



Technical Note

No. 216

COMPUTATION OF HANKEL FUNCTIONS

LESLIE A. BERRY



U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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Issued June 12, 1964

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Foreword

This work was sponsored in part by Rome Air Development Center, Griffiss Air Force Base, New York, in connection with Tasks 4 and 5 of D. O. No. AF-30(602)-2488, and in part by NBS Project No. 85111.

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Computation of Hankel Functions

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Methods of computing solutions to Bessel's equation, $z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - v^2) y = 0$, for complex v and z , and for small, real v , are given in detail. The methods are most suitable for use on an electronic digital computer. With a computer which carries eight decimal digits, the methods presented for small, real v will return at least six correct significant digits for all z . When v and z are both complex, the accuracy depends on $|z|$ (since asymptotic forms are used) and the ratio v/z . Relative error curves are shown as a function of $|v/z|$ parametric in $|z|$. For v sufficiently far from z , seven correct digits are obtained, which for $v \cong z$, the relative error is of the order $10^{-2} / |z|$.

1. Introduction

Solutions to electromagnetic or sound-wave problems in spherical or cylindrical co-ordinates usually involve solutions of Bessel's equation, $z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - v^2) y = 0$. The Hankel functions [Watson, 1958] $H_v^{(1)}(z)$ and $H_v^{(2)}(z)$ are one set of linearly independent solutions. Other solutions can be written in terms of these solutions [see (19)]. The superscript indicates the "kind," v , the "order," and z , the "argument" of the functions, and either v , or z , or both may be complex. The purpose of this paper is to present in one place detailed instructions for computing the Hankel functions on an electronic digital computer, and to investigate quantitatively the relative errors incurred. No new mathematical development is involved, but since considerable algebraic reduction and systemization of scattered results were necessary, publication of the results seems desirable.

Section 2 details methods of computing $H_v(z)$ when v and z are complex. If $v \cong z$, $H_{\frac{1}{2}}(z)$ must be computed accurately, so techniques are given in section 3 for computing $H_v(z)$ when v is real and small and z is unrestricted. The accuracy of the methods is discussed in section 4.

2. Complex Order, v or z Large

In this section, it is assumed only that either v or z is large. What is meant by "large" will be discussed quantitatively in section 4, but the asymptotic expressions used here [(1) and table 1] accurately describe the qualitative behavior of the Hankel functions over most of the v and z planes, since the minimum relative error is about one per cent for $|v| \cong |z| \cong 1$ (figure 5).

Two approximations are used. In the first, which will be called the Hankel approximation [Fok, 1934],

$$H_v^{(k)}(z) \cong \exp [(-1)^{k-1} i\pi/6] \sqrt{1 - \eta \coth \eta} H_{\frac{1}{3}}^{(k)}(-i\xi), \quad k=1, 2, \quad (1)$$

where

$$\xi = v (\tanh \eta - \eta), \quad (2)$$

$\cosh \eta = v/z$, and $|\arg v| < \pi/2$ [Olver, 1954]. For $|\arg v| > \pi/2$, (3) (9) is used.

The principal square root is used; i. e. ,

$$\arg \sqrt{w} = \frac{1}{2} \arg w,$$

and

$$-\pi < \arg w \leq \pi.$$

for $v/z \cong 1$ [Fok, 1946],

$$-i\xi = \frac{z}{3} \left(1 - \frac{v^2}{z^2}\right)^{3/2} \left[1 + \frac{2}{5} \left(1 - \frac{v^2}{z^2}\right) + 0 \left(1 - \frac{v^2}{z^2}\right)^2\right]. \quad (4)$$

The first term on the right has been used extensively as an approximation to $-i\xi$ in radio propagation theory [Fok, 1946; Bremmer, 1949].

The second approximation, called the Debye approximation, will be written in terms of [Watson, 1958]

$$S_v^{(k)}(z) \cong \frac{\exp \left[(-1)^{k-1} \left(\xi - \frac{\pi}{4} i \right) \right]}{\sqrt{-\frac{\pi}{2} i v \tanh \eta}} \sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{(-1)^{(k-1)m} A_m}{\left(\frac{v}{2} \tanh \eta \right)^m}. \quad (5)$$

The combination of $S_v^{(1)}(z)$ and $S_v^{(2)}(z)$ to be used depends on the location of v/z in the complex plane. The first four A_m are [British Association Mathematical Tables, 1952]:

$$A_0 = 1,$$

$$A_1 = \frac{1}{8} - \frac{5}{24} \coth^2 \eta,$$

$$A_2 = \frac{3}{128} - \frac{77}{576} \coth^2 \eta + \frac{385}{3456} \coth^4 \eta,$$

$$A_3 = \frac{5}{1024} - \frac{1521}{25600} \coth^2 \eta + \frac{17017}{138240} \coth^4 \eta - \frac{17017}{248832} \coth^6 \eta. \quad (6)$$

Forms for A_4 -- A_7 can also be found in the above reference. In (5), use the principal square root. The series in (5) can be written

$$\sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{A_m}{\left(\pm \frac{1}{2} v \tanh \eta \right)^m} = 1 \pm \frac{A_1 \coth \eta}{v} + \frac{3 A_2 \coth^2 \eta}{v^2} \pm \frac{15 A_3 \coth^3 \eta}{v^2} + \dots$$

(7)

This is an asymptotic series and should be truncated at the smallest term; i. e., all terms preceding the smallest term should be summed.

The regions of the plane to be considered are shown in figure 1, and the expressions for $H_v(z)$ are given in table 1. These expressions are correct only if $\text{Re}(z) \geq 0$, but if $\text{Re}(z) < 0$, the formulas [Watson, 1958]

$$H_{\nu}^{(1)}(ze^{k\pi i}) = \frac{\sin(1-k)\pi\nu}{\sin\pi\nu} H_{\nu}^{(1)}(z) - e^{-\nu\pi i} \frac{\sin k\pi\nu}{\sin\pi\nu} H_{\nu}^{(2)}(z) \quad (8a)$$

and

$$H_{\nu}^{(2)}(ze^{k\pi i}) = \frac{\sin(1+k)\pi\nu}{\sin\pi\nu} H_{\nu}^{(2)}(z) + e^{\nu\pi i} \frac{\sin k\pi\nu}{\sin\pi\nu} H_{\nu}^{(1)}(z), \quad (8b)$$

where k is any integer, can be used to evaluate the desired function. Although expressions can be derived for $H_{\nu}(z)$ in regions 6b, 7b, and 9, it is simpler to compute $H_{-\nu}(z)$ [so that $-\nu/z$ is in 6a, 7a, or 8] and use [Watson, 1958]

$$H_{\nu}^{(k)}(z) = \exp[(-1)^k \nu\pi i] H_{-\nu}^{(k)}(z), \quad k = 1, 2. \quad (9)$$

2.1. Determination of η

Directions for computing η and for determining the correct region of the plane, figure 1, will now be given. Let $\eta = \alpha + i\beta$, where α and β are real. Then $\alpha + i\beta = \cosh^{-1} \nu/z$. The infinitely many-valued function, \cosh^{-1} , is made single valued by requiring that $0 \leq \beta < \pi$ [Watson, 1958]. $\beta = 0$ if ν/z is real and $\nu/z \geq 1$. Specifically

$$\alpha = -\log \left(\left| \frac{\nu}{z} \pm \sqrt{\frac{\nu^2}{z^2} - 1} \right| \right). \quad (10)$$

The sign is chosen so that α has the same sign as $\text{Im}(\nu/z)$. If $\text{Im}(\nu/z) = 0$, and $|\nu/z| > 1$, the positive sign is chosen; if $|\nu/z| \leq 1$, $\eta = \text{Cos}^{-1} \nu/z$. Then

$$\beta = \text{Cos}^{-1} [\text{Re}(\nu/z)/\cosh \alpha] \quad (11)$$

These directions are exact for use in (5), and are valid for (1) over most of the plane. (For a discussion of the region of validity for the Hankel approximation, consult Olver [1954].) For (1), when $\nu \cong z$, $\arg(-i\zeta)$ can be determined by examining the first term of (4); when ν is not near z , the Debye approximation is superior.

The m in table 1 is the largest integer less than

$$\{(1 - \alpha \tanh \alpha) \tan \beta + \beta\} / \pi.$$

Now, let

$$f(\alpha, \beta) = 1 - \beta \cot \beta - \alpha \tanh \alpha . \quad (12)$$

The signs of $f(\alpha, \beta)$, $f(\alpha, \beta) + \pi \cot \beta$, $\operatorname{Re}(v/z)$, and $\operatorname{Im}(v/z)$ uniquely determine in which of the regions shown in figure 1 v/z is located. Column 2 of table 1 shows how the region can be determined.

3. Real Order

To use (1), a routine for computing $H_{\frac{1}{3}}(z)$ must be available. In this section, methods are given for computing $H_\nu(z)$ for small, real ν . The restriction that ν be real is probably not necessary if z^ν is carefully defined, but it is retained in this section.

3.1. Small $|z|$

If ν is not an integer [Watson, 1958]

$$H_\nu^{(k)}(z) = (-1)^k \{ \exp [(-1)^k \nu \pi i] J_\nu(z) - J_{-\nu}(z) \} / i \sin \nu \pi, \quad k = 1, 2, \quad (13)$$

where $J_\nu(z)$ is the Bessel function of the first kind, defined by [Watson, 1958]

$$\begin{aligned} J_\nu(z) &= \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m}}{m! \Gamma(\nu+m+1)} \\ &= \frac{(z/2)^\nu}{\Gamma(\nu+1)} \left(1 - \frac{(z/2)^2}{1! (\nu+1)} + \frac{(z/2)^4}{2! (\nu+1)(\nu+2)} - \frac{(z/2)^6}{3! (\nu+1)(\nu+2)(\nu+3)} + \dots \right) \end{aligned} \quad (14)$$

$J_\nu(z)$ is made single-valued by specifying $\arg(z/2)^\nu = \nu \arg(z/2)$ for $\Delta - 2\pi < \arg(z/2) \leq \Delta$. Ordinarily, $\Delta = \pi$, but may have different values for some applications.

If ν is an integer, (13) is indeterminate. To compute $H_n(z)$ for integral n and small z , compute $J_0(z)$ and $J_1(z)$ using (14) and then find the Bessel function of the second kind, $Y_0(z)$, from [Watson, 1958]

$$Y_0(z) = \frac{2}{\pi} \left[(\gamma + \log \left(\frac{z}{2} \right)) J_0(z) + \sum_{m=1}^{\infty} \frac{(z/2)^m J_m(z)}{m \cdot m!} \right], \quad (15)$$

where γ is Euler's constant, $\gamma = .5772156649 \dots$. The $J_m(z)$, $m > 1$ are found with the recurrence formula [Watson, 1958].

$$J_{m+1}(z) = \frac{2m}{z} J_m(z) - J_{m-1}(z). \quad (16)$$

Then [Watson, 1958]

$$Y_1(z) = \left\{ J_1(z) Y_0(z) + \frac{2}{\pi z} \right\} / J_0(z) \quad (17)$$

and

$$Y_{m+1}(z) = \frac{2m}{z} Y_m(z) - Y_{m-1}(z). \quad (18)$$

Finally,

$$H_n^{(k)}(z) = J_n(z) + (-1)^{k-1} i Y_n(z). \quad (19)$$

3.2. Large $|z|$

As $|z|$ gets large, (14) converges slowly and loses precision. More serious is the precision lost in (13) and (19), since in half of the z -plane the terms on the right are nearly negatives of each other when $\text{Im}(z)$ is large. For example, four significant digits are lost in (19) when $n = 1$, $z = 5 \exp(i 85^\circ)$. It is therefore necessary to use the asymptotic series [Watson, 1958].

$$H_\nu^{(1)}(z) = \frac{\exp(iz - i(\nu + \frac{1}{2})\pi/2)}{\sqrt{\pi z/2}} T_\nu(-2iz), \quad (-\pi < \arg z < 2\pi), \quad (20a)$$

$$H_\nu^{(2)}(z) = \frac{\exp(-iz + i(\nu + \frac{1}{2})\pi/2)}{\sqrt{\pi z/2}} T_\nu(2iz), \quad (-2\pi < \arg z < \pi), \quad (20b)$$

where

$$T_v(z) = 1 + \sum_{n=1}^{\infty} \frac{(4v^2 - 1^2)(4v^2 - 3^2) \cdots (4v^2 - (2n-1)^2)}{n! (4z)^n}. \quad (21)$$

Unless $v = n + \frac{1}{2}$, for integral n , $T_v(z)$ is an asymptotic series and should be stopped at the smallest term. If $v = n + \frac{1}{2}$, $T_v(z)$ terminates, and (20) is the finite series for $H_{n+\frac{1}{2}}(z)$.

Using a computer which carries eight decimal digits in floating point operations, about four correct significant digits in $H_v(z)$ are obtained over all of the z -plane, using the methods presented above. This accuracy is increased to at least six digits if Dingle's [1958, 1959] convergence factor is used in (21), as described below.

3.3. The Convergence Factor

In a series of papers, R. B. Dingle has developed a general theory [Dingle, 1958] of convergence factors and has applied this theory to many special functions [Dingle, 1958, 1959]. When the product of a convergence factor, $C_n(z)$, and the n th term of an asymptotic series is added to the previous $n - 1$ terms, the sum is the exact sum of the entire series. In practice, of course, the sum is not "exact," but the use of a convergence factor does significantly extend the range of application of an asymptotic series.

In the particular case of the Hankel functions, (21) is written:

$$T_v(z) = 1 + \sum_{n=1}^{N-1} \frac{(4v^2 - 1^2)(4v^2 - 3^2) \cdots (4v^2 - (2n-1)^2)}{n! (4z)^n} + \frac{(4v^2 - 1^2) \cdots (4v^2 - (2N-1)^2)}{N! (4z)^N} C_N(v, z), \quad (22)$$

where N is chosen so that the term corresponding to $N-1$ is the smallest term of (21). Then, [Dingle, 1959] for $s = N - v - \frac{1}{2}$,

$$C_N(v, z) = \sum_{t=0}^{\infty} \frac{\Gamma(v + \frac{1}{2}) \Gamma(N + v - t + \frac{1}{2})}{\Gamma(v - t + \frac{1}{2}) \Gamma(N + v + \frac{1}{2})} (-z)^t \Lambda_s^{(t)}(z)$$

$$= \Lambda_s^{(0)}(z) - \frac{(v - \frac{1}{2})z}{(N + v - \frac{1}{2})} \Lambda_s^{(1)}(z) + \frac{(v - \frac{1}{2})(v - \frac{3}{2})z^2}{(N + v - \frac{1}{2})(N + v - \frac{3}{2})} \Lambda_s^{(2)}(z) - \dots, \quad (23)$$

where [Dingle, 1958]

$$\Lambda_s^{(0)}(z) = \frac{1}{\Gamma(s+1)} \int_0^{\infty} \frac{x^s e^{-x} dx}{1+x/z},$$

$$z \Lambda_s^{(1)}(z) = (s+z+1) \Lambda_s^{(0)}(z) - z,$$

$$2z \Lambda_s^{(2)}(z) = (s+z) \Lambda_s^{(1)}(z) + \Lambda_s^{(0)}(z) - 1,$$

and

$$tz \Lambda_s^{(t)}(z) = (s+z+2-t) \Lambda_s^{(t-1)}(z) + \Lambda_s^{(t-2)}(z), \quad t > 2. \quad (24)$$

When $|s| \gg 1$, or $|z| \gg 1$, and $|\arg z| < 3\pi/4$ [Dingle, 1958],

$$\Lambda_s^{(0)}(z) \cong \frac{z}{s+z} \left(1 - \frac{z}{(s+z)^2} - \frac{z(s-2z)}{(s+z)^4} - \frac{z(s^2-8sz+6z^2)}{(s+z)^6} - \dots \right). \quad (25)$$

Otherwise, $\Lambda_s^{(0)}(z)$ can be found by numerical integration. For the general routine used with this paper, $\Lambda_s^{(0)}(z)$ was found with an eight point Gaussian quadrature formula:

$$\int_0^{\infty} x^s e^{-x} f(x) dx \cong \sum_{k=1}^8 W_k f(x_k), \quad (26)$$

since $s \geq 5$ for use in (24). The W_k and x_k are taken from Rabinowitz and Weiss [1959]. Then $C_N(v, z)$ is approximated by the first four terms of (23).

4. Discussion of Results

Fortran subroutines were written to compute $H_{\frac{1}{3}}(z)$ and $H_{\frac{2}{3}}(z)$ and $H_n(z)$ using the methods of section 3. The values for $H_{\frac{1}{3}}(z)$ and $H_{\frac{2}{3}}(z)$ were checked against the Tables of Modified Hankel Functions of Order One-Third and Their Derivatives [The Staff of the Computation Laboratory, 1945]. The computed values agree with the tabular values to at least six significant digits where the tables give six digits. (The tables give eight decimal places.) The values of $H_n(z)$ computed by the routine were checked against NBS tables for $J_0(z)$ and $J_1(z)$, [Mathematical Tables Project, 1947] and $Y_0(z)$ and $Y_1(z)$ [Computation Laboratory, 1950] and gave comparable results.

The Debye and Hankel approximations were programmed separately and checked for real order by comparing the values they returned with the correct (to at least six significant digits) answers given by the methods of section 3. A subroutine valid for small, complex v and z was obtained from SHARE [Goldstein, Kresge; and Chen, 1960], and the asymptotic routines described here returned answers correct to the expected accuracy, providing an independent check of the routines' logic.

Defining

$$\text{Relative error} = \left| \frac{\text{answer} - \text{approximation}}{\text{answer}} \right|, \quad (27)$$

figures 2, 3, and 4 show relative error curves for both approximations as functions of $|v/z|$, parametric in $|z|$ for $|\arg z| = 0^\circ, 5^\circ, \text{ and } 15^\circ$, respectively. It is apparent from these curves that the Debye approximation is the better except when $v \cong z$. Even when $|v/z| = 1$, if $|\arg v/z|$ and $|z|$ are not small, the Debye approximation is superior, as shown in figure 5. Nevertheless, any general routine for computing $H_v(z)$ must include the Hankel approximation. The relative-error curves all show that, for any z , there is an $\epsilon > 0$ such that, if $|v-z| < \epsilon$, the Debye approximation is useless.

If series (7) is truncated at the smallest term, the relative error in the Debye approximation is of the order of the magnitude of that term. Examination of figures 2 through 5 yields an empirical formula,

$$|\text{relative error}| \cong 10^{-2} / |z|, \quad (28)$$

for the Hankel approximation (1).

A general routine for $H_\nu(z)$ was written, incorporating both the Debye and Hankel approximations. The routine uses the Debye approximation unless the smallest term of series (7) is larger than $10^{-2} / |z|$, in which case the Hankel approximation is used. This routine and other computer routines described in this paper are available from SHARE [Berry, 1963].

5. Conclusions

Using an electronic computer, solutions to Bessel's equation of real order and complex argument can be computed very accurately. Even when the order is complex, computations can be made sufficiently accurate for most applications. The subroutines incorporating the methods should be valuable in radio and sound wave computations.

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7. List of Symbols

$H_\nu^{(k)}(z)$	Hankel function of argument z , order ν , and kind k , $k = 1, 2$.
$J_\nu(z)$	Bessel function of the first kind, of argument z and order ν .
$Y_\nu(z)$	Bessel function of the second kind, of argument z and order ν , sometimes called Neumann's function and denoted $N_\nu(z)$.
$T_\nu(z)$	The series defined in (21).
$\eta =$	$\alpha + i\beta = \cosh^{-1} \frac{\nu}{z}$
$S_\nu^{(k)}(z)$	The series defined in (5).
A_m	The coefficient of $S_\nu^{(k)}(z)$, see (6).
$f(\alpha, \beta) =$	$1 - \beta \cot \beta - \alpha \tanh \alpha$
$m =$	Largest interger less than $\{(1 - \alpha \tanh \alpha) \tan \beta + \beta\} / \pi$.
$C_N(z)$	Convergence factor for asymptotic series, see section 3.3.

Region	Determined by	$H_v^{(1)}(z)$	$H_v^{(2)}(z)$
1	$f(\alpha, \beta) \geq 0$ $f(\alpha, \beta) + \pi \cot \beta \geq 0$	$S_v^{(1)}(z)$	$S_v^{(2)}(z)$
2	$f(\alpha, \beta) < 0$ $f(\alpha, \beta) + \pi \cot \beta \geq 0$ $\text{Im}(\frac{v}{z}) < 0$	$S_v^{(1)}(z) - S_v^{(2)}(z)$	$S_v^{(2)}(z)$
3	$f(\alpha, \beta) < 0$ $f(\alpha, \beta) + \pi \cot \beta \geq 0$ $\text{Im}(\frac{v}{z}) < 0$	$S_v^{(1)}(z)$	$S_v^{(2)}(z) - S_v^{(1)}(z)$
4	$f(\alpha, \beta) \geq 0$ $f(\alpha, \beta) + \pi \cot \beta < 0$ $\text{Im}(v/z) > 0$	$S_v^{(1)}(z)$	$S_v^{(2)}(z) - e^{2v\pi i} S_v^{(1)}(z)$
5	$f(\alpha, \beta) \geq 0$ $f(\alpha, \beta) + \pi \cot \beta < 0$ $\text{Im}(v/z) < 0$	$S_v^{(1)}(z) - e^{-2v\pi i} S_v^{(2)}(z)$	$S_v^{(2)}(z)$
6a	$f(\alpha, \beta) < 0$ $f(\alpha, \beta) + \pi \cot \beta < 0$ $\text{Im}(v/z) > 0$ $\text{Re}(v/z) \geq 0$	$\left[1 + e^{(m+1)v\pi i} \frac{\sin mv\pi}{\sin v\pi} \right] S_v^{(1)}(z) - S_v^{(2)}(z)$	$S_v^{(2)}(z) + e^{(m+1)v\pi i} \frac{\sin mv\pi}{\sin v\pi} S_v^{(1)}(z)$
6b	$f(\alpha, \beta) < 0$ $f(\alpha, \beta) + \pi \cot \beta < 0$ $\text{Im}(v/z) > 0$ $\text{Re}(v/z) < 0$	Compute $H_{-v}^{(2)}(z)$ and use (9)	Compute $H_{-v}^{(1)}(z)$ and use (9)
7a	$f(\alpha, \beta) < 0$ $f(\alpha, \beta) + \pi \cot \beta < 0$ $\text{Im}(v/z) < 0$ $\text{Re}(v/z) < 0$	$S_v^{(1)}(z) - e^{-(m+1)v\pi i} \frac{\sin mv\pi}{\sin v\pi} S_v^{(2)}(z)$	$\left[1 + e^{-(m+1)v\pi i} \frac{\sin mv\pi}{\sin v\pi} \right] S_v^{(2)}(z) - S_v^{(1)}(z)$
7b	$f(\alpha, \beta) < 0$ $f(\alpha, \beta) + \pi \cot \beta < 0$ $\text{Im}(v/z) < 0$ $\text{Re}(v/z) < 0$	Compute $H_{-v}^{(2)}(z)$ and use (9)	Compute $H_{-v}^{(1)}(z)$ and use (9)
8	$\text{Im}(v/z) = 0$ $\text{Re}(v/z) \geq 1$	$\frac{1}{2} S_v^{(1)}(z) - S_v^{(2)}(z)$	$\frac{1}{2} S_v^{(1)}(z) + S_v^{(2)}(z)$
9	$\text{Im}(v/z) = 0$ $\text{Re}(v/z) < -1$	Compute $H_{-v}^{(2)}(z)$ and use (9)	Compute $H_{-v}^{(2)}(z)$ and use (9)

Table 1

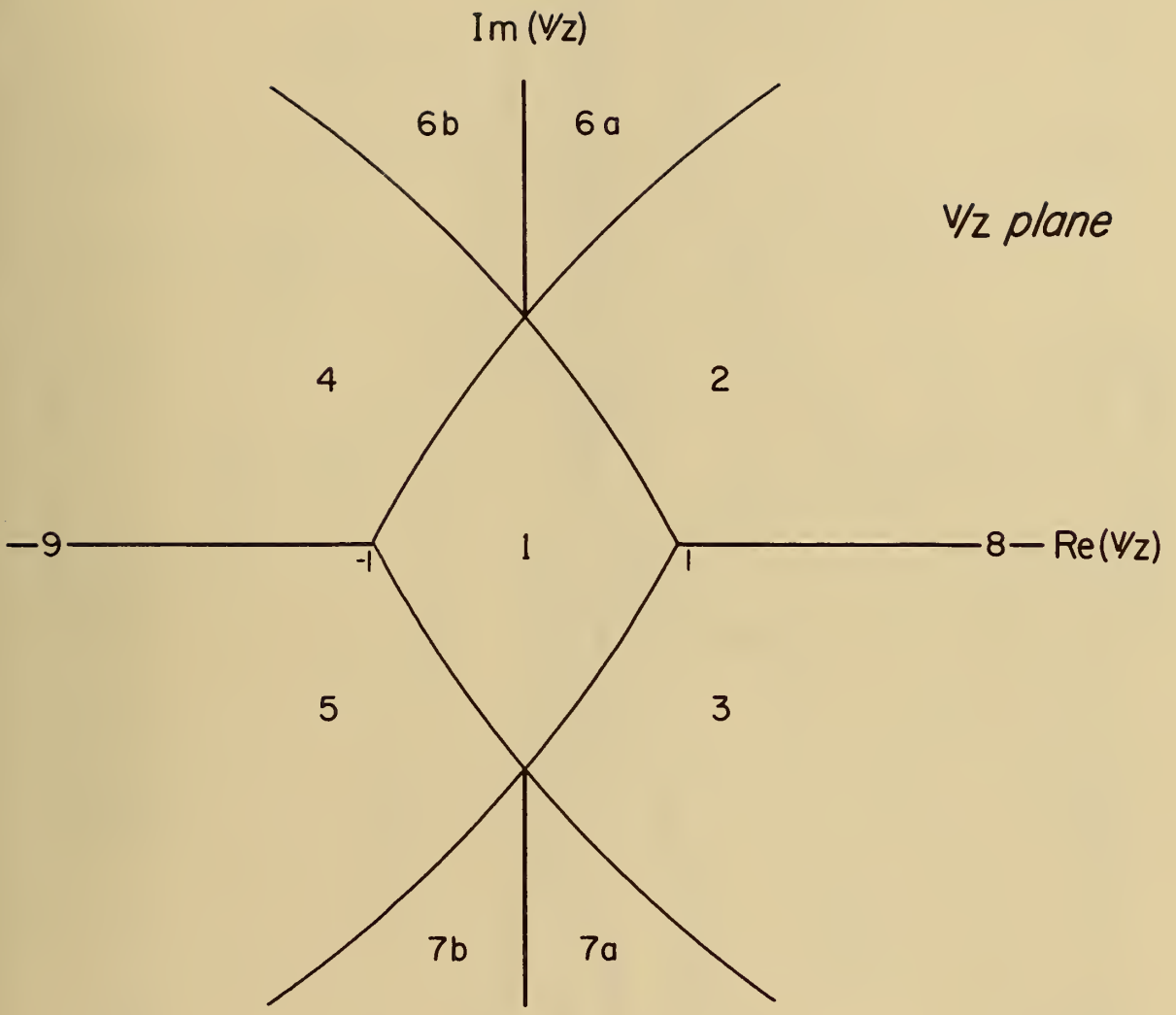


Figure 1. Regions of the v/z plane (table 1).

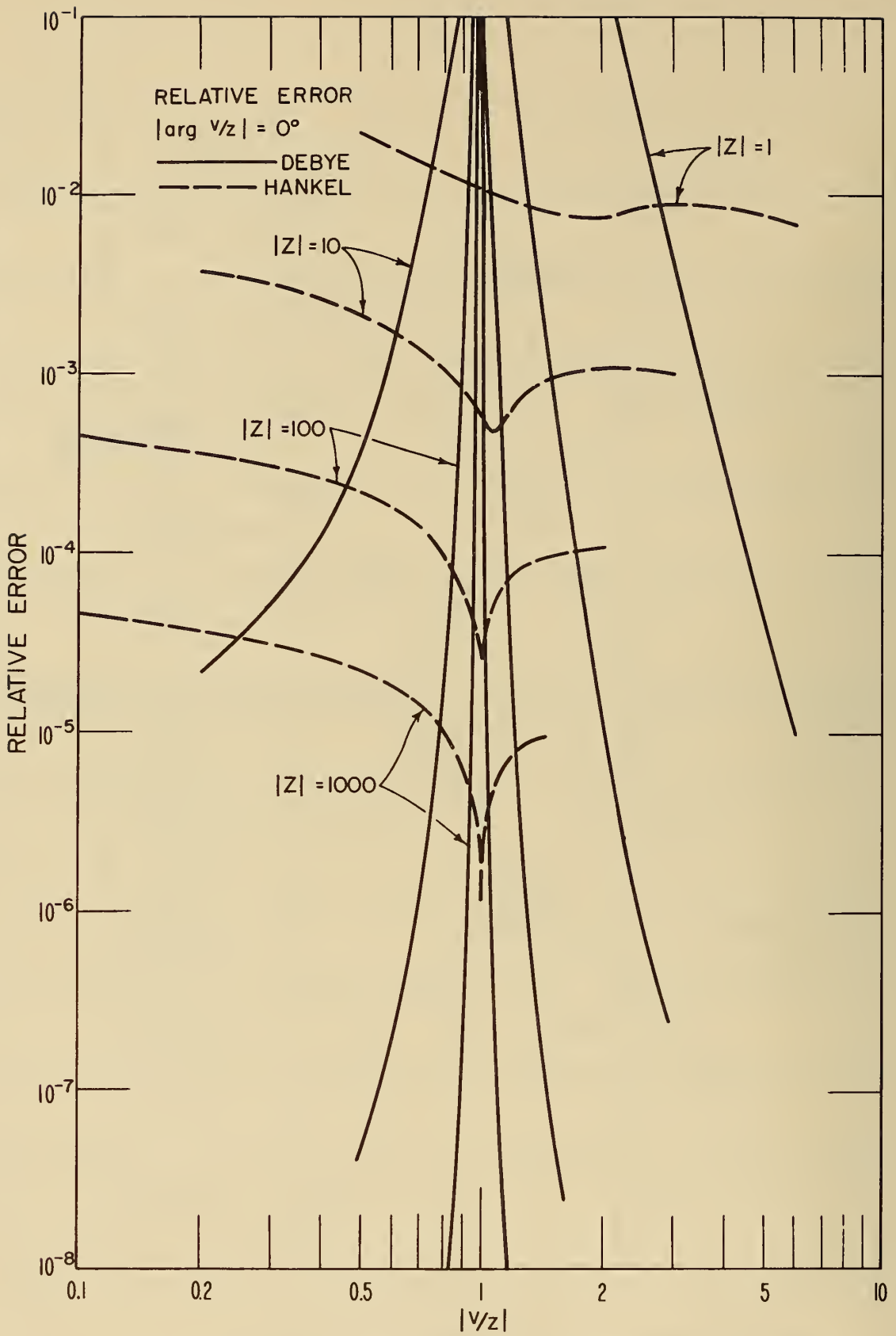


Figure 2. Relative errors of the Debye and Hankel approximations for $|\arg v/z| = 0^\circ$, for various $|z|$.

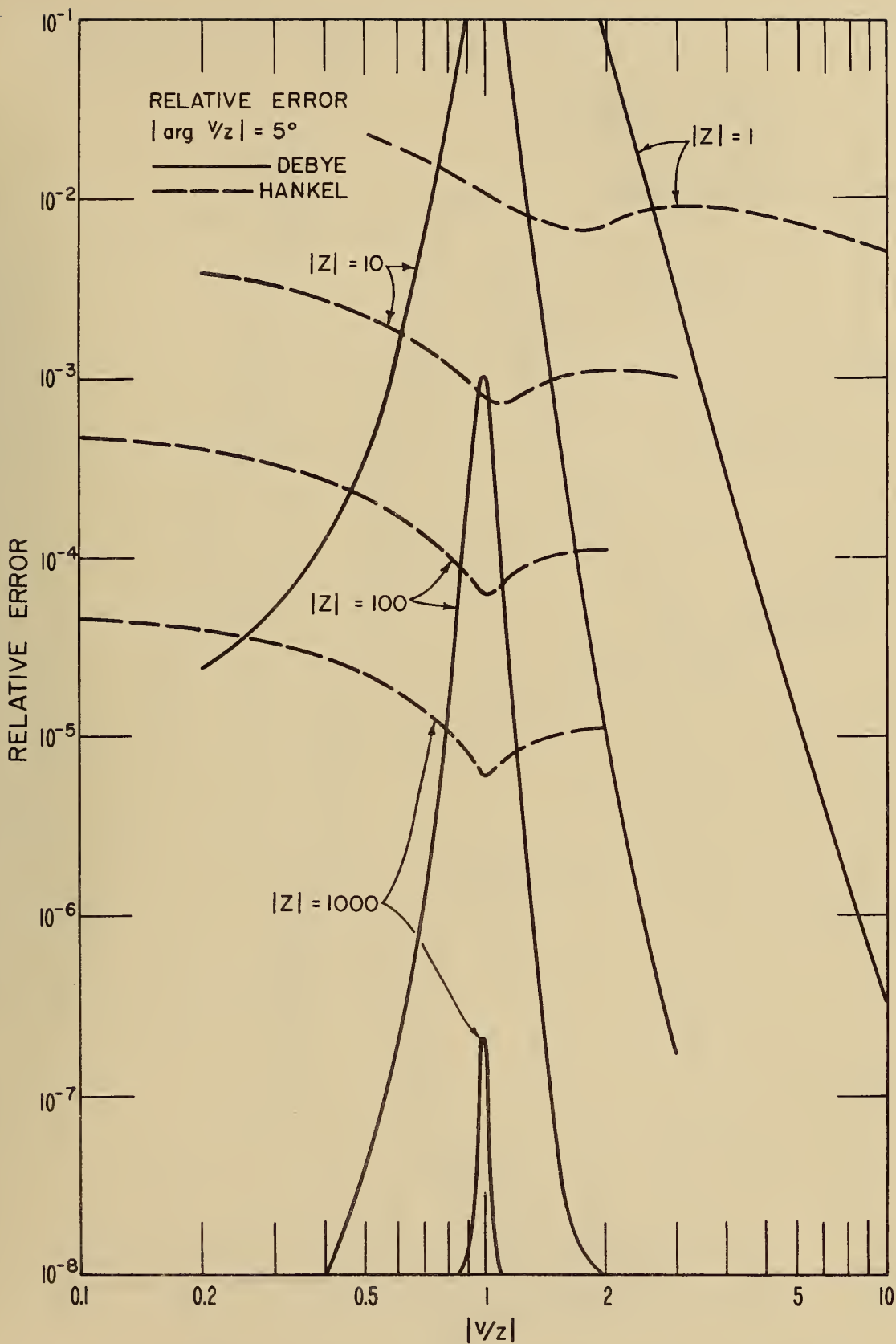


Figure 3. Relative errors of the Debye and Hankel approximations for $|\arg v/z| = 5^\circ$, for various $|z|$.

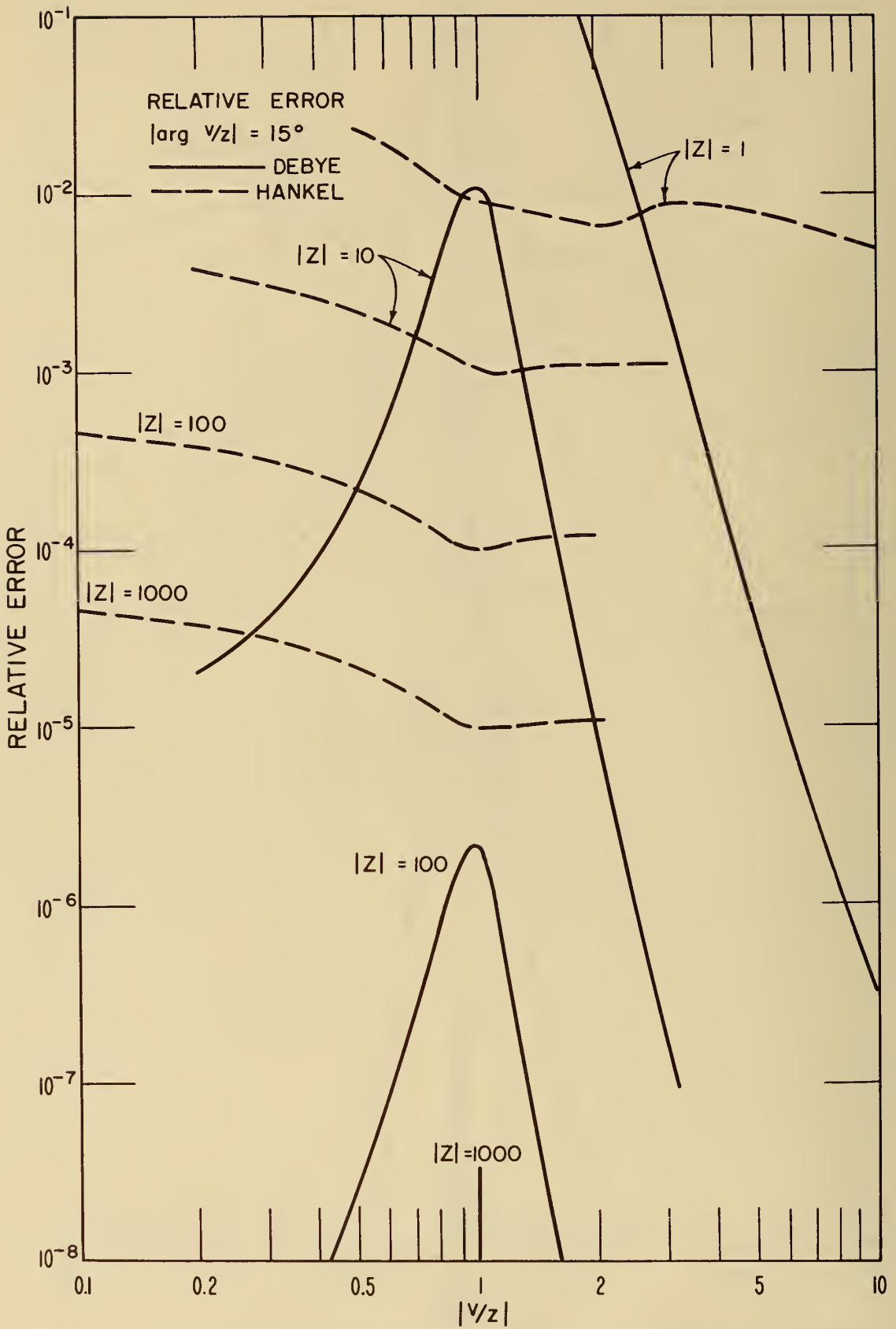


Figure 4. Relative errors of the Debye and Hankel approximations for $|\arg v/z| = 15^\circ$.

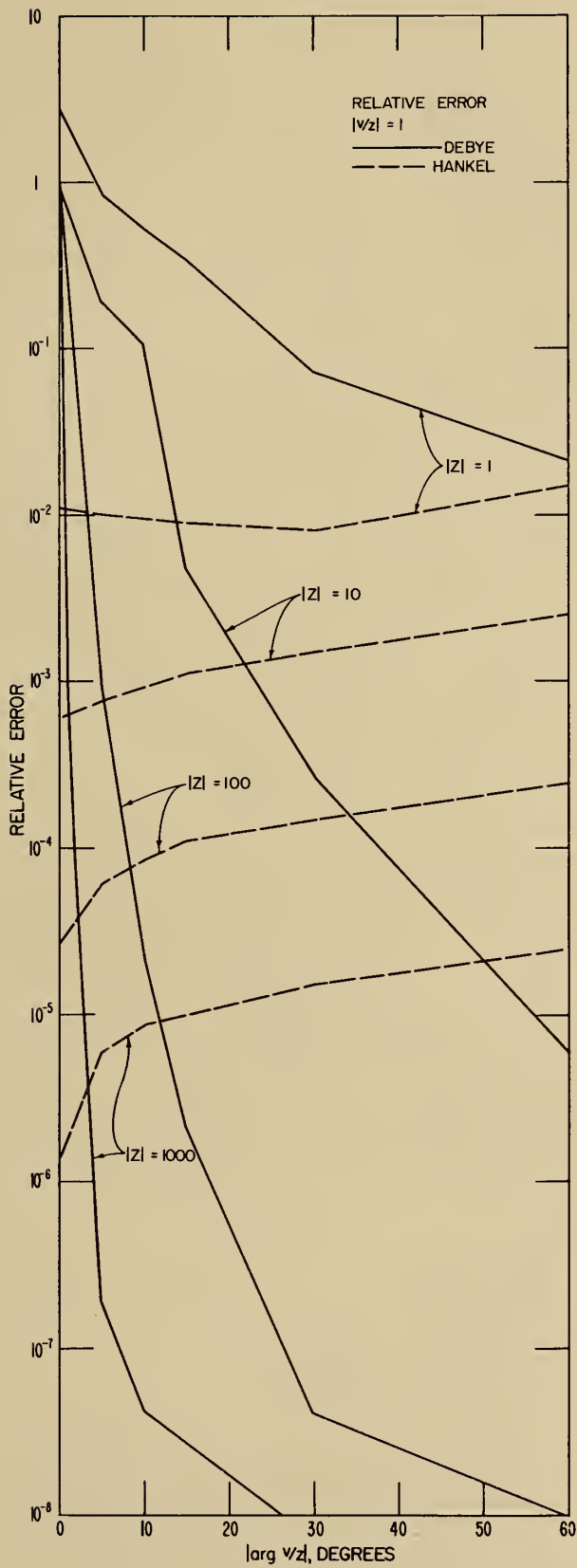


Figure 5. Relative errors of the Debye and Hankel approximations for $|v/z| = 1$, as a function of $|arg v/z|$, for various $|z|$.

