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CONCERNING THE THEORY OF RADIATION FROM A SLOTTED CONDUCTING PLANE IN A PLASMA ENVIRONMENT

James R. Wait

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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A preliminary analysis is given for the radiation into a plasma half-space from an infinitely long slot in a perfectly conducting ground plane. The plasma is anisotropic by virtue of a d-c magnetic field which is parallel to the slot. For a homogeneous plasma, it is found that for such a configuration the radiation pattern will be symmetrical but the excited surface wave is highly asymmetrical. The extension of the theory to an inhomogeneous magneto plasma is outlined briefly.

1. Introduction

The characteristics of an antenna immersed in an ionized medium has been the subject of numerous recent papers such as those presented at a recent symposium on electromagnetic theory [Jordan, 1963]. In most of these, the surrounding plasma has been assumed to be homogeneous. More realistically, the plasma properties will vary significantly in a direction normal to the conducting surface of the antenna and to its supporting structure.

In this note, the formulation is given for a slot antenna radiating into a stratified plasma medium. To simplify matters, the problem is idealized by assuming that the slot is of infinite length and the associated ground plane is perfectly conducting and is of infinite extent. Furthermore, it is assumed that the plasma may be described in terms of conventional magneto-ionic theory. In other words, thermal effects are ignored but collisions between electrons and the immobile heavy particles are accounted for by a constant collision frequency ν.

2. Formulation of the Homogeneous Plasma Problem

A relatively simple preliminary problem is considered first. An infinitely long slot in a perfectly conducting ground plane, at $z = 0$, is assumed to be excited by a voltage V which is constant for all values of y. The region, $z > 0$, above the slot is a cold electron plasma characterized by an angular plasma frequency ω_o and a constant collision frequency ν. To achieve some kind of correspondence with existing configurations used in plasma diagnostic studies, the d-c magnetic field of strength H_o is applied uniformly in the y direction. The resulting angular gyrofrequency is denoted ω_c. The problem is to calculate the fields in the plasma which are produced by the prescribed slot voltage V. A subsidiary problem is the determination of the driving point admittance of the slot. In what follows, the implied factor is $\exp(i\omega t)$.

The details of the mathematical derivation will be suppressed since closely related problems have been treated elsewhere [e.g., Wait, 1962]. For the situation described, the magnetic field of the waves has only a y component, H_y. Then, on using the appropriate tensor representation for the dielectric constant of the plasma, it may be shown that H_y satisfies the Helmholtz equation of the form

$$(\nabla^2 - \Gamma^2) H_y = 0 \quad , \qquad (1)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ is the two-dimensional Laplace operator, and where Γ is the appropriate propagation constant. It was shown previously [Wait, 1962], that for transverse propagation

$$\Gamma^2 = -k_o^2/M_p \quad , \qquad (2)$$

where $k_o = 2\pi/(\text{free space wavelength})$, and

$$M_p = \frac{1 + \frac{\omega_o^2}{i\omega g a}}{\left(1 + \frac{\omega_o^2}{i\omega g a}\right)^2 - \left(\frac{\omega_o^2 \omega_c}{g^2 \omega a}\right)^2} , \tag{3}$$

with

$$g = \nu + i\omega \quad \text{and} \quad a = 1 + \frac{\omega_c^2}{g^2} .$$

The electric field components are found from a direct application of Maxwell's equations. Thus,

$$i \epsilon_o \omega E_x = - M_p \frac{\partial H_y}{\partial z} + i K_p \frac{\partial H_y}{\partial x} , \tag{4}$$

and

$$i \epsilon_o \omega E_z = + M_p \frac{\partial H_y}{\partial x} + i K_p \frac{\partial H_y}{\partial z} , \tag{5}$$

where

$$\epsilon_o = 8.854 \times 10^{-12} ,$$

and

$$K_p = \frac{\frac{\omega_o^2 \omega_c}{g^2 \omega a}}{\left(1 + \frac{\omega_o^2}{i\omega g a}\right)^2 - \left(\frac{\omega_o^2 \omega_c}{g^2 \omega a}\right)^2} . \tag{6}$$

3. Transform Method of Solution

An important step in the solution is now to express the fields in a Fourier transform representation. Thus,

$$H_y = \int_{-\infty}^{\infty} A(\lambda)\, e^{-uz}\, e^{-i\lambda x}\, d\lambda , \tag{7}$$

where $A(\lambda)$ and u are functions of λ. Since H_y satisfies (1), it is not difficult to see that

$$u = (\lambda^2 + \Gamma^2)^{\frac{1}{2}} \quad ,$$

which is chosen to have a positive real part. The corresponding integral representations for the electric fields are found from (4) and (5). In particular, the x component is given by

$$i\,\epsilon_o\,\omega E_x = \int_{-\infty}^{+\infty} A(\lambda)\,[M_p u + K_p \lambda]\, e^{-i\lambda x}\, e^{-uz}\, d\lambda \quad . \tag{8}$$

A formal solution of the problem is obtained by noting that $E_x = 0$ at the ground plane $z = 0$ except over the slot. Thus, the requirement is

$$E_x = 0 \quad \text{for} \quad |x| > b/2 \quad \text{and} \quad z = 0 \quad , \tag{9a}$$

while

$$E_x = e(x) \quad \text{for} \quad |x| < b/2 \quad \text{and} \quad z = 0 \quad , \tag{9b}$$

where $e(x)$ is the electric field in the aperture or slot. Integrating $e(x)$ on the width of the slot, we obtain

$$\int_{-b/2}^{+b/2} e(x)\, dx = V \quad . \tag{10}$$

Now, it follows from (8), and the prescribed conditions at $z = 0$, that

$$i\,\epsilon_o\,\omega\, e(x)\, U\!\left(\frac{b}{2} - x\right) U\!\left(\frac{b}{2} + x\right)$$

$$= \int_{-\infty}^{+\infty} A(\lambda)\,[M_p u + K_p \lambda]\, e^{-i\lambda x}\, d\lambda \quad , \tag{11a}$$

where $U(x) = 1$ for $x > 0$, $= 0$ for $x < 0$, is the positive step function. By the well-known property of the Fourier integral, (11a) may be inverted to give

$$A(\lambda)\,[M_p u + K_p \lambda] = \frac{i \epsilon_o \omega}{2\pi} \int_{-b/2}^{b/2} e(x)\, e^{i\lambda x}\, dx \quad , \tag{11b}$$

which, in essence, is the formal solution of the problem.

To carry out the integration in (11b) requires that $e(x)$ be specified. The simplest assumption is a uniform field across the slot. For example,

if

$$e(x) = V/b \quad , \tag{12}$$

it is seen that

$$\int_{-b/2}^{b/2} e(x)\, e^{i\lambda x}\, dx = V \frac{\sin(\lambda b/2)}{(\lambda b/2)} \ . \tag{13}$$

A second and somewhat more realistic assumption is

$$e(x) = \frac{V}{\pi\,[(b/2)^2 - x^2]^{\frac{1}{2}}} \quad , \tag{14}$$

which both satisfies (9) and has the singular behavior at $|x| = b/2$ which is appropriate for a conducting half-plane or knife edge. Furthermore, the field distribution given by (14) is exact for the electrostatic problem of a slot in a conducting plane [Smythe, 1950]. Thus, using (14), we find that

$$\int_{-b/2}^{b/2} e(x)\, e^{i\lambda x}\, dx = V J_o(\lambda b/2) \quad , \tag{15}$$

where $J_o(Z)$ is the Bessel function of zero order and argument Z. On arriving at (15), we make use of the well-known identity

$$\pi J_o(Z) = \int_0^{\pi} e^{i Z \cos\theta} \, d\theta \quad . \tag{16}$$

Using the $A(\lambda)$'s deduced from (11), it is now found that (7) may be written

$$H_y = \frac{V i \varepsilon_o \omega}{2\pi} \int_{-\infty}^{+\infty} f(\lambda) \frac{e^{-uz} e^{-i\lambda x}}{M_p u + K_p \lambda} \, d\lambda \quad , \tag{17}$$

where

$$f(\lambda) = \frac{\sin(\lambda b/2)}{(\lambda b/2)} \quad \text{when} \quad e(x) = V/b \quad , \tag{18}$$

and

$$f(\lambda) = J_o(\lambda b/2) \quad \text{when} \quad e(x) = (V/\pi)[(b/2)^2 - x^2]^{-\frac{1}{2}} \quad . \tag{19}$$

In general,

$$f(\lambda) = \frac{1}{V} \int_{-b/2}^{b/2} e(x) \, e^{i\lambda x} \, dx \quad . \tag{20}$$

From any of the above representations, it readily follows that $f(\lambda)$ approaches unity when b tends to zero. Thus, in many applications when dealing with excitation by a narrow slot, it is permissible to replace $f(\lambda)$ by unity. However, for the sake of generality, the function will be retained in the integral of (17). Corresponding integral expressions for the electric field components are obtained by differentiation of H_y according to (4) and (5).

4. Asymptotic Expressions for the Field

Of principal interest is the radiation pattern of the slot. This may be obtained by applying the saddle-point method to evaluate (17). The saddle point is located at $\lambda = \lambda_a$ which is located where $\partial[uz + i\lambda x]/\partial\lambda = 0$. In the resulting deformation of the contour to the steepest descent path, one must account for any poles which are crossed. Under most conditions the integrand of (17) has a pole in the fourth quadrant at $\lambda = \lambda_g$ which is a solution of

$$M_p u + K_p \lambda = 0 \quad . \tag{21}$$

Thus, in an asymptotic sense, (17) becomes

$$H_y \cong \frac{V i \epsilon_o \omega}{2\pi}\left[\frac{f(\lambda_a) u_a}{M_p u_a + K_p \lambda_a}\left(\frac{2\pi}{\Gamma R}\right)^{\frac{1}{2}} e^{-\Gamma R} - (2\pi i)\frac{f(\lambda_g)\,\delta}{\left[\frac{\partial}{\partial\lambda}(M_p u + K_p \lambda)\right]_{\lambda=\lambda_g}} e^{-u_g z} e^{-i\lambda_g x}\right], \tag{22}$$

where $u_a = (\lambda_a^2 + \Gamma^2)^{\frac{1}{2}}$, $u_g = (\lambda_g^2 + \Gamma^2)^{\frac{1}{2}}$, and $R = (x^2 + z^2)^{\frac{1}{2}}$. The saddle-point condition, mentioned above, leads readily to the relations

$$\lambda_a = -i\Gamma \sin\theta \quad \text{and} \quad u_a = \Gamma \cos\theta \quad ,$$

where $\sin\theta = x/R$ and $\cos\theta = z/R$. The factor, δ, occurring in (22) is unity or zero depending on whether or not the pole at λ_g is swept over in the deformation of the contour to the steepest descent path.

In (22), the term proportional to $(\Gamma R)^{-\frac{1}{2}} \exp(-\Gamma R)$ is the cylindrical space wave. By definition, the θ dependence of its multiplier is the radiation pattern of the source. Because of the asymptotic nature of (22), terms which contain factors like $(\Gamma R)^{-3/2}$, $(\Gamma R)^{-5/2}$, ... etc., have been neglected. The second term, in the square brackets of (22), has the characteristics of a surface wave. Under some conditions the imaginary part of λ_g is very small in which case the attenuation in the x direction is very small. However, in these cases, the real part of u_g is appreciable, so the attenuation in the z direction (i.e., normal to the ground plane) is quite large.

The situation discussed here bears a close resemblance to one considered by Seshadri [1962]. He dealt with the fields of a magnetic line source in a lossless anisotropic plasma with a perfectly conducting boundary. To compare our results with his, we set $\nu = 0$ and note that

$$M_p = \frac{K_1}{K} \quad \text{and} \quad K_p = -\frac{K_2}{K} ,$$

where

$$K_1 = 1 - \frac{\omega_o^2}{\omega^2}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} , \tag{23}$$

$$K_2 = \frac{\omega_o^2}{\omega^2}\left(\frac{\omega}{\omega_c} - \frac{\omega_c}{\omega}\right)^{-1} , \tag{24}$$

and $K = K_1^2 - K_2^2$. The corresponding integral representation for H_y is given by

$$H_y = \frac{V \epsilon_o \omega K}{2\pi} \int_{-\infty}^{+\infty} f(\lambda) \frac{e^{-ivz} e^{-i\lambda x}}{K_1 v + i K_2 \lambda} d\lambda , \tag{25}$$

where $v = (k_e^2 - \lambda^2)^{\frac{1}{2}}$ with $k_e^2 = k_o^2 K/K_1$. The pole now occurs at $\lambda = \lambda_g$, where

$$\lambda_g = k_e K_1/K^{\frac{1}{2}} = k_o K_1^{\frac{1}{2}} \quad .$$

As indicated by Seshadri [1962], this pole is either on the real or the negative imaginary axis depending on whether K_1 is positive or negative, respectively. However, if the sign of K_2 is changed by replacing ω_c by $-\omega_c$, this pole is the negative real or the positive imaginary axis, depending on whether K_1 is positive or negative, respectively.

The asymptotic evaluation of (25) shows that, for large positive x,

$$H_y \cong - \frac{V \epsilon_o \omega K f(k_e \sin\theta) \cos\theta}{K_1 \cos\theta + i K_2 \sin\theta} \frac{e^{-i(k_e R - \pi/4)}}{(2\pi k_e R)^{\frac{1}{2}}} \tag{26}$$

$$+ V f(k_o K_1^{\frac{1}{2}}) \, \delta \, \epsilon_o |K_2| \exp\left[-i k_o K_1^{\frac{1}{2}} x - k_o |K_2| K_1^{-\frac{1}{2}} z\right] ,$$

where $\delta = 1$ if $\tan\theta = (x/z) > (K)^{\frac{1}{2}}/K_2$ and $= 0$ otherwise.

The latter inequality is the explicit condition that the surface wave pole be swept over in the steepest descent deformation of the contour. It is important to remember, as indicated above, that the excitation of the surface wave for $x > 0$ requires K_2 to be positive. Also, in order that the surface wave be non-attenuating in the x direction, it is understood that $\mathrm{Im}\,\lambda_g = 0$. This requires, in addition to other restrictions, that $\omega_o/\omega < 1$ and

$$-\left[1 - \left(\frac{\omega_o}{\omega}\right)^2\right]^{\frac{1}{2}} < \frac{\omega_c}{\omega} < \left[1 - \left(\frac{\omega_o}{\omega}\right)^2\right]^{\frac{1}{2}} \quad .$$

It is evident from (26) and the conditions imposed on it that if the direction of the d-c magnetic field is reversed, K_2 becomes negative and the surface wave is not present for $x > 0$. However, in this situation, the surface wave is still excited but it propagates in the negative x direction. Thus, the solution given by (26) also holds for negative x if these conventions are used.

5. Some Extensions of the Theory

In the foregoing discussion, the plasma has been assumed homogeneous for the whole half-space $z > 0$. The theory may be extended to a horizontally stratified plasma without too much difficulty. In certain plasma configurations this may be a more realistic assumption. An outline of the theoretical approach is given here for the case when M_p and K_p, as defined by (3) and (6), are functions of z.

Maxwell's equations for the horizontally stratified plasma may be written

$$i \epsilon_o \omega E_x = - M_p \frac{\partial H}{\partial z} + K_p \lambda H \quad , \qquad (27)$$

$$i \epsilon_o \omega E_z = - i \lambda M_p H + i K_p \frac{\partial H}{\partial z} \quad , \qquad (28)$$

$$-i \mu_o \omega H_y = \frac{\partial E_x}{\partial z} + i \lambda E_z \quad , \qquad (29)$$

where λ stands for the operator $i \partial/\partial x$. Using (27) and (28) to eliminate E_x and E_z from (29), it readily follows that

$$\frac{\partial^2 H}{\partial z^2} + \left(\frac{k_o^2}{M_p} - \lambda^2\right) H + \frac{1}{M_p} \frac{\partial M_p}{\partial z} \frac{\partial H}{\partial z} - \frac{\lambda}{M_p} \frac{\partial K_p}{\partial z} H = 0. \qquad (30)$$

As in the previous section, H_y is written in the form of a Fourier integral. Thus,

$$H_y = \int_{-\infty}^{+\infty} B(\lambda) F(\lambda, z) e^{-i \lambda x} d \lambda \quad , \qquad (31)$$

where $B(\lambda)$ is some undetermined function of λ and $F(\lambda, z)$ is a function of λ and z which satisfies (30), and is well behaved at $z \to +\infty$. Using (27) and (30), it follows that

$$i \epsilon_o \omega E_x = \int_{-\infty}^{+\infty} B(\lambda) [-M_p(z) F'(\lambda, z) + K_p(z) \lambda F(\lambda, z)] e^{-i\lambda x} d\lambda , \tag{32}$$

where the prime indicates a differentiation with respect to z.

The formal solution for initial conditions (9a) and (9b) are obtained in almost the same manner as for the homogeneous plasma. In terms of the prescribed electric field, $e(x)$, in the slot, it is readily shown that

$$B(\lambda) [-M_p(0) F'(\lambda, 0) + K_p(0) \lambda F(\lambda, 0)] = \frac{i \epsilon_o \omega}{2\pi} \int_{-b/2}^{b/2} e(x) e^{i\lambda x} dx . \tag{33}$$

It is then found that the resultant magnetic field anywhere in the half-space is given by

$$H_y = \frac{V i \epsilon_o \omega}{2\pi} \int_{-\infty}^{+\infty} f(\lambda) \frac{F(\lambda, z) e^{-i\lambda x}}{[-M_p(0) F'(\lambda, 0) + K_p(0) \lambda F(\lambda, 0)]} d\lambda , \tag{34}$$

where

$$f(\lambda) = \frac{1}{V} \int_{-b/2}^{b/2} e(x) e^{i\lambda x} dx , \tag{35}$$

in terms of the voltage V across the slot.

Equation (34) is a formal solution of the problem. In the present form, it is not very useful since, among other things, it requires that a solution of (30) be found before the integration in (34) may be effected. However, some simplification is achieved when the properties of the plasma are slowly varying in the z direction.

In this case, a WKB-type solution is applicable.

To obtain a WKB solution, it is desirable to introduce a new function $\Phi(\lambda, z)$, which is defined by

$$\Phi = M^{\frac{1}{2}} F \quad , \tag{36}$$

Now, since F satisfies (30) it is not difficult to show that Φ satisfies a wave equation of the form

$$\left[\frac{\partial^2}{\partial z^2} + Q^2\right] \Phi = 0 \quad , \tag{37}$$

where

$$Q^2 = \frac{k_o^2}{M_p} - \lambda^2 - \frac{\lambda^-}{M} \frac{\partial K_p}{\partial z} + \frac{1}{4 M_p^2} \left(\frac{\partial M_p}{\partial z}\right)^2 - \frac{1}{2 M_p} \frac{\partial^2 M_p}{\partial z^2} . \tag{38}$$

The required WKB solution of (37) is given by

$$\Phi \cong \frac{1}{Q^{\frac{1}{2}}} \exp\left[-i \int_o^z Q \, dz\right] \quad , \tag{39}$$

and

$$F(\lambda, z) \cong \frac{1}{M^{\frac{1}{2}}(z)\, Q^{\frac{1}{2}}(\lambda, z)} \exp\left[-i \int_o^z Q(\lambda, z) \, dz\right] \quad , \tag{40}$$

where the explicit dependence on λ and z is noted. This approximate solution is only valid when $Q(\lambda, z)$ does not have any singularities over the significant range of λ and z. In particular, a zero of Q at some value of z will give rise to a turning point, and the solution must be modified [Budden, 1961]. The matter will not be pursued further here.

6. Concluding Remarks

Although the present note contains only a mathematical derivation, some important physical principles are evident. In particular, it is seen that the fields produced by the slot are asymmetrical about the broadside direction (i.e., $\theta = 90^{\circ}$). However, for the lossless homogeneous plasma, the radiation pattern is symmetrical insofar as amplitude of the field is concerned. On the other hand, the surface wave which may be excited in the lossless plasma is uni-directional. Under more general conditions such as in the presence of collisions and stratification in the plasma, the situation becomes much more complicated and further work is obviously required. Recent progress on related problems for isotropic plasma has been reported by Galejs [1964]. It is believed that the methods which he develops are necessary for considering the impedance of the slot.

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