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CONFERENCE ON NON-LINEAR PROCESSES IN THE IONOSPHERE DECEMBER 16-17, 1963

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Volume IV

Table of Contents

Ionospheric Cross-Modulation: A Microscopic Theory

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In 1937 Bailey and Martyn proposed a theory of ionospheric cross-modulation along the following lines. Radiation emitted by the disturbing transmitter is strongly absorbed in a region of the ionosphere where the wanted wave is refracted and attenuated. Absorption of the disturbing radiation raises the temperature of the ionosphere. If the amplitude of the disturbing radiation varies with time, so will the resulting temperature increase and hence the attenuation suffered by the wanted wave. In the simplest experimental arrangement the wanted wave is initially unmodulated and the disturbing radiation is modulated at a definite audio frequency. In traversing the ionosphere, the wanted wave becomes modulated at the frequency of the disturbing radiation and also at twice this frequency.

The Bailey-Martyn theory makes three quantitative predictions concerning the transferred modulation. The first concerns its dependence on the power of the disturbing radiation, and involves only the assumption that the effect is small enough to warrant a linearized description. The remaining two predictions concern the dependence of the transferred modulation and its phase lag on the modulation frequency. These predictions involve the additional assumption of a unique relation, valid for all values of the modulation frequency, between the mean thermal energy and the mean collision frequency of the electrons. A relation of this kind can exist, however, only for modulation frequencies much less than 150 C/s. Since the experimental range of modulation frequencies extends to nearly 1500 C/s, the foundations of the Bailey-Martyn theory need to be reconsidered.

This paper describes a microscopic theory of ionospheric cross-modulation. The velocity-distribution function for the electrons enters explicitly into the theory; it is determined by a differential equation whose form depends on the modulation frequency. Knowing the distribution function, one can calculate the absorption coefficient. The form of the predicted absorption coefficient depends on the assumed form of the electron-molecule interaction law. By numerical methods, we have calculated the transferred modulation and its phase lag as functions of modulation frequency for a few simple interaction laws. The calculations show that the effects of departures from a Maxwellian velocity distribution are indeed significant. The predictions are sensitive to the assumed form of the electron-molecule collision law. Although the present theory is still highly idealized, the results obtained suggest that further theoretical and experimental refinements could lead to an experimental determination of the electron-molecule interaction law in the D region.

1. The Bailey-Martyn Theory

The passage of a radio wave through a region of the ionosphere in which it is partially absorbed raises the electron temperature in the region and hence changes its absorption coefficient. The absorption process is accordingly non-linear: the attenuation suffered by each of several radio waves passing through the same region depends on that suffered by all the others. This is the physical basis of Bailey and Martyn's [1934] theory of ionospheric cross-modulation.

In a typical cross-modulation experiment an initially unmodulated carrier wave (the "wanted" wave) traverses a region in which a second wave at a different carrier frequency (the "disturbing" wave) is heavily absorbed. The disturbing wave is amplitude-modulated at a definite frequency. On reception, the wanted wave is found to be modulated at the same frequency and also at twice this frequency.

The main predictions of the Bailey-Martyn theory do not depend on a detailed description of the physical processes involved, but follow from a few general assumptions, the first of which is that the phenomenon admits a linearized description. The linearized energy-balance equation for the electron gas has the form

$$\frac{d\epsilon}{dt} = \kappa I - \tau^{-1}\epsilon \ . \tag{1.1}$$

Here ϵ denotes the difference

$$\epsilon = \frac{3}{2} k(T_e - T_m) \tag{1.2}$$

between the mean thermal energy of the electrons and the mean thermal energy of the molecules, κ denotes the absorption coefficient of the medium, $\underline{I}$ denotes the intensity of the ambient radiation field, and τ denotes the thermal relaxation time of the electron gas. As (1.1) is linear, we may treat the individual Fourier components of the radiation field separately. The ω-component of (1.1) is

$$i\omega\epsilon_\omega = \kappa I_\omega - \tau^{-1}\epsilon_\omega \ , \tag{1.3}$$

which has the solution[1]

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[1] Equation (1.1) has the same form as that governing the current in an RL circuit; τ represents the time constant of the circuit and $\underline{I}$ the applied e.m.f.

$$\epsilon_\omega = \frac{\kappa\tau I_\omega}{1+i\omega\tau} = \frac{\kappa\tau e^{-i\varphi} I_\omega}{\sqrt{(1+\tau^2\omega^2)}} \ , \qquad \varphi = \tan^{-1}(\omega\tau) \ . \tag{1.4}$$

The angle φ represents the phase difference between the ω-component of the radiation field and the ω-component of the electron temperature. In the D and lower E regions, $\tau \approx 100$ sec, so that the attenuation factor $(1+\tau^2\omega^2)^{-1/2}$ is very small except for audio frequencies.

The assumption of linearity also implies that the variable part of the electron-molecule collision frequency $\nu \propto \epsilon$. According to Lorentz's theory, $\kappa \propto \nu$. Hence, setting

$$\kappa = \kappa_0 + \int \kappa_\omega e^{i\omega t} d\omega \ , \qquad \nu = \nu_0 + \int \nu_\omega e^{i\omega t} d\omega \ , \tag{1.5}$$

we have

$$\kappa_\omega \propto \nu_\omega \propto \epsilon_\omega \ . \tag{1.6}$$

Finally, if the region where appreciable modulation transfer occurs is nearly homogeneous, we have

$$X_\omega \propto \kappa_\omega \ , \tag{1.7}$$

where X_ω describes the depth and phase of the transferred modulation. From (1.4), (1.6), (1.7), we have

$$X_\omega = I_\omega T_\omega e^{-i\varphi} \ ; \quad T_\omega = (1 + \tau^2\omega^2)^{-1/2} \ , \quad \varphi_\omega = \tan^{-1}(\omega\tau) \ . \tag{1.8}$$

The predicted linear dependence of X_ω on I_ω, the predicted variation of the phase lag φ_ω with ω, and the predicted variation of the coefficient T_ω with ω can all be tested separately.

(i) Linear dependence of X_ω on I_ω. If the disturbing wave is modulated to a depth M at a single frequency ω and if P denotes the total power of the disturbing radiation, then

$$I \propto P[(1 + \tfrac{1}{2}M^2) + 2M\cos\omega t + \tfrac{1}{2}M^2\cos 2\omega t] \quad . \tag{1.9}$$

Ratcliffe and Shaw [1948] verified that $X_\omega \propto P$ over a wide range of P. Huxley et al.[1947, 1948] verified the predicted linear dependence of P_ω on M and showed that the predicted quadratic dependence on $X_{2\omega}$ on M was qualitatively in agreement with experiment. These experimental results show that a linearized description of ionospheric cross-modulation is valid over a wide range of experimental conditions.

(ii) The phase lag $\varphi(\omega)$. Ratcliffe and Shaw [1948] and subsequent workers found that they could secure good agreement between predicted and measured phase lags over a wide range of ω by choosing the parameter τ appropriately. The required values of τ seem to be consistent with values derived by extrapolating laboratory measurements [Huxley, 1959].

(iii) The coefficient T_ω. Formula (1.8) for T_ω agrees qualitatively, but not quantitatively, with the experiment

[Ratcliffe and Shaw, 1948; Huxley, 1950]. The discrepancies between theory and experiment, though not large, appear to be significant. Huxley [1950] suggested that the two spatially separated regions in which cross-modulation occurs in a typical experiment -- one on the ascending, the other on the descending branch of the wanted ray -- may have sufficiently different properties to invalidate the assumption of homogeneity underlying (1.6). Calculations based on a two-center model [Huxley, 1950] do not, however, significantly reduce the discrepancies.

The derivation of formula (1.8) actually rests not only on the assumptions of linearity and homogeneity but also on a third major assumption: that, at any given point in the ionosphere, the quantities κ, ν, and τ depend only on the electron temperature and not explicitly on the modulation frequency. Now, the relation between collision frequency (say) and electron temperature depends in general on the form of the velocity distribution of the electrons. If the distribution is Maxwellian, or has any other fixed form depending on a single parameter, it is completely specified by T_e, so that ν becomes a function of T_e. But if the electron temperature does not serve to specify the distribution completely, the relation between ν and T_e may be many-valued. In the present problem, the variable part of the distribution changes appreciably in a time of order ω^{-1}. Since this is short compared with the thermal relaxation time τ, there is no reason to suppose that the distribution remains accurately Maxwellian. It is true that if the intensity of the incident radiation field is sufficiently small, the velocity distribution of the electrons will be approximately Maxwellian. But the phenomenon of cross-modulation depends entirely on the variable component of the velocity distribution, and, as we shall see, this component remains non-Maxwellian even in the limit of vanishing field intensity.

This paper presents a theory of cross-modulation in which the electronic velocity-distribution function figures explicitly. Although the present theory is more realistic than the macroscopic theory sketched above, it is over-simplified in one important respect: It treats collisions between electrons and molecules as if they were perfectly elastic. In reality, cooling of the electron gas in the D and E regions results chiefly from the collisional excitation of molecular nitrogen. Caldirola and De Barbieri [1963] have recently succeeded in extending the present theory to allow, at least approximately, for inelastic collisions.

For the sake of simplicity, the following discussion ignores the effects of the earth's magnetic field. This is permissible only if all the carrier frequencies that figure in the discussion are much greater than the gyro-frequency. The modifications required when this condition is not met are straightforward; they are described in the paper by Caldirola and De Barbieri mentioned in the last paragraph.

2. Reduction of the Boltzmann Equation

The Boltzmann equation for an electron gas in an electromagnetic field has the form

$$\frac{\partial f}{\partial t} + \underset{\sim}{q} \cdot \frac{\partial f}{\partial \underset{\sim}{x}} + \frac{e}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{q}}{c} \times \underset{\sim}{B} \right) \cdot \frac{\partial f}{\partial \underset{\sim}{q}} = \frac{\delta f}{\delta t} . \tag{2.1}$$

The following discussion of this equation is a generalization of the treatment of a weakly ionized gas in a constant electric field given by Chapman and Cowling [1939, pp. 346-352].

We assume at first that the field is that of a linearly polarized plane wave propagating in the z-direction:[2]

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[2] We adopt the convention that when a real quantity, such as E_x or f, is represented by a complex expression, as in (2.2) and (2.6), the real part of this expression is to be understood. Note, however, that the quantities E, F, k, $f^{(2)}$, etc., are all complex, and that both the real and imaginary parts of an equation like (2.11) are significant.

$$E_x = E\, e^{i(kz-pt)} , \qquad B_y = \frac{ck}{p} E\, e^{i(kz-pt)} . \tag{2.2}$$

The distribution function f depends on the six variables z, $\underset{\sim}{q}$, t. Since $|ck/p| \approx 1$, the magnetic force is of order q/c compared with the electric force. Similarly, the second term on the left side of (2.1) is of order q/c compared with the first term. Neglecting terms of this order, we obtain in place of (2.1)

$$\frac{\partial f}{\partial t} + F_x \frac{\partial f}{\partial u} = \frac{\delta f}{\delta t} , \tag{2.3}$$

where

$$\underset{\sim}{F} = \frac{e}{m} \underset{\sim}{E} . \tag{2.4}$$

In this approximation, f depends on only four arguments:

$$f = f(z;u,q;t) . \tag{2.5}$$

We may expand f in Legendre polynomials of the argument u/q. We shall need only the first two terms in the expansion:

$$f = f^{(0)}(q,t) + iFe^{i(kz-pt)}uf^{(1)}(q,t) . \tag{2.6}$$

If the number density n of electrons is sufficiently small compared with the number density of molecules, we may neglect electron-electron and electron-ion collisions. The collision term in the Boltzmann equation, represented by $\delta f/\delta t$, is then a linear functional of f:

$$\frac{\delta f}{\delta t} = \Lambda(f) .$$

Lorentz showed that

$$\Lambda[ug(q)] = -\nu(q)ug(q) + O\left(\frac{m}{M}\right),\tag{2.8}$$

where m/M is the electron-molecule mass ratio, ν is the velocity-dependent collision frequency, and g is an arbitrary function. Davydov [1935] derived the important formula

$$\Lambda[g(q)] = \frac{m}{M}\frac{1}{q^2}\frac{d}{dq}\left[q^3\nu(q)\left(1 + \frac{kT_m}{m}\frac{1}{q}\frac{d}{dq}\right)g(q)\right] + O\left[\left(\frac{m}{M}\right)^2\right].\tag{2.9}$$

By inserting (2.6) into (2.3) and using the formulae of Lorentz and Davydov to evaluate the right side of the resulting equation, we obtain differential equations for the functions $f^{(0)}$ and $f^{(1)}$.[3]

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[3] The differential equation for $f^{(k)}$, the coefficient of $P_k(u/q)$ in the Legendre expansion of f, involves only the functions $f^{(j)}$ with $j \leq k$, so that the sequence of approximations may be terminated at any point. By contrast, the equations for the moments of f of order k involve moments of order k + 1.

The equation for $f^{(1)}$ is

$$uf^{(1)}(p + i\nu) + \frac{\partial f^{(0)}}{\partial u} = 0\tag{2.10}$$

or

$$f^{(1)} = - \frac{1}{p + i\nu} \frac{1}{q} \frac{\partial f^{(0)}}{\partial q} . \tag{2.11}$$

The left side of the equation for $f^{(0)}$ has the form

$$\frac{\partial f^{(0)}}{\partial t} + \left\langle F_x \frac{\partial}{\partial u} \left(iF_x u f^{(1)} \right) \right\rangle . \tag{2.12}$$

Here the brackets indicate a double average: over direction in velocity space (to eliminate terms containing Legendre polynomials $P_k(u/q)$ with $k > 0$), and over time (to eliminate radio-frequency fluctuations in $f^{(0)}$, which are of no physical interest). Since we need to preserve audio-frequency variations of $f^{(0)}$, the time-averaging must be over an interval that is short compared with the reciprocal of the highest modulation frequency of interest as well as long compared with the reciprocal of the carrier frequency p; thus we must have $p >> \omega$. Application of the well-known rule

$$\langle \mathrm{Re}\, Ae^{ipt} \cdot \mathrm{Re}\, Be^{ipt} \rangle_t = \frac{1}{2} \mathrm{Re}\, \bar{A}B , \tag{2.13}$$

where $\bar{A}$ denotes the complex conjugate of A, gives

$$\left\langle F_x \frac{\partial}{\partial u} \left(iF_x u f^{(1)} \right) \right\rangle_t$$

$$= \frac{1}{2} \mathrm{Re} \left\{ \bar{F} \left(\frac{-i}{p + i\nu} \right) F \frac{\partial}{\partial u} \left(\frac{u}{q} \frac{\partial f^{(0)}}{\partial q} \right) \right\}$$

$$= -\frac{1}{2}\frac{\nu}{p^2+\nu^2}|F|^2\left[\frac{1}{q}\frac{\partial f^{(0)}}{\partial q}+\frac{u^2}{q}\frac{\partial}{\partial q}\left(\frac{1}{q}\frac{\partial f^{(0)}}{\partial q}\right)\right]. \tag{2.14}$$

Since

$$u^2 = q^2\left[\frac{1}{3}+\frac{2}{3}P_2\left(\frac{u}{q}\right)\right],$$

averaging over direction in velocity space gives

$$\left\langle F_x\frac{\partial}{\partial u}\left(iF_x uf^{(1)}\right)\right\rangle$$

$$= -\frac{1}{2}\frac{\nu}{p^2+\nu^2}|F|^2\left[\frac{1}{q}\frac{\partial f^{(0)}}{\partial q}+\frac{1}{3}q\frac{\partial}{\partial q}\left(\frac{1}{q}\frac{\partial f^{(0)}}{\partial q}\right)\right]$$

$$= -\frac{1}{6}\frac{\nu}{p^2+\nu^2}|F|^2\frac{1}{q^2}\frac{\partial}{\partial q}\left(q^2\frac{\partial f^{(0)}}{\partial q}\right). \tag{2.15}$$

Combining this result with (2.9) we have, finally,

$$\frac{\partial f^{(0)}}{\partial t}-\frac{m}{M}\frac{1}{q^2}\frac{\partial}{\partial q}\left\{q^3\nu(q)\left[f^{(0)}+\frac{\Theta}{q}\frac{\partial f^{(0)}}{\partial q}\right]\right\}=0, \tag{2.16}$$

where

$$\Theta = \frac{1}{6}\frac{M}{m}\frac{|F|^2}{p^2+\nu^2}+\frac{kT_m}{m}. \tag{2.17}$$

We may immediately generalize these results to an arbitrary radiation field. The electric vector at a given point may be written in the form

$$\underset{\sim}{E} = \sum_j \underset{\sim}{E}_j, \tag{2.18}$$

where each Fourier component $\underset{\sim}{E}_j$ represents the electric field of a linearly polarized plane wave. In place of (2.6) we have

$$f = f^{(0)}(q,t) + \sum_j iF_j(\hat{\underset{\sim}{k}}_j \cdot \underset{\sim}{q}) f_j^{(1)}(q,t)\exp(i\underset{\sim}{k}_j \cdot \underset{\sim}{x} - p_j t) \tag{2.19}$$

and in place of (2.10),

$$f_j^{(1)} = - \frac{1}{p_j + i\nu} \frac{1}{q} \frac{\partial f^{(0)}}{\partial q} \; . \tag{2.20}$$

Finally, the function $f^{(0)}$ satisfies (2.16) with Θ given by

$$\Theta = \frac{1}{6} \frac{M}{m} \sum \frac{|F_j|^2}{p_j^2 + \nu^2} + \frac{kT_m}{m} \tag{2.21}$$

instead of by (2.17).

3. The Dispersion Relation

The electric current density produced by the radiation field (2.2) is given by

$$j_x = \frac{e}{c} \int u f d^3\underset{\sim}{q} = \frac{ieF_x}{c} \int u^2 f^{(1)}(q) d^3\underset{\sim}{q}$$

$$= \frac{4\pi ie}{3c} F_x \int q^4 f^{(1)} dq = - \frac{4\pi ie}{3c} F_x \int \frac{q^3}{p + i\nu} \frac{df^{(0)}}{dq} dq$$

$$= i \frac{e}{c} F_x \int f^{(0)} d \left(\frac{4\pi}{3} \frac{q^3}{p + i\nu} \right) , \tag{3.1}$$

$$j_y = j_z = 0 . \tag{3.1'}$$

From Maxwell's equation,

$$\text{curl } \underset{\sim}{B} = \frac{1}{c} \dot{\underset{\sim}{E}} + 4\pi \underset{\sim}{j} , \tag{3.2}$$

and (3.1), we obtain

$$(ck^2) = p^2 - \frac{4\pi e^2 p}{m} \int f^{(0)} d \left(\frac{4\pi}{3} \frac{q^3}{p + i\nu} \right)$$

$$= p^2 - \frac{4\pi e^2}{m} \int f^{(0)} d \left(\frac{4\pi}{3} \frac{p^2 q^2}{p^2 + \nu^2} \right)$$

$$+ i \frac{4\pi e^2}{m} \int f^{(0)} d \left(\frac{4\pi}{3} \frac{p\nu q^3}{p^2 + \nu^2} \right) . \tag{3.3}$$

In order to write this relation in a more compact form we introduce the plasma frequency p_o, defined by

$$p_o^2 = \frac{4\pi n e^2}{m} , \tag{3.4}$$

and use the abbreviation

$$\langle \Phi \rangle = \frac{1}{n} \int f^{(0)} d \left[\frac{4\pi}{3} q^3 \Phi(q) \right] , \tag{3.5}$$

where n denotes the electron density. The dispersion relation (3.3) then takes the form

$$(ck)^2 = p^2 - p_o^2 \left\langle \frac{p^2}{p^2 + \nu^2} \right\rangle + i p_o^2 \left\langle \frac{p\nu}{p^2 + \nu^2} \right\rangle . \tag{3.6}$$

The absorption coefficient κ is defined by

$$k = k_o + \frac{1}{2} i\kappa , \tag{3.7}$$

where k_o and κ are both real. The following approximate formulae, valid under the conditions stated, are often convenient:

$$c\kappa = \frac{p_o^2}{p} \left\langle \frac{p\nu}{p^2 + \nu^2} \right\rangle \quad (p_o \ll p) , \tag{3.8}$$

$$c\kappa = \left(\frac{p_o^2}{p} \right)^2 \langle \nu \rangle \quad (p_o \ll p, \ \nu \ll p) . \tag{3.9}$$

The results of this section were first obtained by Lorentz.

4. The Absorption of Unmodulated Radio Waves

When $\omega = 0$, (2.16) reduces to an ordinary first-order differential equation, whose solution is

$$f^{(0)} = nC \exp\left| - \int_0^q \Theta^{-1} q dq \right| , \tag{4.1}$$

where Θ is given by (2.21) and the constant C is determined by the normalization condition

$$4\pi \int_0^\infty f^{(0)} q^2 dq = n \ . \tag{4.2}$$

Formula (4.1) is valid for all values of the carrier frequencies p_j occuring in formula (2.21) for Θ. We recall that $f^{(0)}$ represents the isotropic part of the distribution function averaged over a time interval long compared with all the characteristic times p_j^{-1}. If the radiation field has only a single Fourier component, $f^{(0)}$ represents the isotropic part of the distribution function averaged over a single period.

Formula (4.1) remains valid in the limit $p = 0$ ($\underset{\sim}{E}$ = const.). In this case

$$\Theta = \frac{1}{6} \frac{M}{m} \frac{|F|^2}{\nu^2(q)} + \frac{kT_m}{m} \qquad (p = 0) \tag{4.3}$$

and (4.1) coincides with a formula derived by Chapman and

Cowling [1960; eq. (13), p.350].

When the carrier frequency is very large, Θ is given by

$$\Theta = \frac{1}{6}\,\frac{M}{m}\,\frac{|F|^2}{p^2} + \frac{kT_m}{m} \qquad (p \gg \nu)\;; \tag{4.4}$$

$f^{(0)}$ is accordingly Maxwellian, the electron temperature being given by Lorentz's formula,

$$kT_e = kT_m + \frac{1}{6}\,\frac{M|F|^2}{p^2}\;. \tag{4.5}$$

If the molecules are Maxwellian (collision frequency independent) of velocity), the distribution function is Maxwellian for all values of the carrier frequency. Otherwise the distribution function departs markedly from the Maxwellian form when $\Theta \gg kT_m/m$ and $p \approx \nu$.

If the molecules are rigid elastic spheres ($\nu = q/l$, l = const.) the integral in (4.1) may be evaluated explicitly; one obtains

$$f^{(0)} = C'\,(\bar{q}^2 + \bar{p}^2 + A)^A\, e^{-\bar{q}^2}, \tag{4.6}$$

where

$$\bar{q}^2 = \frac{mq^2}{2kT_m}\,, \qquad \bar{p}^2 = \frac{ml^2p^2}{2kT_m}\,, \qquad A = \frac{M|F|^2}{2kT_m}\;. \tag{4.7}$$

5. The Variable Part of $f^{(0)}$

In this section we derive the equations governing small departures from the steady-state velocity distributions discussed in the preceding section.

Let

$$\Theta(q,t) = \Theta_0(q)[1 + \theta(q,t)] \; . \tag{5.1}$$

The function Θ_0 coincides with the time-independent function that was called Θ in the preceding section. We define new dimensionless variables x, X, φ:

$$dx = \frac{qdq}{\Theta_\bullet} \; , \qquad X = \frac{1}{2}\frac{q^2}{\Theta_0} \; , \qquad \varphi(x,t) = 4\pi n^{-1}\int_0^q q^2 f^{(0)}(q,t)dq \; . \tag{5.2}$$

Note that

$$\frac{dx}{X} = \frac{dX}{X} + \frac{d\Theta_0}{\Theta_0} \; . \tag{5.3}$$

The function φ is the <u>probability distribution function</u> of the dimensionless velocity variable x;

$$\varphi(a,t) = \Pr\{x \leqslant a\} \; . \tag{5.4}$$

It follows that

$$\varphi(0,t) = 0 \; , \qquad \varphi(\infty,t) = 1 \; . \tag{5.5}$$

The partial derivative $\partial\varphi/\partial x$ is the <u>probability density</u> associated with x. In terms of the new variables (2.16) takes the form

$$\frac{\partial\varphi}{\partial t} = \frac{2m}{M}\,\nu\left\{ x\Theta_o \frac{\partial}{\partial x}\left(\Theta_o^{-1}\frac{\partial\varphi}{\partial x}\right) + \left(x - \frac{1}{2}\right)\frac{\partial\varphi}{\partial x}\right.$$

$$\left. + \,\theta\left[-\frac{1}{2}\frac{\partial\varphi}{\partial x} + x\Theta_o\frac{\partial}{\partial x}\left(\Theta_o^{-1}\frac{\partial\varphi}{\partial x}\right)\right]\right\} . \tag{5.6}$$

So far we have not made any assumptions about the magnitude of the function $\theta(q,t)$. In order to reduce (5.6) to an ordinary differential equation we now set

$$\varphi = \varphi_0(x) + \varphi_1(x,t) \tag{5.7}$$

and assume that θ and φ_1 are so small that their product may be neglected in (5.6). This will be true if at least one of the conditions $\Theta \approx kT_m/m$, $M \ll 1$, is satisfied. Having linearized (5.6), we can deal separately with the Fourier components of the radiation field, just as in the elementary theory of Section 1. Let

$$\theta(x,t) = \theta_\omega(x)e^{i\omega t} , \qquad \varphi_1(x,t) = \varphi_\omega(x)e^{i\omega t} , \tag{5.8}$$

It is convenient to replace the modulation frequency ω by a dimensionless variable and to separate out the velocity dependence of the function $\nu(x)$. We accordingly define

$$\eta(x) = \frac{\bar{\nu}}{\nu(x)} , \qquad \bar{\nu} = \int_0^\infty \nu(x)\,d\varphi_o , \tag{5.9}$$

$$\alpha = \frac{\omega}{2(m/M)\bar{v}} \, . \tag{5.10}$$

Finally, since φ_0 satisfies the equation

$$X\Theta_0 \frac{\partial}{\partial x} \frac{1}{\Theta_0} \frac{\partial \varphi_0}{\partial x} + \left(X - \frac{1}{2} \right) \frac{\partial \varphi_0}{\partial x} = 0 \, , \tag{5.11}$$

we have

$$\frac{\partial \varphi_0}{\partial x} = C\Theta_0^{3/2} X^{1/2} e^{-X} \, , \tag{5.12}$$

where C is determined by the normalization condition $\varphi_0(\infty) = 1$.

Hence

$$- X \frac{\partial \varphi_0}{\partial x} = - C(\Theta_0 X)^{3/2} e^{-X} \, . \tag{5.13}$$

Omitting second-order terms in (5.6) and using (5.9), (5.10), and (5.13), we obtain

$$X\Theta_0 \frac{\partial}{\partial x} \left(\Theta_0^{-1} \frac{\partial \varphi_\omega}{\partial x} \right) + \left(X - \frac{1}{2} \right) \frac{\partial \varphi_\omega}{\partial x} + i\alpha\eta\varphi_\omega = \theta_\omega C\Theta_0^{3/2} X^{3/2} e^{-X}. \tag{5.14}$$

Since φ and φ_0 both satisfy the boundary conditions (5.5), φ_ω must satisfy the boundary conditions

$$\varphi_\omega(0) = \varphi_\omega(\infty) = 0 \, . \tag{5.15}$$

Equations (5.14) and (5.15) together with the dispersion relation (3.6) represent the formal solution of our problem. The functions Θ_0 and θ are defined by the radiation field.

Given Θ_0, one finds the function $x(q)$ and $X(q)$ from (5.2). Equation (5.14) with the boundary conditions (5.15) may be integrated numerically by the method described in the Appendix. Finally, knowing φ_ω, one can calculate the variable part of the absorption coefficient from the dispersion relation.

Equation (5.14) assumes a simpler form when $\Theta \approx kT_m/m$, so that the unperturbed distribution function is nearly Maxwellian. We may then write

$$X = x \ , \qquad \Theta_0 = \text{const.} \ , \qquad \theta_\omega = \text{const.} \ , \tag{5.16}$$

so that (5.12) becomes

$$\frac{\partial \varphi_0}{\partial x} = \frac{2}{\sqrt{\pi}} x^{1/2} e^{-x} . \tag{5.17}$$

Setting

$$\varphi_\omega(x) = - \frac{2}{\sqrt{\pi}} \theta_\omega y(x) \ , \tag{5.18}$$

we obtain in place of (5.14)

$$xy'' + \left(x - \frac{1}{2} \right) y' - i\alpha\eta y = - x^{3/2} e^{-x} . \tag{5.19}$$

The boundary conditions are

$$y(0) = y(\infty) = 0 \ . \tag{5.20}$$

In the numerical work described in the next section the function $\eta(x)$ is taken to have the form

$$\eta(x) = \frac{2(r + \frac{1}{2})!}{\sqrt{\pi}} x^{-r} \ . \tag{5.21}$$

6. Some Numerical Results

Miss Cara Joy Hughes employed the IBM 7094 computer at the Harvard University Computing Center to integrate (5.19) with the boundary conditions (5.20) and with $\eta(x)$ given by (5.21). Integrations were carried out for every pair of parameter values (α, r) with α in the set (.1, .2, .5, 1, 2, 5, 10, 15) and r in the set (-.5, .5, 1, 1.5). The integration procedure is described in the Appendix.

For r = 0 the solution of (5.19) is

$$y = \frac{x^{3/2}e^{-x}}{1 + i\alpha} \qquad (r = 0) \ . \tag{6.1}$$

The complete distribution function $f^{(0)}$ corresponding to this solution is Maxwellian with a variable electron temperature. The factor $(1 + i\alpha)^{-1}$ in (6.1) corresponds to the factor $(1 + i\omega\tau)^{-1}$ in (1.4).

Figures 1 and 2 show the modulus and argument of the function

$$\tilde{y} = (1 + i\alpha)y \tag{6.2}$$

for $r = -.5$, 1.5 and for $\alpha = 2$, 5. In both figures the curves corresponding to $r = -.5$ and $r = 1.5$ differ markedly from the curve for the Maxwellian case $r = 0$. For $r = -.5$ the peak of the function $|\tilde{y}(x)|$ occurs at a smaller value of x than for r = 0, while for r = 1.5 it occurs at a larger value. In the four cases with $r \neq 0$ the function arg $\tilde{y}$ changes sign near x = 1.5. For large positive values of x, arg $\tilde{y}$ is positive for r = 1.5, negative for r = -.5.

According to (3.9), the absorption coefficient κ varies directly as the mean collision frequency (the mean being defined by (3.5)) if the carrier frequency is sufficiently high. By (3.5), (5.2), (5.7), (5.8), (5.18), and (5.21),

$$\langle \nu_\omega \rangle \propto \int \frac{d\varphi_\omega}{dq} q^{-2} d(q^3 \nu) \propto (2r + 3) \int d\varphi_\omega x^r$$

$$\propto - (2r + 3)r \int_0^\infty \varphi_\omega x^{r-1} dx \propto - (2r + 3) r\theta_\omega \int_0^\infty y x^{r-1} dx \;. \tag{6.3}$$

Just as in Section 1, the assumptions of linearity and homogeneity imply that $X_\omega \propto \kappa_\omega$ (see (1.7)). Hence

$$X_\omega \propto (2r + 3) r\theta_\omega \int_0^\infty y x^{r-1} dx \;. \tag{6.4}$$

Note that X_ω vanishes for Maxwellian molecules (r = 0). This means that the elementary theory becomes rigorously valid in the limit when the phenomenon it describes disappears.

By analogy with (1.8), we write

$$\int_0^\infty y x^{r-1} dx = T(\alpha) e^{-i\varphi(\alpha)} \;. \tag{6.5}$$

The functions $\varphi(\alpha)$ and $T(\alpha)$ are shown in Figures 3 and 4 for $r = -.5, 0, .5, 1, 1.5$. For $r \geqslant 0$ the curves $\varphi(\alpha)$ have roughly the same shape. Compressing the horizontal scale of the curve for $r = .5$ by 25 per cent would bring it into near-coincidence with the curve for $r = 0$, but for $r = 1, 1.5$, the scaling factor increases markedly with x. For $r = -.5$ (the law appropriate to collisions between electrons and ions), the phase lag at first increases more rapidly with increasing modulation frequency than in the Maxwellian case, then more slowly. The shapes of the curves for $r = -.5$ and $r = 0$ are entirely different.

If one were to use the phase-lag curve for $r = 0$ to analyze experimental results relating to a hypothetical gas in which the electron-molecule collisions were elastic and were characterized by a velocity-independent free path (the case $r = .5$), one would over-estimate the mean collision frequency $\bar{\nu}$ by 25 per cent. If a higher value of r were appropriate, the error could be much greater.

Turning now to the coefficient of transferred modulation $T(\alpha)$ (Fig. 4), we see that the curvature of the function $T(\alpha)$ increases with increasing r over the entire range $-.5 \leqslant r \leqslant 1.5$. Compressing the horizontal scale of the curve for $r = .5$ by 25 per cent would make it fall off more steeply at small and moderate values of α than the curve for $r = 0$. In general, one cannot devise a horizontal scale transformation that, for a given value of $r \neq 0$, will make both $T(\alpha)$ and $\varphi(\alpha)$ assume the forms appropriate to the case $r = 0$.

To sum up, the numerical calculations show that significant departures from the predictions of the elementary theory may be expected. Moreover, values of the mean collision frequency derived by using the elementary theory to analyze cross-modulation data may be significantly in error. On the positive side, the sensitivity of the predictions to changes in the collision law suggests that further theoretical and experimental refinements could ultimately lead to an accurate experimental determination not only of the collision frequency but of the form of the collision law.

Much of the material in this paper is described in an unpublished report by the present authors dated March 14, 1959, and supported by the U. S. Air Force.

Appendix

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The following paragraphs describe the method used to integrate (5.19). The same method can be applied to the more general equation (5.14).

Consider the inhomogeneous second-order linear differential equation

$$a(x)y'' + b(x)y' + c(x)y = d(x) \ . \tag{A.1}$$

We may approximate (A.1) by the set of coupled difference equations

$$a_i \Delta^2 y_i + b_i \Delta y_i + c_i y_i = d_i \qquad (i = 1,\ldots,n-1) \tag{A.2}$$

where

$$a_i = a(x_i) \ , \ \text{etc.} \tag{A.3}$$

$$x_i = x_0 + ih \tag{A.4}$$

$$\Delta y_i = \frac{y_{i+1} - y_{i-1}}{2h} \ , \qquad \Delta^2 y_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \ . \tag{A.5}$$

(A.2) is a set of (n-1) equations for the (n+1) variables $y_0, y_1, \ldots, y_n$. The two boundary conditions provide two additional equations, so that in general the set of difference equations together with the boundary conditions have a unique solution.

In the problem at hand, y_0 and y_n are given:

$$y_0 = y_n = 0 \ . \tag{A.6}$$

The equations (A.2) thus have the form

$$\left(\frac{a_2}{h^2}+\frac{b_2}{2h}\right)y_2+\left(-\frac{2a_1}{h^2}+c_1\right)y_1=d_1$$

$$\left(\frac{a_{i+1}}{h^2}+\frac{b_i}{2h}\right)y_{i+1}+\left(-\frac{2a_i}{h^2}+c_i\right)y_i+\left(\frac{a_{i-1}}{h^2}-\frac{b_{i-1}}{2h}\right)y_{i-1}$$

$$=d_i \qquad (2\leq i\leq n-2)$$

$$\left(-\frac{2a_{n-1}}{h^2}+c_{n-1}\right)y_{n-1}+\left(\frac{a_{n-2}}{h^2}-\frac{b_{n-2}}{2h}\right)y_{n-2}=d_{n-1}\ . \tag{A.7}$$

In matrix notation,

$$MY = D \tag{A.8}$$

where M is a tridiagonal square matrix and Y,D are column matrices.

To solve (A.8), we write M in the form

$$M = LR \tag{A.9}$$

where L and R have the forms

$$L=\begin{bmatrix} 1 & 0 & 0\ldots 0 \\ l_2 & 1 & 0\ldots 0 \\ 0 & l_3 & 1\ldots 0 \\ \cdot & \cdot & \cdots\cdots \\ 0 & \cdot & \ldots\ldots l_n\,1 \end{bmatrix} \quad R=\begin{bmatrix} k_1 & r_1 & 0 & \cdot & \ldots 0 \\ 0 & k_2 & r_2 & \cdot & \ldots 0 \\ 0 & 0 & k_3 & r_3 & \ldots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ 0 & \cdot & \cdot & \cdot & \ldots k_n \end{bmatrix} \tag{A.10}$$

The coefficients l_i, k_i, r_i must satisfy the equations

$$M_{11}=k_1, M_{12}=r_1$$

$$M_{i,i-1}=l_i k_{i-1},\ M_{i,i}=l_i r_{i-1}+k_i,\ M_{i,i+1}=r_i \qquad (2\leq i\leq n-1)$$

$$M_{n,n-1}=l_n k_{n-1},\ M_{n,n}=l_n r_{n-1}+k_n \tag{A.11}$$

$$\left(\frac{a_2}{h^2} + \frac{b_2}{2h} \right) y_2 + \left(- \frac{2a_1}{h^2} + c_1 \right) y_1 = d_1$$

$$\left(\frac{a_{i+1}}{h^2} + \frac{b_i}{2h} \right) y_{i+1} + \left(- \frac{2a_i}{h^2} + c_i \right) y_i + \left. \frac{a_{i-1}}{h^2} - \frac{b_{i-1}}{2h} \right) y_{i-1}$$

$$= d_i \qquad (2 \leq i \leq n-2)$$

$$\left(- \frac{2a_{n-1}}{h^2} + c_{n-1} \right) y_{n-1} + \left(\frac{a_{n-2}}{h^2} - \frac{b_{n-2}}{2h} \right) y_{n-2} = d_{n-1} \,. \tag{A.7}$$

In matrix notation,

$$MY = D \tag{A.8}$$

where M is a tridiagonal square matrix of order (n-1), and

Y,D are column matrices.

To solve (A.8), we write M in the form

$$M = LR \tag{A.9}$$

where L and R have the forms

$$L = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ l_2 & 1 & 0 & . & . & . & 0 \\ 0 & l_3 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & l_{n-1} & 1 \end{bmatrix} \qquad R = \begin{bmatrix} k_1 & r_1 & 0 & . & . & . & 0 \\ 0 & k_2 & r_2 & . & . & . & 0 \\ 0 & 0 & k_3 & r_3 & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & k_{n-1} \end{bmatrix} \tag{A.10}$$

The coefficients l_i, k_i, r_i must satisfy the equations

$$m_{1,1} = k_1 \,, \qquad m_{1,2} = r_1$$

$$m_{i,i-1} = l_i k_{i-1} \,, \qquad m_{i,i} = l_i r_{i-1} + k_i \,, \qquad m_{i,i+1} = r_i \qquad (2 \leq i \leq n-1)$$

$$\tag{A.11}$$

which may clearly be solved in serial order for k_1, r_1, l_2, k_2, r_2, etc. Thus L and R are completely specified by M.

Let

$$RY = U \ . \tag{A.12}$$

Then (A.8) becomes

$$LU = D \tag{A.13}$$

or, again equating matrix elements,

$$\left.\begin{aligned} u_1 &= d_1 \\ l_i u_{i-1} + u_i &= d_i \qquad (2 \leqslant i \leqslant n-1) \end{aligned}\right\} \tag{A.13'}$$

which can be solved serially for the u_i. Having found U, we then determine Y from (A.12), giving

$$\left.\begin{aligned} k_i y_i + r_i y_{i+1} &= u_i \qquad (1 \leqslant i \leqslant n-2) \\ k_{n-1} y_{n-1} &= u_{n-1} \end{aligned}\right\} \tag{A.12'}$$

which can be solved in reverse serial order, beginning with the last equation in the set. The solution is now complete.

One can estimate the accuracy of the procedure described, as applied to the differential equation (5.19), by comparing numerical solutions of this equation for $\eta = 1$ with the exact solutions (6.1). The numerical solutions were carried out for a grid spacing $h = 0.05$ and with the approximate boundary condition $y(x = 25) = 0$ in place of $y(x = \infty) = 0$.

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Legends

Fig. 1 The function $|\tilde{y}|$ (see Eqs. 6.2, 5.18) for representative values of r and α

Fig. 2 The function arg $\tilde{y}$ (see Eqs. 6.2, 5.18) for representative values of r and α

Fig. 3 Phase lag of the transferred modulation (see Eq. 6.5) as a function of modulation frequency for various collision laws

Fig. 4 Amplitude of the transferred modulation (see Eq. 6.5) as a function of modulation frequency for various collision laws

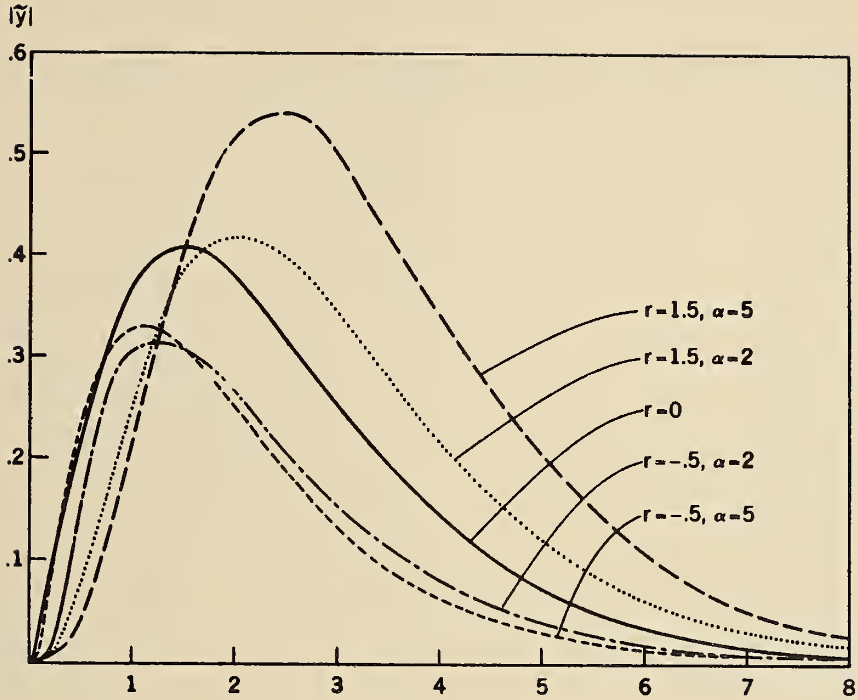

Fig. 1. The function $|\tilde{y}|$ (see Eqs. 6.2, 5.18) for representative values of r and α.

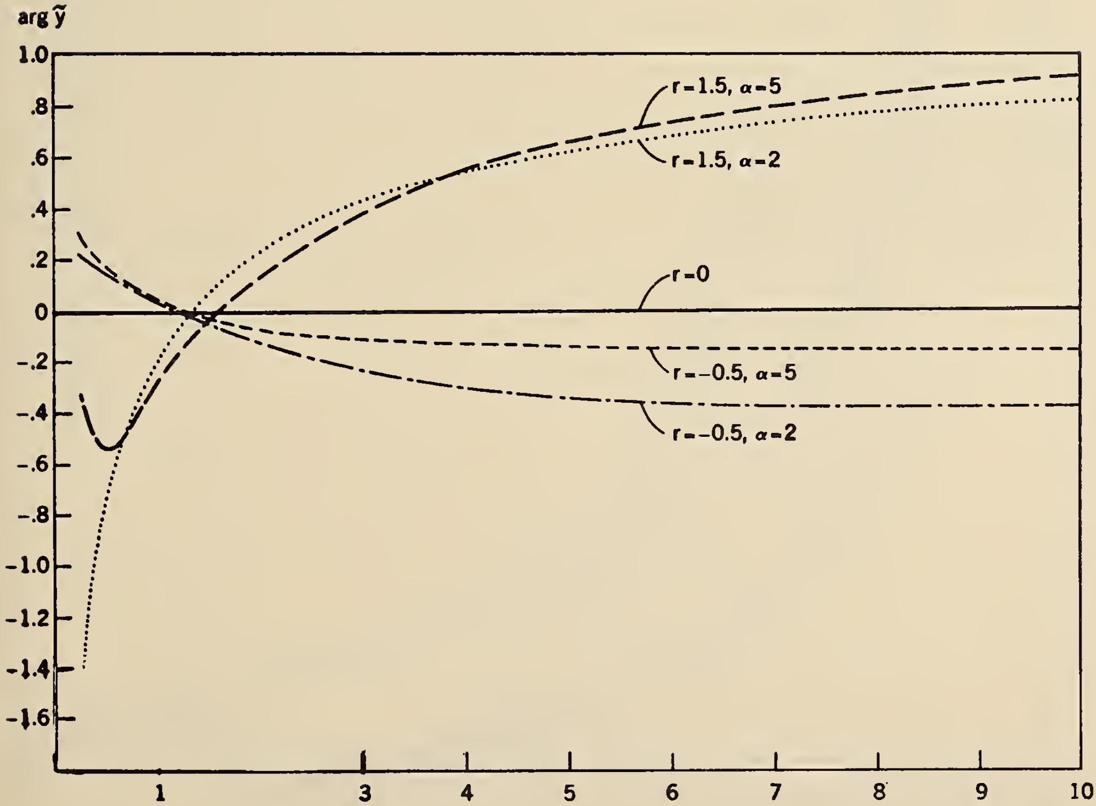

Fig. 2 The function arg $\tilde{\gamma}$ (see Eqs. 6.2, 5.18) for representative values of r and α

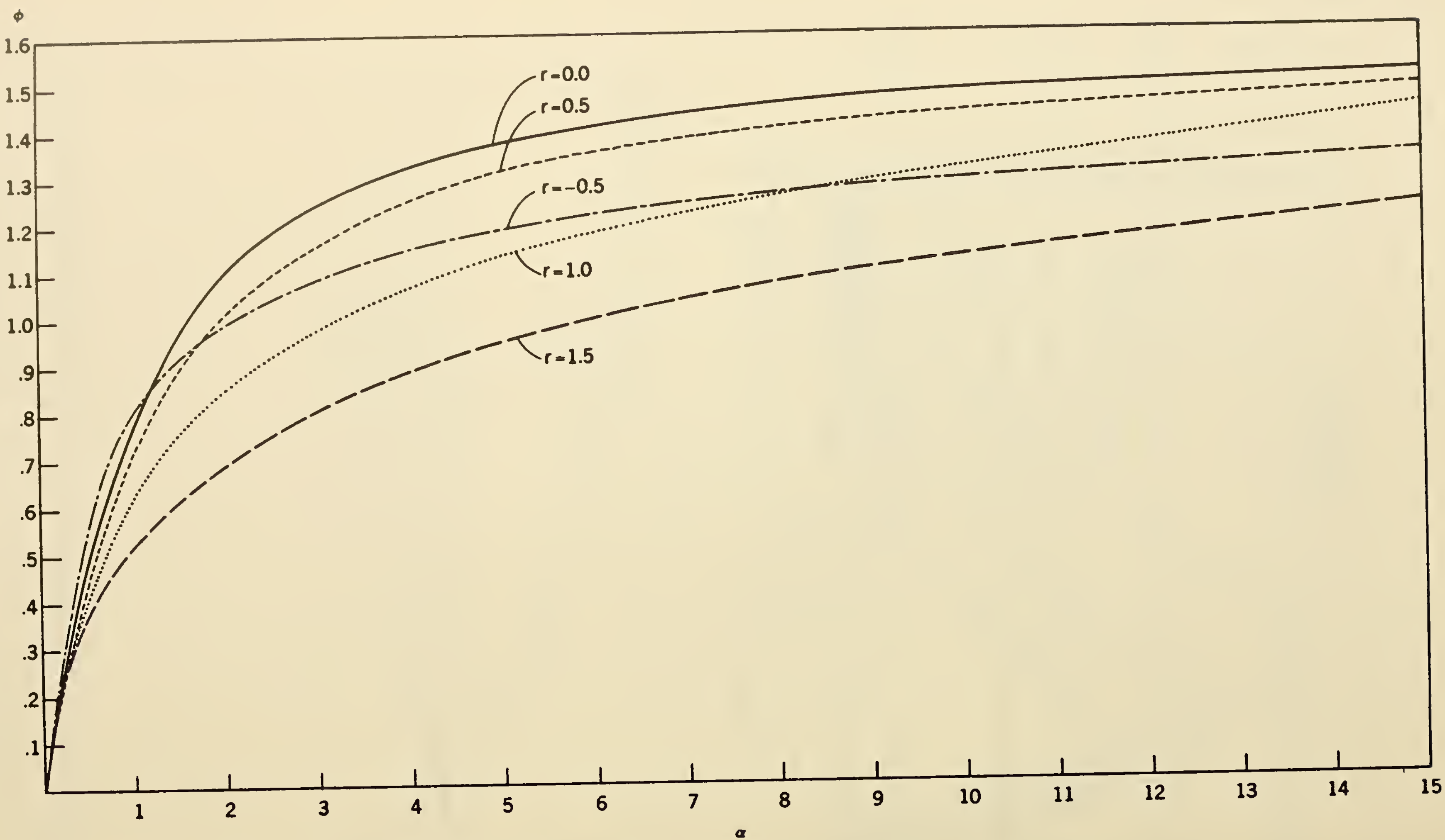

Fig. 3 Phase lag of the transferred modulation (see Eq. 6.5) as a function of modulation frequency for various collision laws.

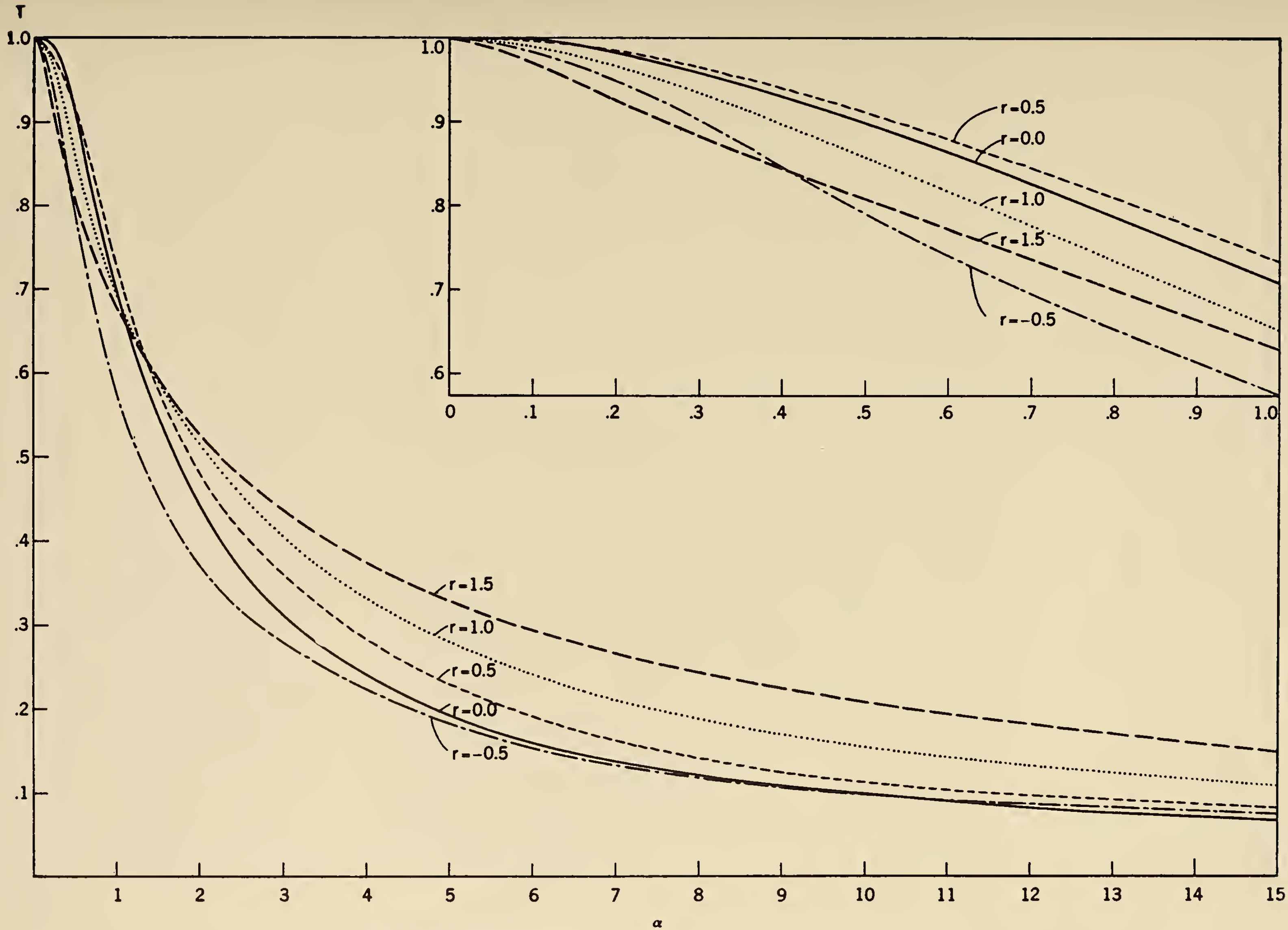

Fig. 4. Amplitude of the transferred modulation (see Eq. 6.5) as a function of modulation frequency for various collision laws.

Wave Interaction Research at the University of New England

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At the University of New England, Armidale, New South Wales, Australia, investigations of radio wave interaction in the ionosphere have been characterised by the use of extraordinary gyro-waves * to enhance the electron heating and by progressive increases of the power flux to levels for which the electron temperature has greatly exceeded the temperature of the gas. Because of the different methods employed to study this heating, the research can be divided into two periods. From 1950 to 1957 the experimental conditions were chosen to conform to the requirements of the theory of gyro-interaction given by Bailey in 1937 and 1938, the disturbing wave being a pulsed gyro-wave transmitted vertically and the wanted wave a low frequency wave which passed at an oblique angle through the region of the ionosphere acted on by the gyro-wave. Since 1959 the wanted wave has also been pulsed and transmitted vertically, the techniques due to Fejer (1955) being used to measure its interaction with the pulsed gyro-wave. The latter experiments have been made both at night and during the day and have added considerably to knowledge of the fundamental processes occurring in radio wave interaction and of the properties of the regions of the ionosphere where this interaction occurs. The principal experimental parameters in each period and the results obtained by the two methods are now summarized.

1950 - 1957

The experiments (Bailey et al., 1951; Smith, 1957) were carried out between 1950 and 1954 in collaboration with a group from the University of Sydney under Professor V. A. Bailey. The disturbing transmitter was located at armidale and generated pulses of 1 ms duration and 36 kw power at a number of frequencies within ± 20% of the gyro-frequency (1.515 Mc/s). Its antenna was a horizontal half-wave dipole suspended between two 150 ft. self-supporting towers.

*The term gyro-wave is here used in the sense of a wave whose frequency is sufficiently close to the gyro-frequency for the absorption of the extraordinary wave mode to be significantly higher than that of the ordinary wave mode.

The wanted wave (0.59 Mc/s), transmitted as an unmodulated carrier wave by the National broadcasting station 4QR, Brisbane, Queensland, was received at Katoomba, New South Wales, 740 km from Brisbane (Armidale is close to the mid-point of a great circle path between Brisbane and Katoomba). The envelope of the impressed modulation pulse was displayed on an oscillograph and photographed at 10-second intervals. Observations were rejected when there was any evidence of selective fading.

The plots of impressed modulation against frequency of the disturbing wave were found to be of resonance form with two peaks separated by a central minimum at the gyro-frequency. The depth of the impressed modulation was about 9%, and its time constant was between 800 and 1200 μs.

In 1957 a complete theory of gyro-interaction was developed (Smith, 1957), which includes a rigorous discussion of the propagation of both the wanted and disturbing waves. For a model of the night-time lower ionosphere based on independent data, the resonance curve of interaction computed using the complete theory agrees well with the observed curves. The complete theory clarifies the physical processes occuring in gyro-interaction and resolved in favour of Bailey the theoretical arguments and experimental results obtained in England between 1947 and 1949 which caused J. A. Ratcliffe, D. F. Martyn, and others to reject his theory. It also shows that the time constant of impressed modulation is dependent on the frequency of the disturbing gyro-wave and is a minimum when this wave is an exact gyro-wave. This behaviour of the time constant was subsequently found on re-examination of the experimental data.

The work in the period 1950-1957 demonstrated that wave interaction in complex situations involving oblique propagation can be predicted with satisfactory accuracy.

<u>1959 - 1963</u>

The precise studies of ionospheric wave interaction experiments using vertically propagated disturbing and wanted waves have the considerable advantage over the gyro-interaction type of experiment that the propagation of both waves, being at a fixed angle to the earth's magnetic field, is much simpler to analyse. Also, the interaction is free of effects from selective fading.

The equipment for the pulse experiments is located at three stations, referred to as A, B, and C. Of these, A is the station from which the earlier disturbing transmissions were made. The stations B and C were constructed under Contract AF19(604)-6177 with the Geophysics Research Directorate, Air Force Cambridge Research Laboratories, Bedford, Mass.

Station A The gyro-transmitter at this station is a self-oscillator having a frequency range of 1.2 to 1.9 Mc/s, a maximum pulse power of 200 kw, and a maximum pulse duration of 1 ms. Its aerial is a circularly-polarized array of 4 horizontal half-wave dipoles arranged in a square pattern. The disturbing beam from this installation is estimated to heat electrons at night to a temperature of about 1000°K, i.e., to about five times their undisturbed temperature.

The station has complete radio probing and receiving equipment for wave interaction experiments of the pulse type.

Station B This station is a high power transmitting installation. The transmitter is x-tal controlled for the frequencies 1.43, 1.515, and 1.60 Mc/s. The maximum power is 500 kw for CW operation and 1 MW for pulse operation. At the power of 500 kw, the pulses may have any duration from 50 μs to CW. The aerial is a circularly-polarized array of 40 horizontal half-wave dipoles suspended in a square pattern from twenty-five 90 ft. wooden poles. The array, which covers an area of 70 acres, is stressed for a power of 4 MW.

The installation is estimated to heat electrons at night to energies approaching 1 ev.

Station C The station contains extensive radio probing and receiving equipment for the experiments with station B. It can, however, also be operated synchronously with A. The station is 1.5 km from B and 9 km from A.

The wanted frequencies used in the pulse wave interaction experiments were 1.78 and 2.12 Mc/s during the night, and 2.12 and 3.85 Mc/s during the day. The night-time wanted frequencies are sufficiently close to the gyro-frequency of 1.515 Mc/s for the modulation depth impressed on the extraordinary mode to be significantly greater (by factors of 20 and higher) than on the ordinary mode.

The experimental data obtained since 1959 using the pulse methods are almost an order of magnitude more accurate than the gyro-interaction observations. Effects have been studied over a range of power flux of the disturbing wave of four decades and decay processes to six times the time constant. Accurate determinations have also been made of the values of the wave interaction coefficient G and electron collision frequency in the lower ionosphere. The most important results are now summarized;

(i) Small-perturbation wave-interaction theory (i.e., theory applicable when the change in temperature of the electrons is small compared to their undisturbed temperature) predicts that the depth of impressed modulation is proportional to power of the disturbing wave. It has been found that this linear relationship holds for changes up to three times the undisturbed temperature. For larger changes of electron temperature, the departure from proportionality to power is not so marked as to cause appreciable error in an order of magnitude estimate of the interaction.

(ii) For theoretical calculations of non-linear phenomena, it is more accurate to use generalized magneto-ionic theory (Sen and Wyller, 1960) than the well-known theory due to Appleton and Hartree. However, the errors due to the latter theory are not large enough to justify revision of published calculations in which it has been used.

(iii) At a height of 85 km above sea level, the electron relaxation time τ, the collision frequency ν_m in the Sen and Wyller magneto-ionic theory, and the energy coefficient G_m (G_m is defined by $\tau = 1/G_m \nu_m$) have the values 640 μs, 2.3×10^5, and 6.8×10^{-3}, respectively. For radio wave interaction interpreted using the Appleton-Hartree formulae, the corresponding values of collision frequency and G are 7.8×10^5 and 2.0×10^{-3}, respectively.

(iv) At a height of 70 km above sea level, the values of τ and ν_m are 40 μs and 4.0×10^6, respectively. The Appleton-Hartree collision frequency at this height is 1.25×10^7.

(v) The values of G (or G_m) have been found to depend on the temperature of the gas. For the variation of temperature with height in the lower ionosphere measured using rockets, the change in G ($\not>$ 10%) is small enough to be neglected in most calculations of wave interaction.

(vi) An electronic synthesizer for interpretation of D-region wave interaction data has been constructed. Extreme accuracy of measurement is required before definite conclusions on the structure of the D-region can be reached.

(vii) The modulation impressed on partial echoes from the D-region has been found to consist of two parts, the first being the normal interaction below the reflecting discontinuity and the second to a change in the reflecting power of the discontinuity. The latter is a new and hitherto unpredicted wave interaction phenomenon with great promise for the study of the lowest levels of the D-region.

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7/1/64

"An experimental study of gyro interaction in the ionosphere, at oblique incidence"

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This paper describes further experiments on the interaction between a gyro disturbing wave and an obliquely incident wanted wave in the night time lower E region. The results fully confirm those reported some years ago concerning the enhanced effect at the gyro frequency. Average values of $G\nu$ of about 800 sec^{-1} were obtained, with a height variation consistent with a scale height of the order of 8 km. The effects of multipath interference and the problem of the energy loss factor G are described in appendices.

*Permanent address

1. Introduction

This paper describes an experimental study of gyro interaction in the ionosphere carried out in Australia by the author, R. A. Smith and V. A. Bailey. Some results in the early part of this study were reported briefly by Bailey et al., (1952). We present here an account of further experiments, with improved precision, that were made following this early work. The experiments were designed primarily to test the gyro resonance effect predicted by Bailey, and were carried out at night using a pulsed gyro disturbing wave and an obliquely incident wanted wave.

The resonance effect in gyro interaction depends in part on the well-known resonance in the absorption of the extraordinary component of the disturbing wave at the gyro frequency and in part on the path of the wanted wave in the disturbed region. The phenomenon was analysed in detail by Bailey (1937, 1938). Although the full mathematical description is somewhat complex, the basic physical ideas can be presented quite simply as follows. Because of the very high absorption near the gyro frequency, the energy in the extraordinary component of the disturbing wave is practically all absorbed after it has travelled some 6 or 7 km upward into the night-time ionosphere. The region of the ionosphere in which the electron energy is perturbed thus consists of a relatively thin horizontal slab, above the disturbing transmitter, of fairly large horizontal extent. The increase in electron energy within this slab is much greater than that which would be produced in the same region by a wave of the same power whose frequency is far removed from the gyro frequency. The way in which the energy I absorbed her unit volume in the slab varies with frequency and with height in the slab is shown in Figure 1. At the lower levels the curves have a maximum at the gyro frequency; at the higher levels there is a subsidiary minimum here because the wave has been severely attenuated before it reaches these levels. The curves in Figure 1 have been computed, for purposes of illustration only, for a typical model of the lower night-time ionosphere in which the electron density N and collision frequency ν vary with height as

$$N = N_o \exp(0.8) \qquad \text{and} \quad \nu = 1.2 \times 10^6 \exp(-0.16\ h),$$

where h is the height in kilometers measured from an arbitrary base height - the bottom of the slab - at which N_o is taken as 1 electron cm^{-3}.

When a wanted-wave travels horizontally through the slab at a given height, the modulation that it acquires over an element of its path length will vary with disturbing frequency in the same manner as the curve in Figure 1 that corresponds to that height. Because the wanted wave takes a finite time to pass through the disturbed region, the phase of the modulation acquired over each successive element of path changes as the wave proceeds. But provided that the time to traverse the whole path is small compared with the time variation of the collision frequency, this phase change will be small and may be neglected. Further, the path within the slab is not exactly horizontal in practice but is slightly concave downwards and so the wave acquires its modulation over a narrow range of heights with the slab. The result of this is that the impressed modulation corresponds to average values of the ionospheric properties over a narrow height range, rather than to values at a sharply defined height. For the very obliquely propagated wanted wave used in our experiments, this height range is probably two or three kilometers.

From the foregoing it will be seen that a large amount of modulation can be impressed on a wanted wave by a gyro disturbing wave when the wanted wave travels nearly horizontally through the slab and so has a long path length in the disturbed region. It is this combination of a gyro disturbing wave, with its concentrated and strongly frequency-dependent effect, and an appropriate trajectory for the wanted wave, that gives rise to the "resonance" in gyro interaction. This situation should be contrasted with those in which (i) an oblique wanted wave is reflected well above the slab and (ii) a vertically incident wanted wave is reflected above the height where most of the energy has been absorbed from the disturbing wave. In neither of these situations would one expect to observe a large effect at the gyro frequency or a marked dependence on disturbing frequency.

2. Experimental Details

In the experiments described here the wanted transmitter was located at Brisbane and radiated a continuous wave. The wanted wave was received at Katoomba, west of Sydney and 720 km south of Brisbane. The disturbing transmitter was located at Armidale on the mid-point between wanted transmitter and receiver. It radiated rectangular pulses on a frequency that was variable about the gyro frequency. Details of the transmitters are as follows:

Wanted transmitter: Continuous wave, frequency 590 kc/s, power 10 kw.

Disturbing transmitter: Pulse length 1 milisecond, pulse repetition frequency 40 per second, peak power 36 kw, frequency variable in steps of 30 kc/s from 1390 to 1690 kc/s, half-wave dipole antenna. Local gyro frequency at height of 90 km is 1530 kc/s, magnetic dip angle is 60°.

Observations were made between 0100 and 0220 hours. The band-width of the wanted-wave receiver was sufficiently wide to avoid distorting the pulse modulation acquired by the wanted wave. Automatic gain control was used in the receiver, and a meter was incorporated in the A.G.C. circuit to indicate the strength of the received signal.

The disturbing transmitter was switched on for one minute on each of the thirteen frequencies from 1390 to 1690 kc/s in succession, with a blank interval of one minute between each disturbing radiation. During this blank interval the wanted carrier was modulated for 30 seconds at its transmitter with an 80 c/s sinusoidal signal to a depth of 5 per cent. This sinusoidal modulation provided a calibration for the measurement of the depth of the modulation impressed on the wanted carrier by interaction.

The rectified output of the receiver was applied to the vertical deflection plates of an oscilloscope, and the length of the trace photographed on horizontally-moving film. Examples of the records obtained are shown in Figures 2(a) and 2(b). Records of this type yielded the amplitude of the impressed modulation as a function of the disturbing frequency.

The rectified output was also displayed on a second oscilloscope with linear sweep and photographed at intervals of 3 or 6 seconds. The signal strength was also recorded on this photograph. Figure 3 is a typical picture of the display. The upper trace shows the increase and decay of the incremental absorption of the wanted wave and hence of the electron collision frequency. The shape of these pulses was used to obtain the time constant for the change in collision frequency. The lower trace shows the disturbing pulse which was received directly on a separate receiver after reflection from the ionosphere, together with numerous multiple reflections. This trace plays no essential part in the observations and was used merely to monitor the disturbing transmitter.

The disturbing transmitter was correctly matched to the antenna at each frequency in turn, and the antenna current measured with calibrated meters. The various matching and coupling settings at the transmitter had been determined by prior calibration. The power actually radiated was calculated from the antenna current and the measured radiation resistance at each frequency.

3. Experimental resonance curves

Because of the effects of multipath fading, meaningful measurements of the amplitude of interaction can only be made when the received wanted wave consists solely of the ray that has been reflected in the vicinity of the disturbed region. Fluctuation of the amplitude of the rectified pulse on the wanted wave and occasional inversions of the pulse occurred sporadically for more than one third of the total observing time. These fluctuations were always accompanied by a decreased and fluctuating signal strength of the wanted wave. When the wanted signal remained strong and reasonably constant (free of the deep fading characteristic of multipath fading) the amplitude of the rectified pulse was remarkably constant and records like those in Figure 2(a) were obtained. On the other hand, when the signal strength fell or fluctuated, the records were like those in Figure 2(b).

For reasons given in Appendix 1,it was concluded that when the records were like those in Figure 2(a), which were obtained when the wanted wave was strong and steady, the only ray present was that which had been reflected once in the E region. Records of this type alone were selected for amplitude measure ment. The depths of impressed modulation obtained at each frequency were normalized to a standard disturbing power of 36 kw. and plotted as a function of disturbing frequency. (Some measurements of the depth of modulation were also made on records of the type of Figure 3. These agreed well with the main measurements).

During the 23 minutes required to traverse the range of disturbing frequencies some multipath fading invariably occurred. In fact, on some nights virtually no useful amplitude measurements were obtainable. It is usually not possible to combine data obtained at widely separated times, because minor changes in the gradient of electron density or of reflection height produce changes in the detailed shape of the resonance curves. Nevertheless, there were sufficient occasions when complete, or almost complete, resonance curves were obtained. Examples of such curves are shown in Figure 4.

It is seen that these curves exhibit the predicted variation with disturbing frequency and that a relatively large depth of modulation, of the order of 5 per cent, is produced with a disturbing power of 36 kW. They thus confirm in detail Bailey's prediction of the resonance effect. The same conclusion was reached in the brief report by Bailey, et al., (1952) of the earlier measurements. The earlier measurements were made on the amplitude of individual pulses, like that in Figure 3, and it was not always as easy to recognize the occurrence of multipath fading as it was in the present experiments.

4. The time constant $1/(G\nu)$

According to the conventional theory of wave interaction (Bailey and Martyn, 1934; Bailey, 1937, 1938) the time constant describing changes in electron collision frequency ν is given by $1/(G\nu)$, where G is a constant related to the mean fractional loss of energy by an electron in a collision with an air molecule. Thus the change in collision frequency produced at a point in the ionosphere by a rectangular disturbing pulse of duration T varies with time as

$$1 - e^{-G\nu t} \qquad \text{for } 0 < t < T$$

and as

$$e^{-G\nu t} \qquad \text{for } T < t < \infty$$

The change in absorption of the wanted wave varies in the same manner. The pulses of modulation impressed on the wanted wave should therefore consist of an exponential rise and decay. This has been found to be the case, as illustrated in Figure 3.

A number of the clearest photographs of pulses in which the noise level was low were very carefully measured, and the shapes were found to be indistinguishable from a pure exponential. Also, no significant difference in exponent could be detected between pulses obtained in the absence or presence of fading, including such severe fading that the rectified pulse was inverted. It is shown in Appendix 1 that no difference is to be expected, and for this reason the measurement of $G\nu$ was not restricted to those occasions when there was no fading. Of course, it can occasionally happen that the relative amplitudes and phases of the interfering components are such that the resultant wave is over-modulated, but this produces a characteristic distortion of the pulse shape which is very easily recognized. It occurred on only a very few occasions.

In order to measure the values of $G\nu$ from a large number of photographs, an exponential pulse was generated from a rectangular pulse of 1 millisecond duration and of adjustable amplitude, by means of a simple resistance-capacitance arrangement. The generated pulse was displayed on an oscilloscope and was matched in amplitude and shape to that in the photographic negative. $G\nu$ was then obtained from the corresponding value of RC. The main limitation on accuracy in the determination of $G\nu$ arises from the presence of noise on the wanted signal. The uncertainty in individual measurements of $G\nu$ is of the order of 10 percent.

Values of $G\nu$ were obtained from all records in which the noise level was sufficiently low, irrespective of whether the occurrence of fading prevented useful amplitude measurements from being obtained. The measured values of $G\nu$ lay between 650 and 1000 sec^{-1}, and within these limits there was considerable scatter at any one disturbing frequency. However, following the suggestion of R. A. Smith, when the values of $G\nu$ were averaged separately for each disturbing frequency, a variation of $G\nu$ with disturbing frequency was found. This is shown in Figure 5. Since the path of the wanted wave is independent of disturbing frequency we conclude that this effect is associated with the fact that more energy is absorbed lower in the slab at the gyro frequency than at disturbing frequencies removed from the gyro frequency.

We can make an estimate the order of magnitude of the scale height from this. By replotting the data in Figure 1 to show the variation with height of the energy absorbed per unit volume for various constant frequencies, it is found that its maximum (or mean) value at the gyro frequency occurs at a height approximately 1 kilometer below that of the maximum (or mean) value for frequencies that are ± 150 kc/s from the gyro frequency. The scale height

H is

$$H = - \nu \, \Delta h / \Delta \nu .$$

From Figure 5 we have that $\Delta\nu/\nu \sim 1/8$ and, from the above, $\Delta h \sim 1$ km. This gives H $\sim$ 8 km, within a factor of about 2. The effect seen in Figure 5 is thus consistent with the known height dependence of the collision frequency at altitudes near 90 km.

With the assumption that the time constant can be identified with $1/G\nu$, we **adopt** the value $G = 1.70 \times 10^{-3}$ suggested in Appendix 2. From Figure 5 we see that $G\nu = 880$ when the disturbing frequency is very close to the gyro frequency. This yields $\nu = 5.2 \times 10^5$ for the collision frequency. There is some uncertainty about the height that this corresponds to but it must be where the electron density has a value of 100 - 400 cm^{-3} and is probably close to 90 km. This value of collision frequency may be compared with that obtained by extrapolating Kane's (1959, 1962) data to 90 km, which is $\nu = 5.0 \times 10^5$ using the Appleton-Hartree magnetoionic theory or 2.4×10^5 using the generalized magneto-ionic theory. The agreement suggests that the assumption concerning the electron energy loss, discussed in Appendix 2, is at least approximately correct.

Appendix 1

Effects of multipath interference fading

In oblique incidence transmission a wave may propagate from transmitter to receiver by several different paths. For the experiments described here it is necessary to examine the result of the simultaneous arrival at the wanted receiver of a wave that has undergone a single reflection in the vicinity of the disturbed part of the E region and other waves, such as those reflected once or more from the F or E_s layers, that have not traversed the disturbed part of the ionosphere. The ray that traverses the disturbed region acquires modulation there by interaction; the other rays remain unmodulated. Interference between the modulated and unmodulated continuous waves produces a resultant modulation whose amplitude and phase depend on the relative amplitudes and phases of the two components. The effects of simultaneous reception of modulated and unmodulated rays can be seen by considering what happens when their R.F. carriers arrive (i) in phase, and (ii) 180 degrees out of phase.

When the carriers are in phase they add, and the modulation depth on the resultant is less than the true depth of modulation impressed on the modulated component by interaction. However the phase of the modulation, or for pulses, the pulse shape, is unchanged by the presence of the unmodulated component.

When the carriers are antiphased, they subtract. There are then three possibilities:

(a) The amplitude of the unmodulated component is less than the amplitude of the modulated component at the modulation troughs: The resultant modulation depth is greater than that of the modulated component alone, but the phase or shape of the modulation is unchanged.

(b) The amplitude of the unmodulated component is greater than that of the modulated component at the modulation peaks: The resultant modulation depth may take any value. The phase of the modulation is shifted by 180 degrees. For pulse modulation this implies that the rectified signal will be inverted, but the shape of the pulses will be otherwise unchanged.

(c) The relative amplitudes of the components are intermediate between these in (a) and (b): The resultant modulation depth exceeds 100 percent, so that the modulation is distorted.

It is thus seen that the true depth of the modulation impressed on the wanted wave can only be obtained when the wanted wave consists solely of the ray that has been reflected in the vicinity of the disturbed region. On the other hand, except when the modulation is distorted (Case (c)), the presence of some unmodulated component does not prevent meaningful measurements of the modulation phase, or pulse shape, from being made.

It is interesting to note that Ratcliffe and Shaw (1948) studied the dependence of the amplitude and phase of sinusoidal modulation impressed on the wanted wave as a function of modulation frequency of the disturbing wave. They found that the phase measurements agreed much better with theory than did the amplitude measurements and were unable to explain this. The observation could well be accounted for if from time to time in their experiments some significant amount of unmodulated wanted wave was received.

Appendix 2

The electron energy loss relation and the value of G for air

The problem of the loss of energy by electrons in collisions with air molecules has received considerable attention in recent years, but it cannot yet be regarded as satisfactorily resolved.

It is desirable, however, to interpret the values of the time constant for changes in the electron energy, as measured from the changes in absorption of the wanted wave. This will be done by retaining the original hypothesis that the mean loss of energy by an electron in a collision is proportional to the difference between the mean energy of the electrons and the mean energy of the air molecules. When this assumption is made the time constant is identified as the quantity $1/(G\nu)$, where G is the constant of proportionality in the above hypothesis.

Information about the hypothesis and the value of G, if it is in fact a constant, can be obtained from laboratory experiments on the diffusion and drift of low energy electrons in gases. Unfortunately, the experimental data for air only extend down to mean electron energies that are about six times greater than thermal, and extrapolations or other assumptions must be made in order to obtain values for the lower energies relevant to ionospheric studies. We introduce the following symbols used in diffusion and drift studies:

Q_o = mean energy of agitation of a gas molecule

Q = mean energy of agitation of an electron

$k = Q/Q_o$

ΔQ = mean energy lost by an electron in a collision with a gas molecule

$\lambda = \Delta Q/Q$ = mean fraction of its energy lost by an electron in a collision

Z/p = ratio of applied electric field to gas pressure

The quantity k represents the mean electron energy expressed in terms of the energy of a molecule as unit. The product λk represents the energy lost by an electron per collision, in the same units, and is directly related to the drift velocity of electrons in the gas. For a given gas, k and λk are each functions of Z/p alone and may be obtained from separate laboratory experiments. By associating corresponding values of k and λk measured at the same Z/p one obtains λk as a function of k. The most reliable values of k for air are those obtained by Crompton, Huxley and Sutton (1953) and, of drift velocity and λk, those obtained by Nielson and Bradbury (1937). The values of λk as a function of k, for the lowest energies, are plotted in Figure 6. The lowest energy represented here is k = 5.9, whereas the region of interest for ionospheric effects is near k = 1.

The original hypothesis of Bailey and Martyn (1934), based on measurements in air by Townsend and Tizard (1913) and on the theoretical behaviour for elastic collisions, is that the mean loss of energy by an electron in a collision is proportional to the difference between the mean energy of the electrons and that of the gas molecules. The hypothesis may be written as

$$\Delta Q = G(Q - Q_o)$$

when G is the constant of proportionality. This is equivalent to

$$\lambda k = G(k - 1)$$

If the hypothesis is correct the graph of λk against k should be a straight line which intersects the k axis at k = 1. In Figure 6 the experimental points from k = 21.7 to k = 10.5 lie very closely on a straight line whose extension meets the k axis at k = 1. The slope of this line yields for G the value of 1.70×10^{-3}. (Points for k greater than about 22 lie above the line, but this effect is presumably associated with electronic excitation and is not relevant to our discussion.)

The fact that the five points below $k = 10.5$ fall increasingly below the line have led to serious doubt about the validity of the G hypothesis for air. It should be noted that the greatest deviation from the line, shown by the lowest points, would correspond either to a deviation of 20 per cent in the value of the drift velocity or a deviation of 45 per cent in the value of k. Whether the results of Figure 6 either establish or disprove the G hypothesis is far from obvious, as indeed also is the more fundamental question of whether there is any thermodynamic or other reason to expect the hypothesis to hold for gases other than the rare gases.

Data for nitrogen are available for considerably lower energies than those for air and arguments have been given that suggest that at very low energies the loss of energy by electrons in collisions with oxygen molecules is much less than those with nitrogen, so that the energy loss in air at very low energies may be estimated from the experimental results for nitrogen. On the other hand the results of Brose (1925) and of Healey and Kirkpatrick (1939) for oxygen at $k \sim 6$ indicate that at this energy the energy loss is much greater in oxygen than nitrogen. There is still uncertainty about these questions. If, for simplicity, one adopts the G hypothesis for air as a tentative approximation, the best value for G would appear to be 1.70×10^{-3}.

Finally, it will be recalled that it was at one time proposed that the energy loss in air was proportional to $(Q - \Omega_o)^2$. This relation was subsequently withdrawn because it was considered to conflict with deductions from the magneto ionic theory. This discarded hypothesis is only mentioned here because it has been revived in a recent report. A simple argument shows that the relation cannot be true in the vicinity of $k = 1$. Because the velocity distribution of each set of particles, electrons and molecules, is Maxwellian or very nearly so near $k = 1$, the suggested relation is equivalent to the statement that the electron energy loss (or transfer of thermal energy) is proportional to $(T - T_o)^2$ where T is the electron temperature and T_o is the gas temperature. This implies that the direction of heat flow does not reverse when the temperature difference changes sign, which contradicts the second law of thermodynamics. The electron energy loss must clearly be an odd function of $Q - Q_o$ in the vicinity of $k = 1$.

The uncertainty about the dependence of the electron energy loss on the electron energy constitutes one of the most serious problems in the theory of wave interaction. It is directly related to the other outstanding problem of the manner in which electron collision frequency depends on the electron energy.

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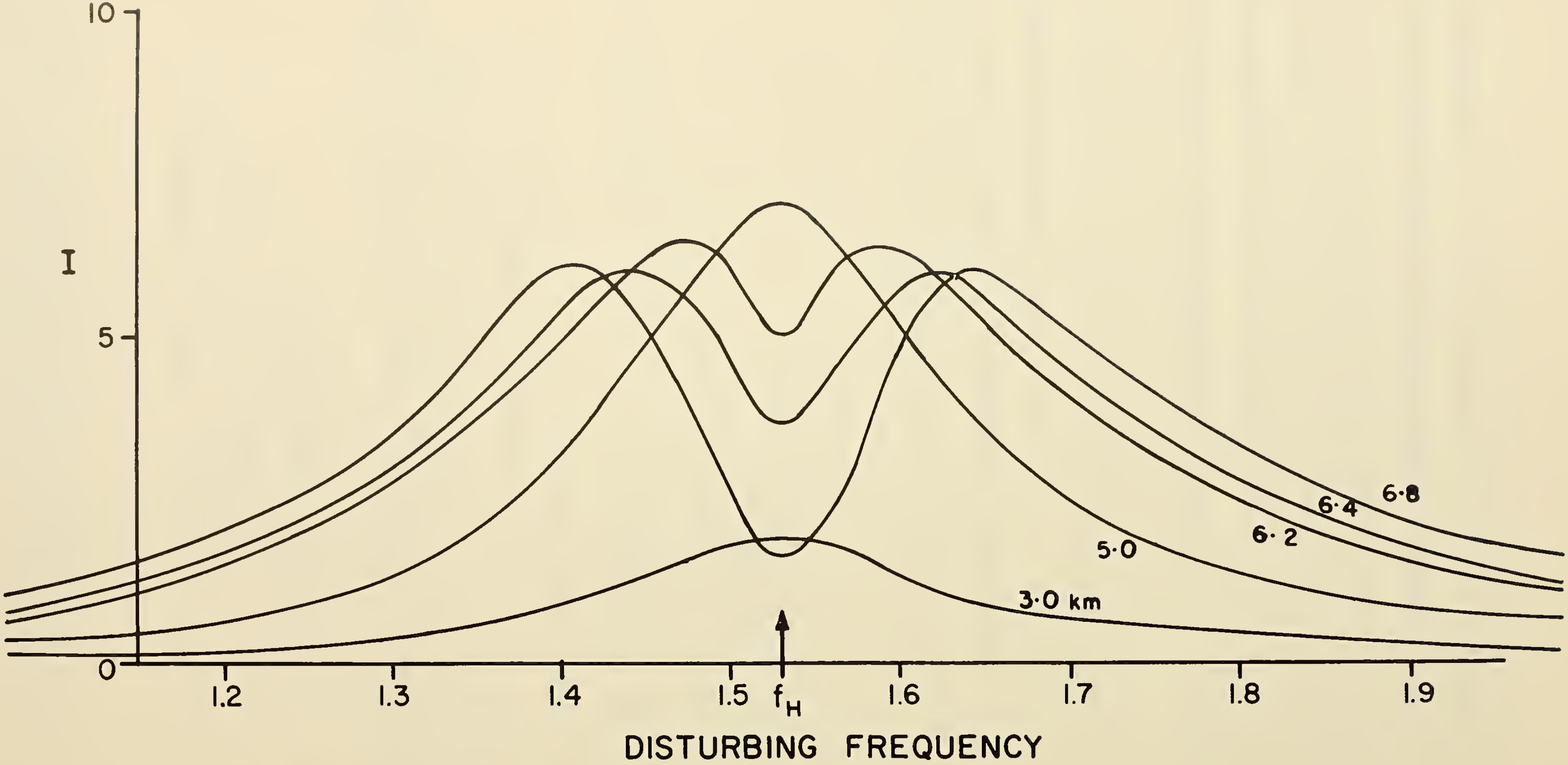

Figure 1 The energy I (in arbitrary units) absorbed per unit volume of the ionosphere as a function of the frequency, in Mc/s, of the disturbing wave in the vicinity of the gyro frequency, computed for a model night-time ionosphere, at various heights above the base of the ionosphere.

Figure 2a The amplitude of modulation impressed on wanted wave (long portion) and of the calibration sinusoidal modulation (short portions), recorded on moving film. This record is typical of those obtained when fading was absent.

Figure 2b Similar record to Figure 2a obtained when multipath fading occurs.

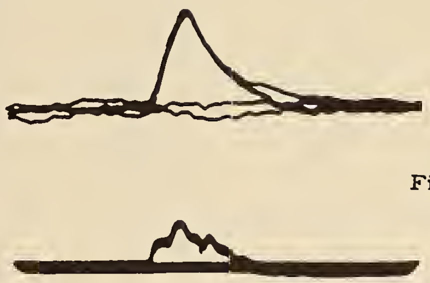

Figure 3 Photograph of the pulse modulation impressed on the wanted wave by a rectangular disturbing pulse. The lower trace shows the directly received disturbing pulse and its multiple echoes.

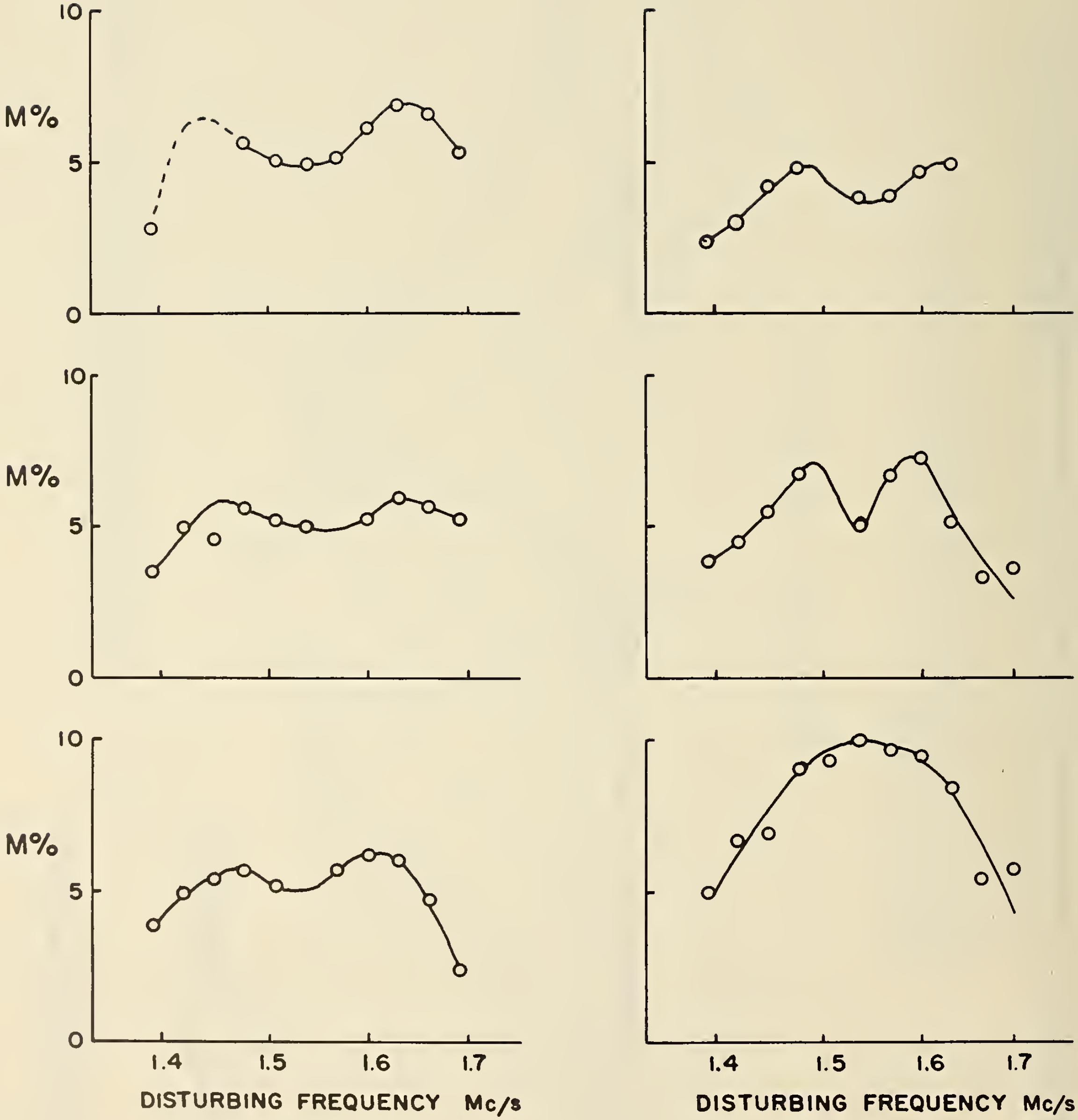

Figure 4 Experimental curves obtained on various occasions, showing the depth of modulation impressed on the wanted wave as a function of disturbing frequency.

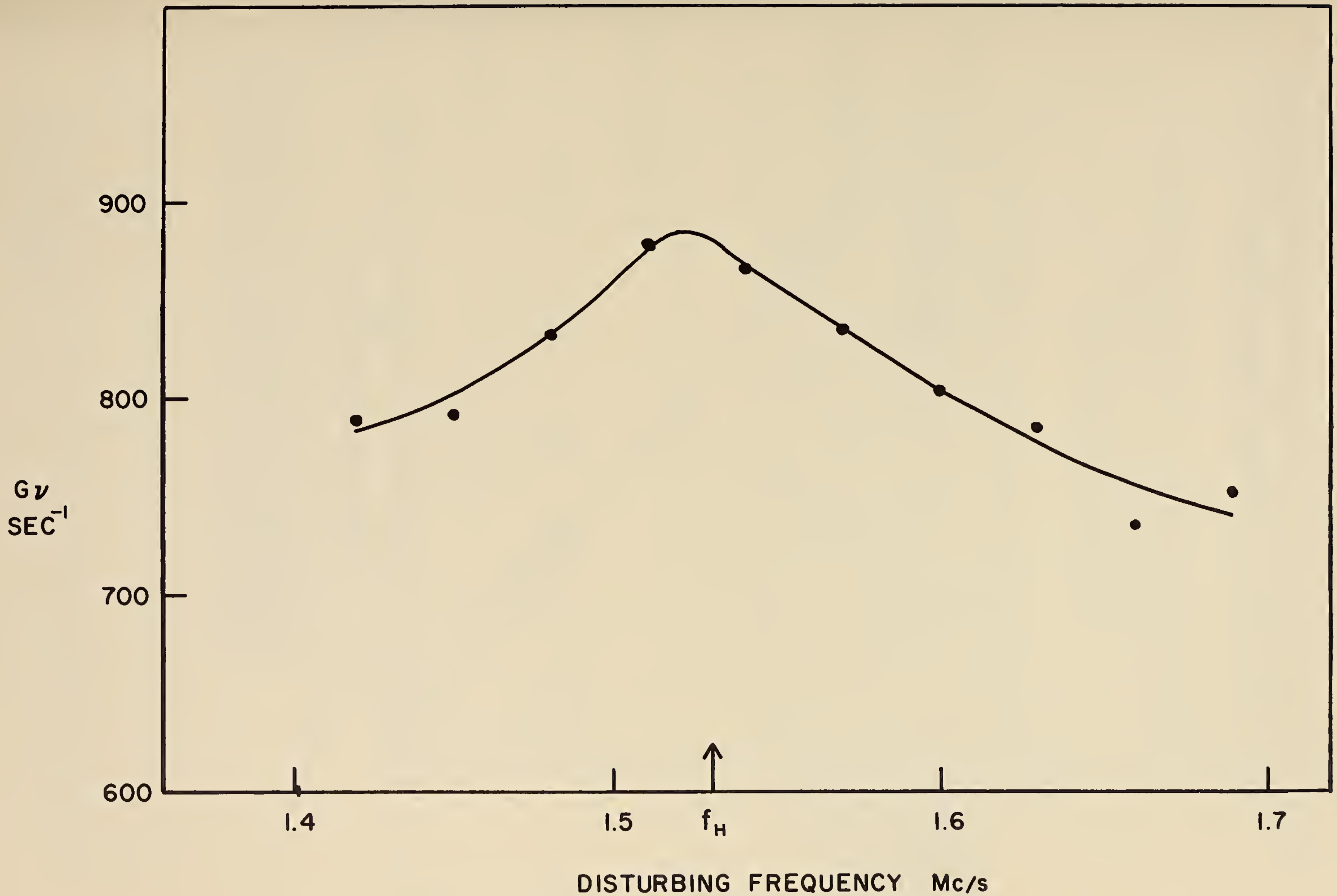

Figure 5 Variation of $G\nu$ as a function of disturbing frequency.

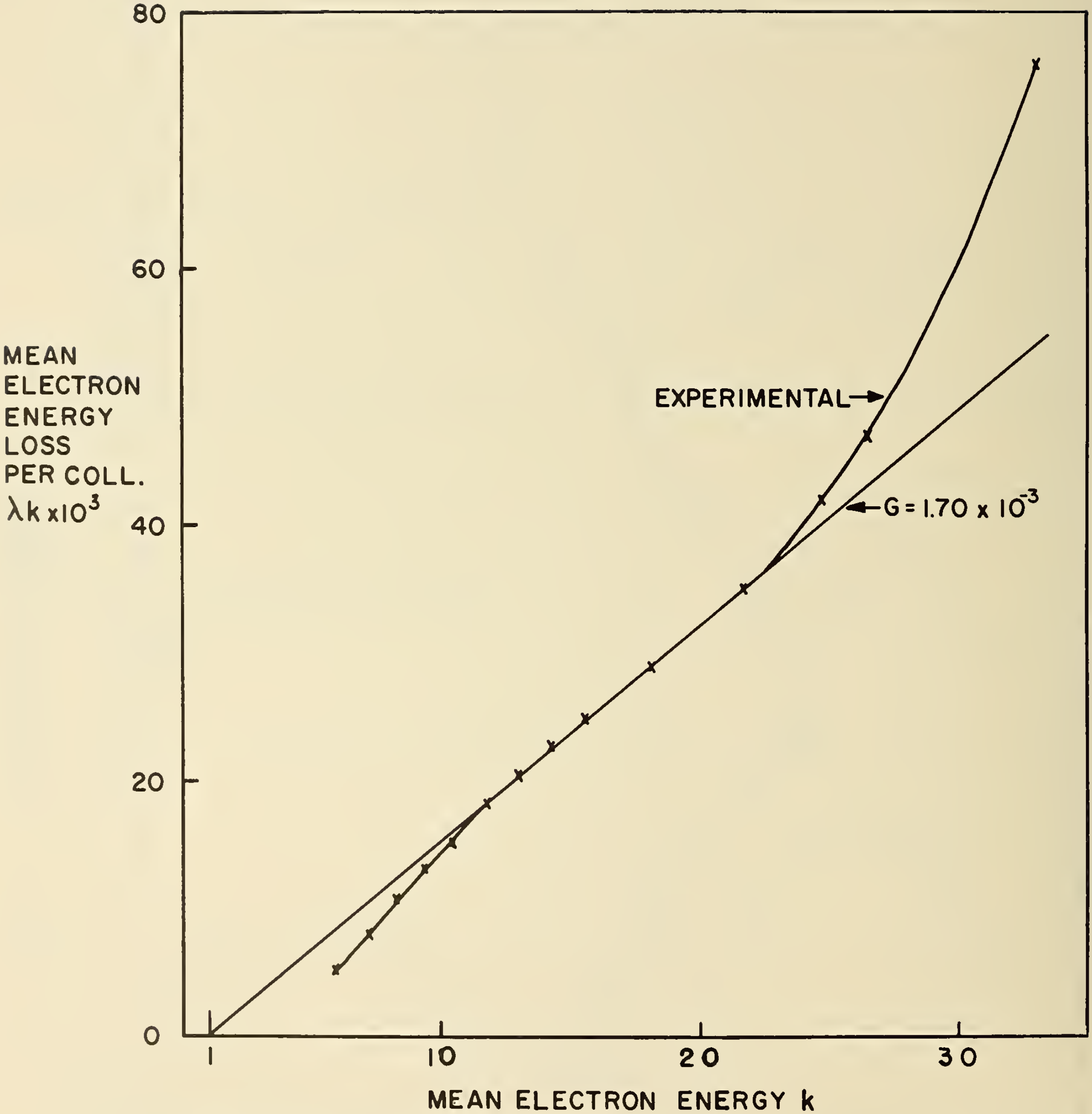

Figure 6 Experimental values for air of the mean energy lost by an electron per collision, $\lambda\kappa$, as a function of the mean electron energy, κ: after Crompton et al (1953). The energies in terms of the mean molecular energy as unit. The straight line corresponds to the hypothesis that the electron energy loss is proportional to the difference between the electron energy and the molecular energy.

NONLINEAR INTERACTION OF ELECTROMAGNETIC WAVES WITH MAGNETOACTIVE PLASMAS II*

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In this paper an expression has been derived for the nonlinear conductivity tensor of a Lorentzian magnetoactive plasma which is correct to quadratic terms of the amplitude of the elective vector. This expression has been substituted into the wave equation and the resulting nonlinear differential equation has been solved to the second order of approximation. Using this technique the propagation of electromagnetic waves in an infinite nonlinear magnetoplasma and the reflection and refraction at the interface of a magnetoactive plasma and a linear isotropic medium have been investigated.

Boltzmann's transfer equation for electrons in a uniform plasma may be written as

$$\frac{\partial f}{\partial t} + \alpha_x' \frac{\partial f}{\partial v_x} + \alpha_y' \frac{\partial f}{\partial v_y} + \alpha_z' \frac{\partial f}{\partial v_z} = \left(\frac{\partial f}{\partial t}\right)_c \tag{1}$$

where the symbols have usual meanings.

The acceleration α' of electrons in the presence of an electric field $\underset{\sim}{E} = \underset{\sim}{E}_0 \exp(i\omega_0 t)$ and magnetic field $\underset{\sim}{B}$ is given by

$$-\underset{\sim}{\alpha}' = \underset{\sim}{a}' \exp(i\,\omega_0 t) + \underset{\sim}{v} \times \underset{\sim}{\omega} \tag{2}$$

where

$$\underset{\sim}{\omega} = \frac{e\underset{\sim}{B}}{mc_0},$$

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$$\underset{\sim}{a}' = \frac{e\underset{\sim}{E}_0}{m},$$

e is the electronic charge,

m is the electronic mass,

and c_0 is the velocity of light in vacuum.

For a Lorentzian plasma it may be shown that [Chapman and Cowling, 1939]

$$\left(\frac{\partial f}{\partial t}\right)_c = -\nu(f - f_0) + \frac{m}{M\upsilon^2}\frac{\partial}{\partial \upsilon}(f_0\upsilon^3\nu) + \frac{k_0 T}{M\upsilon^2}\left(\nu\upsilon^2\frac{\partial f_0}{\partial \upsilon}\right). \quad (3)$$

The symbols have usual meanings.

Proceeding in a manner similar to that given by Sodha [1960],

$$\begin{aligned}
\frac{J_x}{E_{00}} &= (\sigma_{xx}\mathcal{E}_x + \sigma_{xy}\mathcal{E}_y + \sigma_{xz}\mathcal{E}_z)\exp(i\,\omega_0 t),\\
\frac{J_y}{E_{00}} &= (\sigma_{yx}\mathcal{E}_x + \sigma_{yy}\mathcal{E}_y + \sigma_{yz}\mathcal{E}_z)\exp(i\,\omega_0 t),\\
\frac{J_z}{E_{00}} &= (\sigma_{zx}\mathcal{E}_x + \sigma_{zy}\mathcal{E}_y + \sigma_{zz}\mathcal{E}_z)\exp(i\,\omega_0 t),
\end{aligned} \quad (4)$$

where $\underset{\sim}{J}$ is the current density,

E_{00} is an arbitrary normalizing field,

$$\underset{\sim}{\mathcal{E}} = \frac{\underset{\sim}{E}_0}{E_{00}}.$$

The components of the nonlinear conductivity tensor σ may be expressed as

$$\sigma_{\rho\delta} = \sigma_{\rho\delta_0} + \alpha_0 \left\{ \sigma_{\rho\delta_1} (\underset{\sim}{\mathcal{E}} \cdot \tilde{\underset{\sim}{\mathcal{E}}}) + \sigma_{\rho\delta_2} \Phi(\mathcal{E}, \omega) \right\}, \tag{5}$$

where $\alpha = \dfrac{e^2 E_{00}^2}{6m^2 k_0 T}$,

$$\rho, \delta = x, y, z,$$

$$\Phi(\mathcal{E}, \omega) = (\underset{\sim}{\mathcal{E}} \cdot \underset{\sim}{\omega}) \cdot (\tilde{\underset{\sim}{\mathcal{E}}} \cdot \underset{\sim}{\omega}),$$

and $\sigma_{\rho\delta_0}$, $\sigma_{\rho\delta_1}$, and $\sigma_{\rho\delta_2}$ can be expressed in terms of known integrals involving collision frequency.

Substituting $\underset{\sim}{E} = \underset{\sim}{E}_0 \exp(i\,\omega_0 t)$ in the wave equation, and expressing the wave equation in dimensionless form, one obtains, for propagation along the Z axis,

$$\frac{d^2 \mathcal{E}_x}{d\xi^2} = \frac{\epsilon\mu}{\epsilon_0 \mu_0} \left\{ -\mathcal{E}_x + \frac{4\pi i}{\epsilon \omega_0} (\sigma_{xx} \mathcal{E}_x + \sigma_{xy} \mathcal{E}_y + \sigma_{xz} \mathcal{E}_z) \right\}, \tag{6A}$$

$$\frac{d^2 \mathcal{E}_y}{d\xi^2} = \frac{\epsilon\mu}{\epsilon_0 \mu_0} \left\{ -\mathcal{E}_y + \frac{4\pi i}{\epsilon \omega_0} (\sigma_{yx} \mathcal{E}_x + \sigma_{yy} \mathcal{E}_y + \sigma_{yz} \mathcal{E}_z) \right\}, \tag{6B}$$

and
$$\frac{d^2 \mathcal{E}_z}{d\xi^2} = -\beta_z^2 \mathcal{E}_z, \tag{6C}$$

where $\xi = \dfrac{(\epsilon_0 \mu_0)^{\frac{1}{2}} \omega_0 z}{c_0}$

and $$\beta_z^2 = \frac{\epsilon\mu}{\epsilon_0\mu_0}\left\{1 - \frac{4\pi i}{\epsilon\omega_0}\sigma_{zz} - \frac{4\pi i}{\epsilon\omega_0}\frac{\left(\sigma_{zx}\mathcal{E}_y + \sigma_{zy}\mathcal{E}_y\right)}{\mathcal{E}_z}\right\}. \tag{6D}$$

The electric vector may be expanded as

$$\underset{\sim}{\mathcal{E}} = \underset{\sim}{\mathcal{E}}' + \alpha_0\underset{\sim}{\mathcal{E}}''. \tag{7}$$

Combining (6A), (6B), (6D), and (7), putting $\beta_z^2 = 0$ (following the usual procedure [Ginzburg, 1961]) and equating like powers of α_0, we see that the propagation of the two modes $\mathcal{E}_x + \alpha_1\mathcal{E}_y$ and $\mathcal{E}_x + \alpha_2\mathcal{E}_y$ is described by

$$\frac{d^2}{d\xi^2}(\mathcal{E}_x' + \alpha_1\mathcal{E}_y') + \beta_1^2(\mathcal{E}_x' + \alpha_1\mathcal{E}_y') = 0, \tag{8A}$$

$$\frac{d^2}{d\xi^2}(\mathcal{E}_x' + \alpha_2\mathcal{E}_y') + \beta_2^2(\mathcal{E}_x' + \alpha_2\mathcal{E}_y') = 0, \tag{8B}$$

$$\frac{d^2}{d\xi^2}(\mathcal{E}_x'' + \alpha_1\mathcal{E}_y'') + \beta_1^2(\mathcal{E}_x'' + \alpha_1\mathcal{E}_y'') \tag{9A}$$

$$= (a_1\mathcal{E}_x' + a_1\mathcal{E}_y')\,\underset{\sim}{\mathcal{E}}\cdot\underset{\sim}{\tilde{\mathcal{E}}} + (b_1\mathcal{E}_x' + b_1'\mathcal{E}_y')\,\Phi(\mathcal{E}, \omega),$$

and $$\frac{d^2}{d\xi^2}(\mathcal{E}_x'' + \alpha_2\mathcal{E}_y'') + \beta_2^2(\mathcal{E}_x'' + \alpha_2\mathcal{E}_y'') \tag{9B}$$

$$= (a_2\mathcal{E}_x' + a_2'\mathcal{E}_y')\,\underset{\sim}{\mathcal{E}}\cdot\underset{\sim}{\tilde{\mathcal{E}}} + (b_2\mathcal{E}_x' + b_1'\mathcal{E}_y')\,\Phi(\mathcal{E}, \omega),$$

where α_1, α_2, β_1, β_2, a_1, a_2, a_1', a_2', b_1, b_2, b_1', and b_2' can be expressed in terms of $\sigma_{\rho\delta_0}$, $\sigma_{\rho\delta_1}$, and $\sigma_{\rho\delta_2}$.

The expansion is valid when $\psi^2 << 1$. Significant nonlinear effects will occur when ψ is not much less than unity. Hence, the theory is both useful and valid only in the domain when ψ is not much less than unity, but ψ^2 is much less than unity. Even in the case when ψ^2 is not much less than unity, the theory should give qualitatively correct results.

For a given value of the amplitude of the electric vector $\underset{\sim}{E}_0$, ψ is minimum when the magnetic field is along the direction of propagation, because in this case $\underset{\sim}{E}_0 \cdot \underset{\sim}{\omega} = 0$. Hence, for a given combination of the amplitude vector $\underset{\sim}{E}_0$ and the magnetic field $\underset{\sim}{B}$ (or $\underset{\sim}{\omega}$), the nonlinear effects are least pronounced when the direction of propagation coincides with the direction of the magnetic field.

When the direction of the electric vector coincides with that of the magnetic field, $(\underset{\sim}{E}_0 \cdot \underset{\sim}{\omega})$ and hence ψ has a maximum value for a given combination of the amplitude of the electric vector and the magnetic field. Thus, in this case the nonlinear effects are most pronounced. It may be added that in this case the propagation is in a plane polarized mode in contrast to two elliptically polarized mode propagation in general

It may also be seen from the present analysis that the two modes of propagation of electromagnetic waves in a magnetoplasma do not propagate independently of each other; the complex amplitude (and hence

The solutions of (8A), (8B), (9A), and (9B) may be obtained by using the radiation condition and the boundary conditions

$$\mathcal{E}'_x = \mathcal{E}_{x0}, \quad \mathcal{E}'_y = \mathcal{E}_{y0} \qquad \text{at } \xi = 0,$$

$$\mathcal{E}''_x = \mathcal{E}''_y = 0 \qquad \text{at } \xi = 0,$$

and hence
$$\mathcal{E}'_x + \alpha_1 \mathcal{E}'_y = A_1 \exp(i\beta_1 \xi),$$

$$\mathcal{E}'_x + \alpha_2 \mathcal{E}'_y = A_2 \exp(i\beta_2 \xi).$$

The problem of reflection and refraction of an electromagnetic wave at normal incidence on the interface of a linear isotropic medium and nonlinear anisotropic medium has also been solved, and explicit expressions for the reflected and refracted components have been obtained.

The second order nonlinear theory presented in this paper is valid when the expansion

$$(1 + \psi)^{-1} \approx 1 - \psi$$

is valid, where ψ is given by

$$\psi = \frac{e^2 M \Big/ 6 m_0^2 k_0 T}{[\nu^2 + (\omega + \omega_0)^2][\nu^2 + (\omega - \omega_0)^2]}$$

$$\left\{ \underset{\sim}{E}_0 \cdot \underset{\sim}{E}_0 (\nu^2 + \omega_0^2 + \omega^2) + (\underset{\sim}{E}_0 \cdot \underset{\sim}{\omega})(\tilde{\underset{\sim}{E}}_0 \cdot \underset{\sim}{\omega}) \frac{(\nu^2 + \omega^2 - 3\omega_0^2)}{(\nu^2 + \omega_0^2)} \right\}.$$

the intensity and phase) of a given mode does depend on the complex amplitude of the other mode. This nonlinear interaction of the two modes will give rise to cross-modulation, if both of the modes are amplitude modulated in a different manner.

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Collision Effects in Hydromagneto-ionic Theory

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A hydromagneto-ionic theory has been developed within the framework of the Burgers formalism [1958] which is a microscopic theory based on moments of the Boltzmann transport equation. The effects on electromagnetic wave propagation of electron-electron and electron proton collisions have been considered to the order of Chapman and Cowling's second approximation. The present theory is an extension of the magneto-ionic theory derived earlier by one of the authors [Wyller, 1961] for a fully ionized hydrogen plasma, in that it includes the effects of ion motions. The theory is therefore applicable to the low frequency modes of wave propagation, such as the whistler, Alfven and the retarded magneto-acoustic modes. Expressions have been derived for the refractive index, absorptivity, the wave polarization, and the zeros and infinities of the refractive index. Numerical applications have been given for the four characteristic modes of low frequency wave propagation, viz., the whistler mode, lower hybrid frequency, ion gyroresonance, and the hydromagnetic mode. Applications of the theory to the solar corona, and future extensions to the terrestrial ionosphere have been indicated.

I. Introduction

The present paper considers the effect of the ion motions on electromagnetic wave propagation in a uniform plasma with a superposed homogeneous magnetic field. The conventional magneto-ionic theory considers the motions of the electrons only, which is a valid approximation for the high wave frequencies used in ionospheric propagation or for a weakly ionized gas. When the wave frequency goes below the electron gyrofrequency, the extraordinary electromagnetic wave becomes the oblique Alfvén (whistler) mode, and for a further decrease below the ion gyrofrequency the ordinary electromagnetic wave becomes the retarded magneto-acoustic mode. Delcroix and Denisse have given an excellent illustration of this in their book [1961]. We note that if we take account of the pressure tensor, two additional modes will appear, viz., the electron and the ion plasma waves. We have not considered these modes in our paper.

Our work differs from most treatments [Schlüter, 1950, 1951; Dungey 1951; Åström, 1951; Hines, 1953; Piddington, 1956; Gershman, Ginzburg, and Denisov 1957; Lehnert, 1959; Fejer, 1960; Delcroix and Denisse, 1961; Allis, 1963] in that it takes account of collisional damping by a microscopic kinetic theory with a proper averaging over the velocity distribution function, of like particle interactions (proton-proton and electron-electron collisions), and of the heat-flow equations.

We have seen that a microscopic treatment through the Boltzmann equation led to significant departures in the propagation factors for electromagnetic waves in the terrestrial ionosphere [Sen and Wyller, 1950; Leinbach, 1962]. Later on, one of us [Wyller, 1961] applied the Burgers formalism [1958] to find the electrical conductivity for a completely ionized hydrogen plasma. The Burgers formalism proceeds via moments of the Boltzmann transport equation and the associated collision integral. It includes the effects of like particle interactions and of the electron and ion heat flows, and is equivalent to the Chapman and Cowling second approximation [1958]. The present work is an extension of Wyller's [1961] work to include the effects of ion motions.

In a recent work, Kantor [1963] has considered the effects of ion motions on magneto-ionic theory without collisions and considered the properties of different directions of wave propagation with respect to the magnetic field. Our work checks with Kantor's formulae for the case of no collisions.

II. Basic Equations

It is well known that the presence of magnetic field in a plasma introduces an anisotropy in the propagation of electromagnetic waves through the medium. This anisotropy is characterized by the tensorial representation of the associated physical quantities governing the modes of propagation; namely, the electric conductivity and dielectric constant of the medium. After introducing the dielectric tensor (which is easily derived from the conductivity tensor) into the Maxwell equations, one can derive expressions for the refraction, absorption, and polarization of the propagating waves. With this end in view we shall set up in this section the fundamental equations that will enable us to compute the a. c. conductivity tensor of the magneto-ionic medium which may transmit electromagnetic waves associated with the oscillations of the ions as well as electrons. In doing that we shall adopt, as we have already said, Burgers' formalism for d. c. conductivities in the a. c. case.

Choosing a rectangular coordinate system (o, x, y, z) such that the z-axis lies along the uniform applied magnetic field $\vec{H}_o$, the electric conductivity tensor can be put into a very simple form. The electric field due to the propagating wave is represented by an a. c. field in that medium,

$\vec{E} \cos \omega t = (E_x \vec{i} + E_y \vec{j} + E_z \vec{k}) \cos \omega t$, where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors along the x, y and z axes and ω is the excitation frequency of the wave which is assumed to remain constant during the propagation of the wave inside the medium.

For a simple model we shall assume the plasma medium to be homogeneous with no pressure (and density) gradients. We shall consider a fully ionized hydrogen plasma containing electrons and protons in equal number densities, $n_1 = n_2$, throughout the medium. The subscripts 1 and 2 will be used consistently to denote protons and electrons respectively in the following discussion. Furthermore, under these conditions it will be appropriate to assume that the mean velocity of mass flow of the plasma as a whole is zero. In terms of Burgers' notation this would mean that the components of diffusion velocities of the two types of particles reduce to their respective mean velocities, i.e.,

$$W_{1h} = \xi_{1h} \quad ; \qquad W_{2h} = \xi_{2h} \quad ; \qquad h = 1, 2, 3 \; . \tag{1}$$

The assumptions made above bring about a considerable simplification in the Burgers formalism. There is a simple relation between Burgers' "friction coefficient" $K_{12} = K_{21} = K$ and the mean electron-proton collision frequency $\bar{\nu}$, namely,

$$K = n_2 m_2 \bar{\nu} \; , \qquad \bar{\nu} = \frac{\int_0^\infty \nu e^{-\mathcal{L} v_2^2} v_2^4 \, dv_2}{\int_0^\infty e^{-\mathcal{L} v_2^2} v_2^4 \, dv_2} \; , \qquad \mathcal{L} = \frac{m_2}{2kT} \; , \tag{2}$$

$$\bar{\nu} = \frac{2}{3\sqrt{\pi}} \frac{\omega_0^2 e_2^2}{kT} \left(\frac{m_2}{2kT} \right)^{1/2} \ln \Lambda \; , \qquad \Lambda = \frac{3kT}{e_2^2} r_D \; .$$

Here r_D is the Debye length and ω_0 is the electron plasma frequency.

In order to assess the full implications of the momentum and heat transfer equations of the electrons and protons, we state here some of the physical parameters of Burgers' theory that are relevant to the conditions of our problem.

For a binary gas with known relative velocities of the colliding particles "g" the collision cross section is given by :

$$S_{st}^{(\ell)} = S_{ts}^{(\ell)} = 2\pi\int_0^{r_D}(1-\cos^\ell \chi)\, b\, db\,, \qquad (3)$$

where b and χ are Chapman and Cowling's [1958] collision parameters. In particular, $\ell = 1$ for momentum transfer with which we are primarily concerned here. The average collision cross section for all possible relative velocities is

$$Z_{st}^{(\ell j)} = Z_{ts}^{(\ell j)} = \frac{4}{\sqrt{\pi}}\left(\frac{\mu}{2kT}\right)^{j+2}\int_0^\infty g^{2j+3} e^{-\mu g^2/2kT} S_{st}^{(\ell)} dg. \qquad (4)$$

Here the subscripts " s " and " t " refer to the species of the colliding particles and μ is the reduced mass. The superscript "ℓ" refers to the angular dependence, while " j " refers to the velocity dependence of the collision cross section. In terms of these average collision cross sections, the quantity defined by

$$z_{st} = 1 - \frac{2}{5}\,\frac{Z_{st}^{(12)}}{Z_{st}^{(11)}} \qquad (5)$$

reduces to the value 3/5 for Coulomb interactions.

In order to be able to estimate the effects of collisions between like and unlike particles separately, Burgers [Part II pp. 65 and 93] introduces the following collision factors which turn out to be proportional to the average collision frequencies:

$$\mathcal{X} = \frac{4}{5}\frac{m_1 m_2}{(m_1+m_2)^2}\frac{Z_{12}^{(22)}}{Z_{12}^{(11)}} \quad \text{(electron-proton collisions);}$$

$$\mathcal{X}_1 = \frac{2}{5}\left(\frac{m_1+m_2}{2m_2}\right)^{1/2}\frac{Z_{11}^{(22)}}{Z_{12}^{(11)}} \quad \text{(proton-proton collisions),} \quad (6)$$

$$\mathcal{X}_2 = \frac{2}{5}\left(\frac{m_1+m_2}{2m_1}\right)^{1/2}\frac{Z_{22}^{(22)}}{Z_{12}^{(11)}} \quad \text{(electron-electron collisions).}$$

It will be seen later that the auxiliary quantity ξ which incorporates the effects of electron-proton and electron-electron collisions enters into the electrical conductivity tensor. It is defined by

$$\xi = \mathcal{X} + \mathcal{X}_2 + \gamma Y_2 \,; \quad \gamma = \frac{m_2}{m_1} \quad \text{and} \quad (7)$$

Y_2 is a function of $\mathcal{X}$ only.

Though the proton-proton collisions do enter into the equations of the heat flow through the factor $\varkappa_1$, their net effects in the electrical conductivity tensor vanish [Spitzer, 1956; Marshall, 1957]. On the other hand, in thermal conductivity the dissipative effects due to ion-ion encounters is not negligible [Rosenbluth and Kaufman, 1958; Wyller, 1963].

Burgers' momentum and heat flow equations in their time-dependent forms will be our fundamental set of equations for the electrons and protons. We will rewrite Burgers' equations [Part 1, p. 20, eq. (37) and p. 42, eq. (83)].

Momentum equations for electrons:

$$
\begin{aligned}
i\omega W_{2x} - \Gamma_{2x} + s_2 W_{2y} &= \bar{\nu}(W_{1x} - W_{2x}) + z\bar{\nu}r_{2x} - z\gamma\bar{\nu}r_{1x}, \\
i\omega W_{2y} - \Gamma_{2y} - s_2 W_{2x} &= \bar{\nu}(W_{1y} - W_{2y}) + z\bar{\nu}r_{2y} - z\gamma\bar{\nu}r_{1y}, \\
i\omega W_{2z} - \Gamma_{2z} \phantom{{}- s_2 W_{2x}} &= \bar{\nu}(W_{1z} - W_{2z}) + z\bar{\nu}r_{2z} - z\gamma\bar{\nu}r_{1z}.
\end{aligned} \tag{8}
$$

Momentum equations for protons:

$$
\begin{aligned}
i\omega W_{1x} - \Gamma_{1x} - s_1 W_{1y} &= -\gamma\bar{\nu}(W_{1x} - W_{2x}) - z\gamma\bar{\nu}r_{2x} + z\gamma^2\bar{\nu}r_{1x}, \\
i\omega W_{1y} - \Gamma_{1y} + s_1 W_{1x} &= -\gamma\bar{\nu}(W_{1y} - W_{2y}) - z\gamma\bar{\nu}r_{2y} + z\gamma^2\bar{\nu}r_{1y}, \\
i\omega W_{1z} - \Gamma_{1z} \phantom{{}+ s_1 W_{1x}} &= -\gamma\bar{\nu}(W_{1z} - W_{2z}) - z\gamma\bar{\nu}r_{2z} + z\gamma^2\bar{\nu}r_{1z}.
\end{aligned} \tag{9}
$$

Heat flow equations for electrons:

$$
\begin{aligned}
i\omega r_{2x} + s_2 r_{2y} &= -\frac{5}{2} z\bar{\nu}(W_{1x}-W_{2x}) - \xi\bar{\nu} r_{2x} + \frac{27}{10}\gamma\bar{\nu} r_{1x}, \\
i\omega r_{2y} - s_2 r_{2x} &= -\frac{5}{2} z\bar{\nu}(W_{1y}-W_{2y}) - \xi\bar{\nu} r_{2y} + \frac{27}{10}\gamma\bar{\nu} r_{1y}, \\
i\omega r_{2z} &= -\frac{5}{2} z\bar{\nu}(W_{1z}-W_{2z}) - \xi\bar{\nu} r_{2z} + \frac{27}{10}\gamma\bar{\nu} r_{1z}.
\end{aligned}
\tag{10}
$$

Heat flow equations for protons:

$$
\begin{aligned}
i\omega r_{1x} - s_1 r_{1y} &= \frac{5}{2} z\gamma^2\bar{\nu}(W_{1x}-W_{2x}) - \varkappa_1\bar{\nu} r_{1x} + \frac{27}{10}\gamma^2\bar{\nu} r_{2x}, \\
i\omega r_{1y} + s_1 r_{1x} &= \frac{5}{2} z\gamma^2\bar{\nu}(W_{1y}-W_{2y}) - \varkappa_1\bar{\nu} r_{1y} + \frac{27}{10}\gamma^2\bar{\nu} r_{2y}, \\
i\omega r_{1z} &= \frac{5}{2} z\gamma^2\bar{\nu}(W_{1z}-W_{2z}) - \varkappa_1\bar{\nu} r_{1z} + \frac{27}{10}\gamma^2\bar{\nu} r_{2z}.
\end{aligned}
\tag{11}
$$

In the above equations the quantity $i\omega$ appears because of the time dependence of the particle velocity and heat flow velocity of the form $\frac{\partial}{\partial t}(\vec{W}e^{i\omega t})$ and $\frac{\partial}{\partial t}(\vec{r}e^{i\omega t})$ respectively induced by the alternating electric field $\vec{\mathcal{E}} = \vec{\mathcal{E}}_0(e^{i\omega t})$.

Also the accelerations due to the electric field are denoted by

$$\vec{\Gamma}_i = \frac{e_i}{m_i}\vec{\mathcal{E}}; \quad e_1 = +e; \quad e_2 = -e, \tag{12}$$

so that

$$\vec{\Gamma}_1 = -\gamma\vec{\Gamma}_2, \quad \gamma = \frac{m_2}{m_1}. \tag{13}$$

The gyrofrequencies are defined by

$$s_i = \left|\frac{e_i}{m_i} H_z\right| .$$

The above set of 12 equations in 12 unknowns shall now be solved to obtain the current flow components ($W_{1h} - W_{2h}$). Then the conductivity tensor will be found from the usual formula:

$$\vec{I} = -n_2 e_2 (\vec{W}_1 - \vec{W}_2) = \vec{\vec{\sigma}} \cdot \vec{E} . \tag{14}$$

III. Solution for Electrical Conductivity Along the Magnetic Field

The momentum and heat flow equations for both electrons and protons along the z axis are:

$$i\omega W_{2z} - \Gamma_{2z} = \bar{\nu}(W_{1z} - W_{2z}) + z\bar{\nu} r_{2z} - z\gamma\bar{\nu} r_{1z} , \tag{15a}$$

$$i\omega W_{1z} - \Gamma_{1z} = -\gamma\bar{\nu}(W_{1z} - W_{2z}) - z\gamma\bar{\nu} r_{2z} + z\gamma^2\bar{\nu} r_{1z} , \tag{15b}$$

$$i\omega r_{2z} = -\frac{5}{2} z\bar{\nu}(W_{1z} - W_{2z}) - \xi\bar{\nu} r_{2z} + \frac{27}{10}\gamma\bar{\nu} r_{1z} , \tag{16a}$$

$$i\omega r_{1z} = \frac{5}{2} z\gamma\bar{\nu}(W_{1z} - W_{2z}) - \chi_1\bar{\nu} r_{1z} + \frac{27}{10}\gamma\bar{\nu} r_{2z} . \tag{16b}$$

According to Burgers we neglect the terms $z\gamma\bar{\nu} r_{1z}$, $z\gamma^2\bar{\nu} r_{1z}$, and $\frac{27}{10}\gamma\bar{\nu} r_{1z}$ for $\gamma r_{1z} \ll r_{2z}$. This can be verified by solving for r_{1h} and r_{2h} from the basic equations of the last section.

From (16a) we have

$$r_{2z} = -\frac{5}{2} z\bar{\nu} \frac{(W_{1z} - W_{2z})}{(\xi\bar{\nu} + i\omega)} . \tag{17}$$

Upon inserting (17) into equations (15a, b) we find that

$$i\omega W_{2z} - \Gamma_{2z} = \bar{\nu}(W_{1z} - W_{2z}) - \frac{5}{2}\frac{z^2\bar{\nu}^2}{(\xi\bar{\nu} + i\omega)}(W_{1z} - W_{2z}) \tag{18a}$$

$$i\omega W_{1z} - \Gamma_{1z} = -\gamma\bar{\nu}(W_{1z} - W_{2z}) + \frac{5}{2}\frac{\gamma z^2\bar{\nu}^2}{(\xi\bar{\nu} + i\omega)}(W_{1z} - W_{2z}). \tag{18b}$$

Defining the current flow velocity as $\vec{W} = \vec{W_1} - \vec{W_2}$ we have

$$W_z = W_{1z} - W_{2z}. \tag{19}$$

Thus we get from equations (18a, b),

$$i\omega W_z + (1+\gamma)\Gamma_{2z} = -(1+\gamma)\bar{\nu}W_z + (1+\gamma)\frac{5}{2}\frac{z^2\bar{\nu}^2}{(\xi\bar{\nu} + i\omega)}W_z. \tag{20}$$

Now solving for W_z with separation of real and imaginary parts we obtain

$$W_z\left[(1+\gamma)\bar{\nu}\left(1 - \frac{5}{2}\frac{z^2\bar{\nu}^2\xi}{\xi^2\bar{\nu}^2 + \omega^2}\right) + i\omega\left(1 + (1+\gamma)\frac{5}{2}\frac{z^2\bar{\nu}^2}{\xi^2\bar{\nu}^2 + \omega^2}\right)\right] = -(1+\gamma)\Gamma_{2z}. \tag{21}$$

At this stage we introduce simplifying notations to condense the final expression for the conductivity tensor component along the z- direction and to elucidate therein the effects of electron-electron and electron-proton collisions;we set

$$\delta_3 = \frac{5}{2}\frac{z^2}{\xi}\frac{\xi^2\bar{\nu}^2}{\xi^2\bar{\nu}^2+\omega^2} \quad \text{and} \quad \nu_3 = \frac{(1-\delta_3)\bar{\nu}}{\left[1+(1+\gamma)\frac{\delta_3}{\xi}\right]} \tag{22}$$

The component of the current vector in the z-direction is then given by

$$I_z = -n_2 e_2 W_z = \frac{(1+\gamma)\sigma_0 \mathcal{E}_z}{\left[1+(1+\gamma)\frac{\delta_3}{\xi}\right]\left[(1+\gamma)\nu_3+i\omega\right]}, \quad \sigma_0 = \frac{\omega_0^2}{4\pi}. \tag{23}$$

and the conductivity tensor component in the z-direction is finally obtained

$$\sigma_{zz} = \frac{(1+\gamma)\sigma_0}{\left[1+(1+\gamma)\frac{\delta_3}{\xi}\right]\left[(1+\gamma)\nu_3+i\omega\right]}. \tag{24}$$

When we compare this tensor component with the corresponding Lorentz tensor component $\sigma_0/(\bar{\nu}+i\omega)$, we find that the ion-motions affect the conductivity along the z-direction through the factor $(1+\gamma)$ while the electron-proton and electron-electron collisions through δ_3 and ν_3. In particular, the expression (24) checks with Spitzer's d. c. conductivity [1962], when γ and ω are set equal to zero.

IV. Solutions for Conductivities Perpendicular to the Magnetic Field

The magnetic field couples the x and y components of the momentum and heat flow equations, making the solution for the flow components W_{ix} and W_{iy} somewhat awkward to obtain. We may avoid this difficulty by introducing a coordinate system defined by unit bi-vectors [Menzel,1961].

$$\vec{\alpha} = \frac{1}{\sqrt{2}}(\vec{i} + i\vec{j}) \;,\; \vec{\beta} = \frac{1}{\sqrt{2}}(\vec{i} - i\vec{j}) \;,\; \vec{\gamma} = \vec{k} . \tag{25}$$

Now we may express the electric field as, omitting the time dependence for simplicity,

$$\vec{\mathcal{E}} = \mathcal{E}_x \vec{i} + \mathcal{E}_y \vec{j} + \mathcal{E}_z \vec{k} = \frac{1}{\sqrt{2}}(\mathcal{E}_x - i\mathcal{E}_y)\vec{\alpha} + \frac{1}{\sqrt{2}}(\mathcal{E}_x + i\mathcal{E}_y)\vec{\beta} + \mathcal{E}_z \vec{k} . \tag{26}$$

Allis used a similar coordinate system in his formulation of the conductivity tensor for plasmas [Allis,1956]. $\mathcal{E}_l$ corresponds to the left rotating field component and $\mathcal{E}_r$ to the right rotating component.

We also write the flow components in this coordinate system,

$$W_{il} = \frac{1}{\sqrt{2}}(W_{ix} - i W_{iy}) \;,\; W_{ir} = \frac{1}{\sqrt{2}}(W_{ix} + i W_{iy}) \tag{27}$$

$$r_{il} = \frac{1}{\sqrt{2}}(r_{ix} - i\, r_{iy}) \;,\; r_{il} = \frac{1}{\sqrt{2}}(r_{ix} + i\, r_{iy}) .$$

We begin by solving for W_r from the rewritten momentum and heat flow equations:

$$i\omega W_{2r} - \Gamma_{2r} - i s_2 W_{2r} = \bar{\nu}(W_{1r} - W_{2r}) + z\bar{\nu} r_{2r} \tag{28a}$$

$$i\omega W_{1r} - \Gamma_{1r} + i s_1 W_{1r} = -\gamma\bar{\nu}(W_{1r} - W_{2r}) - z\gamma\bar{\nu} r_{2r} \tag{28b}$$

$$i\omega r_{2r} - i s_2 r_{2r} = -\frac{5}{2} z\bar{\nu}(W_{1r} - W_{2r}) - \xi\bar{\nu} r_{2r} \tag{29a}$$

$$i\omega r_{1r} + i s_1 r_{1r} = \frac{5}{2} z\gamma\bar{\nu}(W_{1r} - W_{2r}) - \chi_1\bar{\nu} r_{1r} + \frac{27}{10}\gamma\bar{\nu} r_{2r} . \tag{29b}$$

From equation (28a) we have

$$\bar{\nu}(W_{1r} - W_{2r}) = i\omega W_{2r} - \Gamma_{2r} - i s_2 W_{2r} - z\bar{\nu} r_{2r} \tag{30}$$

which yields r_{2r} when put into eq. (28a)

$$r_{2r} = \frac{5}{2} z \left[\frac{i(s_2 - \omega) W_{2r} + \Gamma_{2r}}{\bar{\nu}(\xi - \frac{5}{2} z^2) - i(s_2 - \omega)} \right] . \tag{31}$$

Putting expression (30) for $\bar{\nu}(W_{1r} - W_{2r})$ into equation (28b) we obtain

$$W_{1r} = \gamma \frac{(s_2 - \omega)}{(s_1 + \omega)} W_{2r} \tag{32}$$

since $\Gamma_{1r} = -\gamma\Gamma_{2r}$. The electric field and heat flow terms have dropped out. At this point we may return to equation (28a) with these expressions for W_{1r} and r_{2r} and solve uniquely for

$$W_{2r} = \frac{-[\xi\bar{\nu} - i(s_2 - \omega)]\Gamma_{2r}}{\left\{ (s_2 - \omega)^2 + [\bar{\nu}(\xi - \frac{5}{2} z^2) - i(s_2 - \omega)][\gamma\bar{\nu}\frac{(s_2 - \omega)}{(s_1 + \omega)} - \bar{\nu}] + i\xi\bar{\nu}(s_2 - \omega) \right\}} . \tag{33}$$

The total flow associated with the right rotating electric field is then the sum of the electron and proton flows

$$W_r = W_{1r} - W_{2r} = \left[\gamma \frac{(S_2-\omega)}{(S_1+\omega)} - 1\right] W_{2r} \, . \tag{34}$$

For subsequent algebraic simplification we introduce the symbols

$$\mathcal{L} = 1 + \gamma \frac{(\omega - S_2)}{(\omega + S_1)} \quad \text{and} \quad A = \left[\bar{\nu}\left(\xi - \frac{5}{2} z^2\right) - i(S_2 - \omega)\right]$$

which gives

$$W_r = \frac{(+\mathcal{L})\left[\xi\bar{\nu} - i(S_2-\omega)\right]\Gamma_{2r}}{\left[(S_2-\omega)^2 - \mathcal{L}\bar{\nu}A + i\xi\bar{\nu}(S_2-\omega)\right]} \tag{35}$$

Now, expression (35) is equivalent to

$$W_r = \frac{(+\mathcal{L})\Gamma_{2r}}{\dfrac{\left[(S_2-\omega)^2 - \mathcal{L}\bar{\nu}A + i\xi\bar{\nu}(S_2-\omega)\right]\left[\xi\bar{\nu} + i(S_2-\omega)\right]}{\left[\xi\bar{\nu} + i(S_2-\omega)\right]\left[\xi\bar{\nu} - i(S_2-\omega)\right]}} \, . \tag{36}$$

On separating the real and imaginary parts in the denominator we have

$$W_r = \frac{(+\mathcal{L})\Gamma_{2r}}{\left\{-\mathcal{L}\bar{\nu}\left[1 - \frac{5}{2}\frac{z^2\bar{\nu}^2}{\xi^2\bar{\nu}^2 + (\omega-S_2)^2}\right] + i(\omega - S_2)\left[1 + \mathcal{L}\frac{5}{2}\frac{z^2\bar{\nu}^2}{\xi^2\bar{\nu}^2 + (\omega-S_2)^2}\right]\right\}} \, , \tag{37}$$

Now we introduce a new symbol δ_1 and the expression for W_r becomes

$$W_r = \frac{(-\mathcal{L})\Gamma_{2r}}{\mathcal{L}\bar{\nu}(1-\delta_1) + i(\omega-S_2)\left(1 + \frac{\mathcal{L}\delta_1}{\xi}\right)} \, , \quad \delta_1 = \frac{5}{2}\frac{z^2}{\xi}\frac{\xi^2\bar{\nu}^2}{\xi^2\bar{\nu}^2 + (\omega-S_2)^2} \, . \tag{38}$$

We can repeat the foregoing analysis and obtain the flow for the case of the left rotating electric field

$$W_\ell = \frac{-\beta\ \Gamma_{2\ell}}{\beta\bar{\nu}(1-\delta_2)+i(\omega+S_2)(1+\beta\frac{\delta_2}{\xi})} \tag{39}$$

where, as before, we have simplified the expression with the symbols

$$\delta_2 = \frac{5}{2}\frac{z^2}{\xi}\frac{\xi^2\bar{\nu}^2}{\xi^2\bar{\nu}^2+(\omega+S_2)^2} \quad \text{and} \quad \beta = 1+\gamma\frac{(\omega+S_2)}{(\omega-S_1)}.$$

From the above, we obtain the total current

$$\vec{I} = -n_2 e_2 (W_\ell \vec{\mathcal{L}} + W_r \vec{\beta} + W_z \vec{\gamma}). \tag{40}$$

Rather than directly solve for W_x and W_y, we derive the electrical conductivity tensor in a form analogous to that of the Lorentz tensor form [Sen and Wyller 1960]. We first introduce the symbols

$$\gamma_\ell = \frac{W_\ell}{-\Gamma_{2\ell}}, \qquad \gamma_r = \frac{W_r}{-\Gamma_{2r}}, \qquad \gamma_z = \frac{W_z}{-\Gamma_{2z}} \tag{41}$$

which let us rewrite expression (40) as

$$\vec{I} = n_2 e_2 \left[\frac{1}{\sqrt{2}}\gamma_\ell\Gamma_{2\ell}(\vec{i}+i\vec{j}) + \frac{1}{\sqrt{2}}\gamma_r\Gamma_{2r}(\vec{i}-i\vec{j}) + \gamma_z\Gamma_{2z}\vec{K}\right] \tag{42}$$

and

$$\vec{I} = I_x\vec{i} + I_y\vec{j} + I_z\vec{k} = \frac{\omega_0^2}{8\pi}\Big[\gamma_\ell(\mathcal{E}_x - i\,\mathcal{E}_y)\vec{i} + \gamma_\ell(\mathcal{E}_y + i\,\mathcal{E}_x)\vec{j} + \gamma_r(\mathcal{E}_x + i\,\mathcal{E}_y)\vec{i} + \gamma_r(\mathcal{E}_y - i\,\mathcal{E}_x)\vec{j} + 2\gamma_z\mathcal{E}_z\vec{k}. \tag{43}$$

or, in matrix form

$$\begin{Bmatrix} I_x \\ I_y \\ I_z \end{Bmatrix} = \frac{\omega_0^2}{8\pi}\begin{Bmatrix} (\gamma_\ell+\gamma_r) & i(\gamma_r-\gamma_\ell) & 0 \\ -i(\gamma_r-\gamma_\ell) & (\gamma_\ell+\gamma_r) & 0 \\ 0 & 0 & 2\gamma_z \end{Bmatrix} \tag{44}$$

Then the complete electrical conductivity tensor in the hydromagneto-ionic theory becomes

$$\overrightarrow{\overrightarrow{\sigma}} = \frac{\omega_0^2}{8\pi}\begin{Bmatrix} (\gamma_\ell+\gamma_r) & i(\gamma_r-\gamma_\ell) & 0 \\ -i(\gamma_r-\gamma_\ell) & (\gamma_\ell+\gamma_r) & 0 \\ 0 & 0 & 2\gamma_z \end{Bmatrix} \tag{45}$$

where

$$\gamma_r = \frac{\mathcal{L}}{\left(1+\mathcal{L}\frac{\delta_1}{\xi}\right)\left[\mathcal{L}\nu_1 + i(\omega - S_2)\right]}, \qquad \mathcal{L} = 1 + \gamma\frac{(\omega - S_2)}{(\omega + S_1)}$$

$$\gamma_\ell = \frac{\beta}{\left(1+\beta\frac{\delta_2}{\xi}\right)\left[\beta\nu_2 + i(\omega + S_2)\right]}, \qquad \beta = 1 + \gamma\frac{(\omega + S_2)}{(\omega - S_1)} \tag{46}$$

$$\gamma_z = \frac{(1+\gamma)}{\left[1+(1+\gamma)\frac{\delta_3}{\xi}\right]\left[(1+\gamma)\nu_3 + i\omega\right]}, \qquad \gamma = \frac{m_2}{m_1}.$$

$$\nu_1 = \frac{(1-\delta_1)}{\left(1+\mathcal{L}\frac{\delta_1}{\xi}\right)}\bar{\nu}; \quad \nu_2 = \frac{(1-\delta_2)}{\left(1+\beta\frac{\delta_2}{\xi}\right)}\bar{\nu}, \quad \nu_3 = \frac{(1-\delta_3)}{\left[1+(1+\gamma)\frac{\delta_3}{\xi}\right]}\bar{\nu}.$$

Thus we have been able to express the electrical conductivity tensor containing ion-motion, electron-electron and electron-proton collisions into a form which is similar to that of the Lorentz tensor [Sen and Wyller, 1960 a].

When γ goes to zero, expression (45) reduces to a form of the electrical conductivity tensor [Wyller, 1961] valid for high frequencies (conventional magneto-ionic theory).

The dielectric tensor is derived from the relation

$$\overset{\rightrightarrows}{\epsilon} = \overset{\rightrightarrows}{1} + \frac{4\pi}{i\omega}\overset{\rightrightarrows}{\sigma} \qquad (47)$$

for which we find

$$\epsilon_{I} = (1-a_{11}) - i\, b_{11}\,, \quad \epsilon_{II} = \frac{1}{2}(f_{11}-d_{11}) + \frac{i}{2}(c_{11}-e_{11})$$
$$\epsilon_{III} = \left[a_{11} - \frac{1}{2}(c_{11}+e_{11})\right] + i\left[b_{11} - \frac{1}{2}(f_{11}+d_{11})\right] \qquad (48)$$

where

$$a_{11} = \frac{\omega_0^2(1+\gamma)}{\left[1+(1+\gamma)\frac{\delta_3}{\xi}\right]\left[(1+\gamma)^2\nu_3^2+\omega^2\right]}\,, \quad b_{11} = \frac{\omega_0^2(1+\gamma)^2\nu_3}{\omega\left[1+(1+\gamma)\frac{\delta_3}{\xi}\right]\left[(1+\gamma)^2\nu_3^2+\omega^2\right]}$$

$$c_{11} = \frac{\omega_0^2(\omega-s_1)\alpha}{\omega(1+\alpha\frac{\delta_1}{\xi})\left[\alpha^2\nu_1^2+(\omega-s_1)^2\right]}\,, \quad d_{11} = \frac{\omega_0^2\alpha^2\nu_1}{\omega(1+\alpha\frac{\delta_1}{\xi})\left[\alpha^2\nu_1^2+(\omega-s_1)^2\right]} \qquad (49)$$

$$e_{11} = \frac{\omega_0^2(\omega+s_2)\beta}{\omega(1+\beta\frac{\delta_2}{\xi})\left[\beta^2\nu_2^2+(\omega+s_2)^2\right]}\,, \quad f_{11} = \frac{\omega_0^2\beta^2\nu_2}{\omega(1+\beta\frac{\delta_2}{\xi})\left[\beta^2\nu_2^2+(\omega+s_2)^2\right]}$$

Retaining the Sen and Wyller formalism [1960] we find that the formulae for the refractive index $\frac{c^2}{u^2}$ polarization R are unaltered in form except that the constants A, B, C., etc. are now functions of ϵ_I, ϵ_{II}, and ϵ_{III} defined in the equations (48) and (49). This leads us to a new formulation of the hydromagneto-ionic theory which permits the study of low frequency modes such as the Alfvén mode and the retarded magneto-acoustical mode with electron-electron and electron-proton collisions included. For convenience we recapitulate [Sen and Wyller, 1960 b]:

$$\frac{c^2}{u^2} = \left(n - i\frac{c\chi}{\omega}\right)^2 = \frac{A + B\sin^2\phi \pm (B^2\sin^4\phi - C^2\cos^2\phi)^{1/2}}{D + E\sin^2\phi} \qquad (50)$$

$$R = -\frac{E_y}{E_z} = -\frac{B\sin^2\phi \mp (B^2\sin^4\phi - C^2\cos^2\phi)^{1/2}}{C\cos\phi}$$

$$A = 2\epsilon_I(\epsilon_I + \epsilon_{III}),\quad B = \epsilon_{III}(\epsilon_I + \epsilon_{III}),\quad C = 2\epsilon_I\epsilon_{II} \qquad (50a)$$

$$D = 2\epsilon_I,\quad E = 2\epsilon_{III}$$

Here n is the real refractive index, $\frac{c\chi}{\omega}$ is the absorptivity and ϕ is the angle between the direction of the magnetic field and the direction of propagation.

V. Zeroes and Infinities of the Refractive Index

In the formula (50) the complex refractive index $\bar{n} = \left(n - i\frac{c\chi}{\omega}\right)$ satisfies a biquadratic equation

$$(\bar{n})^4 + (\bar{n})^2\frac{(A + B\sin^2\phi)}{(D + E\sin^2\phi)} + \frac{A^2 + 2AB\sin^2\phi + C^2\cos^2\phi}{(D + E\sin^2\phi)^2}, \qquad (51)$$

Hence the condition that $\bar{n}$ may have a zero root is that

$$A^2 + 2AB\sin^2\phi + C^2\cos^2\phi = 0 ,$$

provided that $$D + E\sin^2\phi \neq 0$$

After some algebraic simplifications, the above two expressions reduce respectively to

$$\epsilon_{I}\left[(\epsilon_{I}+\epsilon_{III})^2 + \epsilon_{II}^2\right](\epsilon_{I} + \epsilon_{III}\sin^2\phi) = 0 \qquad (52)$$

$$\epsilon_{I} + \epsilon_{III}\sin^2\phi \neq 0 \qquad (53)$$

so by virtue of the restriction (53) all the zero roots of $\bar{n}$ will be given by the following three equations,

(a) $\epsilon_{I} = 0$ (b) $\epsilon_{I} + \epsilon_{III} - i\epsilon_{II} = 0$

(c) $\epsilon_{I} + \epsilon_{III} + i\epsilon_{II} = 0$.

These equations are independent of the propagation angle "ϕ". Assigning the appropriate values to ϵ_{I} from (48) and (49) we find that (a) will hold when $a_{11} = 1$ and $b_{11} = 0$ simultaneously. The latter condition means that either $\bar{\nu} = 0$ i. e. there is no collision or else that $\delta_3 = 1$, which gives an imaginary value to the collision parameter $Z = \bar{\nu}/\omega$. The first condition with $\bar{\nu} = 0$ yields a cutoff for n at

$$X = \frac{1}{1+\gamma} . \qquad (54)$$

We use the symbols X for ω_0^2/ω^2 and Y for s_2/ω .

However this root is attained only in the absence of collision; while we may say that collision removes the cutoff. Similar treatment to equations (b) and (c) yields cutoff values for n without collisions at

$$X = \frac{(1+Y)(1-\gamma Y)}{1+\gamma}$$

and (55)

$$X = \frac{(1-Y)(1+\gamma Y)}{1+\gamma}$$

respectively.

Analagous to the analysis of (a) these values are not attained actually when collision is present.

These cutoff values will be illustrated (with proper qualifications) by the curves for n against x on pages (end of the paper).

By inspection of the eq. (51) it is apparent that $(\bar{n})^2$ will have an infinite root when

$$D + E \sin^2\phi = 0$$

This is equivalent to the condition

$$\epsilon_{I} + \epsilon_{III} \sin^2\phi = 0 \tag{56}$$

For the case of collision the condition (56) turns out to be fairly complicated, but if we neglect collisions it yields a simple relation analogous to that of Hines [1957].

$$\left\{1 - \frac{\omega_0^2}{\omega^2}(1+\gamma)\right\} \cos^2\phi + \left\{1 + \frac{\omega_0^2}{S_2^2-\omega^2} + \frac{\gamma\omega_0^2}{S_1^2-\omega^2}\right\} \sin^2\phi = 0 \tag{57}$$

It may be noted that the condition (57) for infinite root of n is dependent on ϕ . So if $\phi = \frac{\pi}{2}$, n will have an infinite root at

$$X = \frac{(1-Y^2)(1-\gamma^2 Y^2)}{(1+\gamma)(1-\gamma Y^2)}. \tag{58}$$

This value is somewhat different from that obtained by Budden [1961] even if an allowance for his approximation is introduced in our computation. For low frequency end of the whistler spectrum we consider the case $S_2 \gg \omega \gg S_1$. In this limit the condition (56) simplifies down to

$$\left\{1 - \frac{\omega_0^2}{\omega^2}(1+\gamma)\right\}\cos^2\phi + \left\{1 + \frac{\omega_0^2}{S_2^2} - \frac{\gamma\omega_0^2}{\omega^2}\right\}\sin^2\phi = 0 \tag{59}$$

Now if the coefficients of $\cos^2\phi$ and $\sin^2\phi$ in the equation (59) are both negative then the condition for an infinity will not hold, meaning thereby that the wave can propagate in all directions. Analysis shows that the first coefficient will be negative if $\omega < \omega_0$; also the second coefficient will be negative only when $\omega < \omega_0\sqrt{\frac{S_1 S_2}{S_2^2 + \omega_0^2}}$. These bounds on ω will be helpful for the detection of ions with whistlers $(\omega \ll \omega_0)$ which are propagated to a large extent in the upper atmospheric regions where our present model of a fully ionized plasma will have a closer approximation [Hines, 1957].

VI. Numerical Applications

The real refractive index, n , and the absorption factor, $\frac{c\chi}{\omega}$, were evaluated from the relations

$$\frac{c^2}{u^2} = \left(n - i\frac{c\chi}{\omega}\right)^2 = L - iM ,$$

and thus rigorously

$$n = \frac{1}{\sqrt{2}}\sqrt{L+\sqrt{L^2+M^2}}$$

$$\frac{c\chi}{\omega} = \frac{1}{\sqrt{2}}\sqrt{-L+\sqrt{L^2+M^2}}$$

We note that our signs in equation (50) for the complex refractive index are opposite to those of the magneto-ionic theory. We have computed the propagation constants n and $c\chi/\omega$ as a function of X for the following representative values of Y :

(i) $Y = 100$ (whistler mode);

(ii) $Y = \sqrt{1/\gamma}$ (lower hybrid frequency);

(iii) $Y = \frac{1}{\gamma}$ (ion gyro-resonance);

and (iv) $Y = 10^4$ (hydromagnetic mode).

We will now briefly discuss the above cases.

Case (i) $Y = 100$ (whistler mode)

Longitudinal propagation ($\phi = 0$). (Figures 1 and 2)

The introduction of ion motion has little effect on the birefringence, which is only slightly decreased. The important parameter for collisional effects seems to be the ratio Z/Y. Even for small values of this ratio (0.1 in Fig. 2), the cutoff in the ordinary ray is removed. The absorption of the extraordinary ray is increased with increasing Z. The reverse is the case with the ordinary ray.

Transverse propagation $(\phi = \pi/2)$. Figure 3. O_3 is not drawn in the figure. It follows very closely the vertical axis, starting from the value $n = 1$ at $X = 0$, with a cutoff at $X = 0.995$. The effect of ion motion is negligible for the ordinary ray. It is quite marked for the extraordinary. The birefringence of the medium is appreciably decreased. The collisions remove the cutoff for both the ordinary and extraordinary rays. The inclusion of ion motion has no effect on the absorption of the ordinary ray. For the extraordinary ray, it at first increases and then decreases the absorption. The ion motion seems to have appreciable effect for transverse propagation. The effect may well be confined within a narrow cone.

Case (ii) $Y = \sqrt{1/\gamma}$ (lower hybrid frequency) Figures 4 - 7. The behavior of the propagation factors for the longitudinal case is as in the magneto-ionic theory. But for transverse propagation, collisions not only remove the cutoff but reverse the trend of the real refractive index curve (Figs. 6 and 7) for the lower sign (ordinary ray of the magneto-ionic theory). How the introduction of ion motion makes possible all directions of propagation for this case has been discussed in the text.

Case (iii) $Y = 1/\gamma$ (ion gyro-resonance)

Longitudinal propagation $(\phi = 0)$. We shall discuss in the text how for longitudinal propagation the ion-gyro-resonance introduces a singularity in the lower mode. Figure 8 shows that even a small amount of collision removes this singularity. The wave can propagate, but is nevertheless subject to high absorption. With increasing collision frequency, the absorption of the lower mode decreases, whereas that of the upper mode increases (Fig. 9).

Transverse propagation $(\phi = \pi/2)$. The singularity is removed in this case and the wave can propagate. The cutoff frequencies for the two modes for no collisions is discussed in the text. The propagation factors for a high collision case (Z = 1000) are shown in Fig. 10.

Case (iv) $Y = 10^4$ (hydromagnetic mode)

Longitudinal propagation $(\phi = 0)$. Figures 11 and 12. These are the retarded magneto-acoustic mode (lower sign) and the oblique Alfven mode [redacted] (upper sign) of Denisse and Delcroix [1961]. Note in Fig. 12 that the oblique Alfven wave (upper sign) does not suffer absorption. This property will have considerable application in the solar corona.

Transverse propagation ($\phi = \pi/2$). Figures 13 and 14. The lower mode suffers a cutoff at $X \approx 1$, as in the magneto-ionic theory, which is removed by collisions. Note that the non-absorptivity of the Alfven mode (upper sign) persists even in transverse propagation.

We shall now give a brief analysis of the ion gyroresonance referred to in Case (iii) above. From (50) we have for longitudinal propagation ($\phi = 0$): $(\bar{n})^2 = (A \pm iC)/D$. Taking the minus sign and $Z = 0$ (no collisions), we have $(\bar{n})^2 = n^2 = 1 - X[(1+Y)^{-1} + \gamma(1-Y_1)^{-1}]$, where $Y_1 = S_1/\omega$. There is therefore a singularity in n^2 at the ion gyroresonance ($Y_1 = 1$). It can be shown that a finite Z will remove the singularity. For transverse propagation ($\phi = \pi/2$), we have $(\bar{n})^2 = (A+B \pm B)/(D+E)$. Taking the + sign, we get after a little algebra (for $Z = 0$, i.e., no collisions):

$$n^2 = 1 - (1+\gamma)\frac{(1-Y_1)X - (1+\gamma)X^2}{(1-Y^2)(1-Y_1^2) - (1+\gamma)(1-Y_1)X}$$

When $Y_1 = 1$ (ion-gyroresonance), $n^2 \to 2 + \gamma X$. Thus $n^2 \to 2$, as $X \to 0$. It can be shown that with collisions $n^2 \to 1$ as $X \to 0$. (See Fig. 10).

VII. Conclusion

In the present paper we have given a microscopic hydro-magneto-ionic theory, to Chapman and Cowling's second approximation, of a completely ionized hydrogen plasma. As a special case for no collisions, i.e., $\vec{\nu}=0$, our formula (50) reduces to the expressions derived by Åström [1951] for the longitudinal and transverse components. Furthermore the most general dispersion formula derived rigorously from Delcroix and Denisse's general conditions (eq. (7), p. 24, loc. cit.) gives the same transverse and longitudinal modes of propagation as those obtained from our formula. As already stated in the introduction we have further agreements with the results obtained by Kantor (1963) in the limit of no collisions. All these provide an indirect check to the consistency of our theory and correctness of our derivations.

We note that the treatment can easily be extended to a neutral plasma of any degree of ionization. In its present formulation, it should have important applications to hydromagnetic wave dissipation in the solar corona. The treatment of collisions in our hydromagneto-ionic theory is broad enough to include any velocity dependence of the collision cross-sections. For applications to hydromagnetic wave dissipation in the terrestrial ionosphere, our treatment should be extended to a partially ionized gas. Burgers' formalism should be adequate for this extended treatment, as is evidenced by Pipkin's work [1961] on the d.c. conductivity of a partially ionized gas.

The numerical applications that have been given are merely illustrative and by no means exhaustive of the results that can be obtained from the theory.

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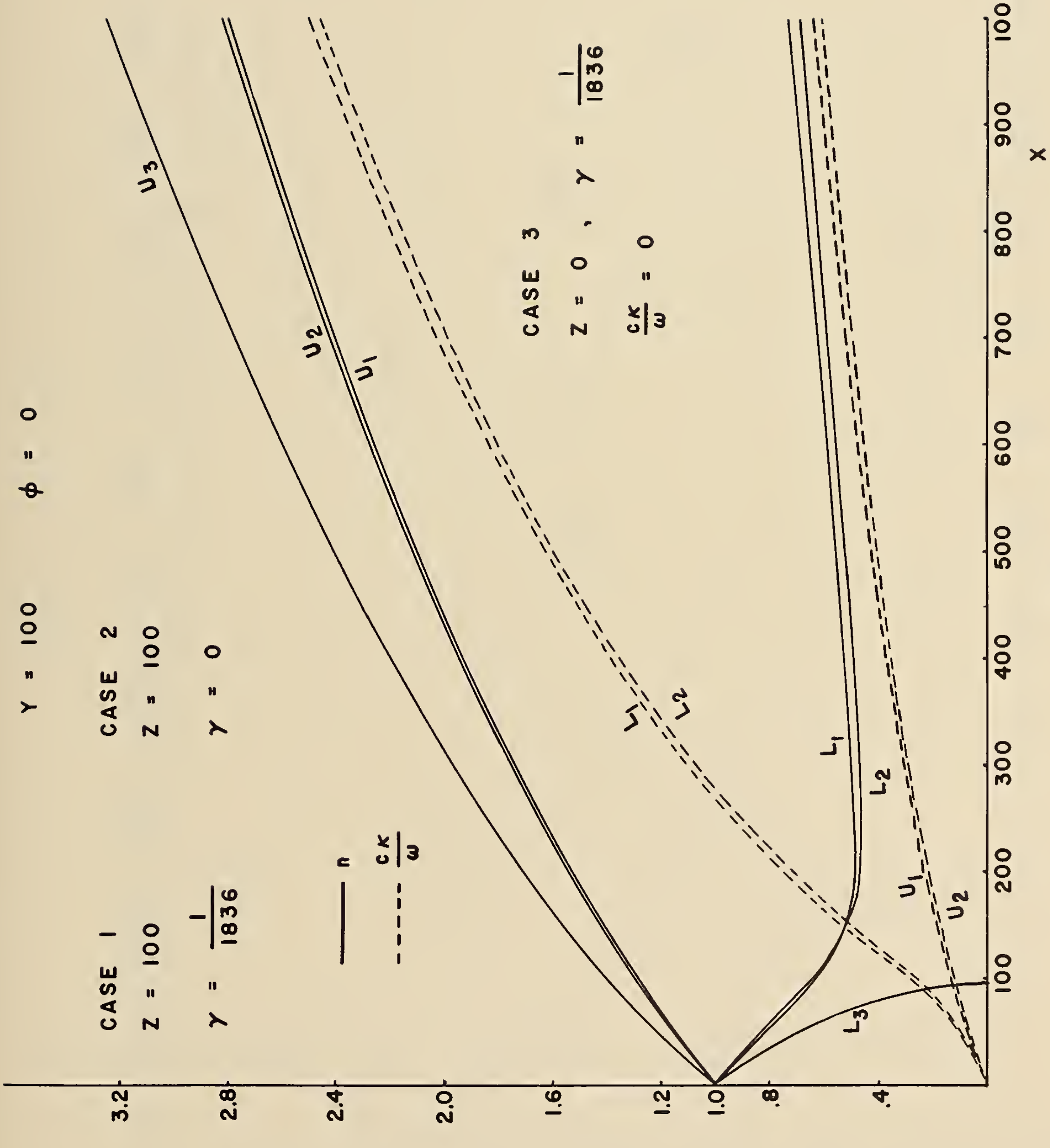
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φ = 0
CASE 1
Z = 100
γ = 1/1836
CASE 2
Z = 100
γ = 0
CASE 3
Z = 0 , γ = 1/1836
cK/ω = 0
n
cK/ω
U3
U2
U1
L1
L2
L3
3.2
2.8
2.4
2.0
1.6
1.2
1.0
.8
.4
100
200
300
400
500
600
700
800
900
1000
X

FIG. 1

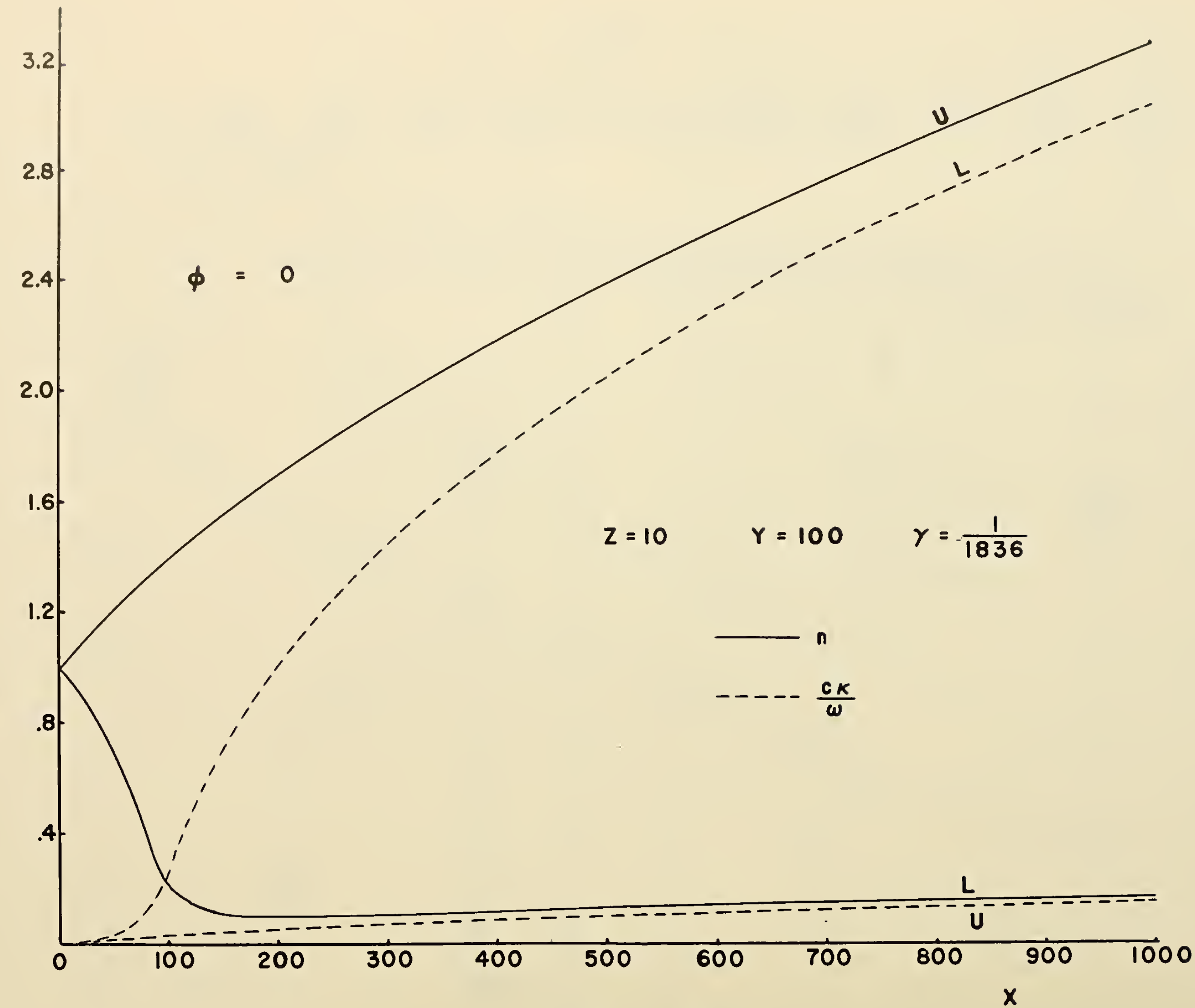
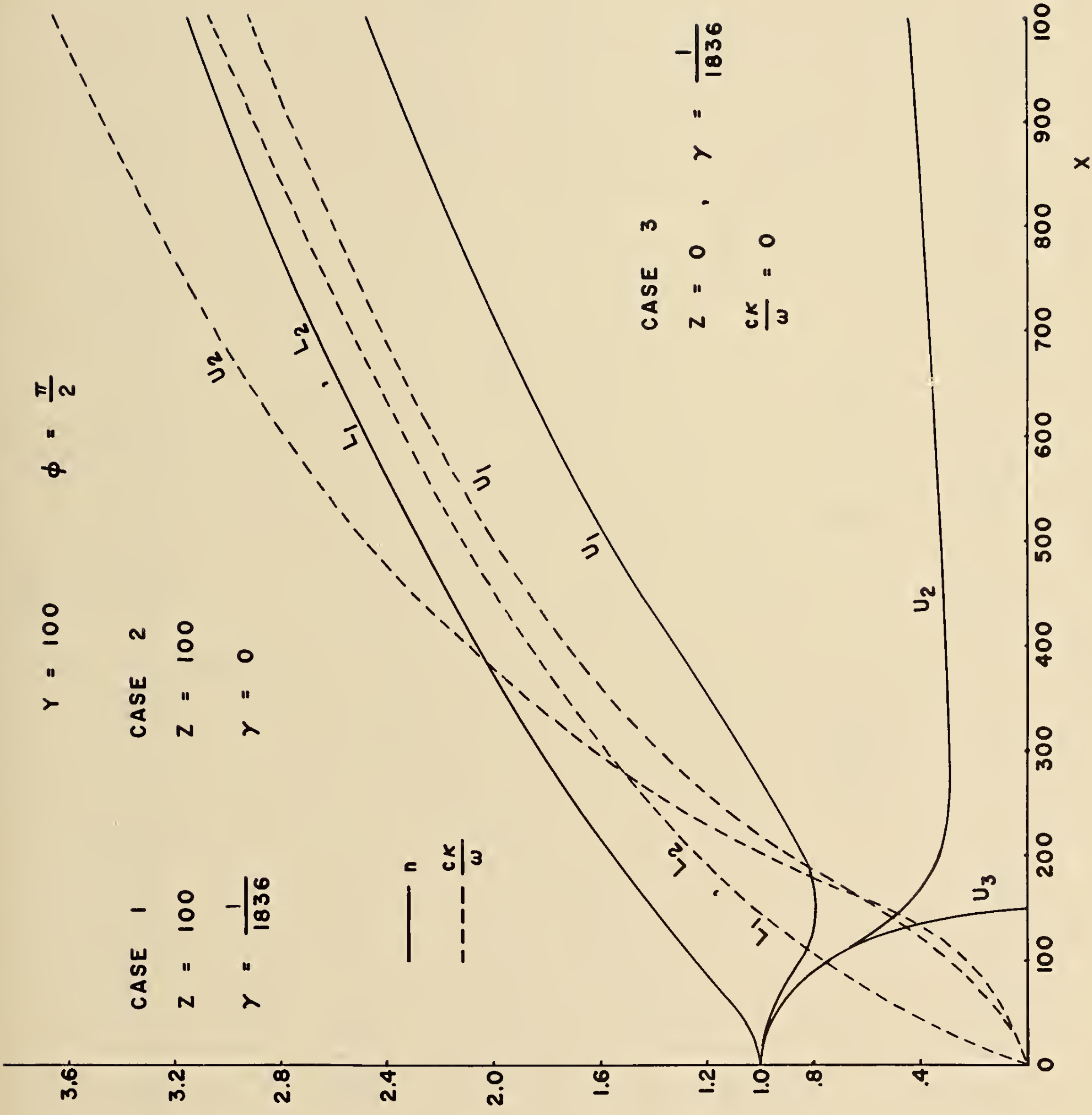

3.2
2.8
2.4
2.0
1.6
1.2
.8
.4
0
100
200
300
400
500
600
700
800
900
1000
X
φ = 0
Z = 10
Y = 100
γ = 1/1836
n
cκ/ω
U
L
L
U

FIG. 2



Y = 100
φ = π/2
CASE 1
Z = 100
γ = 1/1836
CASE 2
Z = 100
γ = 0
CASE 3
Z = 0 , γ = 1/1836
cκ/ω = 0
n
cκ/ω
U2
L2
L1
U1
U1
U2
U3
L2
L1
X
0
100
200
300
400
500
600
700
800
900
1000
.4
.8
1.0
1.2
1.6
2.0
2.4
2.8
3.2
3.6

FIG. 3

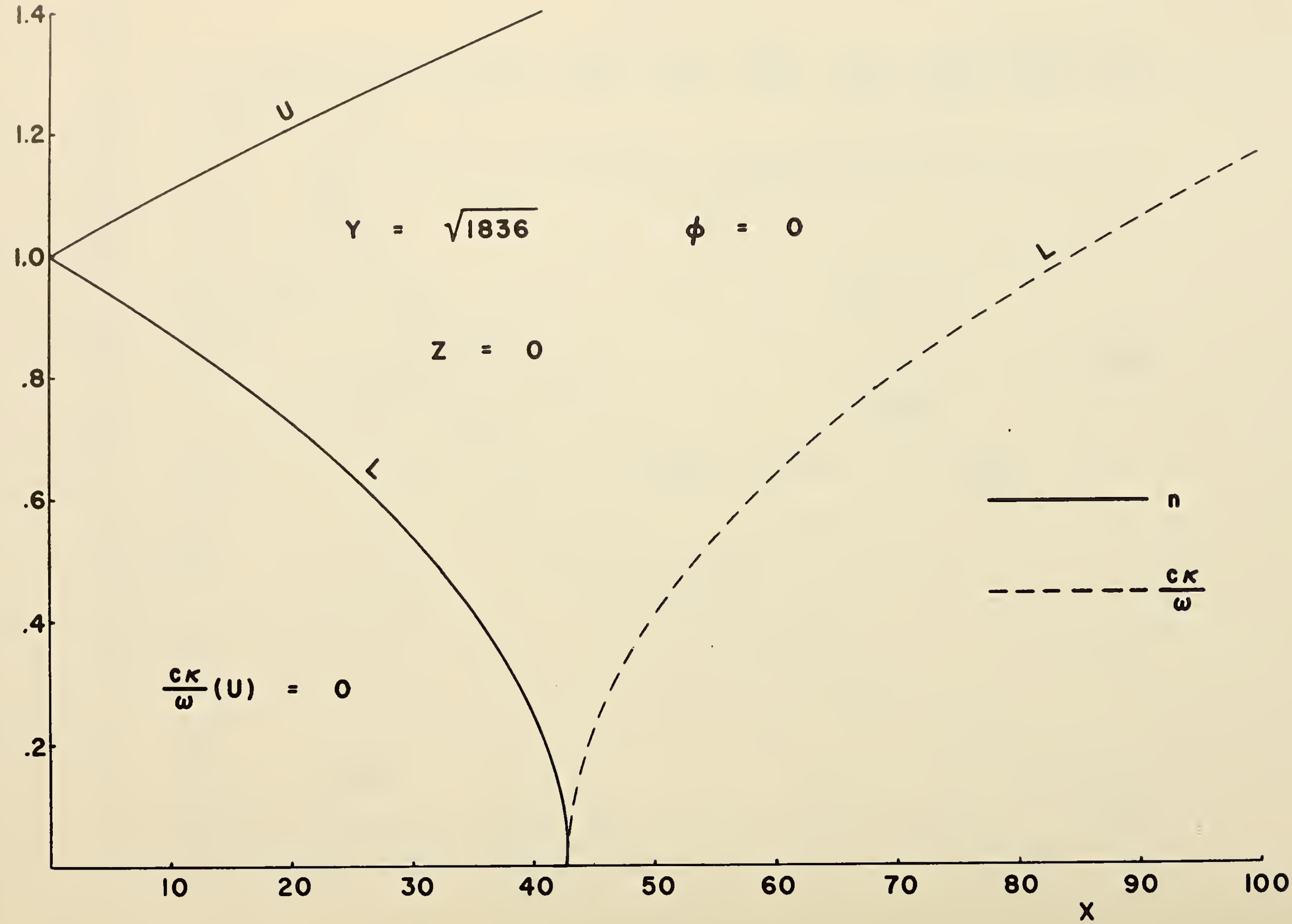
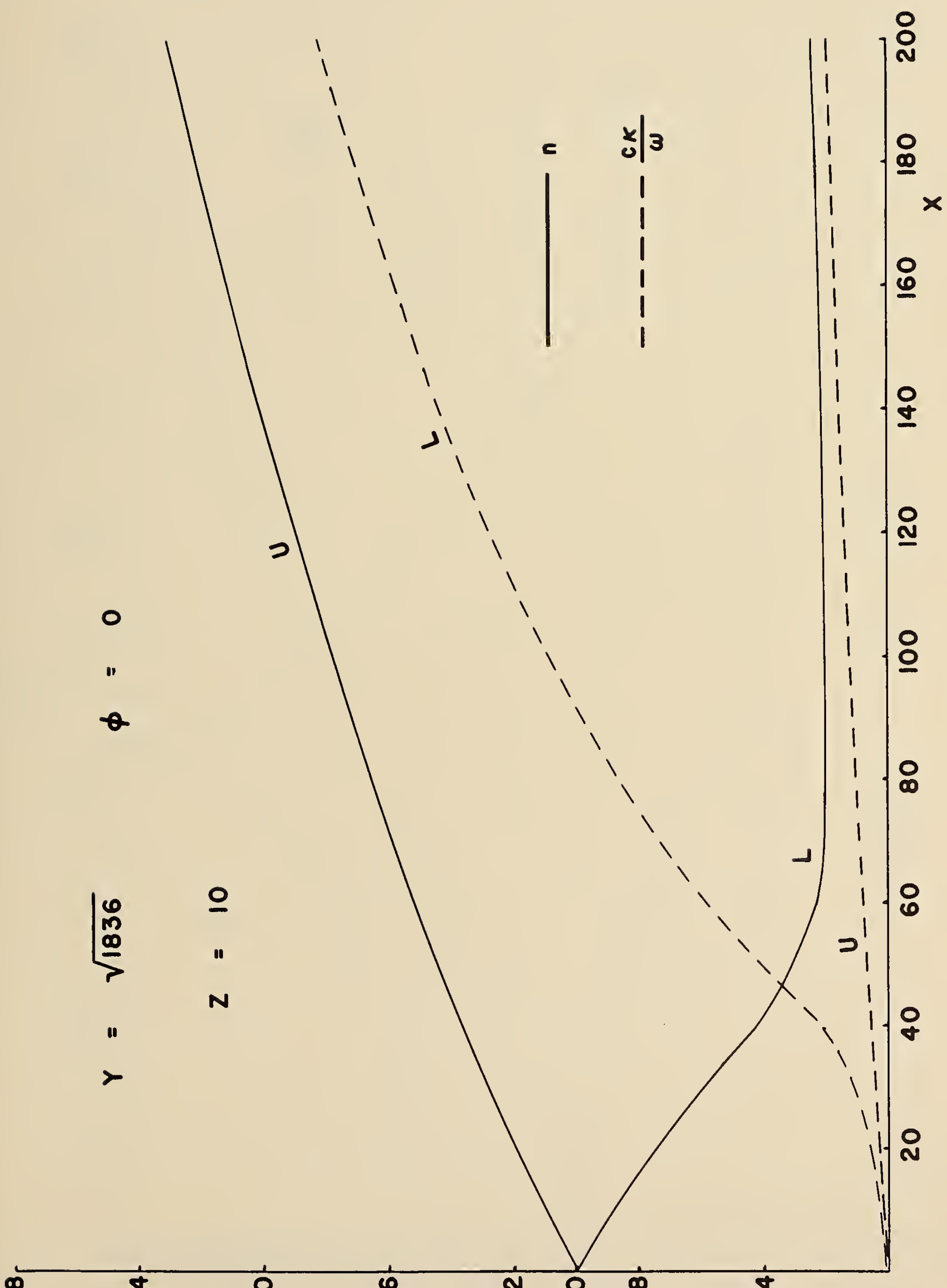

1.4
1.2
1.0
.8
.6
.4
.2
U
L
L
Y = √1836
φ = 0
Z = 0
cκ/ω (U) = 0
n
cκ/ω
10
20
30
40
50
60
70
80
90
100
X

FIG. 4



Y = $\sqrt{1836}$
Z = 10
ϕ = 0
U
L
U
L
n
$\frac{c\kappa}{\omega}$
2.8
2.4
2.0
1.6
1.2
1.0
.8
.4
20
40
60
80
100
120
140
160
180
200
X

FIG. 5

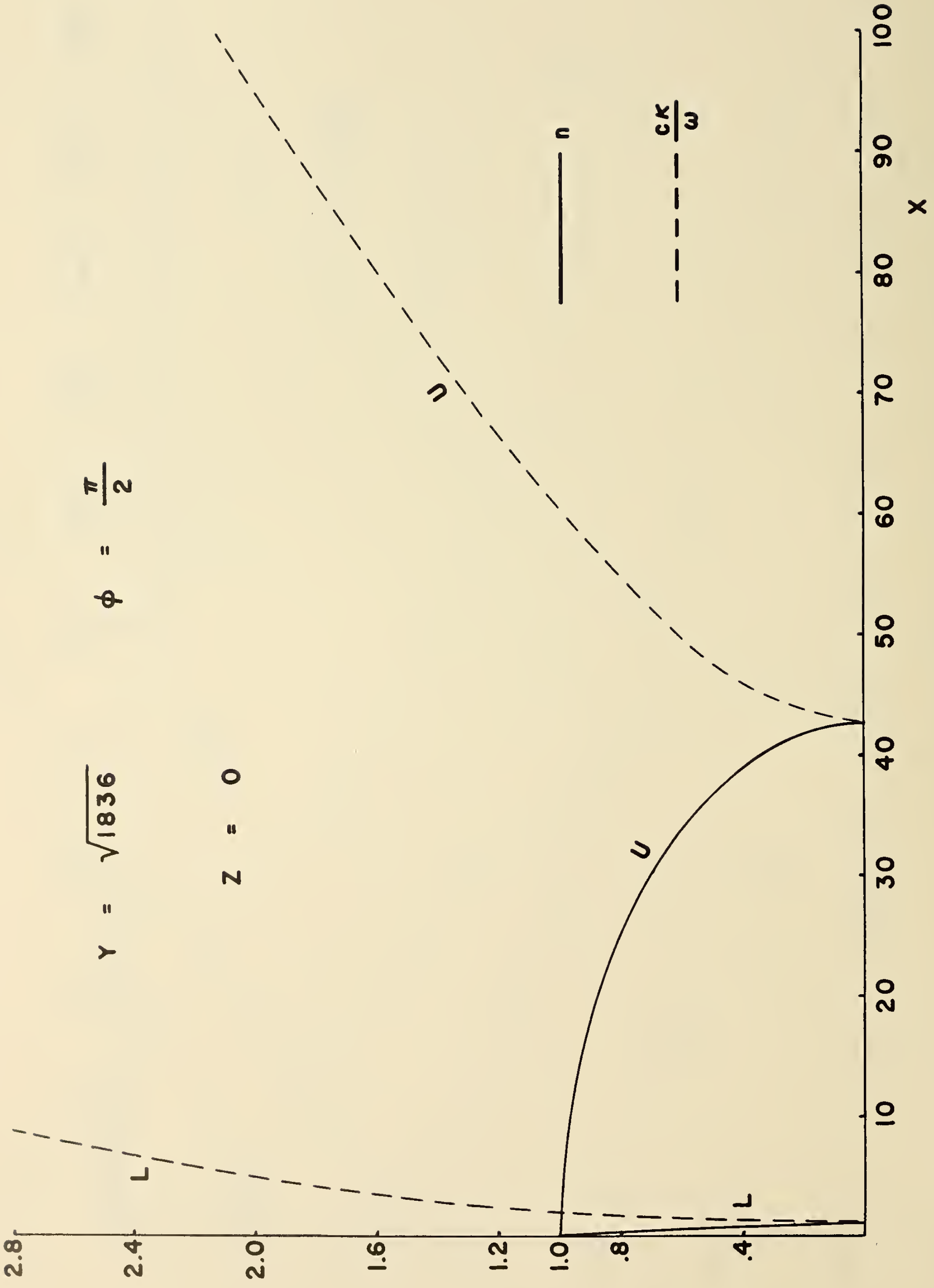

Y = √1836
Z = 0
φ = π/2
n
cκ/ω
U
L
x
2.8
2.4
2.0
1.6
1.2
1.0
.8
.4
10
20
30
40
50
60
70
80
90
100

FIG. 6

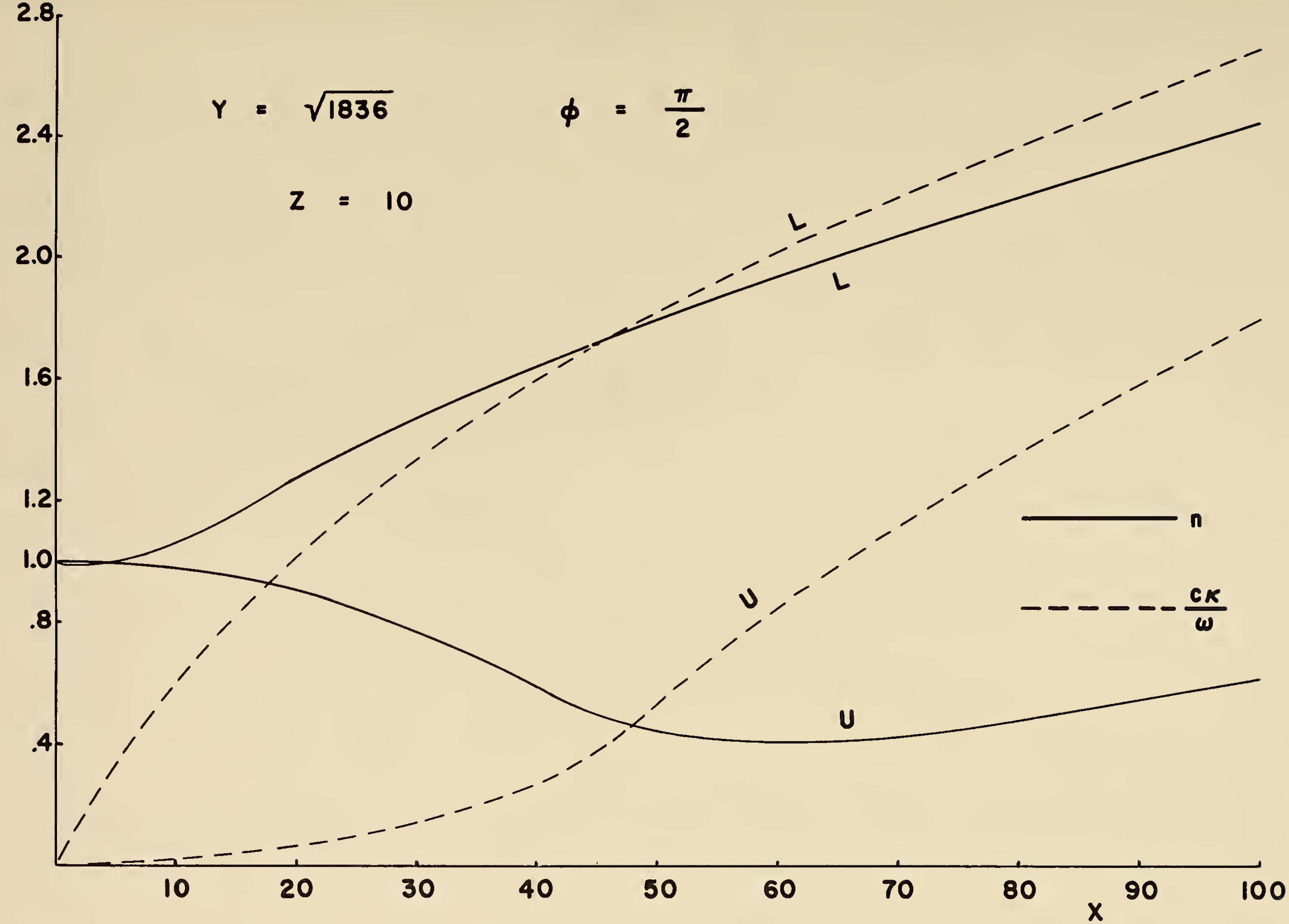
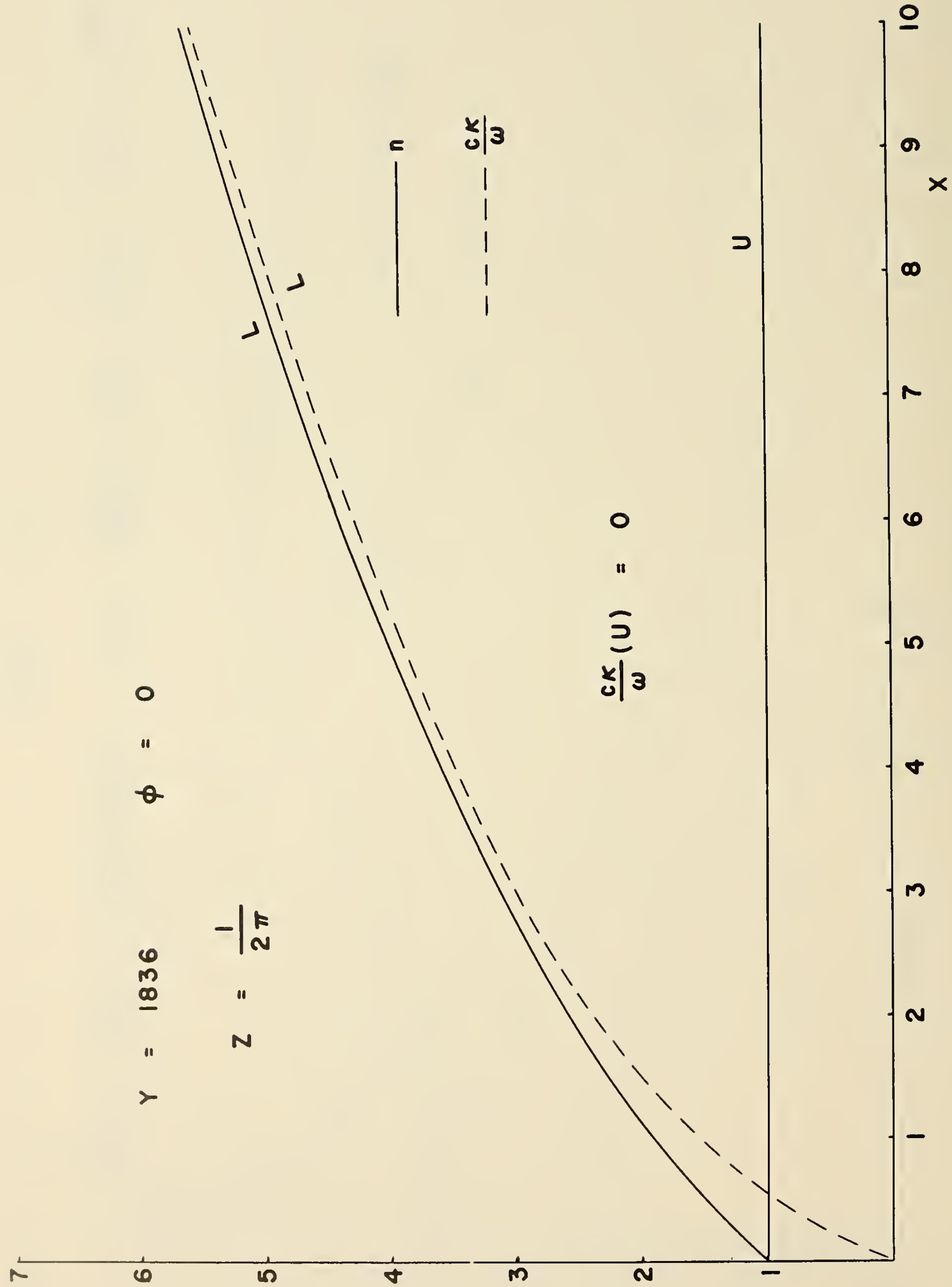
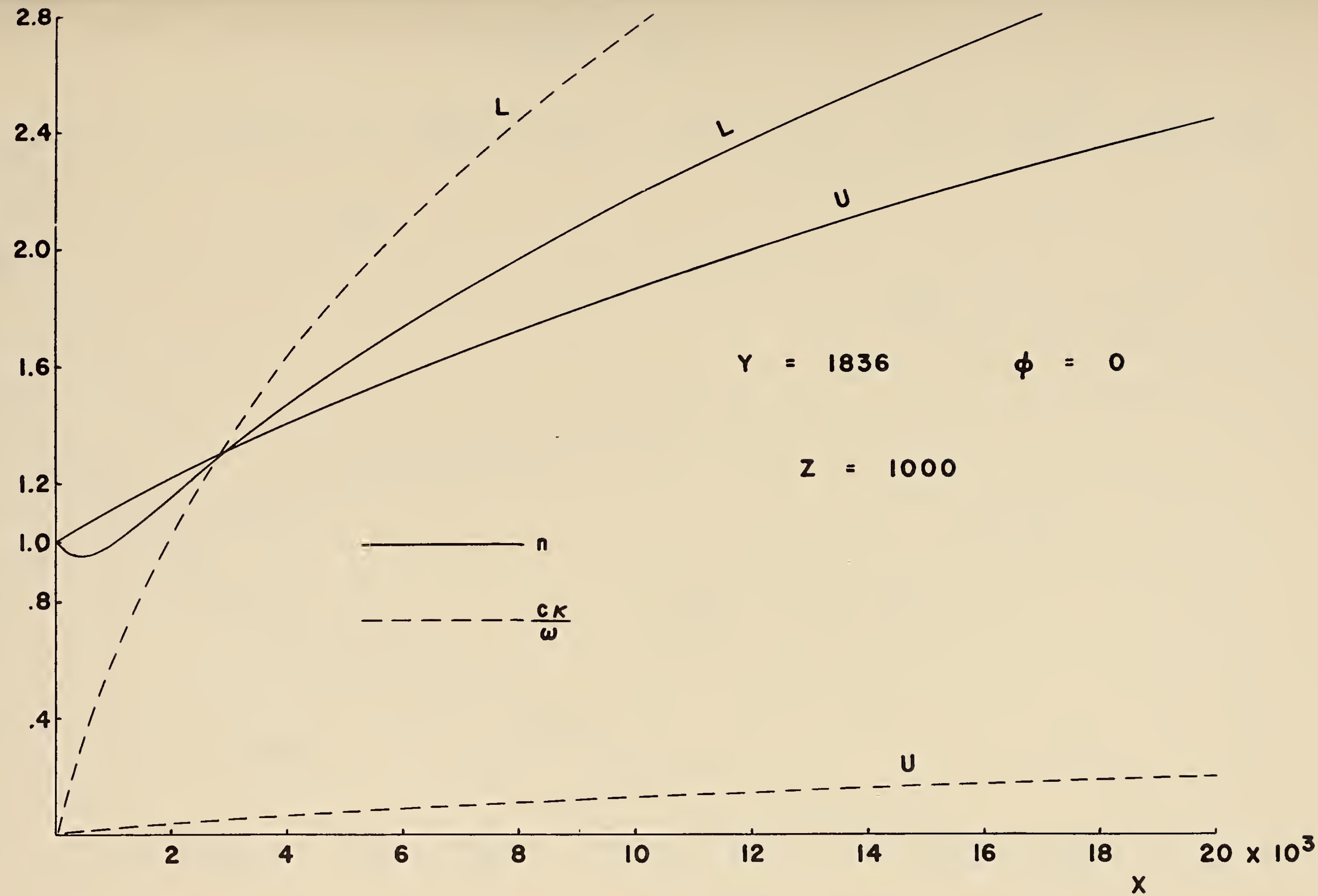

$Y = \sqrt{1836}$
$\phi = \frac{\pi}{2}$
$Z = 10$
L
L
U
U
n
$\frac{c\kappa}{\omega}$
2.8
2.4
2.0
1.6
1.2
1.0
.8
.4
10
20
30
40
50
60
70
80
90
100
X

FIG. 7



Y = 1836
Z = $\frac{1}{2\pi}$
ϕ = 0
$\frac{c\kappa}{\omega}(U)$ = 0
n
$\frac{c\kappa}{\omega}$
L
L
U
X
1
2
3
4
5
6
7
8
9
10
1
2
3
4
5
6
7

FIG. 8



2.8
2.4
2.0
1.6
1.2
1.0
.8
.4
L
L
U
U
Y = 1836
φ = 0
Z = 1000
n
cκ/ω
2
4
6
8
10
12
14
16
18
20 x 10³
X

FIG. 9

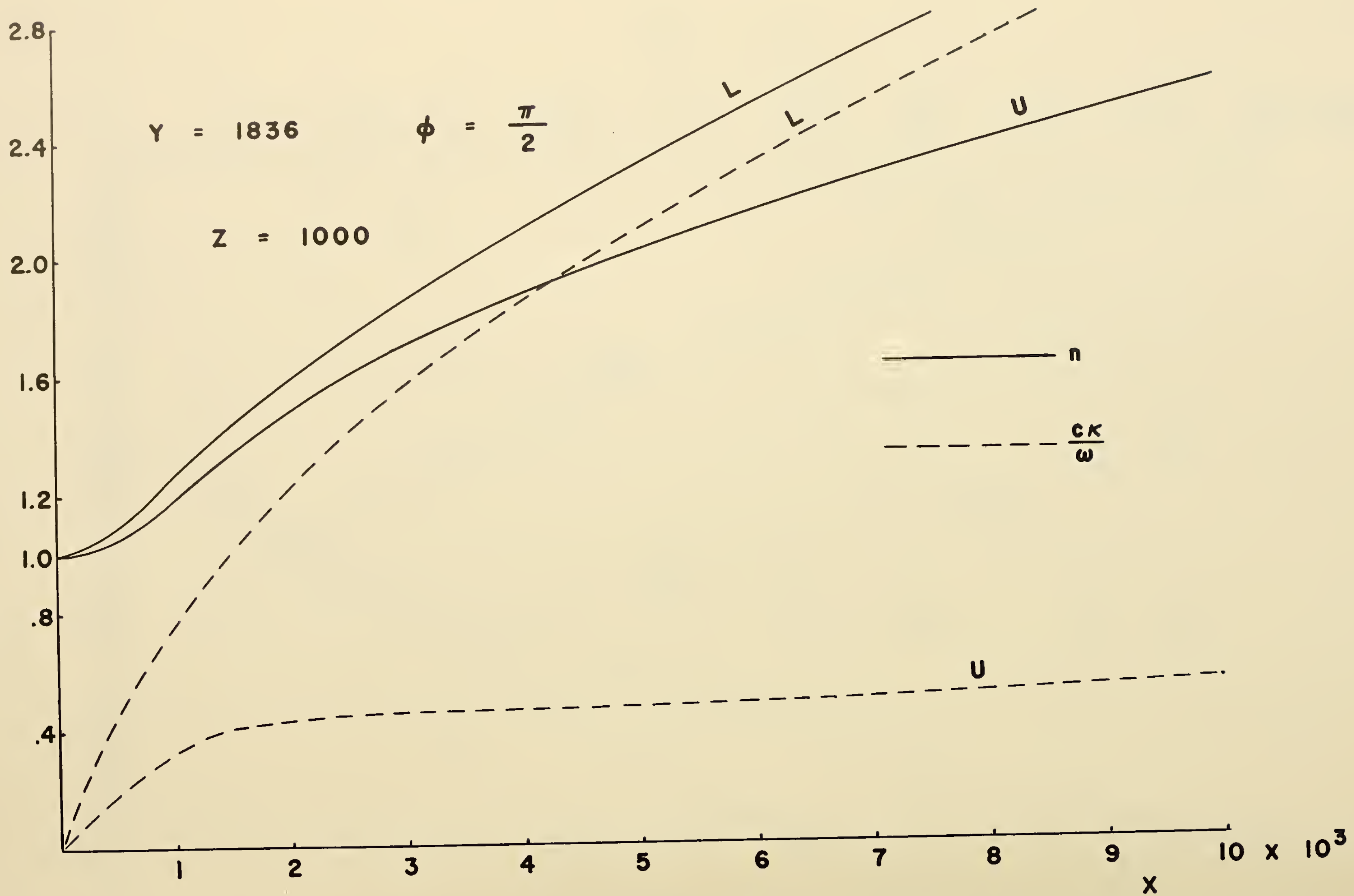

Y = 1836
φ = π/2
Z = 1000
L
L
U
U
n
cκ/ω
2.8
2.4
2.0
1.6
1.2
1.0
.8
.4
1
2
3
4
5
6
7
8
9
10 x 10^3
X

FIG. 10

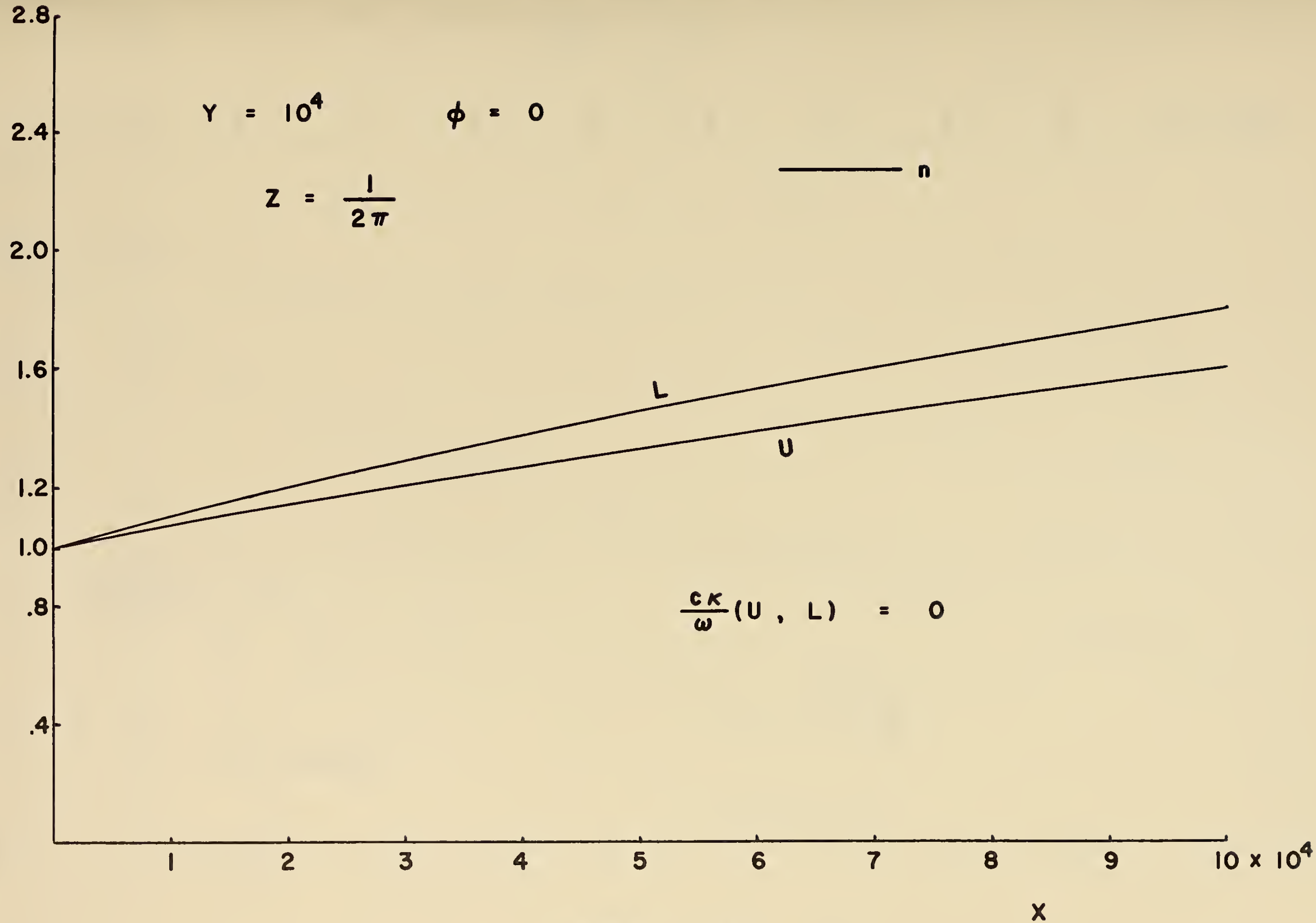
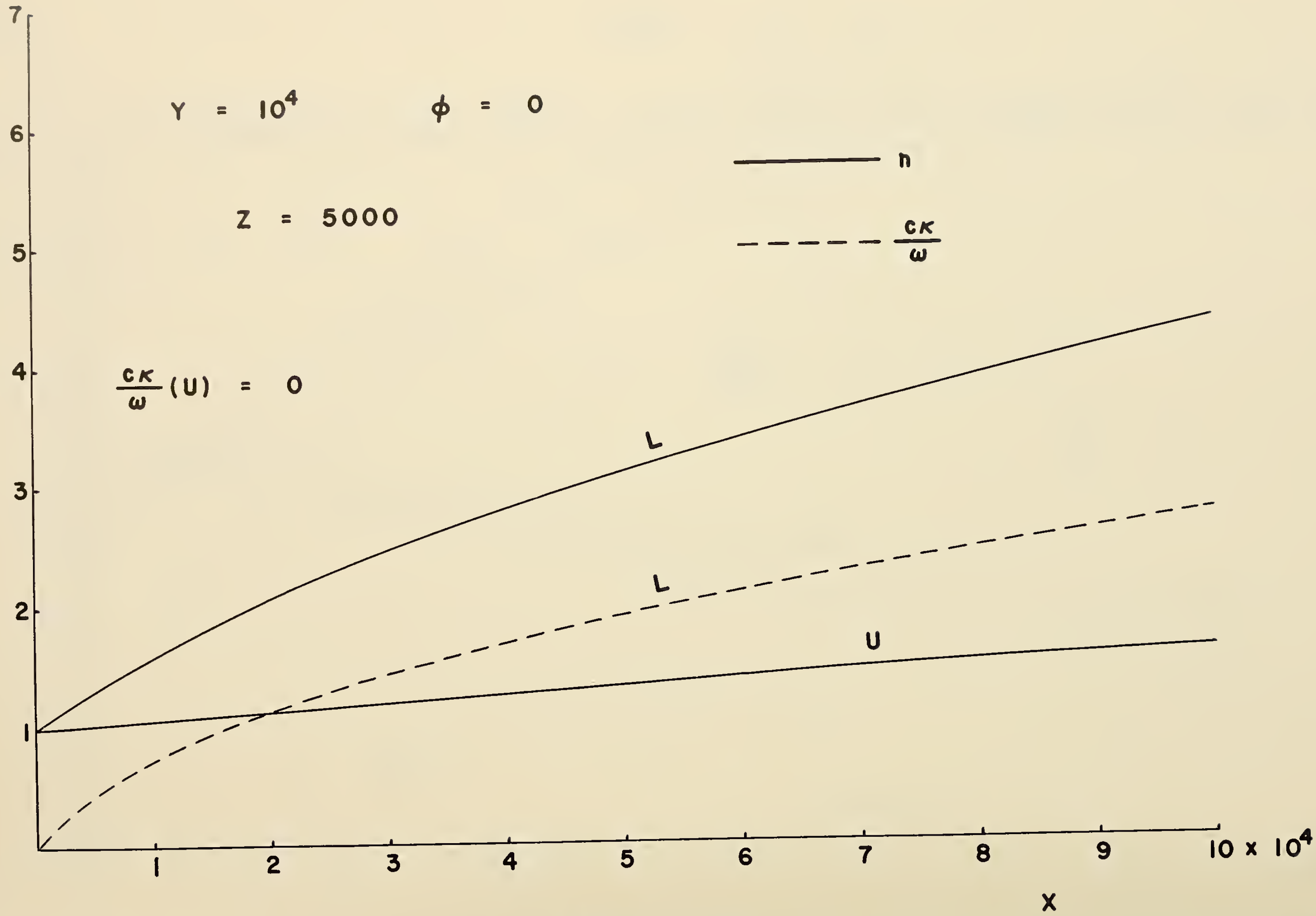
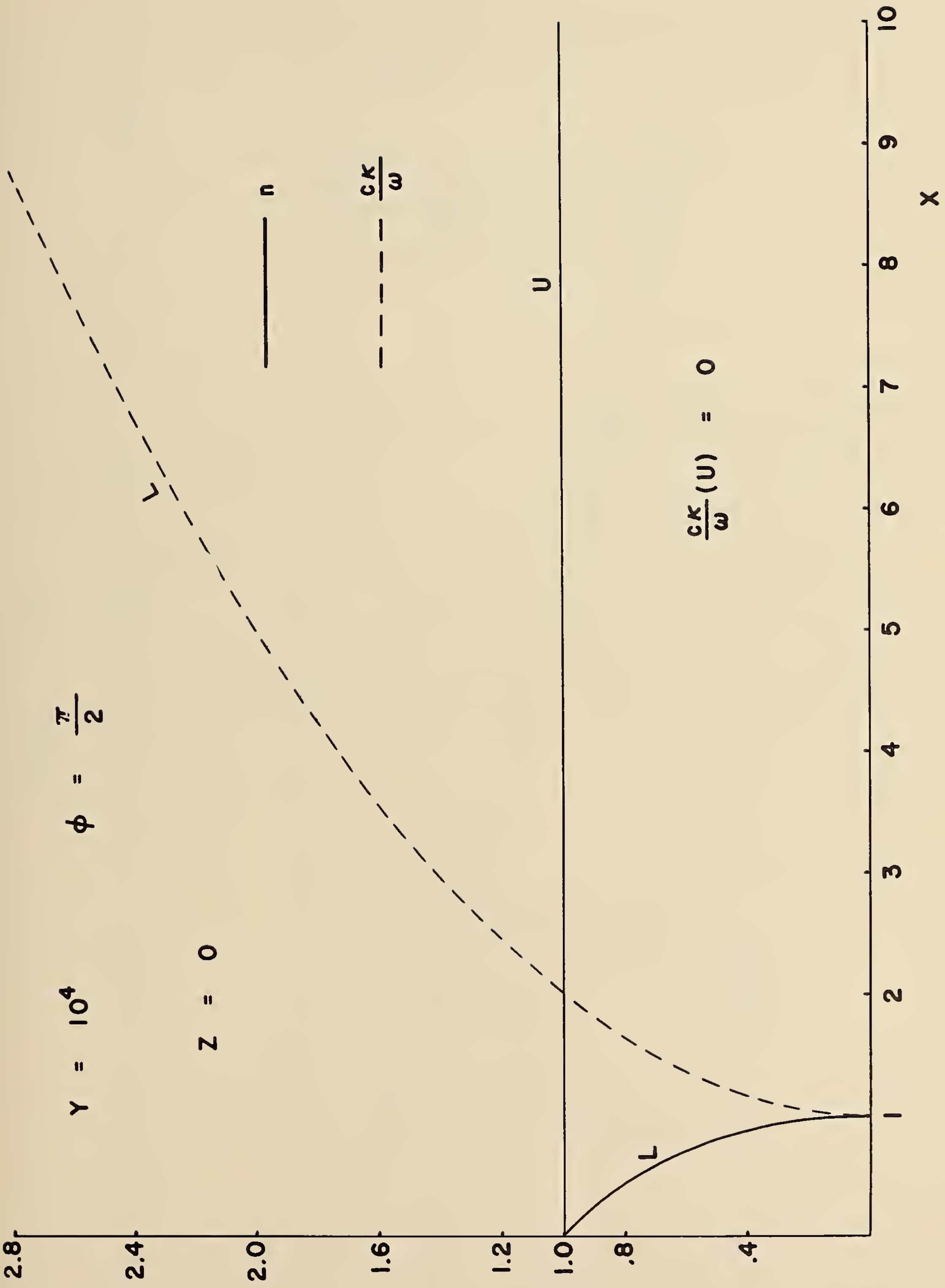


FIG. 11



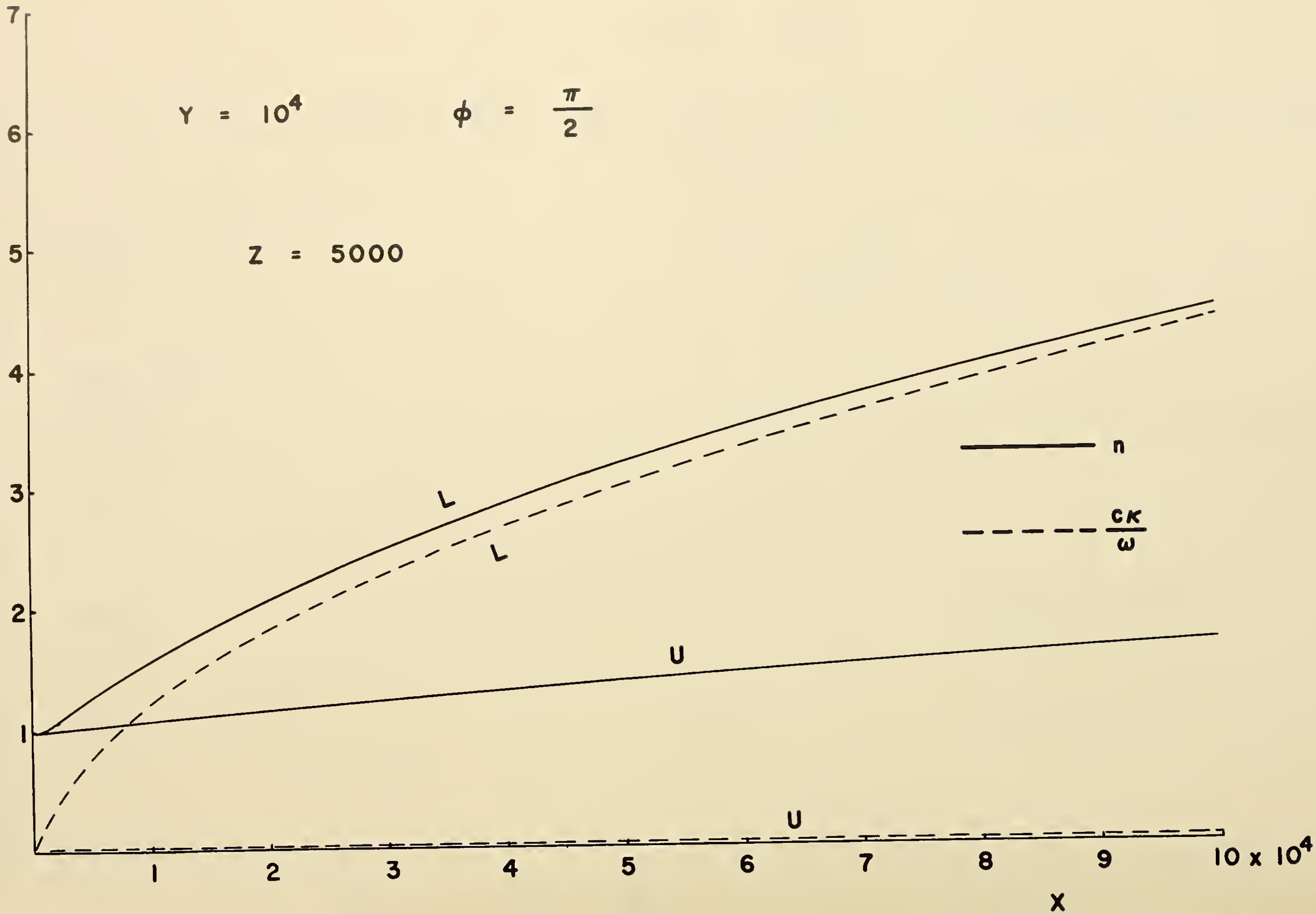
$Y = 10^4$
$\phi = 0$
$Z = 5000$
$\frac{c\kappa}{\omega}(U) = 0$
n
$\frac{c\kappa}{\omega}$
L
L
U
1
2
3
4
5
6
7
8
9
10×10^4
X

FIG. 12



Y = 10^4
Z = 0
$\phi = \frac{\pi}{2}$
n
$\frac{c\kappa}{\omega}$
U
L
L
$\frac{c\kappa}{\omega}(U) = 0$
X
2.8
2.4
2.0
1.6
1.2
1.0
.8
.4
1
2
3
4
5
6
7
8
9
10

FIG. 13



Y = 10^4
$\phi = \frac{\pi}{2}$
Z = 5000
L
L
U
U
n
$\frac{c\kappa}{\omega}$
1
2
3
4
5
6
7
8
9
10 x 10^4
X
1
2
3
4
5
6
7

FIG. 14

A Solution Method for Cold Plasma

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The work we will discuss stems from ideas presented in an unpublished scientific report by J. J. Gibbons.[1] We will consider here an initially neutral plasma cloud set into motion in the presence of a magnetic field. The physical description is idealized to that of the cold plasma approximation for positive and negative fluids. The cold plasma is described by the usual fourteen dependent variables through Maxwell's equations and the dynamical and continuity equations for both positive and negative fluids as follows:

$$
\begin{aligned}
\nabla \cdot \vec{E} &= 4\pi (p - n) \\
\nabla \cdot \vec{H} &= 0 \\
\nabla \times \vec{H} &= \frac{4\pi}{C} (p\vec{v} - n\vec{w}) + \frac{1}{C} \frac{\partial \vec{E}}{\partial t} \\
\nabla \times \vec{E} &= - \frac{1}{C} \frac{\partial \vec{H}}{\partial t} \\
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} &= P(\vec{E} + \frac{1}{C} \vec{v} \times \vec{H}) \\
\frac{\partial \vec{w}}{\partial t} + (\vec{w} \cdot \nabla) \vec{w} &= - N(\vec{E} + \frac{1}{C} \vec{w} \times \vec{H}) \\
\nabla \cdot (p\vec{v}) + \frac{\partial p}{\partial t} &= 0 \\
\nabla \cdot (n\vec{w}) + \frac{\partial n}{\partial t} &= 0
\end{aligned}
\qquad \text{I.}
$$

The solution method we have studied consists of expanding the dependent variables in a power series of time. To satisfy the initial conditions of a neutral plasma immersed in a magnetic field $\vec{H}_o(\vec{r})$ with both positive and negative fluids moving at a constant velocity $\vec{S}_o$ in the absence of an electric field, the power series takes the form

$$\begin{aligned} p(\vec{r},t) &= A_o(\vec{r}) + p_1(\vec{r})t + p_2(\vec{r})t^2 + \ldots.. \\ n(\vec{r},t) &= A_o(\vec{r}) + n_1(\vec{r})t + n_2(\vec{r})t^2 + \ldots.. \\ \vec{v}(\vec{r},t) &= \vec{S}_o + \vec{v}_1(\vec{r})t + \vec{v}_2(\vec{r})t^2 + \ldots.. \\ \vec{w}(\vec{r},t) &= \vec{S}_o + \vec{w}_1(\vec{r})t + \vec{w}_2(\vec{r})t^2 + \ldots.. \\ \vec{E}(\vec{r},t) &= \vec{E}_1(\vec{r})t + \vec{E}_2(\vec{r})t^2 + \ldots.. \\ \vec{H}(\vec{r},t) &= \vec{H}_o(\vec{r}) + \vec{H}_1(\vec{r})t + \vec{H}_2(\vec{r})t^2 + \ldots.. \end{aligned} \qquad \text{II.}$$

The spatially dependent coefficients are solved for by inserting equations II into I and equating each coefficient, of every power of t, to zero. Every coefficient is then expressed in terms of the initial values $A_o(\vec{r})$, $\vec{H}_o(\vec{r})$ and $\vec{S}_o$.

We now consider only one dependent variable, since upon obtaining its solution the other dependent variables can automatically be determined. For no particular reason we choose to study the positive fluid density $p(\vec{r},t)$. It can be shown that the terms of $p(\vec{r},t)$ can be grouped into various sets each of which is identically a three dimensional Taylor series expansion of a representative functional of the initial values $\vec{H}_o(\vec{r})$, $A_o(\vec{r})$ and $\vec{S}_o$. Thus, when the above mentioned sets are summed, the first few terms of $p(\vec{r},t)$ can be written as

$$
\begin{aligned}
p(\vec{r}, t) = {} & A_0(\vec{r} - \vec{S}_0 t) + \nabla \cdot \int_0^t dt' \Bigg[\frac{\tau}{1!} \Big\{ \underset{1}{A_0 \vec{\omega} \times \vec{S}_0} \Big\} + \frac{\tau^2}{2!} \Big\{ \underset{1}{-A_0 \vec{\omega} \times (\vec{\omega} \times \vec{S}_0)} - \underset{1}{2(\vec{S}_0 \cdot \nabla A_0) \vec{\omega} \times \vec{S}_0} \\
& \underset{2}{-A_0 (\vec{S}_0 \cdot \nabla) \vec{\omega} \times \vec{S}_0} \Big\} + \frac{\tau^3}{3!} \Big\{ \underset{1}{A_0 \vec{\omega} \times [\vec{\omega} \times (\vec{\omega} \times \vec{S}_0)]} + \underset{1}{3(\vec{S}_0 \cdot \nabla A_0) \vec{\omega} \times (\vec{\omega} \times \vec{S}_0)} \\
& + \underset{1}{3 \left[\vec{S}_0 \cdot \nabla (\vec{S}_0 \cdot \nabla A) \right] \vec{\omega} \times \vec{S}_0} + \underset{2}{3(\vec{S}_0 \cdot \nabla A_0)(\vec{S}_0 \cdot \nabla) \vec{S}_0 \times \vec{\omega}} + \underset{3}{2A_0 (\vec{\omega} \times \vec{S}_0 \cdot \nabla) \vec{\omega} \times \vec{S}_0} \\
& + \underset{4}{A_0 (\vec{S}_0 \cdot \nabla) \left[(\vec{S}_0 \cdot \nabla) \vec{\omega} \times \vec{S}_0 \right]} + \underset{5}{A_0 (\vec{S}_0 \cdot \nabla) \left[\vec{\omega} \times (\vec{\omega} \times \vec{S}_0) \right]} - \underset{6}{4\pi (P + N) A_0^2 \, \vec{\omega} \times \vec{S}_0} \\
& + \underset{7}{A_0 \vec{\omega} \times \left[(\vec{S}_0 \cdot \nabla) \vec{\omega} \times \vec{S}_0 \right]} + \underset{8}{3 \nabla \cdot (A_0 \vec{\omega} \times \vec{S}_0) \vec{\omega} \times \vec{S}_0} \Big\} + \ldots \Bigg] \qquad \text{III.}
\end{aligned}
$$

where the integrand variables are

$$\tau = t - t', \quad A_0 = A_0(\vec{r} - \vec{S}_0 t') \quad \text{and} \quad \vec{\omega} = \frac{P}{C} \vec{H}_0(\vec{r} - \vec{S}_0 t').$$

We know that if the power series is conditionally convergent then the sum will not be unique for various arrangements of the terms. Therefore, for our purposes, we will assume absolute convergence.

The first term in III represents the undeformed positive fluid density travelling with its initial velocity $\vec{S}_0$ as if no force fields had been encountered. The deformation of the distribution is described through the integral terms which depend upon the entire past history of the plasma.

Solution III will be of definite use if one can discover initial functions $\vec{H}_0(\vec{r})$ and $A_0(\vec{r})$ such that the integral terms can be summed. This would yield a model from which physical intuition can be gained and perturbation to other situations might be possible. At this date we have not discovered such initial conditions. We notice that solution III can be written in the form

$$p(\vec{r}, t) = p(\vec{r} - \vec{S}_o t, O) + \int_o^t f(\vec{r} - \vec{S}_o t', t - t')\, dt' \qquad \text{IV}$$

where $f(\vec{r} - \vec{S}_o t', t - t')$ represents the integrand functions and indicates more clearly the functional relationship of the independent variables. It can be shown that all of the dependent variables can be written in this form. This representation, which we are presently investigating, may be a useful starting point for analysis and may be compared to the expressions obtained by Rosenbluth.[2]

We will now point out several interesting properties born out by equation III. First notice that the various sets of terms indicated by numerals respectively form infinite series which can be immediately summed with the result

$$\begin{aligned}
p(\vec{r}, t) &= A_o(\vec{r} - \vec{S}_o t) \\
&+ \left[A_o(\vec{r} - \vec{S}_o t)\nabla + \left\{\nabla A_o(\vec{r} - \vec{S}_o t)\right\}\right] \cdot \int_o^t dt' \Bigg\{ S_o \frac{|\vec{S}_o x \vec{\omega}|}{|\vec{S}_o||\vec{\omega}|}\Bigg[\sin(\tau\omega) \frac{\vec{\omega} x \vec{S}_o}{|\vec{\omega} x \vec{S}_o|} \\
&+ \left\{\cos(\tau\omega) - 1\right\} \frac{\vec{\omega} x (\vec{\omega} x \vec{S}_o)}{|\vec{\omega} x (\vec{\omega} x \vec{S}_o)|}\Bigg] + \frac{\tau^2}{2!}\left[(\vec{S}_o \cdot \nabla)\vec{S}_o x \vec{\omega}\right] + \frac{\tau^3}{3!}\Big[(\vec{S}_o \cdot \nabla)\left\{(\vec{S}_o \cdot \nabla)\vec{S}_o x \vec{\omega}\right. \\
&+ 4\pi(P + N) A_o \vec{S}_o x \vec{\omega} + 2(\vec{S}_o x \vec{\omega} \cdot \nabla)\vec{S}_o x \vec{\omega} + (\vec{S}_o \cdot \nabla)\vec{\omega} x (\vec{\omega} x \vec{S}_o) \\
&\left. + \vec{\omega} x \left\{(\vec{S}_o \cdot \nabla)\vec{\omega} x \vec{S}_o\right\}\right\}\Big] + \ldots \Bigg\} \qquad \text{V.} \\
&+ \nabla \cdot \int_o^t dt' \left\{ S_o \frac{|\vec{S}_o x \vec{\omega}|}{|\vec{S}_o||\vec{\omega}|} \frac{\tau^2}{2!}\left\{(\vec{\omega} x \vec{S}_o \cdot \nabla)A_o\right\}\left[\sin(\tau\omega)\frac{\vec{\omega} x \vec{S}_o}{|\vec{\omega} x \vec{S}_o|} + (\cos(\tau\omega) - 1)\frac{\vec{\omega} x (\vec{\omega} x \vec{S}_o)}{|\vec{\omega} x (\vec{\omega} x \vec{S}_o)|}\right] + \ldots \right.
\end{aligned}$$

This process can probably be continued indefinitely as suits ones purpose, and each remaining term of a given hierarchy becomes more informative than its predecessors.

Approximate solutions can obtain, depending upon the situation on hand, through the neglect of appropriate terms. For example, if we have a very tenuous plasma and choose to neglect the self-consistent nature of the above description, we may choose to neglect the interaction terms. In this case the dynamical equations become

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = P\,\frac{\vec{v} \times \vec{H}_o}{C}$$

$$\frac{\partial \vec{w}}{\partial t} + (\vec{w} \cdot \nabla)\vec{w} = -N\,\frac{\vec{w} \times \vec{H}_o}{C} \qquad \text{VI.}$$

For this approximation it is found that the terms with the coefficients (P + N) are eliminated. This term obviously indicates the effect of the presence of the negative fluid upon the positive fluid. Similar approximations may be made when the applied magnetic fluid $\vec{H}_o$ is weak or strong.

Another property, evident through direct observation of equation III, is: if the initial distribution $A_o(\vec{r})$ and the applied magnetic field $\vec{H}_o(\vec{r})$ are both dependent upon only one or two spatial variables, then at all future times the plasma will be dependent only upon one or two spatial variables, respectively.

As a brief example of the latter property let us consider a tenuous plasma immersed in a strong but decaying magnetic field

$$\vec{H}_o = H\,e^{-t/T}\,\hat{k} \qquad \text{VII.}$$

In this event we assume the non-interaction equations VI are valid. If the initial distribution is assumed to be a one dimensional Gaussian

$$A_o = Ae^{-x^2/L^2} \qquad \text{VIII}$$

then, employing the above property, the continuity equation takes the form

$$v_x \frac{\partial p}{\partial x} + \frac{\partial p}{\partial t} = 0 . \qquad \text{IX.}$$

The solution of IX is immediately

$$p = Ae^{-[x-g(t)]^2/L^2} \qquad \text{X}$$

where

$$g(t) = S_o \int_o^t \sin\left[\frac{PHT}{C}\left(e^{-t'/T} - 1\right)\right] dt' ,$$

The density asymptotically becomes

$$p \rightarrow Ae^{-[x+vt-\phi]^2/L^2} , \qquad \text{XI}$$

as is intuitively expected. Of course a solution for a three dimensional Gaussian is similarily immediate.

The above approach has been useful in studying the dynamics of a plasma cloud covering half of space and travelling toward a magnetic dipole. Here we are interested in temporal values immediately following initiation to correspond to the sudden commencement of a magnetic storm. We thus need to consider only the first few terms in the power series.

The geometry for this problem is shown in figure 1.[3] The cloud boundary is initially a plane parallel to the x - z plane and approaches a magnetic dipole, located at the origin, from a point on the negative y axis toward the origin with constant velocity S_{oy}. The magnetic dipole is pointing in the negative z direction.

To lowest order, the total charge density at the equator is

$$\rho = \frac{1}{6} S_{oy} \frac{P^2 - N^2}{C} \left\{ H^2_{oz} \frac{\partial A_o}{\partial y} + A_o \frac{\partial H^2_{oz}}{\partial y} \right\} t^3 + \ldots \qquad \text{XII.}$$

Examination of the terms indicates that there is a positive charge density everywhere near the surface with a maximum at x = o. Behind the surface we have a negative volume charge which satisfies the requirement of charge conservation.

The velocities to second order are

$$\vec{v} = \vec{S}_o + \frac{P}{C} \vec{S}_o \times \vec{H}_o t + \left[-\frac{P}{2C} (\vec{S}_o \cdot \nabla) \vec{S}_o \times \vec{H}_o - \frac{P^2}{2C^2} \vec{H}_o \times (\vec{S}_o \times \vec{H}_o) \right] t^2 + \ldots$$

$$\vec{w} = \vec{S}_o - \frac{N}{C} \vec{S}_o \times \vec{H}_o t + \left[\frac{N}{2C} (\vec{S}_o \cdot \nabla) \vec{S}_o \times \vec{H}_o - \frac{N^2}{2C^2} \vec{H}_o \times (\vec{S}_o \times \vec{H}_o) \right] t^2 + \ldots$$

XIII.

The first order terms show that positive charges are deflected to the right and negative to the left. The second term of the second order coefficient for both positive and negative is a velocity component in the negative y direction with a maximum value at x = o. This indicates that the oncoming positive and negative fluids are slowed up in such a manner that the cloud front bends about the origin. The remaining term modifies the x displacement of the first order term. When we consider velocities away from the equatorial plane a current pattern as indicated in figure 2 is observed on the cloud surface.

The electric field to second order is

$$\vec{E} = -2\pi A_o (P + N) \frac{\vec{S}_o \times \vec{H}_o}{C} t^2 + \ldots \qquad \text{XIV.}$$

The generated electric field is opposite to the current, as usual in a generator of e. m. f. This clearly indicates the error in employing Ohms law.

FIGURE 1

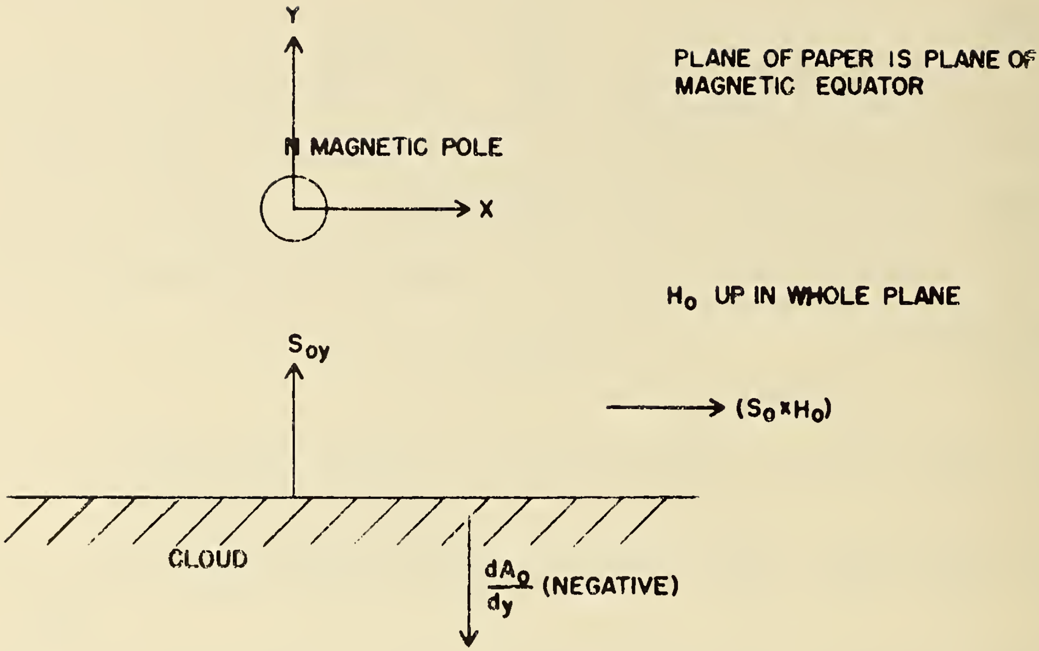
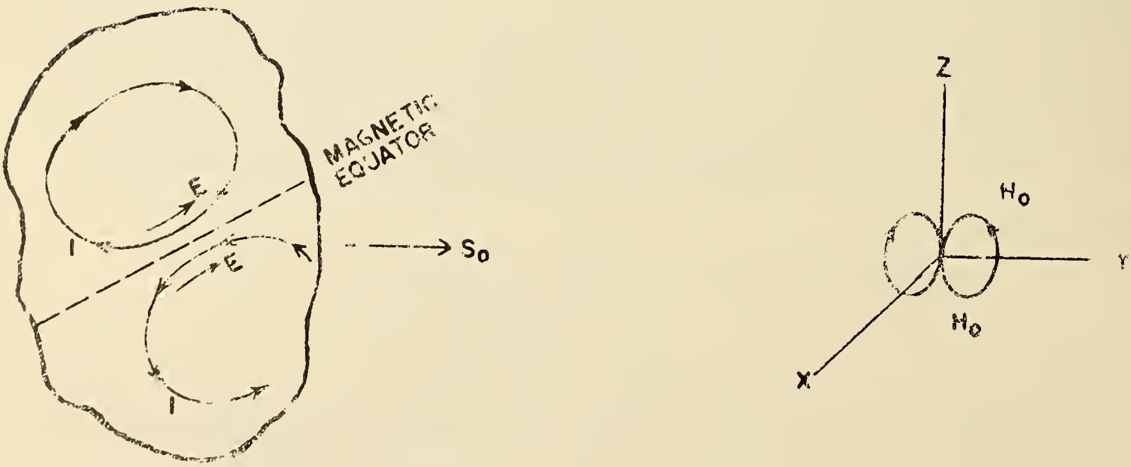


FIGURE 2



Acknowledgements

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1. J. J. Gibbons, Investigation of Initial Transcient Behavior of the Totally Neutral Ion Cloud, Ionosphere Research Laboratory Scientific Report 30, The Pennsylvania State University, 1951.

2. Rolf K. M. Landshoff, The Plasma in a Magnetic Field, Stanford University Press, Stanford, California, 1960.

3. The figures appear in reference 1.

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