NBS

Technical Note 250

A NOMOGRAM FOR COMPUTING $\dfrac{a+jb}{c+jd}$ AND A NOMOGRAM FOR COMPUTING $\left|\dfrac{a+jb}{c+jd}\right|$

HOWARD S. BOWMAN

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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Contents

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A Nomogram for Computing $\left|\frac{a + jb}{c + jd}\right|$

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This report gives two nomograms designed and constructed for use in computing complex ratios of two dimensional vector quantities. One nomogram can be used to compute the real and imaginary parts of a vector ratio, and the other to compute the absolute value of such a function. Whenever a large number of vector ratios are required in data reduction, the use of these nomograms saves time in manual processing.

Part One. A Nomogram for Computing $\frac{a + jb}{c + jd}$

1. Inroduction

Repeated solutions of intricate problems are often facilitated by the construction and use of time saving nomograms or abacs as they are sometimes called. The theory of nomograms is found in many publications [1,2,3,4,5]* The problem which initiated this study was that of repeated computations of the difference between two complex ratios required for the evaluation of the mechanical impedance of the human forehead and mastoid [6]. The calibration constant, K, and the mechanical impedance, Z, involved in these determinations are written as follows:

$$\left.\begin{aligned} K &= \frac{1}{\omega^2 m}\left(\frac{E_{fs}}{E_{ds}} - \frac{E_{fo}}{E_{do}}\right) \\ Z &= \frac{1}{\omega^2 K}\left(\frac{E_{fx}}{E_{dx}} - \frac{E_{fo}}{E_{do}}\right) \end{aligned}\right\} \qquad (1)$$

where the E's are measured complex quantities.

*Figures in brackets indicate the literature references at the end of this paper.

Each complex ratio inside the brackets of (1) above can be represented by an expression

$$\frac{a + jb}{c + jd} = \frac{1}{N}\left[\left(\frac{a}{d} + \frac{b}{c}\right) + j\left(\frac{b}{d} - \frac{a}{c}\right)\right] \qquad (2)$$

where $N = \frac{c}{d} + \frac{d}{c}$.

The above expression (2) reduces the number of operations for the solution of K or Z in that one abac can be used to compute an expression such as

$$\frac{1}{N}\left(\frac{a'}{d'} \pm \frac{b'}{c'}\right)$$

for either real or imaginary part of the complex ratio regardless of the signs of the variables. Tables I and II given below were made for use with the nomogram to determine the particular process and the proper sign for the final answers when one is given the signs of the variables.

Other methods of computing the ratio of complex numbers in the rectangular form, such as conversion to magnitude and phase and then back again, by slide rule or tables [7] are time consuming and tedious. The slide rule computation of equation (2) is fairly rapid but more tedious than using the nomogram and is not an automatic process.

2. Construction Details

In order to perform the products and ratios in the computation, three logarithmic scales were constructed. The lines are straight, vertical, parallel and equidistant. The left vertical line is a five cycle ascending logarithmic scale while the right vertical line is a seven cycle descending logarithmic scale. One cycle of the center ascending logarithmic scale is half as long as one cycle on either of the two other logarithmic scales. The scales are aligned so that a line drawn through 10 on each of the outer scales is horizontal and intersects the center line at 1. If a straight line is drawn so that it intersects the three logarithmic scales, the point of intersection of the center scale equals the ratio of the points of intersection of the first and third scales. From points corresponding to c/d on the center logarithmic scale, straight lines are drawn to the left vertical to points corresponding to computed values of 1/N for given values of c and d. (To prevent overlapping lines, capital letters are used in some cases. See nomogram and instructions.)

Three linear scales are used for the addition and subtraction, subsequently called the summation and difference processes. In this construction the three vertical straight lines are drawn parallel and equidistant. The right and left linear verticals are divided into twenty equal divisions using horizontal lines labeled in a descending order of tenths. The center linear scale has twice as many units as either of the other linear scales and is labeled accordingly. Guide lines connect corresponding points of the left linear scale to the right logarithmic scale.

On the model nomogram the left logarithmic scale is 12 1/2 inches long and 8 inches away from the right logarithmic scale which is 17 1/2 inches long. The linear scales are 7 1/2 inches long and 2 inches apart. The distance between the sets of logarithmic and linear scales is 2 1/2 inches. For convenient usage the drawing is mounted on an 18" x 18" stiff paper board and covered with a sheet of transparent vinyl acetate which can be marked and easily erased.

Each scale is graduated carefully and labeled with numbers and/or letters so that the points can be located with a considerable degree of ease and accuracy. (Figure 1 shows the nomogram in reduced size.)

3. Application

The abac was used for computations using data taken from actual measurements and also sample data for two-dimensional vectors in all four quadrants. The results were compared with those obtained by use of the log log vector slide rule, desk calculator and N.B.S. 704 computer. The average percentage error was less than 3% for 75 computations made by three inexperienced users of the nomograms. It must be borne in mind that the accuracy of the results depends on the care taken in drawing and using the abac.

4. Directions for Using the Nomogram for Finding the Real and Imaginary Parts of the Complex Ratio $\frac{a + jb}{c + jd}$

For the Real Part:

1. From (c) on I to (d) on III a straight line intersects II. From this intersection follow the diagonal line, guidelines, or lettered notation to the corresponding point on I and label that point X_o. $\left[X_o = \frac{1}{N}\right]$

2. From (d) on I to (a) on III a straight line intersects II. A straight line through this intersection and the point X_o intersects III. Label this new intersection X_1. $\left[X_1 = \frac{1}{N} \cdot \frac{a}{d}\right]$

3. From (c) on I to (b) on III a straight line intersects II. A straight line through this intersection and the point X_o intersects III. Label this new intersection X_2. $\left[X_2 = \frac{1}{N} \cdot \frac{b}{c}\right]$ (Consult Table I and determine the process and sign of the real part of the ratio according to the signs of the variables a, b, c, & d.)

Summation Process:

(Note: If the ratio of X_1 to X_2 or X_2 to X_1 is greater than 20,000 then the quantities are beyond the range of the following summation and difference processes of this nomogram and must be added mentally.)

1. Follow the diagonal line or guide lines from X_1 to the corresponding point on IV. Label that point X_1'. $\left[X_1' = \frac{1}{N} \cdot \frac{a}{d}\right]$
2. Then follow the diagonal line or guide lines from X_2 to the corresponding point on VI and label that point X_2'. $\left[X_2' = \frac{1}{N} \cdot \frac{b}{c}\right]$
3. A straight line from X_1' to X_2' intersects V at the result (real or imaginary part of the complex ratio).

Difference Process:

1. Locate the point on V corresponding to the larger $|X|$ on III, and mark that point X_1' or X_2' respectively.
2. From the smaller $|X|$ on III, follow the line or guide lines to the corresponding point on IV and mark that point X_1' or X_2' respectively.
3. A straight line through X_1' and X_2' intersects VI at the results (real or imaginary part of the complex ratio).

For the Imaginary Part:

1. If the real part was determined, first erase X_1, X_2, X_1', and X_2'. If X_o has not been determined, follow step one under the real part solution to locate X_o
2. From (d) on I to (b) on III a straight line intersects II. A straight line through this intersection and the point X_o intersects III. Label this new intersection X_1.
3. From (c) on I to (a) on III a straight line intersects II. A straight line through this intersection and the point X_o intersects III. Label this new intersection X_2.
4. Consult Table II and determine the process and sign of the imaginary part of the ratio and follow directions given under the summation or difference process described above.

5. Limitations

With this model nomogram the range of the variables used is .01 < a, b, c, d, > 1000 with the following limitations:

$$0.01 < \frac{c}{d}, \frac{a}{c}, \frac{a}{b}, \frac{b}{c}, \frac{b}{d}, \frac{a}{d}, < 100.$$

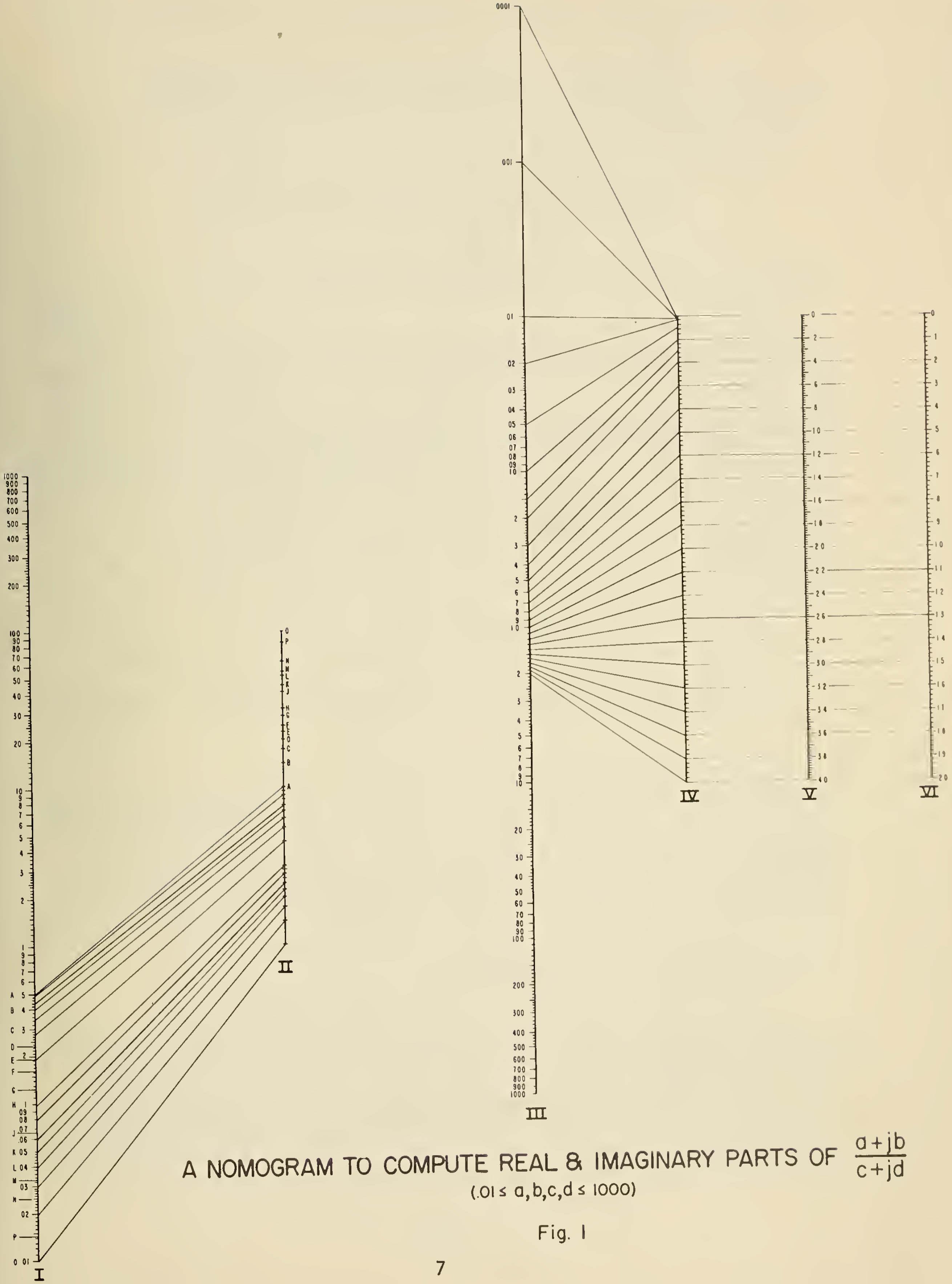
By making the linear scales and the projected lines from the logarithmic scale movable, the range on the results can be extended. In moving the linear scale in order to cover certain ranges on X_1 and X_2, the decimal point on the linear scale is shifted to correspond to the particular number associated with each projected line. The results of the computations can be read accurately to two places with the model nomogram.

Table I. Real Part

Conditions on signs of the variables	Process	Sign of result
All four signs the same	Summation	+
Three signs the same, one different:		
a or c of different sign	Difference	+ if $\|X_1\|<\|X_2\|$ - if $\|X_1\|>\|X_2\|$
b or d of different sign	Difference	+ if $\|X_1\|>\|X_2\|$ - if $\|X_1\|<\|X_2\|$
Two signs the same:		
Signs alternate in the order a, b, c, d.	Summation	+
Two consecutive signs the same in the order a, b, c, d.	Summation	-

Table II. Imaginary Part

Conditions on signs of the variables	Process	Sign of results
All four signs the same	Difference	+ if $\|X_1\|>\|X_2\|$ - if $\|X_1\|<\|X_2\|$
Three signs the same, one different:		
a or d of different sign	Summation	+
b or c of different sign	Summation	-
Two signs the same:		
Signs alternate in the order a, b, d, c.	Difference	+ if $\|X_1\|>\|X_2\|$ - if $\|X_1\|<\|X_2\|$
Two consecutive signs alike in the order a, b, d, c.	Difference	+ if $\|X_1\|<\|X_2\|$ - if $\|X_1\|>\|X_2\|$



A NOMOGRAM TO COMPUTE REAL & IMAGINARY PARTS OF $\dfrac{a+jb}{c+jd}$

$(.01 \leq a,b,c,d \leq 1000)$

Fig. 1

Part Two. A Nomogram for Computing $\left|\frac{a + jb}{c + jd}\right|$

The nomogram and instructions given below can be used to compute the absolute value of the ratio of two vector quantities.

1. Construction Details

This nomogram has three vertical lines parallel and equidistant. The left vertical is an ascending logarithmic scale while the right vertical is a descending logarithmic scale of the same cycle and length. The center vertical line is graduated into an ascending logarithmic scale having cycles one fourth as long as one cycle of either of the two outer logarithmic scales. The scales are aligned so that a horizontal line through the 10 on the outer scales intersects the center at 1. The diagonal lines are drawn from points corresponding to numbers y on the center vertical to points corresponding to $\sqrt{1 + y}$ on the left vertical and to points corresponding to $\sqrt{1 + 1/y}$ on the right vertical. The diagonals on the right are linear extensions of the diagonals on the left.

The range on the variables as well as the accuracy of the results have limitations only due to the practical size of the drawings.

2. Instructions for Using the Nomogram for Computing $\left|\frac{a + jb}{c + jd}\right|$

From a point corresponding to (b) on the left vertical scale to the point corresponding to (a) on the right vertical scale, a straight line intersects the center vertical scale. From this intersection follow the diagonal line or diagonal guide lines to a corresponding point on the left vertical scale. And from this point to the point corresponding to (c) on the right vertical scale, a straight line intersects the center vertical. From this intersection follow the horizontal line or horizontal guide lines to a corresponding point on the left vertical scale and circle that point.

From a point corresponding to (d) on the right vertical scale to the point corresponding to (c) on the left vertical scale, a straight line intersects the center vertical scale. From this intersection follow the diagonal line or diagonal guide lines to a corresponding point on the right vertical scale. And from this point to the point corresponding to (a) on the left vertical scale, a straight line intersects the center vertical. From this intersection follow the horizontal line or horizontal guide lines to a corresponding point on the right vertical scale and circle that point. A straight line through the circled points intersects the center vertical at the answer.

The accuracy of this nomogram depends on the care in drawing, and in using it; also the over all size and the range of the scale on which the answer is found contribute to the accuracy. The drawing Figure 2, in this report is condensed from the model which yielded two place accuracy.

3. Justification

Let L represent the horizontal line extending from the point corresponding to 10 on the left axis to the point corresponding to 10 on the right axis.

Let the distance from L to the point corresponding to 100 on the left axis be one unit.

The scales are so constructed that the directed distance (considering the ascending direction as positive) from L to a point corresponding to a number x on the left axis is

$$\ell(x) = \log x - 1,$$

from L to a point corresponding to a number y on the center axis is

$$m(y) = \frac{\log y}{4},$$

and from L to a point corresponding to a number Z on the right axis is

$$r(Z) = 1 - \log Z.$$

The diagonals on the right are linear extensions of the diagonals on the left since

$$\frac{\ell(\sqrt{1 + y}) \ + \ r(\sqrt{1 + 1/y})}{2} = \frac{(\log\sqrt{1 + y} - 1) \ + \ 1 - \log\sqrt{1 + 1/y})}{2}$$

$$= \frac{1}{2} \log\left(\sqrt{\frac{1 + y}{1 + 1/y}}\right) = \frac{1}{4} \log\left(\frac{1 + y}{1 + 1/y}\right) = \frac{1}{4} \log\left(\frac{y(1 + y)}{y + 1}\right)$$

$= \frac{1}{4} \log y = m(y)$ so that $m(y)$ is the average of $\ell(\sqrt{1 + y})$ and $r(\sqrt{1 + 1/y})$.

The following is a proof that the results obtained from the foregoing instructions is the correct one:

A straight line drawn from the point corresponding to (b) on the left axis to the point corresponding to (a) on the right axis will intersect the center axis at a point corresponding to some number y whose directed distance m(y) from L is the average of $\ell(b)$ and r(a) so that

$$\frac{\log y}{4} = m(y) = \frac{\ell(b) + r(a)}{2} = \frac{(\log b-1) + (1-\log a)}{2}$$

$$= \frac{1}{2} \log \frac{b}{a}$$

$$\text{or } \log y = 2 \log \frac{b}{a} = \log\left(\frac{b}{a}\right)^2 .$$

Hence $y = \frac{b^2}{a^2}$.

The diagonal from this point of intersection will meet the left axis at a point corresponding to $\sqrt{1 + y} = \sqrt{1 + b^2/a^2}$, whose directed distance from L will be $\ell\left(\sqrt{1 + b^2/a^2}\right)$. The straight line from that point to the point on the right axis corresponding to c will intersect the center axis at a point P whose directed distance D_1 from L will be the average of $\ell\left(\sqrt{1 + b^2/a^2}\right)$ and r(c), <u>i.e.</u>

$$D_1 = \frac{\ell\left(\sqrt{1 + b^2/a^2}\right) + r(c)}{2} = \frac{\left(\log\sqrt{1 + b^2/a^2} - 1\right) + (1 - \log c)}{2}$$

$$= \frac{1}{2} \log\left(\frac{\sqrt{1 + (b/a)^2}}{c}\right) .$$

The point to be circled according to the first paragraph of the instructions is on the same horizontal line as P and hence will have the same directed distance D_1 from L.

Similarly, it may be shown that the point to be circled according to the second paragraph of the instructions will have a directed distance

$$D_2 = 1/2 \log\left(\frac{a}{\sqrt{1 + (d/c)^2}}\right) \text{ from L.}$$

The straight line joining the circled points will intersect the center axis at a point corresponding to some number $m(\alpha)$ from L will be the average of D_1 and D_2 so that

$$1/4 \log \alpha = m(\alpha)$$

$$= \frac{D_1 + D_2}{2} = 1/2 \left[1/2 \log\left(\frac{\sqrt{1 + (b/a)^2}}{c}\right) + 1/2 \log\left(\frac{a}{\sqrt{1 + (d/c)^2}}\right) \right]$$

$$= \frac{1}{4} \log \frac{a\sqrt{1 + (b/a)^2}}{c\sqrt{1 + (d/c)^2}}$$

Hence,

$$\alpha = \frac{a\sqrt{1 + (b/a)^2}}{c\sqrt{1 + (d/c)^2}} = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}} = \frac{|a + jb|}{|c + jd|} .$$

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A NOMOGRAM TO COMPUTE $\left|\frac{a+jb}{c+jd}\right|$

$(1 \leq a, b, c, d \leq 200)$

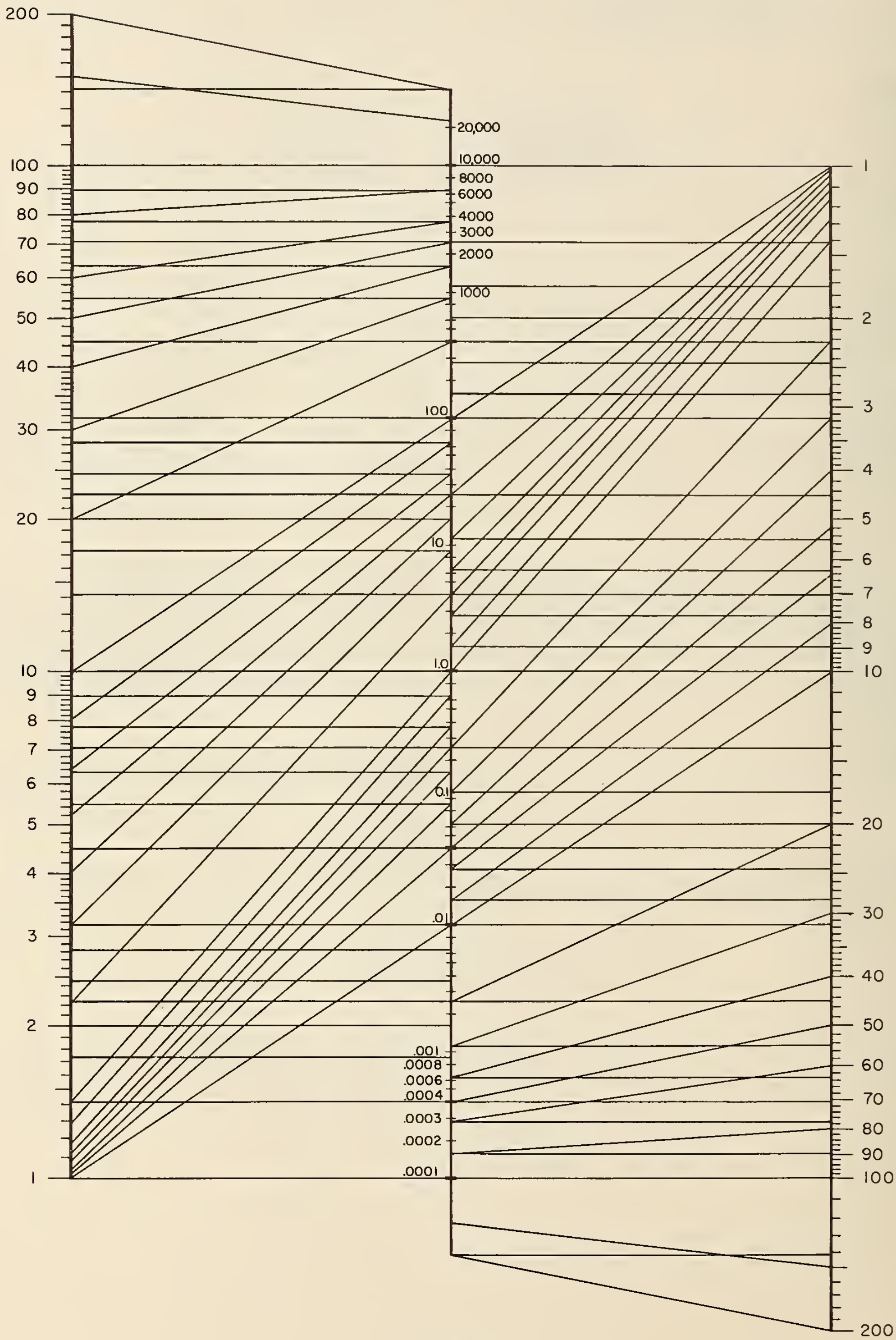

Fig 2

References

[1] A.S. Levens, Nomography, 2nd Edition, John Wiley & Sons, Inc., New York, 1959.

[2] H.J. Allcock, et.al., The Nomogram, 4th Edition, Pitman Publishing Corporation, New York, 1950.

[3] A.S. Levens, Nomography, John Wiley & Sons, Inc., New York, 1948.

[4] L.H. Johnson, Nomography and Empirical Equations, John Wiley & Sons, Inc., New York, 1952.

[5] L. Ivan Epstein, Nomography, Interscience Publishers, Inc., New York, 1958.

[6] E.L.R. Corliss, et.al., "Hearing by Bone Conduction", (not yet published, NBS, Sound Section).

[7] John Todd, Table of Arctangents of Rational Numbers, U.S. Department of Commerce, National Bureau of Standards, Applied Series II, 1951.

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35