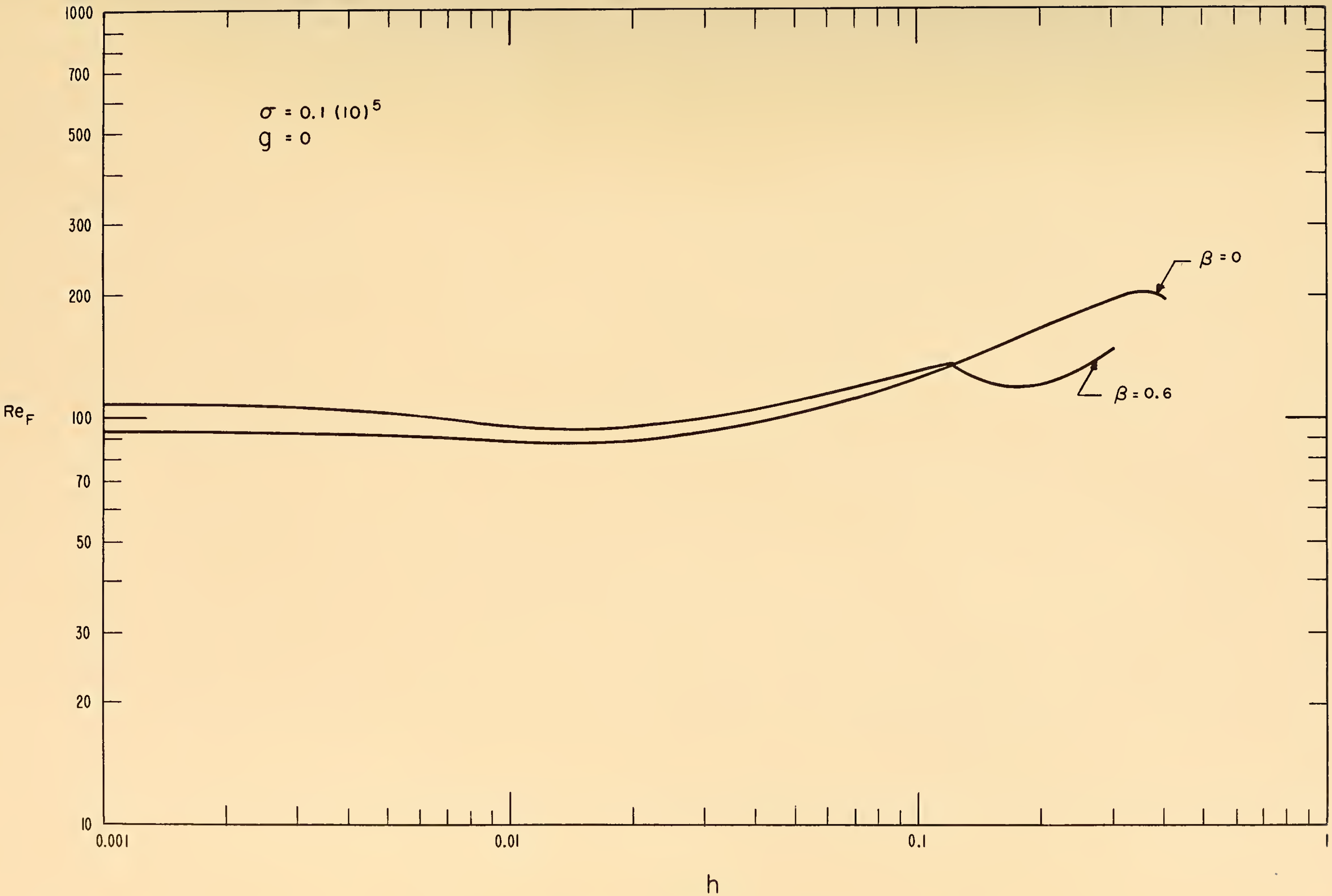

FIG. 3 CRITICAL CURVES

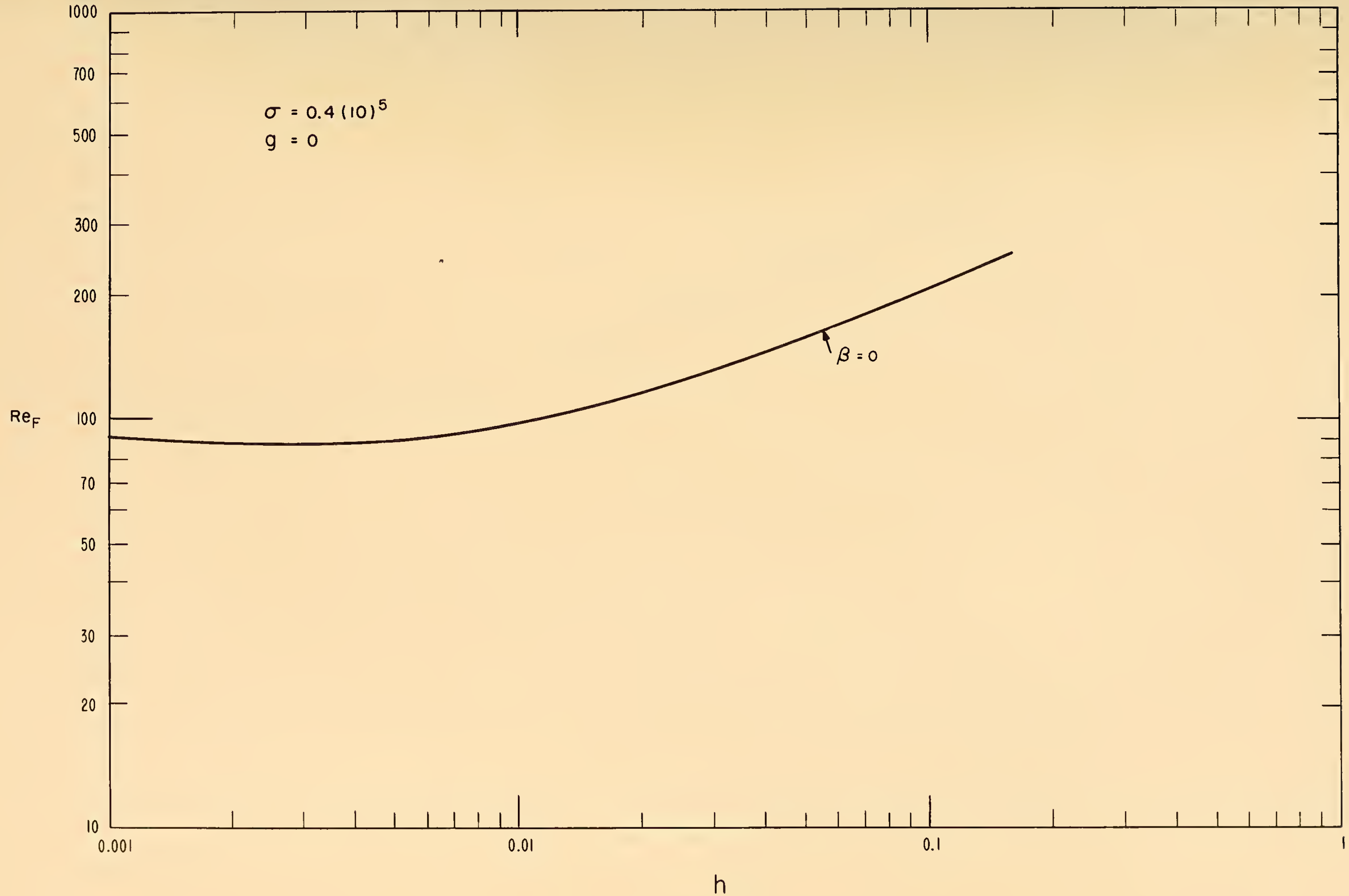

FIG. 4 CRITICAL CURVE

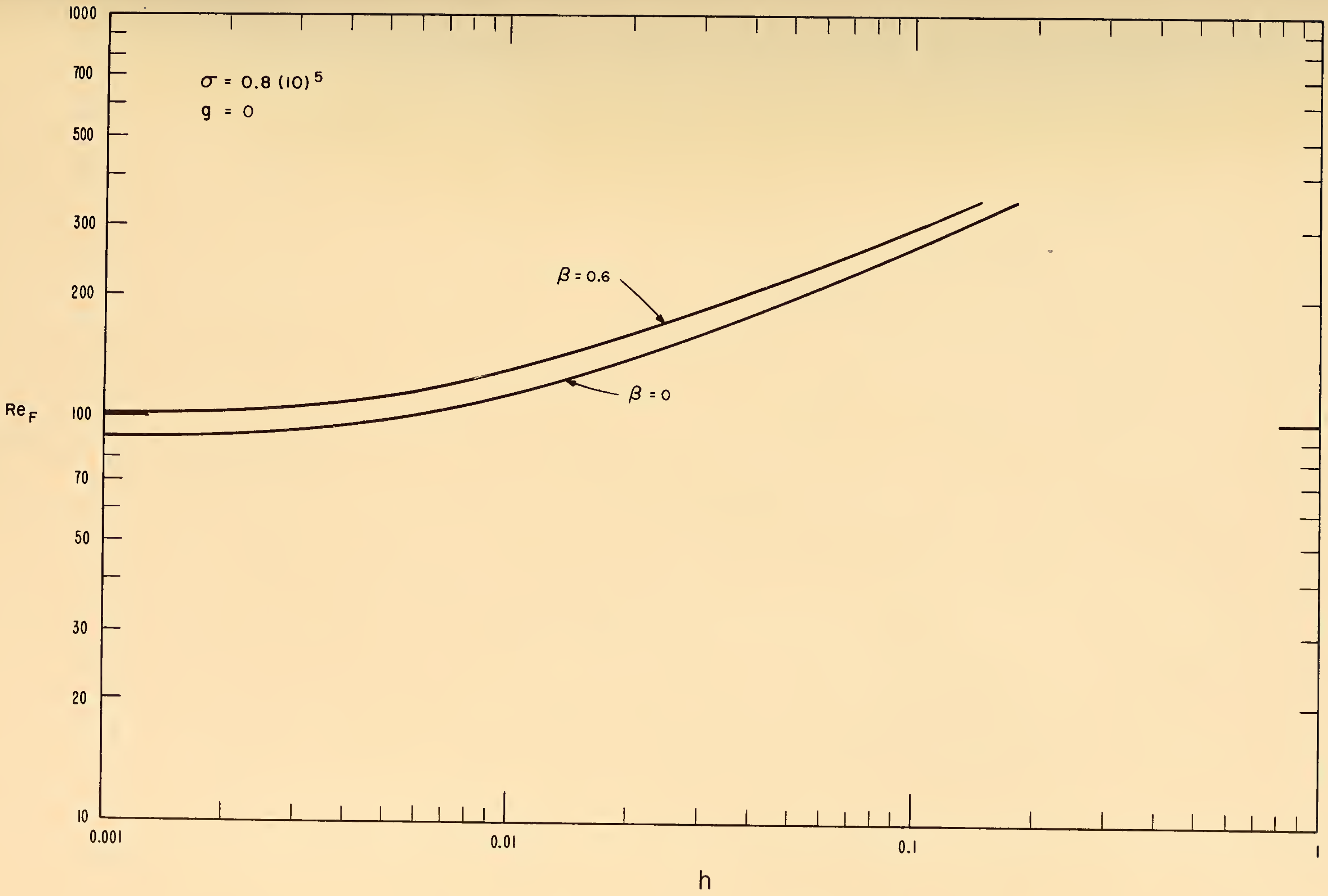

FIG. 5 CRITICAL CURVES

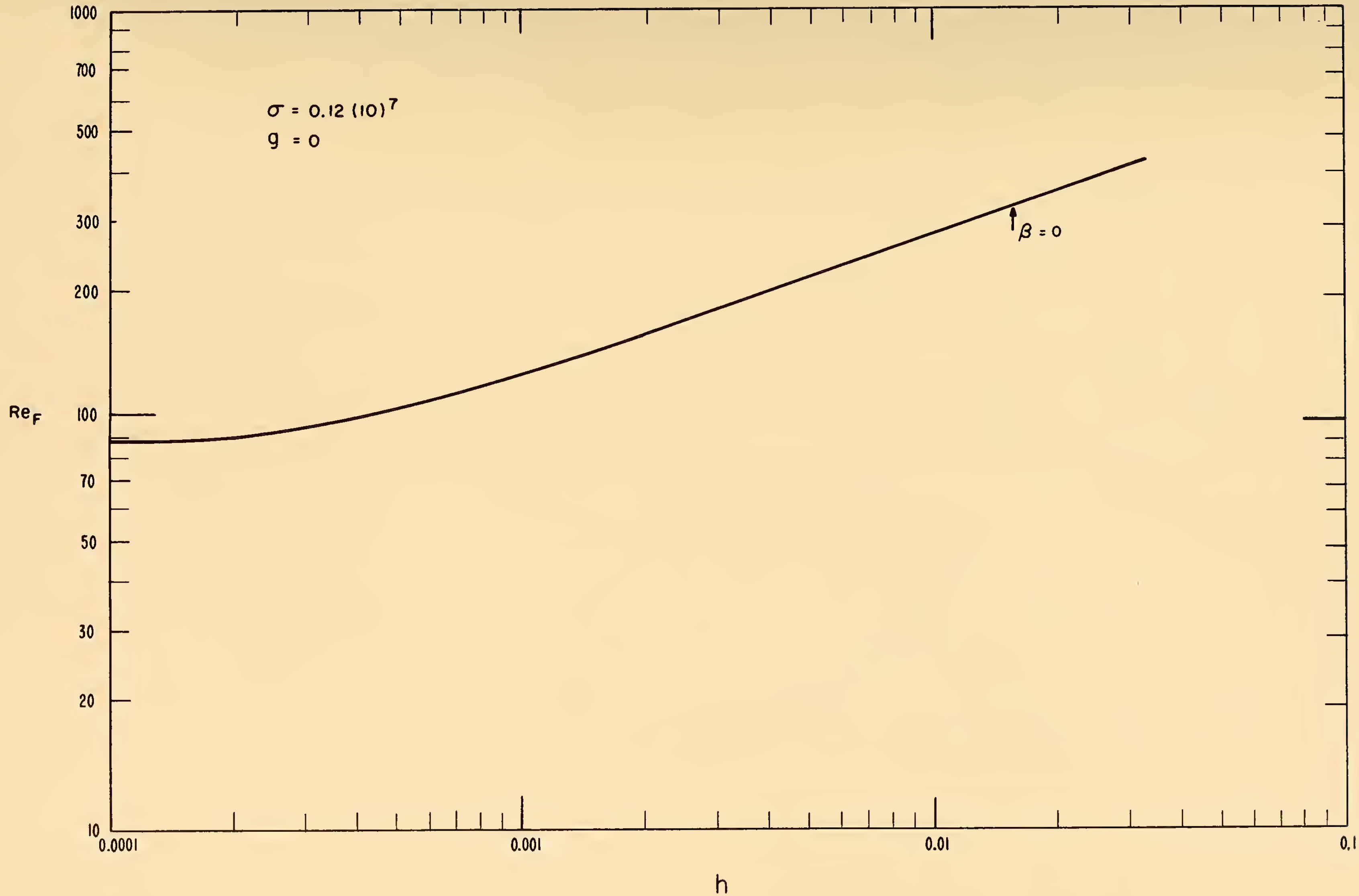

FIG. 6 CRITICAL CURVE

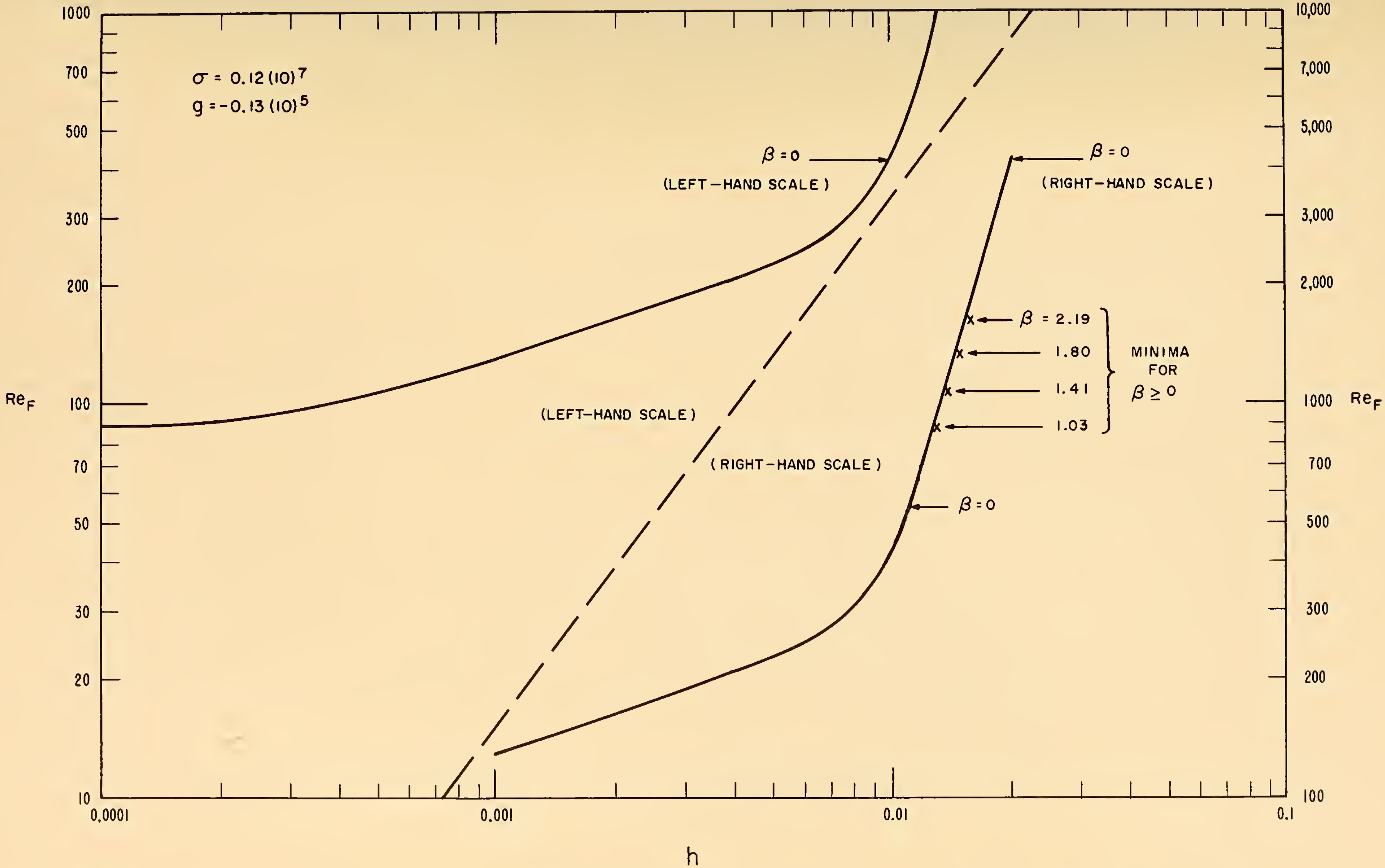

FIG.7 CRITICAL CURVES

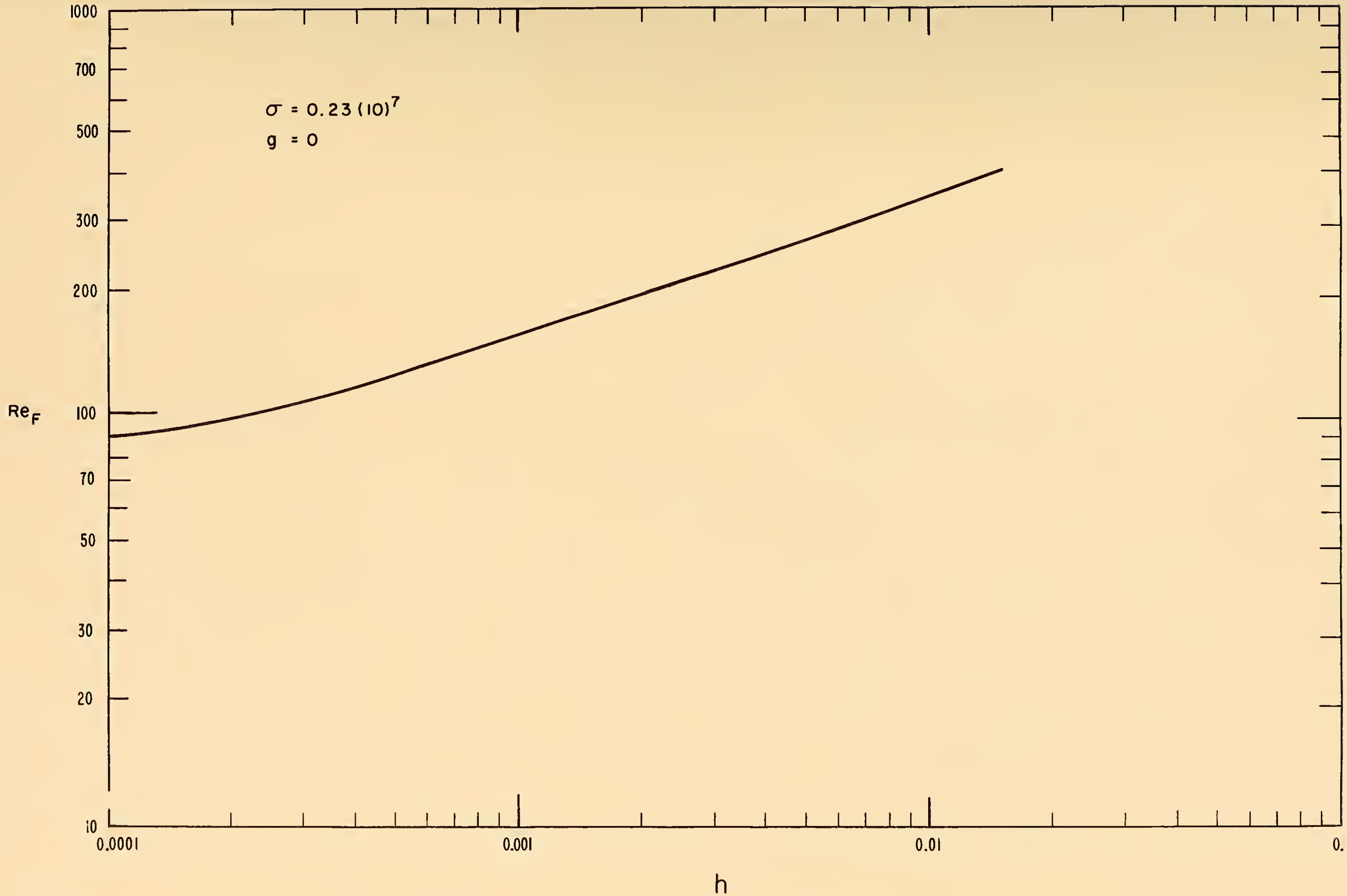

FIG. 8 CRITICAL CURVE

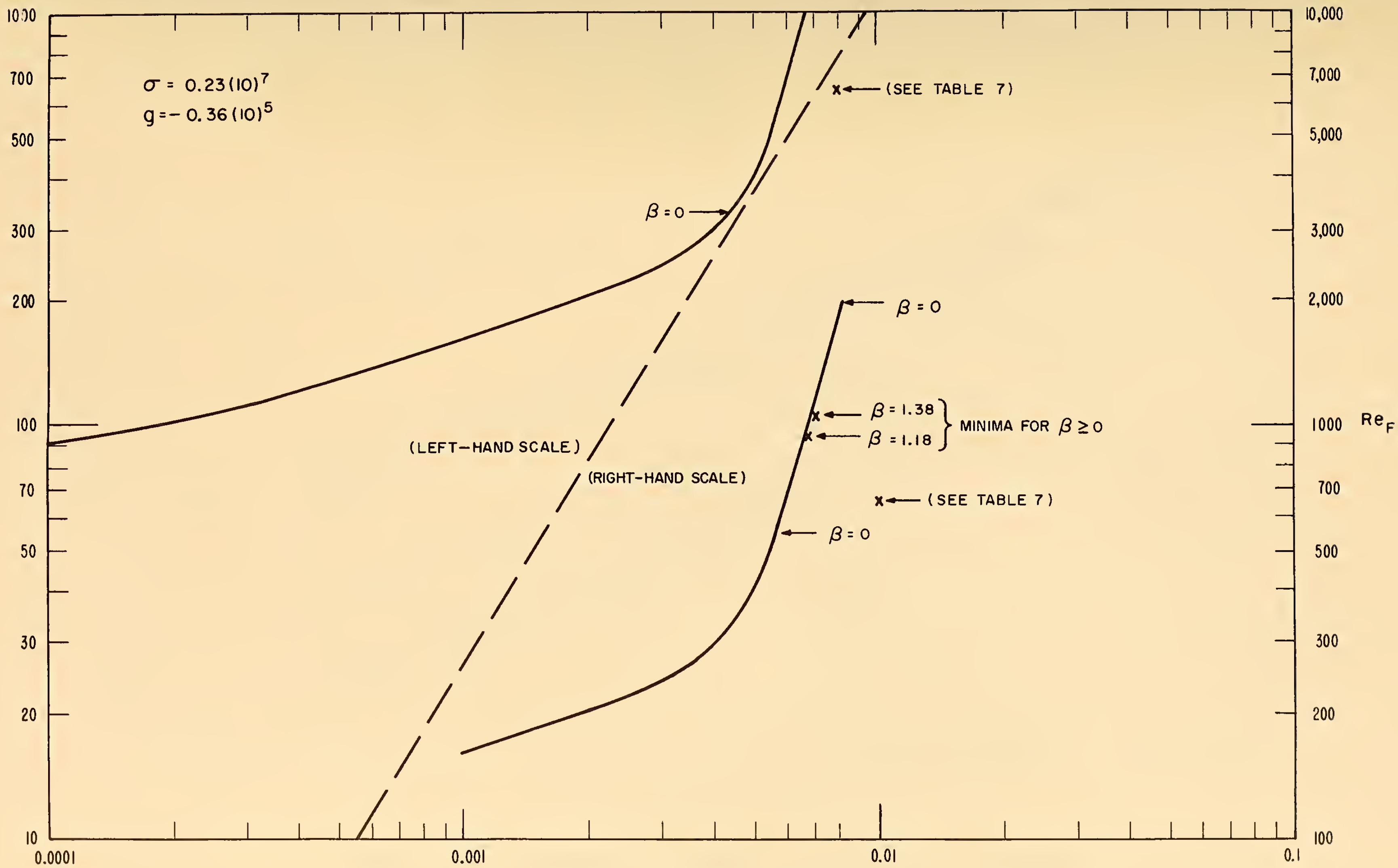

FIG. 9 CRITICAL CURVES

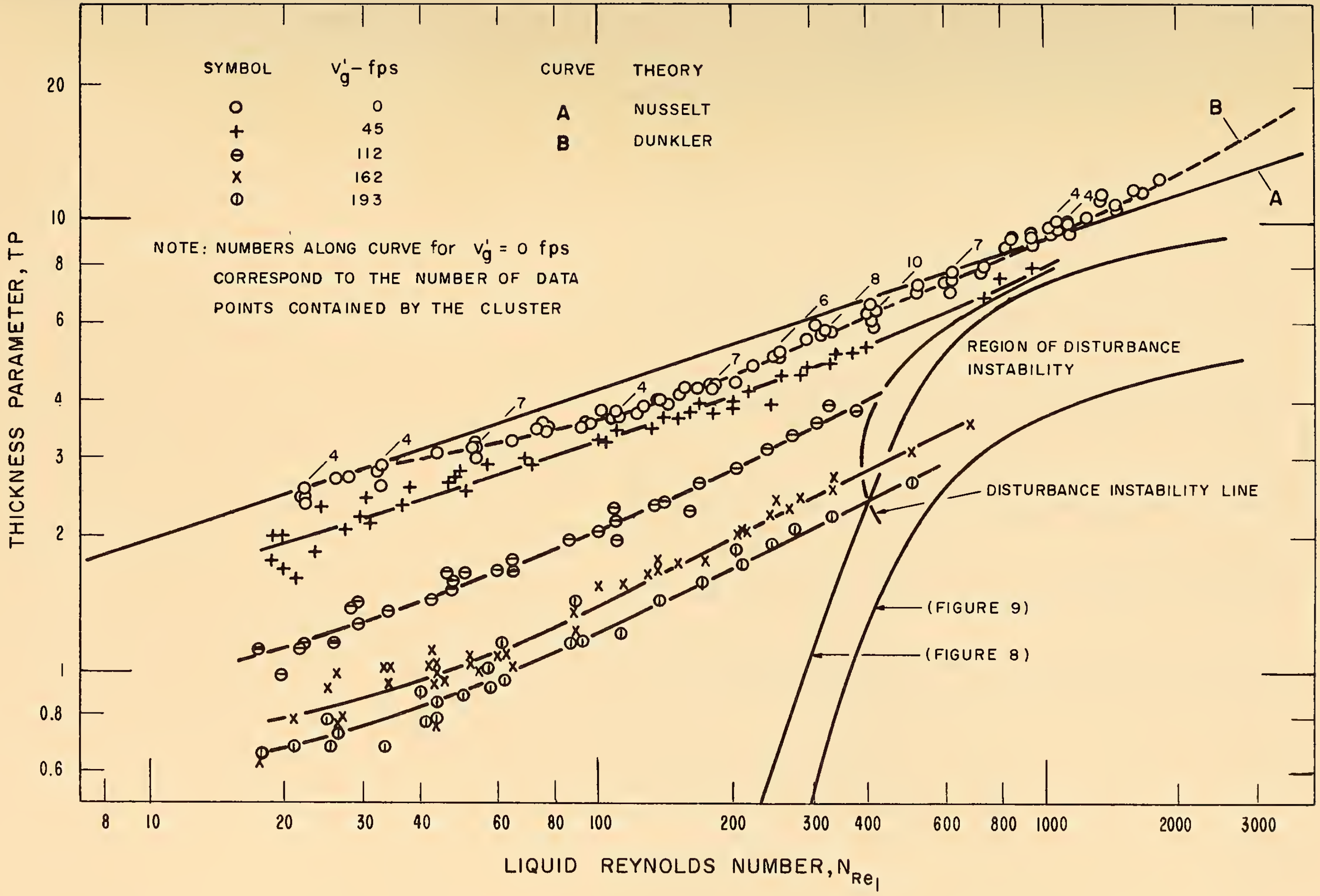

FIG. 10 EXPERIMENTAL CRITICAL CURVE OF CHARVONIA WITH FIGURES 8 AND 9 SUPERIMPOSED

Table 1. $\sigma = 0$

g	β	h	$(\mathcal{R}_F)_{cr}$	α_F	c	$\frac{\partial c_I}{\partial \mathcal{R}_F}$	$\frac{\partial \mathcal{R}_F}{\partial h}$	$\frac{\partial \mathcal{R}_F}{\partial m_0}$	$\frac{\partial \mathcal{R}_F}{\partial m_1}$	$\frac{\partial \mathcal{R}_F}{\partial \sigma}$	$\frac{\partial \mathcal{R}_F}{\partial g}$	$\frac{\partial \mathcal{R}_F}{\partial \beta}$
0	0	.00100	93.5	1.91	.583	.42(-3)	-.60(+2)	.22(+1)	-.66(-1)			
		.00432	92.9	2.01	.589	.33(-3)	-.22(+3)	.27(+1)	-.80(-1)			
		.00898	91.9	2.01	.589	.35(-3)	-.22(+3)	.26(+1)	-.78(-1)			
		.0154	90.5	2.01	.588	.37(-3)	-.20(+3)	.24(+1)	-.71(-1)			
		.0227	89.1	2.01	.588	.39(-3)	-.19(+3)	.22(+1)	-.67(-1)			
		.0320	87.3	2.01	.587	.43(-3)	-.17(+3)	.20(+1)	-.60(-1)			
		.0428	85.2	2.12	.592	.37(-3)	-.21(+3)	.22(+1)	-.68(-1)			
		.0546	82.6	2.12	.591	.42(-3)	-.11(+3)	.19(+1)	-.58(-1)			
		.0697	79.8	2.12	.590	.48(-3)	-.17(+3)	.16(+1)	-.49(-1)			
		.0848	76.5	2.23	.594	.44(-3)	-.19(+3)	.17(+1)	-.49(-1)			
		.103	72.9	2.34	.598	.42(-3)	-.21(+3)	.16(+1)	-.49(-1)			
		.119	69.5	2.34	.595	.52(-3)	-.21(+3)	.12(+1)	-.39(-1)			
		.138	65.5	2.47	.598	.51(-3)	-.19(+3)	.11(+1)	-.34(-1)			
		.160	60.7	2.59	.597	.64(-3)	-.12(+3)	.69(0)	-.21(-1)			
	.2	.00100	95.2	1.90	.585	.38(-3)		.22(+1)	-.69(-1)	-.51(-4)		.21(+2)
	.4	.00100	100.	1.81	.585	.36(-3)		.24(+1)	-.74(-1)	-.56(-4)		.40(+2)
	.6	.00100	109.	1.72	.589	.28(-3)		.30(+1)	-.94(-1)	-.80(-4)		.74(+2)
		.00432	106.	1.72	.589	.31(-3)	-.74(3)	.27(+1)	-.86(-1)	-.32(-3)	.50(-3)	.59(2)
		.00898	102.	1.72	.588	.35(-3)	-.70(3)	.23(1)	-.76(-1)	-.58(-3)	.94(-3)	.41(2)
		.0154	97.2	1.81	.594	.33(-3)	-.81(3)	.24(1)	-.81(-1)	-.13(-2)	.19(-2)	.40(2)
		.0227	90.9	1.81	.593	.42(-3)	-.73(3)	.18(1)	-.66(-1)	-.14(-2)	.24(-2)	.12(2)
		.0319	83.0	1.90	.596	.46(-3)	-.78(3)	.14(1)	-.61(-1)	-.19(-2)	.36(-2)	-.56(1)
		.0428	73.4	1.99	.598	.58(-3)	-.79(3)	.84(0)	-.51(-1)	-.12(-2)	.48(-2)	-.31(2)
		.0546	63.2	2.09	.597	.85(-3)	-.70(3)	.20(0)	-.37(-1)	.24(-2)	.53(-2)	-.50(2)

Table 2. $\sigma = 0.5(10)^4$

g	β	h	$(\mathcal{R}_F)_{cu}$	α_F	c	$\frac{\partial c_F}{\partial \mathcal{R}_{uF}}$	$\frac{\partial \mathcal{R}_F}{\partial h}$	$\frac{\partial \mathcal{R}_B}{\partial m_0}$	$\frac{\partial \mathcal{R}_F}{\partial m_1}$	$\frac{\partial \mathcal{R}_F}{\partial \sigma}$	$\frac{\partial \mathcal{R}_B}{\partial g}$	$\frac{\partial \mathcal{R}_F}{\partial \beta}$
0	0	.00432	91.6	2.00	.586	.35(-3)	-.43(3)	.23(1)	-.72(-1)	-.23(-3)	.49(-3)	.42(1)
		.00898	89.5	2.00	.582	.38(-3)	-.34(3)	.20(1)	-.62(-1)	-.34(-3)	.92(-3)	.24(1)
		.0154	87.3	2.10	.581	.34(-3)	-.31(3)	.21(1)	-.64(-1)	-.44(-3)	.19(-2)	.23(1)
		.0227	85.4	2.10	.574	.40(-3)	-.13(3)	.16(1)	-.47(-1)	+.34(-4)	.21(-2)	.53(0)
		.0320	84.9	2.10	.567	.47(-3)	+.46(2)	.11(1)	-.32(-1)	.10(-2)	.25(-2)	-.19(0)
		.0428	86.1	2.10	.559	.54(-3)	.17(3)	.74(0)	-.21(-1)	.22(-2)	.22(-2)	-.35(0)
		.0546	88.7	2.10	.553	.59(-3)	.25(3)	.50(0)	-.13(-1)	.36(-2)	.25(-2)	+.16(0)
		.0697	92.6	2.00	.543	.70(-3)	.26(3)	.26(0)	-.61(-2)	.42(-2)	.15(-2)	-.54(0)
		.0848	96.9	1.91	.536	.77(-3)	.25(3)	.14(0)	-.20(-2)	.49(-2)	.18(-2)	-.12(1)
		.103	102.	1.91	.531	.77(-3)	.26(3)	.68(-3)	+.24(-2)	.59(-2)	.76(-3)	-.37(0)
		.119	107.	1.82	.524	.80(-3)	.24(3)	-.70(-1)	.49(-2)	.66(-2)	.19(-2)	-.11(1)
		.138	112.	1.82	.521	.78(-3)	.25(3)	-.17(0)	.81(-2)	.73(-2)	-.10(-3)	-.28(0)
		.160	118.	1.73	.514	.79(-3)	.23(3)	-.24(0)	.11(-1)	.84(-2)	+.29(-2)	-.95(0)
		.185	124.	1.65	.508	.77(-3)	.21(3)	-.32(0)	.13(-1)	.81(-2)	-.32(-3)	-.17(1)
		.214	131.	1.65	.503	.73(-3)	.20(3)	-.44(0)	.17(-1)	.98(-2)	+.10(-2)	+.24(0)
		.248	139.	1.57	.496	.69(-3)	.18(3)	-.54(0)	.20(-1)	.95(-2)	-.24(-2)	.40(0)
		.273	144.	1.57	.492	.65(-3)	.16(3)	-.63(0)	.23(-1)	.11(-1)	.69(-3)	.31(1)
		.301	149.	1.50	.484	.61(-3)	.11(3)	-.70(0)	.26(-1)	.10(-1)	-.37(-2)	.35(1)
		.332	153.	1.49	.477	.56(-3)	.69(2)	-.79(0)	.29(-1)	.12(-1)	-.25(-3)	.86(1)
		.366	156.	1.37	.463	.54(-3)	.74(0)	-.96(0)	.34(-1)	.13(-1)	+.50(-2)	.40(2)
		.404	155.	1.37	.447	.47(-3)	-.14(3)	-.10(1)	.36(-1)	.74(-2)	-.18(-1)	.80(2)
		.445	143.	1.30	.415	.39(-3)	-.47(3)	-.10(1)	.29(-1)	-.15(0)	-.60(0)	.13(3)
		.491	102.	1.24	.326	.39(-3)	-.11(4)	-.20(0)	.44(-2)	-.51(-1)	-.12(0)	.21(3)
		.501	78.8	1.24	.264	.76(-3)	-.70(3)	.55(0)	.21(-1)	.28(1)	.55(1)	.49(2)
		.511	63.8	1.34	.197	.78(-3)	-.88(3)	-.43(-1)	-.16(-2)	.30(-2)	.16(-1)	.76(2)
		.521	52.6	1.37	.123	.13(-2)	-.96(3)	-.14(0)	+.33(-4)	-.13(0)	-.19(0)	.13(3)
		.532	43.2	1.40	.050	.20(-2)	-.44(4)	-.12(2)	-.69(0)	-.13(3)	-.16(3)	.80(2)
	.6	.00432	104.	1.73	.586	.31(-3)	-.10(4)	.26(1)	-.81(-1)	-.27(-3)	.49(-3)	.54(2)
		.00898	99.6	1.73	.582	.38(-3)	-.81(3)	.20(1)	-.65(-1)	-.34(-3)	.87(-3)	.32(2)
		.0154	93.4	1.82	.579	.39(-3)	-.67(3)	.16(1)	-.55(-1)	-.14(-3)	.15(-2)	.27(2)
		.0227	89.1	1.91	.574	.44(-3)	-.34(3)	.10(1)	-.39(-1)	+.10(-2)	.21(-2)	.32(2)
		.0320	87.4	1.82	.561	.70(-3)	-.17(2)	.35(0)	-.18(-1)	.22(-2)	.14(-2)	.16(2)
		.0428	88.3	1.82	.551	.87(-3)	+.18(3)	-.11(-1)	-.72(-2)	.40(-2)	.13(-2)	.26(2)
		.0546	90.8	1.73	.541	.11(-2)	.23(3)	-.24(0)	-.14(-2)	.50(-2)	.72(-3)	.26(2)
		.0697	94.6	1.65	.531	.13(-2)	.25(3)	-.45(0)	+.33(-2)	.62(-2)	.39(-3)	.29(2)
		.0848	98.1	1.50	.521	.14(-2)	.21(3)	-.64(0)	.45(-2)	.67(-2)	-.36(-4)	.21(2)
		.103	102.	1.29	.509	.15(-2)	.13(3)	-.96(0)	.33(-2)	.74(-2)	-.52(-3)	.14(1)
		.119	103.	1.12	.498	.15(-2)	.48(2)	-.14(1)	.88(-3)	.82(-2)	-.14(-2)	-.25(2)
		.138	95.6	.592	.451	.80(-3)	-.21(3)	-.33(1)	.10(-1)	.25(-2)	-.10(-1)	-.11(3)
		.160	91.9	.622	.431	.11(-2)	-.57(2)	-.25(1)	.79(-2)	.69(-2)	-.45(-2)	-.59(2)
		.185	91.2	.622	.411	.13(-2)	+.22(2)	-.21(1)	.79(-2)	.31(-2)	-.88(-2)	-.29(2)
		.214	92.6	.592	.388	.14(-2)	.74(2)	-.18(1)	.94(-2)	.14(-2)	-.98(-2)	-.82(1)
		.248	96.1	.592	.373	.15(-2)	.12(3)	-.18(1)	.87(-2)	.15(-2)	-.11(-1)	+.66(1)
		.273	99.9	.592	.365	.15(-2)	.16(3)	-.19(1)	.82(-2)	.79(-2)	-.55(-2)	.16(2)
		.301	106.	.622	.368	.15(-2)	.22(3)	-.22(1)	.66(-2)	.86(-2)	-.91(-2)	.30(2)
	1.2	.103	94.8	.568	.402	.15(-2)	.74(2)	-.16(1)	.96(-2)	.44(-2)	-.28(-2)	.42(2)
		.119	93.6	.568	.384	.17(-2)	.13(3)	-.14(1)	.10(-1)	.59(-2)	-.14(-2)	.48(2)
		.138	99.2	.541	.364	.18(-2)	.15(3)	-.13(1)	.12(-1)	.40(-2)	-.32(-2)	.51(2)
		.160	103.	.541	.350	.18(-2)	.18(3)	-.13(1)	.13(-1)	.63(-2)	-.14(-2)	.60(2)
		.185	108.	.515	.333	.18(-2)	.20(3)	-.13(1)	.14(-1)	.32(-2)	-.49(-2)	.64(2)
		.214	115.	.515	.214	.18(-2)	.23(3)	-.13(1)	.14(-1)	.71(-2)	-.22(-2)	.77(2)

Table 2. $\sigma = 0.5(10)^4$ - Cont'd

g	β	h	$(R_F)_{cr}$	α_F	c	$\frac{\partial c_I}{\partial R_F}$	$\frac{\partial R_F}{\partial h}$	$\frac{\partial R_F}{\partial m_0}$	$\frac{\partial R_F}{\partial m_1}$	$\frac{\partial R_F}{\partial \sigma}$	$\frac{\partial R_F}{\partial g}$	$\frac{\partial R_F}{\partial \beta}$
		.248	124.	.491	.313	.17(-2)	.26(3)	-.14(1)	.14(-1)	.91(-3)	-.11(-1)	.85(2)
		.273	132.	.445	.298	.16(-2)	.26(3)	-.14(1)	.15(-1)	-.27(-2)	-.13(-1)	.80(2)
		.301	141.	.445	.302	.14(-2)	.32(3)	-.16(1)	.13(-1)	-.42(-2)	-.19(-1)	.97(2)
		.332	154.	.404	.294	.12(-2)	.36(3)	-.18(1)	.13(-1)	-.82(-2)	-.24(-1)	.94(2)
		.366	170.	.404	.309	.10(-2)	.52(3)	-.25(1)	.90(-2)	-.92(-2)	-.37(-1)	.12(3)
		.404	197.	.366	.317	.76(-3)	.75(3)	-.35(1)	.65(-2)	-.11(-1)	-.48(-1)	.13(3)
		.445	244.	.349	.341	.47(-3)	.14(4)	-.64(1)	.61(-3)	-.20(-1)	-.99(-1)	.18(3)
$0.5(10)^5$	0	.00163	112.	1.73	.575	.37(-3)	.29(5)	.26(1)	-.80(-1)	-.54(-4)	.83(-3)	.46(1)
		.00241	142.	1.42	.555	.37(-3)	.43(5)	.30(1)	-.93(-1)	-.38(-4)	.17(-2)	.43(1)
		.00339	201.	1.06	.526	.29(-3)	.70(5)	.45(1)	-.14(0)	-.22(-4)	.36(-2)	.46(1)
		.00454	313.	.719	.493	.18(-3)	.18(6)	.88(1)	-.28(0)	-.12(-4)	.10(-1)	.52(1)
$-0.5(10)^5$	0	.00163	23.0	3.97	.524	.30(-2)	-.60(6)	-.18(2)	-.12(1)	-.13(-2)	-.84(-3)	-.50(3)

Table 3. σ = 0.1(10)[5]

g	β	h	$(R_F)_{cr}$	α_F	c	$\frac{\partial c_I}{\partial R_F}$	$\frac{\partial R_F}{\partial h}$	$\frac{\partial R_F}{\partial m_0}$	$\frac{\partial R_F}{\partial m_1}$	$\frac{\partial R_F}{\partial \sigma}$	$\frac{\partial R_F}{\partial g}$	$\frac{\partial R_F}{\partial \beta}$
0	0	.00432	90.5	1.99	.582	.38(-3)	-.56(3)	.21(1)	-.65(-1)	-.18(-3)		.34(1)
		.00898	88.1	2.09	.580	.34(-3)	-.40(3)	.21(1)	-.64(-1)	-.22(-3)		.36(1)
		.0154	86.8	2.09	.569	.41(-3)	+.44(2)	.14(1)	-.42(-1)	+.28(-3)		.21(1)
		.0227	88.4	2.09	.560	.48(-3)	.36(3)	.91(0)	-.26(-1)	.11(-2)		.23(1)
		.0320	92.3	1.99	.548	.60(-3)	.45(3)	.50(0)	-.13(-1)	.17(-2)		.17(1)
		.0428	98.0	1.90	.538	.69(-3)	.46(3)	.27(0)	-.59(-2)	.23(-2)		.15(1)
		.0546	104.	1.81	.530	.73(-3)	.44(3)	.12(0)	-.10(-2)	.27(-2)		.13(1)
		.0697	112.	1.72	.522	.75(-3)	.42(3)	-.25(0)	+.38(-2)	.32(-2)		.12(1)
		.0848	119.	1.72	.518	.72(-3)	.44(3)	-.18(0)	.88(-2)	.39(-2)		.24(1)
		.103	128.	1.64	.512	.71(-3)	.40(3)	-.30(0)	.13(-1)	.46(-2)		.23(1)
		.119	135.	1.56	.506	.70(-3)	.37(3)	-.39(0)	.16(-1)	.46(-2)		.17(1)
		.138	142.	1.49	.501	.68(-3)	.35(3)	-.49(0)	.19(-1)	.54(-2)		.12(1)
		.160	151.	1.49	.498	.63(-3)	.35(3)	-.66(0)	.25(-1)	.55(-2)		.29(1)
		.185	160.	1.42	.492	.59(-3)	.32(3)	-.80(0)	.30(-1)	.77(-2)		.30(1)
		.214	171.	1.35	.486	.54(-3)	.28(3)	-.99(0)	.36(-1)	.62(-2)		.32(1)
		.248	182.	1.29	.479	.49(-3)	.24(3)	-.12(1)	.44(-1)	.12(-1)		.45(1)
		.273	189.	1.28	.475	.44(-3)	.22(3)	-.14(1)	.52(-1)	.67(-2)		.84(1)
		.301	196.	1.22	.468	.41(-3)	.15(3)	-.16(1)	.59(-1)	.18(-1)		.10(2)
		.332	202.	1.17	.458	.36(-3)	.64(2)	-.19(1)	.68(-1)	.82(-2)		.13(2)
		.366	204.	1.11	.446	.31(-3)	-.95(2)	-.22(1)	.78(-1)	.12(-1)		.19(2)
		.404	197.	1.11	.427	.26(-3)	-.39(3)	-.24(1)	.82(-1)	.63(-2)		.36(2)
	0.6	.00432	103.	1.73	.583	.33(-3)	-.11(4)	.24(1)	-.75(-1)	-.21(-3)		.48(2)
		.00898	97.9	1.82	.579	.33(-3)	-.83(3)	.20(1)	-.65(-1)	-.15(-3)		.47(2)
		.0154	94.6	1.82	.567	.45(-3)	-.66(2)	.10(1)	-.35(-1)	.67(-3)		.36(2)
		.0227	95.9	1.82	.556	.57(-3)	+.39(3)	.43(0)	-.17(-1)	.18(-2)		.44(2)
		.0320	100.	1.73	.543	.75(-3)	.52(3)	.36(-1)	-.50(-2)	.26(-2)		.43(2)
		.0428	106.	1.57	.531	.91(-3)	.45(3)	-.20(0)	+.39(-3)	.29(-2)		.35(2)
		.0546	112.	1.50	.522	.98(-3)	.45(3)	-.42(0)	.56(-2)	.36(-2)		.40(2)
		.0697	119.	1.36	.513	.10(-2)	.39(3)	-.65(0)	.87(-2)	.40(-2)		.35(2)
		.0848	126.	1.29	.506	.11(-2)	.38(3)	-.88(0)	.13(-1)	.47(-2)		.40(2)
		.103	132.	1.12	.497	.11(-2)	.28(3)	-.13(1)	.13(-1)	.50(-2)		.17(2)
		.119	137.	1.01	.490	.11(-2)	.23(3)	-.17(1)	.13(-1)	.56(-2)		.20(1)
		.138	125.	.464	.455	.48(-3)	-.32(3)	-.57(1)	.23(-1)	.87(-3)		-.20(3)
		.160	120.	.464	.432	.65(-3)	-.67(2)	-.41(1)	.20(-1)	.45(-2)		-.11(3)
		.185	120.	.464	.413	.78(-3)	.59(2)	-.34(1)	.18(-1)	-.19(-2)		-.66(2)
		.214	123.	.421	.388	.85(-3)	.15(3)	-.32(1)	.16(-1)	.47(-2)		-.40(2)
		.248	130.	.464	.385	.92(-3)	.25(3)	-.33(1)	.14(-1)	-.19(-2)		-.24(2)
		.273	137.	.442	.375	.89(-3)	.32(3)	-.34(1)	.16(-1)	-.12(-2)		-.18(2)
		.301	149.	.442	.372	.84(-3)	.45(3)	-.40(1)	.15(-1)	-.26(-2)		-.15(2)
0.4(10)[5]	0	.001	96.1	1.91	.583	.38(-3)	.82(4)	.23(1)	-.70(-1)	-.46(-4)	.26(-3)	.43(1)
0.2(10)[5]	0	.001	93.9	1.91	.582	.41(-3)	.15(4)	.21(1)	-.64(-1)	-.42(-4)	.17(-3)	.36(1)
-0.2(10)[5]	0	.001	92.0	2.01	.587	.34(-3)	-.34(4)	.24(1)	-.75(-1)	-.61(-4)	.62(-4)	.51(1)
-0.4(10)[5]	0	.001	89.2	2.01	.586	.38(-3)	-.11(5)	.21(1)	-.67(-1)	-.54(-4)	.76(-4)	.42(1)

Table 4. $\sigma = 0.4(10)^5$

g	β	h	$(Re_F)_{cr}$	α_F	c	$\frac{\partial c_r}{\partial Re_F}$	$\frac{\partial Re_F}{\partial h}$	$\frac{\partial Re_F}{\partial m_0}$	$\frac{\partial Re_F}{\partial m_1}$	$\frac{\partial Re_F}{\partial \sigma}$	$\frac{\partial Re_F}{\partial g}$	$\frac{\partial Re_F}{\partial \beta}$
0	0	.001	91.3	1.99	.583	.36(-3)		.22(1)	-.68(-1)	-.45(-4)		.43(1)
		.00432	88.6	2.09	.586	.38(-3)	.66(3)	.22(1)	-.64(-1)	.99(-4)		.81(1)
		.00898	95.6	1.99	.547	.54(-3)	.19(4)	.57(0)	-.15(-1)	.49(-3)		.46(1)
		.0154	108.	1.81	.530	.65(-3)	.17(4)	.15(0)	-.18(-2)	.73(-3)		.46(1)
		.0227	121.	1.56	.517	.70(-3)	.12(4)	.72(-3)	.32(-2)	.77(-3)		.34(1)
		.0319	135.	1.49	.509	.67(-3)	.12(4)	-.26(0)	.12(-1)	.10(-2)		.45(1)
		.0428	149.	1.42	.502	.63(-3)	.11(4)	-.51(0)	.21(-1)	.13(-2)		.56(1)
		.0546	163.	1.35	.497	.59(-3)	.10(4)	-.76(0)	.29(-1)	.15(-2)		.64(1)
		.0697	179.	1.22	.491	.55(-3)	.88(3)	-.97(0)	.36(-1)	.15(-2)		.56(1)
		.0848	194.	1.17	.486	.50(-3)	.82(3)	-.12(1)	.45(-1)	.20(-2)		.62(1)
		.103	210.	1.11	.482	.45(-3)	.77(3)	-.16(1)	.57(-1)	.19(-2)		.69(1)
		.119	224.	1.06	.478	.42(-3)	.72(3)	-.19(1)	.67(-1)	.30(-2)		.71(1)

Table 5. $\sigma = 0.8(10)^5$

g	β	h	$(R_F)_{cr}$	α_F	c	$\frac{\partial c_I}{\partial R_F}$	$\frac{\partial R_F}{\partial h}$	$\frac{\partial R_F}{\partial m_0}$	$\frac{\partial R_F}{\partial m_1}$	$\frac{\partial R_F}{\partial \sigma}$	$\frac{\partial R_F}{\partial g}$	$\frac{\partial R_F}{\partial \beta}$
0	0	.001	89.8	1.99	.577	.39(-3)		.19(1)	-.58(-1)	-.25(-4)		.35(1)
		.00432	95.1	1.99	.548	.53(-3)	.37(4)	.64(0)	-.17(-1)	.22(-3)	.20(-3)	.49(1)
		.00899	112.	1.72	.525	.67(-3)	.31(4)	.96(-1)	+.98(-4)	.38(-3)	.14(-3)	.49(1)
		.0154	133.	1.56	.511	.65(-3)	.26(4)	-.26(0)	.12(-1)	.55(-3)	.69(-4)	.63(1)
		.0227	151.	1.42	.502	.62(-3)	.22(4)	-.55(0)	.22(-1)	.66(-3)	-.36(-4)	.71(1)
		.0319	170.	1.28	.494	.57(-3)	.18(4)	-.85(0)	.32(-1)	.75(-3)	-.19(-3)	.76(1)
		.0428	190.	1.17	.489	.52(-3)	.15(4)	-.12(1)	.42(-1)	.83(-3)	-.37(-3)	.77(1)
		.0546	208.	1.06	.484	.47(-3)	.13(4)	-.14(1)	.51(-1)	.87(-3)	-.56(-3)	.70(1)
		.0697	230.	1.01	.479	.42(-3)	.12(4)	-.19(1)	.68(-1)	.11(-2)	-.56(-3)	.86(1)
		.0848	250.	.960	.475	.37(-3)	.11(4)	-.24(1)	.84(-1)	.12(-2)	-.14(-2)	.97(1)
		.103	272.	.870	.471	.33(-3)	.98(3)	-.28(1)	.97(-1)	.11(-2)	-.17(-2)	.81(1)
		.119	290.	.829	.468	.31(-3)	.92(3)	-.30(1)	.11(0)	.17(-2)	+.12(-3)	.85(1)
		.138	311.	.789	.464	.26(-3)	.87(3)	-.39(1)	.14(0)	.13(-2)	-.28(-2)	.93(1)
		.160	334.	.752	.460	.24(-3)	.82(3)	-.48(1)	.16(0)	.29(-2)	+.39(-2)	.11(2)
		.185	360.	.752	.455	.19(-3)	.78(3)	-.62(1)	.21(0)	.15(-2)	-.50(-2)	.14(2)
		.214	388.	.682	.449	.15(-3)	.73(3)	-.83(1)	.27(0)	.88(-2)	.31(-1)	.24(2)
		.248	422.	.648	.443	.10(-3)		-.13(2)	.44(0)	.16(-2)		.34(2)
		.260	434.	.623	.440	.90(-4)	.68(3)	-.15(2)	.50(0)	.30(-2)	-.75(-2)	.37(2)
		.273	447.	.611	.437	.78(-4)	.66(3)	-.17(2)	.60(0)	.15(0)	.81(0)	.50(2)
		.287	465.	.587	.434	.55(-4)	.78(3)	-.26(2)	.87(0)	.67(-2)	.31(-3)	.62(2)
		.301	486.	.576	.433	.39(-4)	.11(4)	-.36(2)	.13(1)	.11(1)	.59(1)	.12(3)
	0.2	.001	91.4	1.99	.579	.35(-3)		.21(1)	-.64(-1)	-.30(-4)		.23(2)
	0.4	.001	96.3	1.90	.579	.32(-3)		.23(1)	-.70(-1)	-.33(-4)		.43(2)
	0.6	.001	104.	1.72	.577	.33(-3)		.22(1)	-.69(-1)	-.30(-4)		.49(2)
		.00432	110.	1.72	.548	.47(-3)	.44(4)	.65(0)	-.18(-1)	.30(-3)	.19(-3)	.68(2)
		.00898	130.	1.49	.524	.61(-3)	.36(4)	-.88(-2)	+.23(-2)	.47(-3)	.12(-3)	.70(2)
		.0154	152.	1.29	.509	.63(-3)	.27(4)	-.40(0)	.14(-1)	.59(-3)	.44(-4)	.76(2)
		.0227	173.	1.17	.500	.60(-3)	.23(4)	-.78(0)	.26(-1)	.73(-3)	-.76(-4)	.89(2)
		.0320	194.	1.06	.492	.56(-3)	.19(4)	-.12(1)	.38(-1)	.86(-3)	-.26(-3)	.10(3)
		.0428	214.	.960	.486	.52(-3)	.16(4)	-.17(1)	.50(-1)	.97(-3)	-.50(-3)	.11(3)
		.0546	234.	.870	.481	.48(-3)	.14(4)	-.22(1)	.62(-1)	.11(-2)	-.74(-3)	.11(3)
		.0697	256.	.789	.476	.45(-3)	.13(4)	-.28(1)	.75(-1)	.12(-2)	-.12(-2)	.11(3)
		.0848	276.	.716	.472	.42(-3)	.11(4)	-.34(1)	.86(-1)	.13(-2)	-.15(-2)	.10(3)
		.103	299.	.650	.468	.38(-3)	.11(4)	-.42(1)	.10(0)	.14(-2)	-.25(-2)	.97(2)
		.119	318.	.619	.464	.35(-3)	.11(4)	-.50(1)	.12(0)	.19(-2)	-.15(-2)	.11(3)
		.138	340.	.561	.460	.33(-3)	.10(4)	-.60(1)	.13(0)	.17(-2)	-.33(-2)	.94(2)
		.160	365.	.485	.457	.31(-3)	.10(4)	-.75(1)	.13(0)	.20(-2)	-.47(-2)	.40(2)
		.185	343.	1.92	.431	.17(-3)	.10(4)	-.20(2)	.10(0)	.27(-2)	-.14(-1)	-.68(3)

Table 6. $\sigma = 0.12(10)^7$

g	β	h	$(R_F)_{cu}$	α_F	c	$\frac{\partial c_x}{\partial R_F}$	$\frac{\partial R_F}{\partial h}$	$\frac{\partial R_F}{\partial m_0}$	$\frac{\partial R_F}{\partial m_1}$	$\frac{\partial R_F}{\partial \sigma}$	$\frac{\partial R_F}{\partial g}$	$\frac{\partial R_F}{\partial \beta}$
0	0	.0001	88.9	2.04	.573	.38(-3)	-.94(4)	.18(1)	-.54(-1)	-.87(-6)	.12(-4)	.42(1)
		.000161	89.40	2.040	.5639	.43(-3)	.25(5)	.13(1)	-.38(-1)	.38(-5)	.14(-4)	.42(1)
		.000259	93.62	1.981	.5508	.53(-3)	.49(5)	.74(0)	-.21(-1)	.12(-4)	.13(-4)	.46(1)
		.000418	102.6	1.867	.5366	.61(-3)	.53(5)	.34(0)	-.79(-2)	.20(-4)	.11(-4)	.52(1)
		.000673	116.3	1.708	.5228	.65(-3)	.46(5)	.34(-1)	.23(-2)	.28(-4)	.75(-5)	.60(1)
		.001	131.	1.54	.512	.66(-3)	.37(5)	-.20(0)	.10(-1)	.34(-4)	.34(-5)	.65(1)
		.002	164.	1.28	.497	.59(-3)	.24(5)	-.67(0)	.27(-1)	.45(-4)	-.82(-5)	.81(1)
		.004	207.	1.05	.486	.48(-3)	.16(5)	-.14(1)	.52(-1)	.58(-4)	-.46(-4)	.10(2)
		.007	251.	.880	.479	.40(-3)	.11(5)	-.22(1)	.78(-1)	.68(-4)	-.11(-3)	.12(2)
		.015	324.	.692	.472	.30(-3)	.65(4)	-.36(1)	.13(0)	.84(-4)	-.31(-3)	.15(2)
		.030	415.	.562	.465	.22(-3)	.45(4)	-.56(1)	.19(0)	.11(-3)	-.74(-3)	.18(2)
$-0.13(10)^5$	0	.0001	88.9	2.04	.573	.38(-3)	-.94(4)	.18(1)	-.54(-1)	-.87(-6)	.12(-4)	.42(1)
		.000161	89.40	2.040	.5639	.43(-3)	.25(5)	.13(1)	-.38(-1)	.38(-5)	.14(-4)	.42(1)
		.000259	93.62	1.981	.5508	.53(-3)	.49(5)	.74(0)	-.21(-1)	.12(-4)	.13(-4)	.46(1)
		.000418	102.6	1.867	.5366	.61(-3)	.53(5)	.34(0)	-.79(-2)	.20(-4)	.11(-4)	.52(1)
		.000673	116.3	1.708	.5228	.65(-3)	.46(5)	.34(-1)	.23(-2)	.28(-4)	.75(-5)	.60(1)
		.001	131.	1.54	.512	.66(-3)	.37(5)	-.20(0)	.10(-1)	.34(-4)	.34(-5)	.65(1)
		.002	164.	1.28	.497	.59(-3)	.25(5)	-.72(0)	.28(-1)	.45(-4)	-.95(-5)	.81(1)
		.004	210.	1.03	.486	.48(-3)	.18(5)	-.16(1)	.57(-1)	.59(-4)	-.36(-3)	.10(2)
		.007	274.	.796	.482	.34(-3)	.24(5)	-.37(1)	.12(0)	.73(-4)	-.41(-2)	.11(2)
		.010	416.	.556	.488	.14(-3)	.87(5)	-.13(2)	.44(0)	.77(-4)	-.35(-1)	.71(1)
		.011	529.	.448	.489	.89(-4)	.14(6)	-.23(2)	.78(0)	.62(-4)	-.67(-1)	.72(0)
		.012	700.	.340	.487	.57(-4)	.20(6)	-.38(2)	.13(1)	.38(-4)	-.11(0)	-.59(1)
		.013	924.	.256	.485	.41(-4)	.24(6)	-.54(2)	.18(1)	.17(-4)	-.14(0)	-.97(1)
		.014	1190.	.198	.483	.31(-4)	.28(6)	-.68(2)	.23(1)	.14(-4)	-.17(0)	-.11(2)
		.015	1500.	.158	.483	.24(-4)	.32(6)	-.91(2)	.30(1)	.40(-5)	-.23(0)	-.14(2)
		.016	1840.	.127	.482	.20(-4)	.36(6)	-.11(3)	.37(1)	.94(-5)	-.28(0)	-.16(2)
		.017	2240.	.105	.481	.16(-4)	.41(6)	-.14(3)	.46(1)	-.11(-5)	-.33(0)	-.19(2)
		.018	2680.	.0877	.481	.14(-4)	.46(6)	-.17(3)	.55(1)	.27(-5)	-.40(0)	-.22(2)
		.019	3170.	.0742	.481	.11(-4)	.52(6)	-.20(3)	.66(1)	.48(-5)	-.48(0)	-.26(2)
		.020	3720.	.0629	.480	.98(-5)	.56(6)	-.23(3)	.76(1)	.13(-4)	-.54(0)	-.32(2)
	1.0	.0130	875.	.265	.487	.51(-4)	.22(6)	-.44(2)	.14(1)	.72(-4)	-.13(0)	-.90(1)
	1.1		874.	.265	.487	.52(-4)	.21(6)	-.42(2)	.14(1)	.83(-4)	-.12(0)	+.17(2)
	1.0	.0131	896.	.257	.487	.49(-4)	.22(6)	-.45(2)	.15(1)	.69(-4)	-.13(0)	-.19(2)
	1.1		894.	.257	.487	.51(-4)	.21(6)	-.43(2)	.14(1)	.81(-4)	-.12(0)	+.59(1)
	1.1	.0134	960.	.239	.486	.46(-4)	.23(6)	-.49(2)	.16(1)	.69(-4)	-.14(0)	-.22(2)
	1.2		958.	.239	.487	.47(-4)	.22(6)	-.47(2)	.15(1)	.80(-4)	-.13(0)	+.28(1)
	1.4	.0140	1090.	.208	.486	.41(-4)	.24(6)	-.55(2)	.18(1)	.88(-4)	-.16(0)	-.35(1)
	1.5		1090.	.208	.486	.42(-4)	.24(6)	-.53(2)	.17(1)	.97(-4)	-.15(0)	+.22(2)
	1.8	.0150	1350.	.166	.486	.34(-4)	.28(6)	-.67(2)	.22(1)	.97(-4)	-.19(0)	+.23(0)
	1.9		1350.	.166	.486	.35(-4)	.27(6)	-.66(2)	.21(1)	.12(-3)	-.18(0)	.22(2)
	2.2	.0160	1640.	.137	.485	.27(-4)	.32(6)	-.85(2)	.27(1)	.12(-3)	-.24(0)	.12(1)
	2.3		1640.	.135	.485	.29(-4)	.31(6)	-.81(2)	.25(1)	.13(-3)	-.22(0)	.19(2)

Table 7. $\sigma = 0.23(10)^7$

g	β	h	$(R_F)_{max}$	α_F	c	$\frac{\partial c_I}{\partial R_F}$	$\frac{\partial R_F}{\partial h}$	$\frac{\partial R_F}{\partial m_0}$	$\frac{\partial R_F}{\partial m_1}$	$\frac{\partial R_F}{\partial \sigma}$	$\frac{\partial R_F}{\partial g}$	$\frac{\partial R_F}{\partial \beta}$
0	0	.0001	90.4	2.03	.560	.46(-3)	.71(5)	.11(1)	-.32(-1)	.34(-5)	.84(-5)	.44(1)
		.000161	96.4	1.95	.546	.55(-3)	.10(6)	.59(0)	-.16(-1)	.78(-5)	.72(-5)	.49(1)
		.000259	107.	1.79	.531	.65(-3)	.92(5)	.24(0)	-.44(-2)	.11(-4)	.54(-5)	.51(1)
		.000418	122.	1.63	.518	.66(-3)	.79(5)	-.64(-1)	.56(-2)	.16(-4)	.33(-5)	.62(1)
		.000673	142.	1.41	.506	.66(-3)	.57(5)	-.32(0)	.14(-1)	.18(-4)	.71(-6)	.64(1)
		.001	162.	1.29	.498	.60(-3)	.47(5)	-.64(0)	+.25(-1)	.23(-4)	-.33(-5)	.79(1)
		.002	204.	1.06	.486	.49(-3)	.30(5)	-.13(1)	.49(-1)	.29(-4)	-.21(-4)	.10(2)
		.004	257.	.854	.479	.39(-3)	.19(5)	-.23(1)	.82(-1)	.36(-4)	-.66(-4)	.13(2)
		.005	277.	.797	.477	.36(-3)	.16(5)	-.27(1)	.95(-1)	.38(-4)	-.90(-4)	.13(2)
		.006	294.	.749	.475	.33(-3)	.14(5)	-.29(1)	.10(0)	.38(-4)	-.11(-3)	.13(2)
		.008	323.	.690	.473	.30(-3)	.12(5)	-.36(1)	.13(0)	.44(-4)	-.17(-3)	.15(2)
		.010	348.	.644	.471	.27(-3)	.10(5)	-.41(1)	.14(0)	.46(-4)	-.22(-3)	.16(2)
		.014	389.	.583	.468	.24(-3)	.83(4)	-.50(1)	.17(0)	.52(-4)	-.34(-3)	.18(2)
$-0.36(10)^5$		.000161	96.4	1.95	.546	.55(-3)	.10(6)	.59(0)	-.16(-1)	.78(-5)	.72(-5)	.49(1)
		.000259	107.	1.79	.531	.65(-3)	.92(5)	.24(0)	-.44(-2)	.11(-4)	.54(-5)	.51(1)
		.000418	122.	1.63	.518	.66(-3)	.79(5)	-.64(-1)	.56(-2)	.16(-4)	.33(-5)	.62(1)
		.000673	142.	1.41	.506	.66(-3)	.57(5)	-.32(0)	.14(-1)	.18(-4)	.71(-6)	.64(1)
		.001	162.	1.29	.498	.60(-3)	.49(5)	-.68(0)	.26(-1)	.23(-4)	-.23(-5)	.81(1)
		.002	206.	1.04	.487	.49(-3)	.35(5)	-.15(1)	.54(-1)	.29(-4)	-.11(-3)	.99(1)
		.004	297.	.743	.484	.29(-3)	.65(5)	-.48(1)	.17(0)	.40(-4)	-.27(-2)	.12(2)
		.005	399.	.579	.489	.16(-3)	.16(6)	-.12(2)	.40(0)	.41(-4)	-.11(-1)	.81(1)
		.0051	416.	.560	.489	.14(-3)	.18(6)	-.13(2)	.45(0)	.40(-4)	-.13(-1)	.72(1)
		.0052	434.	.540	.489	.13(-3)	.20(6)	-.15(2)	.50(0)	.39(-4)	-.15(-1)	.62(1)
		.0053	455.	.515	.489	.12(-3)	.22(6)	-.17(2)	.56(0)	.37(-4)	-.17(-1)	.49(1)
		.0054	477.	.495	.490	.11(-3)	.24(6)	-.19(2)	.63(0)	.35(-4)	-.19(-1)	.36(1)
		.0055	502.	.470	.489	.98(-4)	.26(6)	-.21(2)	.70(0)	.33(-4)	-.21(-1)	.22(1)
		.0056	529.	.450	.489	.88(-4)	.29(6)	-.24(2)	.80(0)	.32(-4)	-.25(-1)	.93(0)
		.0057	558.	.426	.489	.81(-4)	.31(6)	-.26(2)	.87(0)	.29(-4)	-.27(-1)	-.53(0)
		.0058	590.	.404	.489	.73(-4)	.34(6)	-.29(2)	.97(0)	.26(-4)	-.30(-1)	-.19(1)
		.0059	624.	.384	.488	.66(-4)	.36(6)	-.34(2)	.11(1)	.25(-4)	-.35(-1)	-.32(1)
		.0060	660.	.360	.488	.62(-4)	.38(6)	-.35(2)	.12(1)	.21(-4)	-.36(-1)	-.42(1)
		.0062	739.	.324	.487	.52(-4)	.42(6)	-.42(2)	.14(1)	.16(-4)	-.42(-1)	-.61(1)
		.0064	826.	.291	.486	.45(-4)	.43(6)	-.45(2)	.15(1)	.12(-4)	-.45(-1)	-.69(1)
		.0066	919.	.258	.485	.41(-4)	.48(6)	-.54(2)	.18(1)	.10(-4)	-.52(-1)	-.80(1)
		.0068	1020.	.234	.485	.36(-4)	.53(6)	-.62(2)	.21(1)	.72(-5)	-.60(-1)	-.88(1)
		.0070	1120.	.210	.484	.33(-4)	.54(6)	-.66(2)	.22(1)	.51(-5)	-.63(-1)	-.89(1)
		.00720	1240.	.192	.483	.30(-4)	.58(6)	-.75(2)	.25(1)	.40(-5)	-.70(-1)	-.93(1)
		.00740	1350.	.175	.483	.27(-4)	.62(6)	-.83(2)	.28(1)	.45(-5)	-.76(-1)	-.95(1)
		.00750	1410.	.167	.483	.26(-4)	.62(6)	-.85(2)	.29(1)	.35(-5)	-.78(-1)	-.96(1)
		.00770	1540.	.153	.482	.24(-4)	.65(6)	-.93(2)	.31(1)	.16(-5)	-.85(-1)	-.99(1)
		.00790	1680.	.140	.482	.23(-4)	.67(6)	-.99(2)	.33(1)	.24(-5)	-.89(-1)	-.10(2)
		.00810	1820.	.129	.482	.21(-4)	.70(6)	-.11(3)	.36(1)	.59(-6)	-.97(-1)	-.10(2)
		.00800	6460.	.520	.257	.11(-5)	-.58(6)	.68(3)	-.20(2)	.89(-5)	.91(-1)	.24(3)
		.0100	652.	1.92	.365	.14(-3)	-.33(4)	-.11(1)	-.68(-1)	-.16(-3)	-.23(-2)	-.17(2)
	.1	.0060	660.	.360	.488	.62(-4)	.37(6)	-.34(2)	.12(1)	.21(-4)	-.35(-1)	-.12(2)
	.2		658.	.364	.488	.62(-4)	.38(6)	-.35(2)	.12(1)	.22(-4)	-.36(-1)	-.18(2)
	.3		657.	.364	.488	.63(-4)	.38(6)	-.34(2)	.11(1)	.25(-4)	-.35(-1)	-.19(2)
	.4		655.	.364	.488	.65(-4)	.37(6)	-.33(2)	.11(1)	.27(-4)	-.34(-1)	-.15(2)
	.5		653.	.364	.489	.67(-4)	.36(6)	-.32(2)	.11(1)	.30(-4)	-.33(-1)	-.55(1)

Table 7. $\sigma = 0.23(10)^{1}$ Cont'd

g	β	h	$(R_F)_{cr}$	α_F	c	$\frac{\partial c_I}{\partial R_F}$	$\frac{\partial R_F}{\partial h}$	$\frac{\partial R_F}{\partial m_0}$	$\frac{\partial R_F}{\partial m_1}$	$\frac{\partial R_F}{\partial \sigma}$	$\frac{\partial R_F}{\partial g}$	$\frac{\partial R_F}{\partial \beta}$
	.6		653.	.360	.488	.70(-4)	.33(6)	-.30(2)	.10(1)	.34(-4)	-.31(-1)	+.92(1)
	.5	.0061	689.	.342	.488	.62(-4)	.37(6)	-.34(2)	.11(1)	.26(-4)	-.36(-1)	-.19(2)
	.6		687.	.343	.488	.64(-4)	.36(6)	-.33(2)	.11(1)	.31(-4)	-.34(-1)	-.60(1)
	.7		687.	.343	.488	.66(-4)	.35(6)	-.32(2)	.11(1)	.35(-4)	-.33(-1)	+.13(2)
	.6	.0062	724.	.326	.488	.59(-4)	.39(6)	-.37(2)	.12(1)	.28(-4)	-.38(-1)	-.20(2)
	.7		722.	.326	.488	.61(-4)	.37(6)	-.35(2)	.12(1)	.33(-4)	-.36(-1)	-.42(1)
	.8		722.	.326	.488	.63(-4)	.36(6)	-.34(2)	.11(1)	.37(-4)	-.35(-1)	+.17(2)
	.9	.0065	836.	.280	.488	.52(-4)	.41(6)	-.41(2)	.14(1)	.32(-4)	-.42(-1)	-.15(2)
	1.0		834.	.280	.488	.54(-4)	.40(6)	-.40(2)	.13(1)	.38(-4)	-.41(-1)	+.76(1)
	1.2	.00675	937.	.250	.488	.48(-4)	.44(6)	-.46(2)	.15(1)	.43(-4)	-.47(-1)	.10(2)
	1.3		937.	.248	.488	.50(-4)	.42(6)	-.43(2)	.14(1)	.48(-4)	-.44(-1)	.35(2)
	1.4	.0070	1050.	.223	.487	.43(-4)	.48(6)	-.51(2)	.17(1)	.47(-4)	-.53(-1)	.12(2)
	1.5		1050.	.221	.487	.45(-4)	.45(6)	-.48(2)	.16(1)	.53(-4)	-.50(-1)	.35(2)

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