NBS

Technical Note 238

MISCELLANEOUS STUDIES IN PROBABILITY AND STATISTICS:

Distribution Theory, Small-Sample Problems, and Occasional Tables

THE STATISTICAL ENGINEERING LABORATORY

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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The Statistical Engineering Laboratory

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FOREWORD

This note makes generally available the results of a number of special investigations that have been made at various times, to provide answers to specific questions raised in connection with consulting work or to supply specific needs for tables in special forms. Thus, this publication consists of a collection of heretofore unpublished notes, reports, or working papers prepared by various members of the staff of the Statistical Engineering Laboratory, NBS. Some of the notes, while complete in themselves, provide only partial answers to the questions which motivated the studies. They are published now, for convenient reference, since there is no plan for continuation of the studies of which they formed a part. Others of the notes remained unpublished simply because they were believed to be of very limited interest.

The date of preparation of the original unpublished note is given for each item included in this publication. We are grateful to E. L. Crow, NBS Boulder Laboratories, for suggesting a number of editorial and technical improvements. The authors have made at most very minor changes.

March 1964 Churchill Eisenhart

CONTENTS

ABSTRACT

This publication makes available some notes and tables prepared at various times by the staff of the Statistical Engineering Laboratory.

Contents: (1) Distribution of the ratio of two F variates having n-1 and n degrees of freedom, by J. M. Cameron and Cyrus Derman. (2) Some notes on the Cauchy distribution, by Cyrus Derman. Includes variance of the sample mean for truncated Cauchy distribution; the Cauchy distribution whose cumulative distribution function deviates least from the standard normal c.d.f. (3) The better one out of two, by E. P. King. Variance of the observation closest to the population mean. (4) Variance of medians and pseudo-medians, by Mary G. Natrella. For sample sizes m up to 10, gives the variances (5D) of the median (m odd), pseudo-median (m even), and average of two values on either side of the median (m odd), for the normal and rectangular distributions and ($m \leq 6$) for the extreme-value distribution. (5) Probability points of order statistics in random samples of size n from a uniform distribution over (0,1), by Churchill Eisenhart and Lola S. Deming. Gives probability points (4S) of each order statistic for probabilities α = .001, .005, .01, .025, .05, .10, .20, .25, .50 and n = 2(1)10.

DISTRIBUTION OF THE RATIO OF TWO F VARIATES HAVING n-1 AND n DEGREES OF FREEDOM

J. M. Cameron and Cyrus Derman*

Problem: If F_1 and F_2 are independently distributed as F with n-1 and n degrees of freedom, show that $w = \sqrt{F_1/F_2}$ has the F distribution for 2n-2 and 2n-2 degrees of freedom.

Solution: The density function of F having n-1 and n degrees of freedom is given by

$$g(F) = \frac{\Gamma(\frac{2n-1}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2})} \left(\frac{n-1}{n}\right)^{\frac{1}{2}(n-1)} \frac{F^{\frac{1}{2}(n-3)}}{\left[1 + (\frac{n-1}{n})F\right]^{\frac{1}{2}(2n-1)}} \qquad F>0$$

$$= 0 \qquad F \leq 0$$

If F_1 and F_2 are independent random variables having the above distribution, then their joint density function is

$$g(F_1)g(F_2) = \frac{K^2 (F_1F_2)^{\frac{1}{2}(n-3)}}{\left[1 + (\frac{n-1}{n})(F_1+F_2) + (\frac{n-1}{n})^2 F_1F_2\right]^{\frac{1}{2}(2n-1)}} \qquad F_1, F_2 > 0$$

$$= 0 \qquad \text{otherwise}$$

Letting $F_1 = yz$ and $F_2 = y$ the joint density function of y and z is

$$h(y,z) = \frac{K^2 \, z^{\frac{1}{2}(n-3)} y^{n-2}}{\left[1 + \frac{n-1}{n} y(z+1) + (\frac{n-1}{n})^2 y^2 z\right]^{\frac{1}{2}(2n-1)}} \qquad y,z>0$$

$$= 0 \qquad \text{otherwise}$$

Using formula 5(a), p. 34 of [1], it can be shown that

$$\int_0^{\infty} h(y,z)dy = \frac{K' \, z^{\frac{1}{2}(n-3)}}{(1 + \sqrt{z})^{2(n-1)}}$$

where
$$K' = \frac{1}{2} \frac{\Gamma(2n-2)}{\Gamma(n-1)\,\Gamma(n-1)}$$

By letting $w = \sqrt{z}$ the density function becomes

$$g(w) = \frac{\Gamma(2n-2)}{\Gamma(n-1)\,\Gamma(n-1)} \frac{w^{n-2}}{(1+w)^{2n-2}} \qquad w>0$$

$$= 0 \qquad \text{otherwise}$$

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It can be seen that $w = \sqrt{F_1/F_2}$ has the distribution of an F variate with 2n-2 and 2n-2 degrees of freedom.

REFERENCES

[1] Gröbner, W. and Hofreiter, N., Integraltafel, Zweiter Teil, Bestimmte Integrale, Springer-Verlag, Wien und Innsbruck, 1950.

February 1953

SOME NOTES ON THE CAUCHY DISTRIBUTION*

Cyrus Derman**

1. A critical point of truncation.

It is a well known fact that the distribution of the arithmetic mean of n Cauchy variates is independent of the value of n. For this reason the arithmetic mean is an unacceptable estimator of the location parameter. The sample median is usually used for this purpose. However, the arithmetic mean of variates with a truncated Cauchy distribution no longer has this undesirable property. One might then ask the following questions. What is the relative efficiency of the two estimators? What degree of truncation is necessary in order to make the arithmetic mean more efficient than the sample median?

The probability density of a Cauchy variate is given by

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2} \qquad \text{for } -\infty < x < \infty \,. \tag{1}$$

Let

$$g_z(x) = \frac{f(x)}{2\int_0^z f(x)dx} = \frac{1}{2\tan^{-1}z}\frac{1}{1+x^2} \qquad \text{for } -z < x < z \,, \tag{2}$$

$$= 0 \qquad \text{otherwise;} \tag{3}$$

i.e., $g_z(x)$ is the distribution of x truncated at z and $-z$.

We also have

$$E[x(z)] = 0 \tag{4}$$

and

$$\sigma^2_{x(z)} = \frac{1}{2\tan^{-1}z}\int_{-z}^{z}\frac{x^2}{1+x^2}dx = \frac{z-\tan^{-1}z}{\tan^{-1}z} \,, \tag{5}$$

where $x(z)$ is a random variable with a Cauchy distribution truncated at z and $-z$.

The asymptotic variance of the sample median is given by

$$\frac{1}{4ng_z^2(0)} = \frac{(\tan^{-1}z)^2}{n} \,. \tag{6}$$

Thus, since the asymptotic variance of the arithmetic mean is $\sigma^2_{x(z)}/n$, the

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*This investigation was suggested by Dr. C. Eisenhart. W. Knight was of assistance in connection with some of the work with probability tables.

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efficiency of the arithmetic mean relative to the sample median is

$$\frac{(\tan^{-1}z)^3}{z - \tan^{-1}z} \tag{7}$$

By using the tables of $\tan^{-1}z$ we can find a value of z that makes (7) approximately unity. This occurs for $z = 3.41$. Since (7) is monotonic for $z > 0$, the arithmetic mean is more efficient than the sample median for $z < 3.41$. (See Note 1.)

The probability that $z = 3.41$ is exceeded in the original population is given by $\frac{1}{2} - [\tan^{-1}(3.41)]/\pi = .0908$. Thus one sees that an appreciable truncation of the Cauchy distribution is required in order to put the arithmetic mean on an equal basis with the sample median. (See Note 2.)

2. The distribution of s and s^2 in samples of two.

If $x_1, \ldots, x_n$ are independent random variables with probability density function given by (1), the probability density functions of $s^2 = \sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{n-1}$ and s where $\bar{x} = \frac{x_1 + \ldots + x_n}{n}$ are, in general, unknown. However, in the case of n=2, it is possible to take advantage of the above-mentioned property of the arithmetic mean of Cauchy variates, i.e. the distribution of $\bar{x}$ is the same as x.

In the case of n=2 we use the identity

$$s^2 = (x_1-x_2)^2/2 \ . \tag{8}$$

Since the Cauchy distribution given by (1) is symmetrical about $x = 0$, x and $-x$ are identically distributed. Thus we see that

$$s = (x_1-x_2)/\sqrt{2} \tag{9}$$

is distributed as $\sqrt{2}\ \bar{x}$ which in turn has a distribution given by

$$g(s) = \frac{1}{\pi\sqrt{2}} \frac{1}{1 + s^2/2} \qquad \text{for } 0 \leq s < \infty \ . \tag{10}$$

Thus on making the transformation $t = s^2$, the distribution of t is given by

$$h(t) = \frac{1}{2\sqrt{2}\ \pi} \frac{1}{\sqrt{t}(1+t/2)} \qquad \text{for } 0 < t < \infty \ . \tag{11}$$

3. The normal-like Cauchy distribution.

A more general form of the Cauchy distribution is given by

$$f(x,\theta) = \frac{1}{\pi\theta}\,\frac{1}{1 + (x/\theta)^2} \,. \tag{12}$$

In statistical practice, one usually assumes an underlying normal distribution and then worries about the amount of deviation from this assumption. If we have an underlying Cauchy distribution, we might then be interested in knowing what normal distribution is closest to it in some sense. Equivalently, we might want to know what Cauchy distribution best approximates a given normal distribution.

We shall define a distribution from a class of distributions to be closest in that class to another distribution if the maximum absolute difference of the cumulative distribution functions is a minimum for all distributions in the class. In our case this means finding that value of θ in (12) such that

$$\operatorname*{Max}_{y} \left| \int_{-\infty}^{y} f(x,\theta)dx - \int_{-\infty}^{y} g(x)dx \right| \tag{13}$$

is minimized when $g(x) = 1/\sqrt{2\pi}\, e^{-\frac{1}{2}x^2}$.

Since $\int_{-\infty}^{y} f(x,\theta)dx$ is essentially tabulated in arc tangent tables and $\int_{-\infty}^{y} g(x)dx$ can be found in the normal probability tables, it is possible to find a value of θ which approximately satisfies the condition (13). This occurs for $\theta = .51$.

Thus we say that the normal-like Cauchy distribution is given by (12) where $\theta = .51\sigma$ and where σ is the standard deviation of the normal distribution. (See Note 3.)

September 1952

Referee's Comments:

Note 1. That (7) is monotonic for $z > 0$ may be shown by letting $y = \tan^{-1} z$. Then $\dfrac{(\tan^{-1} z)^3}{z - \tan^{-1} z} = \dfrac{3}{1 + \frac{2y^2}{5} + \ldots}$, where all terms of the power series are positive. Hence the (asymptotic) relative efficiency decreases steadily from $z = 0^+$ to $z = \infty$ and approaches an upper bound of 3 as z approaches zero.

Note 2. In J. Amer. Stat. Assoc. 55, June 1960, pp. 322-3, P. R. Rider gave the exact variances of the Cauchy sample median for n = 1, 3, ..., 31. They are uniformly larger than the asymptotic variances. Hence the exact variance of the truncated Cauchy sample median might be expected to be larger than the asymptotic variance, and hence the sample mean more efficient than shown by Derman. For $n = 1$ the mean and median are identical; hence the relative efficiency for $n = 1$ is unity for all finite z but undefined for infinite z.

Note 3. It would be of interest to know how close the two cumulative distribution functions can be made. It appears that an error less than about 0.12 cannot be guaranteed; i.e., the minimax error is 0.12.

The normal-like Cauchy density function (with $\theta = .51$) has its maximum equal to 0.62 in contrast to the normal maximum of 0.40.

THE BETTER ONE OUT OF TWO

E. P. King*

Problem: Let x_1 and x_2 be independent observations on a $N(0,1)$ population. Find the variance of the observation closer to the true mean, i.e., closer to zero.

Solution: Define

$$Y = x_1 \qquad \text{when } |x_1| < |x_2|$$
$$\;\;= x_2 \qquad \text{when } |x_1| > |x_2|$$

and let $F(y) = \Pr(Y \le y)$. Then we obtain

$$F(y) = \Pr(x_1 \le y, |x_1| < |x_2|) + \Pr(x_2 \le y, |x_1| > |x_2|) \; . \qquad (1)$$

Clearly $\Pr(x_1 \le y, |x_1| < |x_2|) = \Pr(x_2 \le y, |x_1| > |x_2|)$, so that (1) can be written as

$$F(y) = 2\Pr(x_1 \le y, |x_1| < |x_2|) \; . \qquad (2)$$

The appropriate region of integration is composed of two parts of equal probability, one above and one below the x_1-axis. Integrating over the appropriate region above this axis and doubling the result yields,

$$\begin{aligned} F(y) &= 4\left[\int_{-\infty}^{0} d\Phi(x_1)\int_{-x_1}^{\infty} d\Phi(x_2) + \int_{0}^{y} d\Phi(x_1)\int_{x_1}^{\infty} d\Phi(x_2)\right] && \text{for } y>0 \\ &= 4\int_{-\infty}^{y} d\Phi(x_1)\int_{-x_1}^{\infty} d\Phi(x_2) && \text{for } y\le 0 \end{aligned} \qquad (3)$$

where $\Phi(x)$ denotes the normal distribution function. Letting $u = \Phi(x_1)$, $v = \Phi(x_2)$, this expression can be integrated readily to give

$$\begin{aligned} F(y) &= 4\Phi(y) = 2\Phi^2(y) - 1 && \text{for } y>0 \\ &= 2\Phi^2(y) && \text{for } y\le 0 \end{aligned} \qquad (4)$$

We differentiate (4) and obtain the following density function for Y

$$\begin{aligned} f(y) &= 4\varphi(y)[1-\Phi(y)] && y>0 \\ &= 4\varphi(y)\Phi(y) && y\le 0 \end{aligned} \qquad (5)$$

where $\varphi(y) = \frac{d}{dy}\Phi(y)$.

Clearly $E(Y) = 0$. The variance of Y becomes

$$V(Y) = 4\int_{-\infty}^{0} y^2\varphi(y)\Phi(y)dy + 4\int_{0}^{\infty} y^2\varphi(y)dy - 4\int_{0}^{\infty} y^2\varphi(y)\Phi(y)dy \; . \qquad (6)$$

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Observing that $\int_0^{\infty} y^2\varphi(y)dy = 1/2$, and that

$$\int_{-\infty}^{0} y^2\varphi(y)\Phi(y)dy = \int_0^{\infty} y^2\varphi(y)[1-\Phi(y)]dy = \frac{1}{2} - \int_0^{\infty} y^2\varphi(y)\Phi(y)dy$$

we can simplify (6) to

$$V(Y) = 4 - 8\int_0^{\infty} y^2\varphi(y)\Phi(y)dy \ . \tag{7}$$

The integral on the right side of (7) can be integrated by parts. Letting $U = \Phi(y)$ and $dV = y^2\varphi(y)dy$, we find

$$\int_0^{\infty} y^2\varphi(y)\Phi(y)dy = \frac{1}{2} - \int_0^{\infty}\varphi(y)dy\int_0^{y} t^2\varphi(t)dt \ .$$

Substituting this result in (7) yields

$$V(Y) = 8\int_0^{\infty}\varphi(y)dy\int_0^{y} t^2\varphi(t)dt \ . \tag{8}$$

The second integral in (8) can be integrated by parts to give

$$\int_0^{y} y^2\varphi(t)dt = - y\varphi(y) + \Phi(y) - \frac{1}{2} \ .$$

Hence (8) becomes

$$V(Y) = 8\left[- \int_0^{\infty} y\varphi^2(y)dy + \int_0^{\infty}\Phi(y)\varphi(y)dy - \frac{1}{2}\int_0^{\infty}\varphi(y)dy\right] . \tag{9}$$

The three terms in (9) can be integrated easily. The final result is

$$\begin{aligned} V(Y) &= 1 - 2/\pi \\ &\cong 0.363 \end{aligned} \tag{10}$$

July 1953

VARIANCE OF MEDIANS AND PSEUDO-MEDIANS

Mary G. Natrella

Problem: When the median is to be used as an estimator, and there is some possibility for choice of the sample size m, is it better to have m an odd number or an even number? For m odd, the median is a unique value. For m even, the median is taken as the average of the two central observations, and is called the pseudo-median here. The better estimate is defined as the one with the smaller variance.

The question would arise when the total number of observations to be made is fairly large, the average of the medians of small groups is to be used as an estimator, and the choice between similar small values of m is a matter of discretion. For example, if a total of 60 observations are to be made, would it be better to take 10 groups of 6(m=6) or 12 groups of 5(m=5)?

Conclusion: It seems that no general conclusion is possible--the choice of m odd or m even may be better depending on the circumstances.

Investigation:

a. Normal distribution.

Table 1 was obtained from Table 2 of [1]. Col. 1 of Table 1 ("Variance of median for m odd") is copied directly from this source. Col. 2 ("Variance of pseudo-median for m even") is the computed variance of the average of the two appropriate order statistics. Col. 3 ("Variance of average of two on either side of median for m odd") was also computed from this source for comparison with Col. 1. In comparing Col. 1 with Col. 2, it is noted that for $m \leq 10$ the variance decreases with sample size m and therefore that the larger m is always a better choice. Col. 3 is also always smaller than Col. 1, but this fact does not affect choice of sample size.

b. Rectangular distribution.

Table 2 was obtained from Table III of [2] in the same way as described above for the normal. In comparing Col. 1 with Col. 2, it is noted that the variance for m even is smaller than that for next larger m (odd). This would be expected for very small samples, since then the average of extremes from the rectangular distribution is an efficient estimator, and here the two central order statistics are sufficiently close to the extremes. For $m \leq 10$, the better choice would be m even.

c. Extreme value distribution.

Table 3 was obtained from Table II of [3]. As in the normal, the larger m is the better choice. In the extreme value distribution, the extremes (especially the larger one--i.e. sample maximum) also have a large weight, but the effect is opposite to that of the rectangular, i.e. the extremes have larger variance.

Note: For the rectangular distribution (Pearson curve with coefficients $\beta_1 = 0$, $\beta_2 = 1.8$) and for the normal distribution (Pearson curve with coefficients $\beta_1 = 0$, $\beta_2 = 3.0$), standard errors of the median are given in [4] and [5] as multiples of the standard error of the mean $(\sigma/\sqrt{n})$ for sample sizes from 2 to 20.

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[1] Godwin, H. J., "Some Low Moments of Order Statistics", Annals of Math. Stat. vol. XX No. 2, June 1949, p. 279 (Table 2).

[2] Hastings, C. Jr., Mosteller, F., Tukey, J. W., and Winsor, C. P., "Low Moments for Small Samples: A Comparative Study of Order Statistics", Annals of Math. Stat. vol. XVIII No. 3, Sept. 1947, p. 413 (Table III).

[3] Lieblein, Julius, A New Method of Analyzing Extreme Value Data, NACA Technical Note 3053, January 1954, (Table II).

[4] Pearson, E. S. and Adyanthaya, N. K., "The Distribution of Frequency Constants in Small Samples from Symmetrical Populations", Biometrika, vol. XXA, 1928, pp. 356-360.

[5] Pearson, Karl, Tables for Statisticians and Biometricians Part II, Cambridge University Press, 1931. (See CXIX of Introduction and Table XXIII).

Table 1

Variance of Median and Pseudo-Median for the Normal Distribution

Sample Size m	Variance of median (m odd)	Variance of pseudo-median (m even)	Variance of avg. of 2 on either side of median (m odd)
2		.50000	
3	.44867		.36217
4		.29820	
5	.28683		.23073
6		.21474	
7	.21045		.17466
8		.16818	
9	.16610		.14162
10		.13832	

Table 2

Variance of Median and Pseudo-Median for the Rectangular Distribution

Sample Size m	Variance of median (m odd)	Variance of pseudo-median (m even)	Variance of avg. of 2 on either side of median (m odd)
2		.50000	
3	.60000		.30000
4		.40000	
5	.42857		.28571
6		.32143	
7	.33333		.25000
8		.26667	
9	.27273		.21818
10		.22727	

Table 3

Variance of Median and Pseudo-Median for the Extreme Value Distribution

Sample Size m	Variance of median (m odd)	Variance of pseudo-median (m even)	Variance of avg. of 2 on either side of median (m odd)
2		.82247	
3	.65852		.64524
4		.43544	
5	.40598		.34553
6		.30295	

March 1954

PROBABILITY POINTS OF ORDER STATISTICS IN RANDOM SAMPLES OF SIZE n FROM A UNIFORM DISTRIBUTION OVER (0,1)

Churchill Eisenhart and Lola S. Deming

Let $H_{r/n}(x)$ denote the cumulative distribution function of the r-th order statistic $x_r (x_1 \leq x_2 \leq \ldots \leq x_n)$ in random samples of size n from the uniform distribution with probability density function $f(x) = 1$, $0 \leq x \leq 1$, then the α-probability level of $x_r \equiv x_{r/n}$ is, by definition, the value of x for which

$$H_{r/n}(x) = \alpha,$$

and is given implicitly by

$$\alpha = \sum_{s=r}^{n} \binom{n}{s} x^s (1-x)^{n-s} = I_x(r, n-r+1), \qquad (1)$$

where $I_x(p,q)$ is the incomplete beta-function ratio. See Curtiss [1].

Values of x for which $I_x(p,q) = .005, .01, .025, .05, .10, .25$, are given by Catherine M. Thompson [2] in terms of the argument $\nu_1 = 2q = 1(1)10, 12, 15, 20, 24, 30, 40, 60, 120$; $\nu_2 = 2p = 1(1)30, 40, 60, 120, \infty$.

The table which appears below gives the values of x for which $H_{r/n}(x) = \alpha$, where $\alpha = .001, .005, .01, .025, .05, .10, .20, .25, .50$; $n = 2(1)10$; $r = 1(1)n$. The original version of this table was prepared in 1952 from several sources. Entries for the probability levels .005, .01, .025, .05, .10, .25, .50 were taken from the Thompson table, either directly or through interpolation. Many of the entries for the .20 level were taken from an unpublished table prepared by Horace W. Norton III [3]; others at this level and most of the entries at the .001 level were computed on the SEAC by Lambert S. Joel and Alan J. Hoffman, National Bureau of Standards. The Tables of Fractional Powers [4] prepared under the sponsorship of the National Bureau of Standards provided values and checks for x_1 and x_n. In 1963 the table was checked by Roy H. Wampler against a manuscript copy of H. Leon Harter's 7-S table, "Percentage Points of the Beta Distribution" [5], which is to be published shortly. Wherever discrepancies were found, Harter's values were used in the present table. Some entries were obtained by direct calculation, using formula (1) above. These were cases where either it was impossible to determine the fourth significant digit from Harter's table or the entry was originally obtained by interpolation and Harter's table did not provide a check. The table, as it now stands, is believed to be accurate to within half a unit in the last digit. Credit is due to Mrs. Marion T. Carson for carrying out the direct calculations and assisting in the checking.

It will be noted that Table 1 of Robert E. Clark, "Percentage Points of the Incomplete Beta Function", 1953 [6], for samples of size n = 10(1)50, gives directly to 4 significant figures the .005, .01, .025, and .05 probability points of all order statistics x_r, $1 \le r \le n$.

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[2] Thompson, Catherine M., "Tables of Percentage Points of the Incomplete Beta-Function", Biometrika, XXXII, Oct. 1941, pp. 151-181.

[3] Norton, Horace W., III, "Table of the Upper Limit of the Incomplete Beta Integral, for Area 0.2", dated 26 March 1950, unpublished.

[4] National Bureau of Standards, Tables of Fractional Powers, Columbia University Press, 1946.

[5] Harter, H. Leon, "Percentage Points of the Beta Distribution", manuscript copy, Oct. 1963, to be published shortly.

[6] Clark, Robert E., "Percentage Points of the Incomplete Beta Function", Journal of the American Statistical Assn., 48, 1953, pp. 831-843.

January 1964

n	r	Probability Level α								
		.001	.005	.01	.025	.05	.10	.20	.25	.50
2	1	$.0^{3}5001$	$.0^{2}2503$	$.0^{2}5013$	.01258	.02532	.05132	.1056	.1340	.2929
	2	.03162	.07071	.1000	.1581	.2236	.3162	.4472	.5000	.7071
3	1	$.0^{3}3334$	$.0^{2}1669$	$.0^{2}3345$	$.0^{2}8404$	.01695	.03451	.07168	.09144	.2063
	2	.01837	.04140	.05890	.09430	.1354	.1958	.2871	.3264	.5000
	3	.1000	.1710	.2154	.2924	.3684	.4642	.5848	.6300	.7937
4	1	$.0^{3}2501$	$.0^{2}1252$	$.0^{2}2509$	$.0^{2}6309$	.01274	.02600	.05426	.06940	.1591
	2	.01302	.02944	.04200	.06759	.09761	.1426	.2123	.2430	.3857
	3	.06404	.1109	.1409	.1941	.2486	.3205	.4175	.4563	.6143
	4	.1778	.2659	.3162	.3976	.4729	.5623	.6687	.7071	.8409
5	1	$.0^{3}2001$	$.0^{2}1002$	$.0^{2}2008$	$.0^{2}5051$	.01021	.02085	.04365	.05591	.1294
	2	.01010	.02288	.03268	.05274	.07644	.1122	.1686	.1938	.3138
	3	.04755	.08283	.1056	.1466	.1893	.2466	.3266	.3594	.5000
	4	.1220	.1851	.2221	.2836	.3426	.4161	.5098	.5458	.6862
	5	.2512	.3466	.3981	.4782	.5493	.6310	.7248	.7579	.8706
6	1	$.0^{3}1667$	$.0^{3}8351$	$.0^{2}1674$	$.0^{2}4211$	$.0^{2}8512$	.01741	.03651	.04682	.1091
	2	$.0^{2}8255$	.01872	.02676	.04327	.06285	.09260	.1399	.1612	.2644
	3	.03792	.06628	.08473	.1181	.1532	.2009	.2686	.2969	.4214
	4	.09395	.1436	.1731	.2228	.2713	.3332	.4146	.4468	.5786
	5	.1814	.2540	.2943	.3588	.4182	.4897	.5776	.6105	.7356
	6	.3162	.4135	.4642	.5407	.6070	.6813	.7647	.7937	.8909
7	1	$.0^{3}1429$	$.0^{3}7158$	$.0^{2}1435$	$.0^{2}3610$	$.0^{2}7301$	.01494	.03137	.04026	.09428
	2	$.0^{2}6982$	.01584	.02267	.03669	.05338	.07882	.1195	.1380	.2285
	3	.03156	.05530	.07080	.09899	.1288	.1696	.2283	.2531	.3641
	4	.07665	.1177	.1423	.1841	.2253	.2786	.3501	.3788	.5000
	5	.1438	.2030	.2363	.2904	.3413	.4038	.4832	.5139	.6359
	6	.2375	.3151	.3566	.4213	.4793	.5474	.6291	.6593	.7715
	7	.3728	.4691	.5179	.5904	.6518	.7197	.7946	.8203	.9057
8	1	$.0^{3}1251$	$.0^{3}6264$	$.0^{2}1256$	$.0^{2}3160$	$.0^{2}6391$	.01308	.02751	.03532	.08300
	2	$.0^{2}6049$	.01374	.01966	.03185	.04639	.06863	.1044	.1206	.2011
	3	.02704	.04746	.06084	.08523	.1111	.1469	.1986	.2206	.3205
	4	.06483	.09987	.1210	.1570	.1929	.2397	.3032	.3291	.4402
	5	.1196	.1697	.1982	.2449	.2892	.3446	.4163	.4445	.5598
	6	.1927	.2578	.2932	.3491	.4003	.4618	.5379	.5668	.6795
	7	.2887	.3685	.4101	.4735	.5293	.5938	.6696	.6973	.7989
	8	.4217	.5157	.5623	.6306	.6877	.7499	.8178	.8409	.9170
9	1	$.0^{3}1112$	$.0^{3}5568$	$.0^{2}1116$	$.0^{2}2809$	$.0^{2}5683$	.01164	.02449	.03146	.07413
	2	$.0^{2}5337$	.01212	.01736	.02814	.04102	.06077	.09263	.1072	.1796
	3	.02366	.04158	.05335	.07485	.09775	.1295	.1757	.1955	.2862
	4	.05621	.08679	.1053	.1370	.1688	.2104	.2675	.2910	.3931
	5	.1025	.1461	.1710	.2120	.2514	.3010	.3661	.3920	.5000
	6	.1629	.2191	.2500	.2993	.3449	.4006	.4709	.4980	.6069
	7	.2388	.3074	.3437	.3999	.4504	.5099	.5823	.6095	.7138
	8	.3349	.4150	.4560	.5175	.5709	.6316	.7022	.7277	.8204
	9	.4642	.5550	.5995	.6637	.7169	.7743	.8363	.8572	.9259
10	1	$.0^{3}1000$	$.0^{3}5011$	$.0^{2}1005$	$.0^{2}2529$	$.0^{2}5116$	.01048	.02207	.02836	.06697
	2	$.0^{2}4774$	.01085	.01554	.02521	.03677	.05453	.08326	.09640	.1623
	3	.02104	.03701	.04751	.06674	.08726	.1158	.1576	.1756	.2586
	4	.04963	.07677	.09321	.1216	.1500	.1876	.2394	.2609	.3551
	5	.08981	.1283	.1504	.1871	.2224	.2673	.3268	.3507	.4517
	6	.1413	.1909	.2183	.2624	.3035	.3542	.4191	.4445	.5483
	7	.2046	.2649	.2971	.3475	.3934	.4483	.5163	.5423	.6449
	8	.2815	.3518	.3883	.4439	.4931	.5504	.6191	.6446	.7414
	9	.3763	.4557	.4956	.5550	.6058	.6632	.7290	.7526	.8377
	10	.5012	.5887	.6310	.6915	.7411	.7943	.8513	.8706	.9330

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