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FOREWORD

The principal objective of this research was the formulation of a many body theory of electronic charge transport for uniformly magnetized systems which could be utilized with some degree of practicality to predict quantum effects of the electron-electron interactions.

This objective has been achieved to the point where quantum exchange contributions may be calculated in addition to the "self-consistent field" approximation already considered. As a by-product of this research, expressions were obtained for the spin magnetization with quantum exchange interaction included. The results which have been obtained are applicable to the solid state as a model semiconductor and to the gaseous state as a model fully ionized gas (non-relativistic) with stationary positive charges.

Quantum Field Theoretic Techniques
and the Electromagnetic Properties of
a Uniformly Magnetized Electron Gas

L. A. Steinert

Field theoretic operators for the charge, ρ, and current, $\vec{j}$, densities of the electron gas were obtained using the Darwin Hamiltonian in "second quantized" form. Temperature dependent Green's functions were formed from certain averaged combinations of the field creation and annihilation operators. Equations of motion were obtained for these Green's functions, and ρ and $\vec{j}$ were written in terms of the Green's functions.

In order that ρ and $\vec{j}$ might be expressed explicitly in terms of the perturbing field, formal functional series expansions were obtained for ρ and $\vec{j}$ in terms of the potentials. Connecting relations obtained from the requirements of gauge invariance and charge conservation were used to show the general result that ρ and $\vec{j}$ reduce to functionals of the electric field alone.

Using the "self-consistent field" approximation, calculations were performed determining $\vec{j}$ explicitly to an order linear in the perturbing electric field. Specific calculations are given for a zero temperature gas and for classical high temperatures. Discussed briefly was the "dielectric screening" as induced by a magnetized electron gas. The Green's functions equation of motion for the "spin magnetization" was obtained. Brief mention is made of diagrammatic techniques.

INTRODUCTION

Formal functional relations are derived for the charge and current densities of macroscopic systems acted upon by electromagnetic fields. Quantum field theoretical techniques are combined with the results of equilibrium statistical mechanics for a "Many-Body" theory of the electromagnetic properties of a uniformly magnetized electron gas in the non-relativistic limit. The functional relations for charge and current density are utilized for the case of an "additional" electromagnetic disturbance acting upon the electron gas. The results comprise a theory of charge transport and electromagnetic fluctuations for the electron gas.

The method of approach used here for the electron gas involves the Martin-Schwinger techniques concerning temperature dependent Green's functions (reference (1)). Some discussions and examples of the concept of temperature dependent Green's functions are given in the references (1) through (24) with applications to various physical problems. The references (1) through (8) specifically involve the "causal" Green's functions defined by Martin and Schwinger (1), while the review papers listed as (20), (22), and (23) include discussions of related Green's functions ("advanced" and "retarded" Green's functions).

A brief review of electromagnetic field theory as related to macroscopic media is given in Chapter I. It is assumed that Maxwell's electromagnetic equations are valid for quantum mechanical systems with electromagnetic field quantities suitably defined as "probabilistic averages" or "expectation values" of appropriate operators (see reference (25) concerning this point). A formal development of the charge and current densities of macroscopic systems as functional series expansions in the disturbing electromagnetic fields is obtained in Chapter II. This derivation constitutes one of the new results of the present work. Charge and current densities are defined to include polarization charges and currents of material media. A "conductance tensor" is defined which includes both the usual conductivity due to the motions of charges and the effects of polarization currents. The quantum field theoretic Hamiltonian for the electron gas interacting with applied electromagnetic fields is given in Chapter III, along with a derivation of the charge and current density operators for the gas. In Chapter IV the definition of macroscopic "expectation values" of quantum field theoretic operators is discussed in terms of the density matrix formalism. Temperature dependent Green's functions are defined in Chapter V; these functions are essentially space-time "correlation functions" formed from the "expectation values" of certain combinations of the electron field creation and annihilation operators. The general relationship of the Green's functions to the charge and current densities of the electron gas is developed.

Equations of motion are derived for the Green's functions, and the Green's functions are "renormalized" from functionals of the applied electromagnetic fields to functionals of the total electromagnetic fields. Functional derivatives are obtained in the "self-consistent field approximation" for the charge and current densities to the order linear in the disturbing electromagnetic fields in terms of the "one-particle" Green's functions. In Chapter VI a solution is developed for the "one-particle" Green's function in the "self-consistent field approximation." The chemical potential of the electron gas appears as a parameter in the Grand Canonical Ensemble and in Chapter VII the relation of the chemical potential to the equilibrium electron number density is considered briefly. The equilibrium electron energy density is also discussed in Chapter VII. In Chapter VIII the current density of the electron gas to the order linear in the disturbing electromagnetic fields is considered in some detail in terms of the Fourier transforms (equivalent to plane waves) of the current density and of the electromagnetic field (this also is given in the "self-consistent field approximation"). Attention is given to the Fourier transform of the "conductance tensor," the dispersion relations for propagation, plane wave propagation in the direction of the applied static magnetic field, and complex frequencies. The "dielectric screening" of a static Coulomb charge by a magnetized electron gas is discussed briefly in Chapter IX. In Chapter X the "spin magnetization" of the electron gas is discussed. The equation of motion for the "spin magnetization" is formally developed directly from the "Many-Body" point of view utilized here. A discussion of some topological methods relating to the solution of integral equations appearing here is given in the final chapter (XI).

The c. g. s. Gaussian system of units is used throughout this paper.

CHAPTER I

ELECTROMAGNETIC THEORY AND MACROSCOPIC SYSTEMS

The discussion in this chapter is a brief summary of classical electromagnetic theory for macroscopic media. The formal development of this theory may be found in a number of excellent standard textbooks (for example, see reference (26)). It is assumed here that the Maxwell equations are applicable to quantum mechanical systems, in which case the electromagnetic field quantities must be defined as "probabilistic averages" or "expectation values" of the corresponding operators for the quantized electromagnetic field (see reference (25) for discussion related to this point).

The fields in a macroscopic medium are given by Maxwell's electromagnetic equations:

$$\begin{aligned}
\nabla \cdot \vec{E}^T(\vec{r},t) &= 4\pi\,\rho^T(\vec{r},t) \\
\nabla \cdot \vec{B}^T(\vec{r},t) &= 0 \\
\nabla \times \vec{E}^T(\vec{r},t) &= -\frac{1}{c}\,\frac{\partial \vec{B}^T(\vec{r},t)}{\partial t} \qquad \text{(I-1)} \\
\nabla \times \vec{B}^T(\vec{r},t) &= \frac{1}{c}\,\frac{\partial \vec{E}^T(\vec{r},t)}{\partial t} + \frac{4\pi}{c}\,\vec{j}^T(\vec{r},t)\,.
\end{aligned}$$

In the equations (I-1) $\vec{E}^T$ and $\vec{B}^T$ are the basic electric and magnetic field vectors, respectively, while the quantities ρ^T and $\vec{j}^T$ are the total charge density and the total current density, respectively. These charge and current densities include the polarization charges and polarization currents of the material media. The non-linear contributions to the electromagnetic field equations (I-1)(see reference (27)) are incorporated in the quantities ρ^T and $\vec{j}^T$.

The field vectors $\vec{E}^T$ and $\vec{B}^T$ may be obtained from vector and scalar field potentials as follows:

$$\vec{E}^T = -\nabla U^T - \frac{1}{c}\frac{\partial \vec{A}^T}{\partial t}$$
$$\vec{B}^T = \nabla \times \vec{A}^T , \qquad \text{(I-2)}$$

where $\vec{A}^T(\vec{r}, t)$ is the vector potential and $U^T(\vec{r}, t)$ is the scalar potential. The introduction of the field potentials, together with the Lorentz condition,

$$\nabla \cdot \vec{A}^T + \frac{1}{c}\frac{\partial U^T}{\partial t} = 0 , \qquad \text{(I-3)}$$

results in the reduction of the four Maxwell equations (I-1) to a pair of wave equations for U^T and $\vec{A}^T$:

$$-\nabla^2 U^T(\vec{r}, t) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} U^T(\vec{r}, t) = 4\pi \rho^T(\vec{r}, t)$$
$$-\nabla^2 \vec{A}^T(\vec{r}, t) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \vec{A}^T(\vec{r}, t) = \frac{4\pi}{c} \vec{j}^T(\vec{r}, t) . \qquad \text{(I-4)}$$

The field potentials are not unique; the "gauge transformation"

$$U^T = U_1^T + \frac{1}{c}\frac{\partial \Lambda^T}{\partial t}$$
$$\vec{A}^T = \vec{A}_1^T - \nabla \Lambda^T , \qquad \text{(I-5)}$$

leaves the equations (I-2, 3, 4) invariant if Λ^T is an arbitrary function satisfying the equation

$$-\nabla^2 \Lambda^T + \frac{1}{c^2} \frac{\partial^2 \Lambda^T}{\partial t^2} = 0 \; . \qquad \text{(I-6)}$$

The charge density ρ^T and the current density $\vec{j}^T$ are connected by the equation of continuity,

$$\frac{\partial \rho^T}{\partial t} + \nabla \cdot \vec{j}^T = 0 \; , \qquad \text{(I-7)}$$

which follows from equations (I-1) through (I-6).

The equations (I-1) through (I-7) apply both to a given phvsical system and to its environment. If there exist fields not produced by this given system, then these fields are designated as "externally" applied fields. Also, if there exist field producing sources (charges and currents) not considered as properly a part of the given system, these sources are designated as "external" sources. The sources considered as part of the given system and the fields produced by them are designated by the term "internal" (or "intrinsic").

The total charge density ρ^T and the total current density $\vec{j}^T$ may each be expressed as a sum of two parts:

$$\begin{aligned} \rho^T(\vec{r}, t) &= \rho^A(\vec{r}, t) + \rho(\vec{r}, t) \\ \vec{j}^T(\vec{r}, t) &= \vec{j}^A(\vec{r}, t) + \vec{j}(\vec{r}, t) \; , \end{aligned} \qquad \text{(I-8)}$$

where ρ^A and $\vec{j}^A$ are the charge and current densities arising from the "external" sources, while ρ and $\vec{j}$ are the "internal" (or "intrinsic") charge and current densities of the system. Correspondingly, the field potentials, the field vectors, and the gauge function Λ^T may all be defined as sums or a linear superposition of terms arising from the environment of the system (superscript A) and from the system itself (no superscript):

$$U^T(\vec{r}, t) = U^A(\vec{r}, t) + U(\vec{r}, t)$$

$$\vec{A}^T(\vec{r}, t) = \vec{A}^A(\vec{r}, t) + \vec{A}(\vec{r}, t)$$

$$\vec{E}^T(\vec{r}, t) = \vec{E}^A(\vec{r}, t) + \vec{E}(\vec{r}, t) \qquad \text{(I-9)}$$

$$\vec{B}^T(\vec{r}, t) = \vec{B}^A(\vec{r}, t) + \vec{B}(\vec{r}, t)$$

$$\Lambda^T(\vec{r}, t) = \Lambda^A(\vec{r}, t) + \Lambda(\vec{r}, t)$$

This splitting (I-8, 9) of quantities into external and intrinsic parts is by no means unique nor even meaningful in general, but in many applications the procedure will be clear. All the equations (I-1) through (I-7) are duplicated for the "external" and "internal" fields and sources separately. For the "external" fields and sources,

$$\nabla \cdot \vec{E}^A = 4\pi \rho^A$$

$$\nabla \cdot \vec{B}^A = 0$$

$$\nabla \times \vec{E}^A = -\frac{1}{c} \frac{\partial \vec{B}^A}{\partial t}$$

$$\nabla \times \vec{B}^A = \frac{1}{c} \frac{\partial \vec{E}^A}{\partial t} + \frac{4\pi}{c} \vec{j}^A$$

$$\vec{E}^A = -\nabla U^A - \frac{1}{c} \frac{\partial \vec{A}^A}{\partial t}$$

$$\vec{B}^A = \nabla \times \vec{A}^A$$

$$\nabla \cdot \vec{A}^A + \frac{1}{c} \frac{\partial U^A}{\partial t} = 0 \qquad \text{(I-10)}$$

$$-\nabla^2 U^A + \frac{1}{c^2} \frac{\partial^2 U^A}{\partial t^2} = 4\pi \rho^A$$

$$-\nabla^2 \vec{A}^A + \frac{1}{c^2} \frac{\partial^2 \vec{A}^A}{\partial t^2} = \frac{4\pi}{c} \vec{j}^A$$

$$U^A = U_1^A + \frac{1}{c} \frac{\partial \Lambda^A}{\partial t}$$

$$\vec{A}^A = \vec{A}_1^A - \nabla \Lambda^A$$

$$-\nabla^2 \Lambda^A + \frac{1}{c^2} \frac{\partial^2 \Lambda^A}{\partial t^2} = 0$$

$$\frac{\partial \rho^A}{\partial t} + \nabla \cdot \vec{j}^A = 0 ,$$

while for the "intrinsic" fields and sources of the system,

$$\nabla \cdot \vec{E} = 4\pi \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{j}$$

$$\vec{E} = -\nabla U - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A} \qquad \text{(I-11)}$$

$$\nabla \cdot \vec{A} + \frac{1}{c} \frac{\partial U}{\partial t} = 0$$

$$-\nabla^2 U + \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = 4\pi \rho$$

$$-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{j}$$

$$U = U_1 + \frac{1}{c}\frac{\partial \Lambda}{\partial t}$$

$$\vec{A} = \vec{A}_1 - \nabla\Lambda$$

$$-\nabla^2 \Lambda + \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \ .$$

The fact that all the equations (I-1) through (I-7) have been separated into the two sets of equations (I-10) and (I-11) does not necessarily imply that they are independent of each other. These two sets of equations, (I-10) and (I-11), are generally coupled together through the quantities ρ^A, $\vec{j}^A$, ρ, and $\vec{j}$, each of which is in general influenced by all the electromagnetic fields and all other physical forces present.

The field potentials produced by the sources are given by the following integrals:

$$U^{(s)}(\vec{r}_1, t_1) = \int d^3\vec{r}_2 \, \frac{\rho^{(s)}(\vec{r}_2, t_1 - \frac{r_{12}}{c})}{r_{12}}$$

$$\vec{A}^{(s)}(\vec{r}_1, t_1) = \int d^3\vec{r}_2 \, \frac{\vec{j}^{(s)}(\vec{r}_2, t_1 - \frac{r_{12}}{c})}{c r_{12}} \,, \qquad \text{(I-12)}$$

where the superscript (s) applies generally either to the "total" fields and sources, to the "applied" fields and sources, or to the "intrinsic" fields and sources of the given system, and where $r_{12} \equiv |\vec{r}_1 - \vec{r}_2|$.

This integral form of the relationship between the potentials and sources is equivalent to the differential form (I-4). Note, however, that while in the differential form (I-4) the sources are "local" in the time, in the integral form (I-12) on the other hand the sources contain a "time retardation" factor $\left(\frac{r_{12}}{c}\right)$ which is due to the fact that electromagnetic signals propagate with finite velocity, i.e., the electromagnetic interaction occurring between two charges at positions of finite spatial separation requires a finite temporal interval between cause and effect experienced by the charges.

For stationary media, or more precisely, for systems considered to be in a rest frame of reference relative to an observer, the total charge and current densities intrinsic to the material are given by

$$\rho = \rho_F + \rho_P + \rho_M$$

$$\vec{j} = \vec{j}_F + \vec{j}_P + \vec{j}_M ,$$

with

$$\begin{aligned} \rho_P &\equiv - \nabla \cdot \vec{P} \\ \vec{j}_P &\equiv \frac{1}{c} \frac{\partial \vec{P}}{\partial t} \\ \rho_M &\equiv 0 \\ \vec{j}_M &\equiv c \nabla \times \vec{M} , \end{aligned} \qquad \text{(I-13)}$$

and where

ρ_F = charge density of "unbound" charges

$\vec{j}_F$ = current density of "unbound" charges

$\vec{P}$ = electric polarization density from "bound" charges and polarization charges

$\vec{M}$ = magnetic polarization density from "bound" charges and "intrinsic" magnetic dipole moments.

For systems considered to be in a frame of reference moving with constant velocity relative to an observer, ρ and $\vec{j}$ are again given by the equations (I-13); in this case the quantities ρ_F, $\vec{j}_F$, $\vec{P}$, and $\vec{M}$ are related to their "rest system" values by appropriate Lorentz transformations (see reference (26), Chapter 18). It is clear from the equations (I-13) that the total "intrinsic" charge density ρ and the total "intrinsic" current density $\vec{j}$ for macroscopic systems concern all the electric and magnetic properties, both induced and permanent, of the system.

CHAPTER II

FUNCTIONAL SERIES EXPANSIONS FOR THE CHARGE AND CURRENT DENSITIES OF MACROSCOPIC MEDIA

In this chapter formal relationships are derived using electromagnetic field theory for the charge and current densities of macroscopic systems under the action of an electromagnetic disturbance. While it is generally accepted that there exist in nature other physical forces between particles besides the electromagnetic interactions, only the electromagnetic forces are considered here.

For dynamical systems involving only electromagnetic interactions the charge density ρ and the current density $\vec{j}$ respond directly to the total field potentials U^T and $\vec{A}^T$. Mathematically speaking, ρ and $\vec{j}$ are functionals of U^T and $\vec{A}^T$:

$$\begin{aligned} \rho &\equiv \rho[\vec{r}, t; U^T, \vec{A}^T] \\ \vec{j} &\equiv \vec{j}[\vec{r}, t; U^T, \vec{A}^T]. \end{aligned} \qquad \text{(II-1)}$$

From the discussion in Chapter I it is apparent that the potentials U^A, $\vec{A}^A$, U, and $\vec{A}$ are functionals of U^T and $\vec{A}^T$ (and vice versa); therefore, since ρ and $\vec{j}$ respond indirectly to U^A, $\vec{A}^A$, U, and $\vec{A}$, ρ and $\vec{j}$ might, in a formal sense only, be regarded as functionals of U^A and $\vec{A}^A$:

$$\begin{aligned} \rho &\equiv \rho[\vec{r}, t; U^A, \vec{A}^A] \\ \vec{j} &\equiv \vec{j}[\vec{r}, t; U^A, \vec{A}^A], \end{aligned} \qquad \text{(II-2)}$$

or ρ and $\vec{j}$ could be considered as functionals of U and $\vec{A}$:

$$\rho \equiv \rho[\vec{r}, t; U, \vec{A}]$$
$$\vec{j} \equiv \vec{j}[\vec{r}, t; U, \vec{A}]. \qquad \text{(II-3)}$$

The three sets of equations (II-1, 2, 3) do not imply that a change in the potentials necessarily produces changes in the charge and current densities, but they do imply that all changes in the charge and current densities are certainly produced by the field potentials.

It is desirable to consider the "externally" applied fields $\vec{E}^A$ and $\vec{B}^A$ (or U^A and $\vec{A}^A$), or their sources, ρ^A and $\vec{j}^A$, as idependently known. This permits the examination of the behavior of the given system in an arbitrary physical environment.

The state of the system is presumably known for all times t prior to some given time t_o, at which time some electromagnetic perturbation is applied. For the initial state (subscripts (o)) the following equations apply:

$$\nabla \cdot \vec{E}_o^T = 4\pi \rho_o^T = 4\pi (\rho_o + \rho_o^A)$$
$$\nabla \cdot \vec{B}_o^T = 0$$
$$\nabla \times \vec{E}_o^T = -\frac{1}{c} \frac{\partial \vec{B}_o^T}{\partial t}$$
$$\nabla \times \vec{B}_o^T = \frac{1}{c} \frac{\partial \vec{E}_o^T}{\partial t} + \frac{4\pi}{c} \vec{j}_o^T \qquad \text{(II-4)}$$
$$= \frac{1}{c} \frac{\partial \vec{E}_o^T}{\partial t} + \frac{4\pi}{c} (\vec{j}_o + \vec{j}_o^A)$$
$$\vec{E}_o^T = -\nabla U_o^T - \frac{1}{c} \frac{\partial \vec{A}_o^T}{\partial t}$$
$$\vec{B}_o^T = \nabla \times \vec{A}_o^T .$$

After the perturbation is applied ($t \geq t_o$):

$$
\begin{aligned}
U^T &= U_o^T + \delta U^T \\
\vec{A}^T &= \vec{A}_o^T + \delta \vec{A}^T \\
\vec{E}^T &= \vec{E}_o^T + \delta \vec{E}^T \\
\vec{B}^T &= \vec{B}_o^T + \delta \vec{B}^T \\
\rho^T &= \rho_o^T + \Delta\rho^A + \Delta\rho \\
\vec{j}^T &= \vec{j}_o^T + \Delta\vec{j}^A + \Delta\vec{j} ,
\end{aligned}
\qquad \text{(II-5)}
$$

where $\Delta\rho$ and $\Delta\vec{j}$ are the induced charge and current densities of the macroscopic system, while $\Delta\rho^A$ and $\Delta\vec{j}^A$ are the "externally" applied perturbing sources. From equations (I-1, 2) and (II-4, 5) it is easily seen that the following equations apply:

$$
\begin{aligned}
\nabla \cdot \delta \vec{E}^T &= 4\pi (\Delta \rho^A + \Delta \rho) \\
\nabla \cdot \delta \vec{B}^T &= 0 \\
\nabla \times \delta \vec{E}^T &= -\frac{1}{c} \frac{\partial}{\partial t} \delta \vec{B}^T \\
\nabla \times \delta \vec{B}^T &= \frac{1}{c} \frac{\partial}{\partial t} \delta \vec{E}^T + \frac{4\pi}{c} (\Delta\vec{j}^A + \Delta\vec{j}) \\
\delta \vec{E}^T &= -\nabla \delta U^T - \frac{1}{c} \frac{\partial}{\partial t} \delta \vec{A}^T \\
\delta \vec{B}^T &= \nabla \times \delta \vec{A}^T
\end{aligned}
\qquad \text{(II-6)}
$$

The third and fourth equations in (II-6) may be combined to result in a wave equation relating $\delta \vec{E}^T$ and $\Delta\vec{j}$:

$$
\nabla \times [\nabla \times \delta \vec{E}^T] + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \delta \vec{E}^T + \frac{4\pi}{c^2} \frac{\partial}{\partial t} [\Delta\vec{j}^A + \Delta\vec{j}] = 0.
$$

(II-7)

For future reference the following notations and summation convention are introduced:

$$
\begin{aligned}
Q(1) &\equiv Q(\vec{r}_1, t_1) \\
Q^*(1) &\equiv \text{complex conjugate of } Q(1) \\
d(1) &\equiv d^3\vec{r}_1\, dt_1 \\
\delta^4(1,2) &\equiv \delta^3(\vec{r}_1 - \vec{r}_2)\, \delta(t_1 - t_2) \\
q_{\ell m} a_m &\equiv \sum_{m=1}^{3} q_{\ell m} a_m
\end{aligned}
\qquad \text{(II-8)}
$$

where $Q(\vec{r}, t)$ represents any function of the space-time co-ordinates $\vec{r}$ and t.

The quantities ρ and $\vec{j}$ are now written in a functional series expansion (Taylor series; see reference (28) and Mathematical Appendix I) with ρ and $\vec{j}$ considered as functionals of the potentials U^T and $\vec{A}^T$ (see equations (II-1)):

$$
\rho = \rho_o[U_o^T, \vec{A}_o^T] + \Delta\rho[\delta U^T, \delta\vec{A}^T]
$$

$$
\vec{j} = \vec{j}_o[U_o^T, \vec{A}_o^T] + \Delta\vec{j}[\delta U^T, \delta\vec{A}^T],
$$

where

$$
\Delta\rho = \delta\rho + \delta^2\rho + \cdots
$$

$$
\Delta\vec{j} = \delta\vec{j} + \delta^2\vec{j} + \cdots,
$$

with

$$
\delta\rho(1) = \int d(2) \left[\frac{\delta\rho(1)}{\delta U^T(2)}\bigg|_o \delta U^T(2) + \frac{\delta\rho(1)}{\delta A_\ell^T(2)}\bigg|_o \delta A_\ell^T(2) \right]
$$

$$\delta\vec{j}(1) = \int d(2)\left[\frac{\delta\vec{j}(1)}{\delta U^T(2)_o}\,\delta U^T(2) + \frac{\delta\vec{j}(1)}{\delta A_\ell^T(2)_o}\,\delta A_\ell^T(2)\right]$$

$$\delta^2\rho(1) = \frac{1}{2!}\int d(2)d(3)\left[\frac{\delta^2\rho(1)}{\delta U^T(2)\delta U^T(3)_o}\,\delta U^T(2)\delta U^T(3)\right.$$

$$+ 2\,\frac{\delta^2\rho(1)}{\delta U^T(2)\delta A_\ell^T(3)_o}\,\delta U^T(2)\delta A_\ell^T(3) \qquad \text{(II-9)}$$

$$\left.+ \frac{\delta^2\rho(1)}{\delta A_\ell^T(2)\delta A_m^T(3)_o}\,\delta A_\ell^T(2)\delta A_m^T(3)\right]$$

$$\delta^2\vec{j}(1) = \frac{1}{2!}\int d(2)d(3)\left[\frac{\delta^2\vec{j}(1)}{\delta U^T(2)\delta U^T(3)_o}\,\delta U^T(2)\delta U^T(3)\right.$$

$$+ 2\,\frac{\delta^2\vec{j}(1)}{\delta U^T(2)\delta A_\ell^T(3)_o}\,\delta U^T(2)\delta A_\ell^T(3)$$

$$\left.+ \frac{\delta^2\vec{j}(1)}{\delta A_\ell^T(2)\delta A_m^T(3)_o}\,\delta A_\ell^T(2)\delta A_m^T(3)\right],$$

etc., where the expansion coefficients are "functional derivatives" (or "variational derivatives"), and where the following defining relations apply:

$$\frac{\delta U^T(1)}{\delta U^T(2)} = \delta^4(1,2)$$

$$\frac{\delta U^T(1)}{\delta A_m^T(2)} = 0 \qquad \text{(II-10)}$$

$$\frac{\delta A_\ell^T(1)}{\delta U^T(2)} = 0$$

$$\frac{\delta A_\ell^T(1)}{\delta A_m^T(2)} = \delta_{\ell m}\, \delta^4(1, 2) \; .$$

The integrations in (II-9) are taken over all space and time. It is quite possible that there should be some restrictions on the magnitudes of δU^T and $\delta \vec{A}^T$ for the series (II-9) to converge, however, the mathematical criteria for convergence of such expansions are not discussed here and in actual calculations in this paper convergence is simply assumed.

The dependence of $\delta\rho$ and $\delta\vec{j}$ as functionals of $\delta\vec{E}^T$ and $\delta\vec{B}^T$ was accomplished by Ashby (reference (3), Chapter III) for the case of isotropic media with the aid of several connecting relations between the expansion coefficients (or functional derivatives) in the equations (II-9). A similar procedure is utilized here in this discussion, resulting in the development of relations for $\delta\rho$ and $\delta\vec{j}$ of greater generality. These connecting relations for the functional derivatives of ρ and $\vec{j}$ are obtained from the equation of continuity and from gauge transformations (see Mathematical Appendix II). For the first order "variations," these relations are given by the following:

$$\frac{\partial}{\partial t_1} \frac{\delta\rho(1)}{\delta U^T(2)_o} + \nabla_1 \cdot \frac{\delta\vec{j}(1)}{\delta U^T(2)_o} = 0$$

$$\frac{\partial}{\partial t_1} \frac{\delta\rho(1)}{\delta A_\ell^T(2)_o} + \nabla_1 \cdot \frac{\delta\vec{j}(1)}{\delta A_\ell^T(2)_o} = 0 \qquad \text{(II-11)}$$

$$-\frac{1}{c}\frac{\partial}{\partial t_2}\frac{\delta\rho(1)}{\delta U^T(2)_o}+\nabla_{2\ell}\frac{\delta\rho(1)}{\delta A_\ell^T(2)_o}=0$$

$$-\frac{1}{c}\frac{\partial}{\partial t_2}\frac{\delta\vec{j}(1)}{\delta U^T(2)_o}+\nabla_{2\ell}\frac{\delta\vec{j}(1)}{\delta A_\ell^T(2)_o}=0\,.$$

Similar relations may be obtained for the higher order variations. The Fourier transforms for δU^T, $\delta\vec{A}^T$, $\delta\vec{E}^T$, $\delta\vec{B}^T$, and the functional derivatives are now defined in order to make use of equations (II-9). The required Fourier expansions are given as follows:

$$\delta U^T(1)\equiv\int\frac{d^3\vec{k}\,d\omega}{(2\pi)^4}\,u^T(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{r}_1-\omega t_1)}$$

$$\delta\vec{A}^T(1)\equiv\int\frac{d^3\vec{k}\,d\omega}{(2\pi)^4}\,\vec{a}^T(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{r}_1-\omega t_1)}$$

$$\delta\vec{E}^T(1)\equiv\int\frac{d^3\vec{k}\,d\omega}{(2\pi)^4}\,\vec{e}^T(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{r}_1-\omega t_1)}\qquad\text{(II-12)}$$

$$\delta\vec{B}^T(1)\equiv\int\frac{d^3\vec{k}\,d\omega}{(2\pi)^4}\,\vec{b}^T(\vec{k},\omega)\,e^{i(\vec{k}\cdot\vec{r}_1-\omega t_1)}$$

and

$$\frac{\delta\rho(1)}{\delta U^T(2)_o}\equiv\int\frac{d^3\vec{k}_1\,d\omega_1\,d^3\vec{k}_2\,d\omega_2}{(2\pi)^8}\,f(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2)\,e^{i(\vec{k}_1\cdot\vec{r}_1-\omega_1t_1-\vec{k}_2\cdot\vec{r}_2+\omega_2t_2)}$$

$$\frac{\delta\rho(1)}{\delta A^T_m(2)_o} \equiv \int \frac{d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2}{(2\pi)^8} g_m(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) e^{i(\vec{k}_1\cdot\vec{r}_1-\omega_1 t_1-\vec{k}_2\cdot\vec{r}_2+\omega_2 t_2)}$$

(II-13)

$$\frac{\delta j_\ell(1)}{\delta U^T(2)_o} \equiv \int \frac{d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2}{(2\pi)^8} p_\ell(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) e^{i(\vec{k}_1\cdot\vec{r}_1-\omega_1 t_1-\vec{k}_2\cdot\vec{r}_2+\omega_2 t_2)}$$

$$\frac{\delta j_\ell(1)}{\delta A^T_m(2)_o} \equiv \int \frac{d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2}{(2\pi)^8} q_{\ell m}(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) e^{i(\vec{k}_1\cdot\vec{r}_1-\omega_1 t_1-\vec{k}_2\cdot\vec{r}_2+\omega_2 t_2)}$$

The use of the equations (II-12) together with equations (II-6) results in the equivalent equations for the Fourier transforms:

$$\vec{e}^{\,T}(\vec{k},\omega) = \frac{i\omega}{c}\,\vec{a}^{\,T}(\vec{k},\omega) - i\vec{k}\,u^T(\vec{k},\omega)$$

(II-14)

$$\vec{b}^{\,T}(\vec{k},\omega) = i\,\vec{k}\times\vec{a}^{\,T}(\vec{k},\omega)\ .$$

Substitution of the equations (II-13) into the equations (II-11) results in the Fourier equivalents of the connecting relations:

$$-i\omega_1 f(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) + i k_{1\ell}\, p_\ell(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) = 0$$

$$-i\omega_1 g_m(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) + i k_{1\ell}\, q_{\ell m}(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) = 0$$

(II-15)

$$-\frac{i\omega_2}{c} f(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) - i k_{2m} g_m(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) = 0$$

$$-\frac{i\omega_2}{c} p_\ell(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) - i k_{2m} q_{\ell m}(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) = 0\ .$$

Substitution of the equations (II-12) and (II-13) into the equations for $\delta\rho(1)$ and $\delta\vec{j}(1)$ in (II-9) leads to

$$\delta\rho(1) = \int \frac{d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2}{(2\pi)^8} \Big[f(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2)\, u^T(\vec{k}_2, \omega_2)$$

$$+ g_m(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2)\, a_m^T(\vec{k}_2, \omega_2) \Big] e^{i(\vec{k}_1 \cdot \vec{r}_1 - \omega_1 t_1)}$$

and

$$\delta j_\ell(1) = \int \frac{d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2}{(2\pi)^8} \Big[p_\ell(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2)\, u^T(\vec{k}_2, \omega_2)$$

$$+ q_{\ell m}(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2)\, a_m^T(\vec{k}_2, \omega_2) \Big] e^{i(\vec{k}_1 \cdot \vec{r}_1 - \omega_1 t_1)}. \tag{II-16}$$

By combining the last pair of connecting relations (II-15) with the equations (II-16), one obtains

$$\delta\rho(1) = \int \frac{d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2}{(2\pi)^8} \left(\frac{c}{i\omega_2}\right) g_m(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2) \Big[\frac{i\omega_2}{c} a_m^T(\vec{k}_2, \omega_2)$$

$$- i k_{2m} u^T(\vec{k}_2, \omega_2) \Big] e^{i(\vec{k}_1 \cdot \vec{r}_1 - \omega_1 t_1)} \tag{II-17}$$

$$\delta j_\ell(1) = \int \frac{d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2}{(2\pi)^8} \left(\frac{c}{i\omega_2}\right) q_{\ell m}(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2) \Big[\frac{i\omega_2}{c} a_m^T(\vec{k}_2, \omega_2)$$

$$- i k_{2m} u^T(\vec{k}_2, \omega_2) \Big] e^{i(\vec{k}_1 \cdot \vec{r}_1 - \omega_1 t_1)}.$$

The substitution of the first equation (II-14) into (II-17) results in the simplification

$$\delta\rho(1) = \int \frac{d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2}{(2\pi)^8} \left(\frac{c}{i\omega_2}\right) g_m(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2)\, e^T_m(\vec{k}_2, \omega_2)\, e^{i(\vec{k}_1 \cdot \vec{r}_1 - \omega_1 t_1)}$$

$$\delta j_\ell(1) = \int \frac{d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2}{(2\pi)^8} \left(\frac{c}{i\omega_2}\right) q_{\ell m}(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2)\, e^T_m(\vec{k}_2, \omega_2)\, e^{i(\vec{k}_1 \cdot \vec{r}_1 - \omega_1 t_1)} . \qquad \text{(II-18)}$$

Now, by substituting the equations for the Fourier coefficients from (II-12) and (II-13) into the equations (II-18), one obtains

$$\delta\rho(1) = \int K^T_m(1,2)\, \delta E^T_m(2)\, d(2)$$

$$\delta j_\ell(1) = \int S^T_{\ell m}(1,2)\, \delta E^T_m(2)\, d(2) \qquad \text{(II-19)}$$

where

$$K^T_m(1,2)$$

$$\equiv \int \frac{d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2}{(2\pi)^8} \left(\frac{c}{i\omega_2}\right) g_m(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2)\, e^{i(\vec{k}_1 \cdot \vec{r}_1 - \omega_1 t_1 - \vec{k}_2 \cdot \vec{r}_2 + \omega_2 t_2)}$$

$$= \int \frac{d^3\vec{k}\, d\omega}{(2\pi)^4}\, d(3) \left(\frac{c}{i\omega}\right) \frac{\delta\rho(1)}{\delta A^T_m(3)}\bigg|_o\, e^{i\vec{k}\cdot(\vec{r}_3 - \vec{r}_2) - i\omega(t_3 - t_2)}$$

and

$$S^T_{\ell m}(1,2)$$

$$\equiv \int \frac{d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2}{(2\pi)^8} \left(\frac{c}{i\omega_2}\right) q_{\ell m}(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2)\, e^{i(\vec{k}_1 \cdot \vec{r}_1 - \omega_1 t_1 - \vec{k}_2 \cdot \vec{r}_2 + \omega_2 t_2)}$$

$$= \int \frac{d^3\vec{k}\, d\omega}{(2\pi)^4}\, d(3) \left(\frac{c}{i\omega}\right) \frac{\delta j_\ell(1)}{\delta A^T_m(3)}\bigg|_o\, e^{i\vec{k}\cdot(\vec{r}_3 - \vec{r}_2) - i\omega(t_3 - t_2)} .$$

The equations (II-19) form the results upon which this thesis is based; they are formal relations for the $\delta\rho$ and $\delta\vec{j}$ of arbitrary systems in terms of the basic field vectors, and they give $\Delta\rho$ and $\Delta\vec{j}$ up to terms which are linear in the fields. That only the electric field vector should appear in (II-19) is an interesting result, and is due to the fact that $\delta\vec{E}^T$ and $\delta\vec{B}^T$ are not independent, but are linked together by Maxwell's equations (or by the field potentials δU^T and δA^T). In principle, the equations (II-19) show that the relations between the charge density $\delta\rho$ and the fields, and between the current density $\delta\vec{j}$ and the fields are "nonlocal" in the space-time co-ordinates.

It is possible to achieve a formal simplification of the equations (II-19) in the case where the time dependence of $(\delta\rho(1))/(\delta A_m^T(2)_o)$ and $(\delta j_\ell(1))/(\delta A_m^T(2)_o)$ is of the form of the difference $(t_1 - t_2)$ of their time co-ordinates t_1 and t_2. The time derivatives of $\delta\rho$ and $\delta\vec{j}$ in (II-19) are

$$\frac{\partial}{\partial t_1}\delta\rho(1) = \int\left[\frac{\partial}{\partial t_1}K_m^T(1,2)\right]\delta E_m^T(2)\,d(2)$$

$$\frac{\partial}{\partial t_1}\delta j_\ell(1) = \int\left[\frac{\partial}{\partial t_1}S_{\ell m}^T(1,2)\right]\delta E_m^T(2)\,d(2) \qquad \text{(II-20)}$$

where

$$\frac{\partial}{\partial t_1}K_m^T(1,2)$$

$$= -c\int\frac{d^3\vec{k}_1\,d\omega_1\,d^3\vec{k}_2\,d\omega_2}{(2\pi)^8}\left(\frac{\omega_1}{\omega_2}\right)g_m(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2)\,e^{i(\vec{k}_1\cdot\vec{r}_1-\omega_1 t_1-\vec{k}_2\cdot\vec{r}_2+\omega_2 t_2)}$$

$$\frac{\partial}{\partial t_1} S^T_{\ell m}(1,2)$$

$$= -c\int \frac{d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2}{(2\pi)^8}\left(\frac{\omega_1}{\omega_2}\right) q_{\ell m}(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2)\, e^{i(\vec{k}_1\cdot\vec{r}_1-\omega_1 t_1-\vec{k}_2\cdot\vec{r}_2+\omega_2 t_2)}.$$

The condition that $(\delta\rho(1))/(\delta U^T(2)_o)$, $(\delta\vec{j}(1))/(\delta U^T(2)_o)$, $(\delta\rho(1))/(\delta A^T_m(2)_o)$ and $(\delta\vec{j}(1))/(\delta A^T_m(2)_o)$ depend upon the time difference $(t_1 - t_2)$ requires that f, g_m, p_ℓ, and $q_{\ell m}$ be of the form:

$$f(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) = (2\pi)\,\delta(\omega_1-\omega_2)\,\bar{f}(\vec{k}_1,\vec{k}_2,\omega_1)$$

$$g_m(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) = (2\pi)\,\delta(\omega_1-\omega_2)\,\bar{g}_m(\vec{k}_1,\vec{k}_2,\omega_1)$$

$$p_\ell(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) = (2\pi)\,\delta(\omega_1-\omega_2)\,\bar{p}_\ell(\vec{k}_1,\vec{k}_2,\omega_1) \qquad \text{(II-21)}$$

$$q_{\ell m}(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2) = (2\pi)\,\delta(\omega_1-\omega_2)\,\bar{q}_{\ell m}(\vec{k}_1,\vec{k}_2,\omega_1).$$

By substituting the equations (II-21) into (II-13) and (II-20), one obtains

$$\frac{\delta\rho(1)}{\delta U^T(2)_o} = \int \frac{d^3\vec{k}_1 d^3\vec{k}_2 d\omega}{(2\pi)^7}\,\bar{f}(\vec{k}_1,\vec{k}_2,\omega)\,e^{i(\vec{k}_1\cdot\vec{r}_1-\vec{k}_2\cdot\vec{r}_2)-i\omega(t_1-t_2)}$$

$$\frac{\delta\rho(1)}{\delta A^T_m(2)_o} = \int \frac{d^3\vec{k}_1 d^3\vec{k}_2 d\omega}{(2\pi)^7}\,\bar{g}_m(\vec{k}_1,\vec{k}_2,\omega)\,e^{i(\vec{k}_1\cdot\vec{r}_1-\vec{k}_2\cdot\vec{r}_2)-i\omega(t_1-t_2)}$$

$$\frac{\delta j_\ell(1)}{\delta U^T(2)_o} = \int \frac{d^3\vec{k}_1 d^3\vec{k}_2 d\omega}{(2\pi)^7}\,\bar{p}_\ell(\vec{k}_1,\vec{k}_2,\omega)\,e^{i(\vec{k}_1\cdot\vec{r}_1-\vec{k}_2\cdot\vec{r}_2)-i\omega(t_1-t_2)} \qquad \text{(II-22)}$$

$$\frac{\delta j_\ell(1)}{\delta A^T_m(2)_o} = \int \frac{d^3\vec{k}_1 d^3\vec{k}_2 d\omega}{(2\pi)^7}\,\bar{q}_{\ell m}(\vec{k}_1,\vec{k}_2,\omega)\,e^{i(\vec{k}_1\cdot\vec{r}_1-\vec{k}_2\cdot\vec{r}_2)-i\omega(t_1-t_2)}$$

and

$$\frac{\partial}{\partial t_1} K^T_m(1,2) = -c \int \frac{d^3\vec{k}_1 d^3\vec{k}_2 d\omega}{(2\pi)^7} \bar{g}_m(\vec{k}_1, \vec{k}_2, \omega)\, e^{i(\vec{k}_1\cdot\vec{r}_1 - \vec{k}_2\cdot\vec{r}_2) - i\omega(t_1 - t_2)}$$

(II-23)

$$\frac{\partial}{\partial t_1} S^T_{\ell m}(1,2) = -c \int \frac{d^3\vec{k}_1 d^3\vec{k}_2 d\omega}{(2\pi)^7} \bar{q}_{\ell m}(\vec{k}_1, \vec{k}_2, \omega)\, e^{i(\vec{k}_1\cdot\vec{r}_1 - \vec{k}_2\cdot\vec{r}_2) - i\omega(t_1 - t_2)}$$

From a comparison of the equations (II-22) with the equations (II-23), it follows that

$$\frac{\partial}{\partial t_1} K^T_m(1,2) = -c \frac{\delta\rho(1)}{\delta A^T_m(2)_o}$$

(II-24)

$$\frac{\partial}{\partial t_1} S^T_{\ell m}(1,2) = -c \frac{\delta j_\ell(1)}{\delta A^T_m(2)_o} ,$$

and that

$$\frac{\partial}{\partial t_1} . \delta\rho(1) = -c \int \frac{\delta\rho(1)}{\delta A^T_m(2)_o} \delta E^T_m(2)\, d(2)$$

(II-25)

$$\frac{\partial}{\partial t_1} \delta j_\ell(1) = -c \int \frac{\delta j_\ell(1)}{\delta A^T_m(2)_o} \delta E^T_m(2)\, d(2) .$$

Since the functional derivatives enter explicitly in the equations (II-25), these equations (II-25) are of value in calculations involving systems for which the Hamiltonian is time-independent.

The higher order "variations" of ρ and $\vec{j}$ may be developed by a procedure similar to that which resulted in the equations (II-19) and (II-25). Thus the series may be considered to any desired order in the fields. Also, the results obtained here are not strictly dependent upon the use of Fourier expansions. Any complete orthonormal set of basis functions will lead to the same results. Therefore, a representation most convenient to a specific problem may be chosen.

It may be seen without too much difficulty that if one considers ρ and $\vec{j}$ as functionals of U^A and $\vec{A}^A$ (II-2), the end result is

$$\begin{aligned} \delta\rho(1) &= \int K_m^A(1,2)\, \delta E_m^A(2)\, d(2) \\ \delta j_\ell(1) &= \int S_{\ell m}^A(1,2)\, \delta E_m^A(2)\, d(2) \end{aligned} \tag{II-26}$$

where $K_\ell^A(1,2)$ and $S_{\ell m}^A(1,2)$ have definitions similar to the corresponding quantities in (II-19). Similarly, if one begins with (II-3) the final result is

$$\begin{aligned} \delta\rho(1) &= \int K_m(1,2)\, \delta E_m(2)\, d(2) \\ \delta j_\ell(1) &= \int S_{\ell m}(1,2)\, \delta E_m(2)\, d(2) \end{aligned} \tag{II-27}$$

with $K_m(1,2)$ and $S_{\ell m}(1,2)$ again defined similarly to the corresponding quantities in (II-19). From the requirements of causality, one may expect the following condition:

$$\frac{\delta\rho(1)}{\delta U^A(2)_o} = \frac{\delta\rho(1)}{\delta A_m^A(2)} = \frac{\delta j_\ell(1)}{\delta U^A(2)_o} = \frac{\delta j_\ell(1)}{\delta A_m^A(2)_o} \equiv 0 \tag{II-28}$$

for $c^2(t_1 - t_2)^2 - (\vec{r}_1 - \vec{r}_2)^2 \leq 0$ and $(t_1 - t_2) < 0$. Thus, in particular, the time integration in equations (II-26) extends only over the region $t_1 > t_2$.

Now, from the equations (II-9) and (II-19) it is clear that $\delta\rho(1)$ and $\delta\vec{j}(1)$ are linear in δU^T and $\delta\vec{A}^T$, and the equations (II-9) also show that the second order variations $\delta^2\rho$ and $\delta^2\vec{j}$ are bilinear in δU^T and $\delta\vec{A}^T$, and so forth. Thus, to the first order the equation (II-7) and the first of equations (II-6) reduce to

$$\nabla_1 \times [\nabla_1 \times \delta\vec{E}^T(1)] + \frac{1}{c^2}\frac{\partial^2}{\partial t_1^2}\delta\vec{E}^T(1) + \frac{4\pi}{c^2}\frac{\partial}{\partial t_1}\delta\vec{j}(1) + \frac{4\pi}{c^2}\frac{\partial}{\partial t_1}\Delta\vec{j}^A(1) = 0, \tag{II-29}$$

and

$$\nabla_1 \cdot \delta\vec{E}^T(1) = 4\pi\,\delta\rho(1) + 4\pi\,\Delta\rho^A(1)$$

or (II-30)

$$\nabla_1 \cdot \frac{\partial}{\partial t_1}\delta\vec{E}^T(1) = 4\pi\frac{\partial}{\partial t_1}\delta\rho(1) + 4\pi\frac{\partial}{\partial t_1}\Delta\rho^A(1),$$

with $\delta\rho$ and $\delta\vec{j}$ given either by the equations (II-19) or by equations (II-25) as appropriate. Since the quantities $\delta\rho$ and $\delta\vec{j}$ are generally nonlocal in $\delta\vec{E}^T$, the equations (II-29) and (II-30) are inhomogeneous linear integro-differential equations for the field $\delta\vec{E}^T$. This linear equation is justified only for small values of $\delta\vec{E}^T$.

The quantity $S^T_{\ell m}(1, 2)$ is called the "conductance tensor." In the case that the system is translationally invariant, $S^T_{\ell m}(1, 2) \equiv S^T_{\ell m}(\vec{r}_1 - \vec{r}_2, t_1 - t_2)$, and one may define the Fourier transform $s^T_{\ell m}(\vec{k}, \omega)$ by the equation

$$S^T_{\ell m}(1, 2) \equiv \int \frac{d^3\vec{k}\,d\omega}{(2\pi)^4}\, s^T_{\ell m}(\vec{k}, \omega)\, e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)-i\omega(t_1-t_2)}. \tag{II-31}$$

The considerations involved in this present chapter apply to the individual terms ρ_F, $\vec{j}_F$, ρ_P, $\vec{j}_P$, ρ_M and $\vec{j}_M$ (I-13) as well as to ρ and $\vec{j}$ since the gauge transformations and the equation of continuity hold true in each individual case

$$\frac{\partial\rho_F}{\partial t} + \nabla\cdot\vec{j}_F = 0$$

$$\frac{\partial\rho_P}{\partial t} + \nabla\cdot\vec{j}_P = 0 \tag{II-32}$$

$$\frac{\partial\rho_M}{\partial t} + \nabla\cdot\vec{j}_M = 0\,.$$

CHAPTER III

NON-RELATIVISTIC QUANTUM ELECTRODYNAMICS AND THE ELECTRON GAS

The techniques of quantum field theory are utilized here in the investigation of the behavior of a uniformly magnetized electron gas which is unbounded and infinite in extent and is acted upon by electromagnetic disturbances. A background of static positive charge is superimposed upon the electron gas for overall charge neutrality and stability. The system is assumed to be initially in a state of thermodynamic equilibrium at all times t prior to some time t_o. The references (1), (2), (3), (9), (29) and (30) constitute general sources on matters related to quantum field theory and its application to statistical physics.

This "Many-Body" system of electrons constitutes a particle field with the creation operator $\psi^{\dagger}(\vec{r}, t)$ and the annihilation operator $\psi(\vec{r}, t)$ in the Heisenberg representation. The electrons interact with each other, with the positive charge background, and with an "externally" applied electromagnetic field. The interactions between the particles are characterized by "direct interaction at a distance" (local in time) to the order $\left(\frac{v}{c}\right)^2$. (See references (31) through (37) regarding "direct interactions" and "velocity-dependent potentials.") This formulation of the quantum electrodynamics of the many-electron system has its limitations, but it is advantageous in avoiding many of the complexities and difficulties of a quantized field theory of electromagnetic radiation. The "nonlinearities" of the electromagnetic field (reference (27)) and the effects of creation and annihilation of particles are not within the scope of the present paper.

The following definitions and the summation convention for sums over the spin indices (Greek letter subscripts) are listed for future reference:

$$e \equiv |e| \quad \text{(absolute value of the electronic charge)}$$

$$e\,n(\vec{r}) \equiv \text{charge density of the positive charge background}$$

$$\vec{P}(\vec{r}_1, t_1) \equiv \frac{\hbar}{i} \nabla_1 + \frac{e}{c} \vec{A}^A(\vec{r}_1, t_1)$$

$$\vec{P}^*(\vec{r}_1, t_1) \equiv -\frac{\hbar}{i} \nabla_1 + \frac{e}{c} \vec{A}^A(\vec{r}_1, t_1) \qquad \text{(III-1)}$$

$$\vec{r}_{12} \equiv -\vec{r}_{21} \equiv \vec{r}_1 - \vec{r}_2$$

$$r_{12} \equiv r_{21} \equiv |\vec{r}_{12}|$$

$$\sum_{\beta=1}^{2} Q_{\alpha\beta} A_\beta \equiv Q_{\alpha\beta} A_\beta$$

$$\mu \equiv 1 + 2g ,$$

where $2g$ is the "radiative correction" for the anomalous magnetic moment of the electron ($g > 0$, $g << 1$, $\mu = 1.00116$; see reference (38)).

The Hamiltonian $\hat{H}(t)$ that we use here is essentially the second quantized form of the classical Darwin Hamiltonian (reference (31)) generalized to the many-electron system. The Pauli spin contributions (reference (35)) are included. In a straightforward manner (see Mathematical Appendix III) one obtains, in the non time retarded limit (reference (36)), the following expression for $\hat{H}(t)$:

$$\hat{H}(t) = \hat{H}_1(t) + \hat{H}_2(t) + \hat{H}_3(t) , \qquad \text{(III-2)}$$

where

$$\hat{H}_1(t) \equiv \int d^3\vec{r}_1 \, \psi^\dagger_\alpha(\vec{r}_1, t) \Big\{ \delta_{\alpha\beta} \frac{1}{2m} \left[\vec{P}(\vec{r}_1, t \right]^2 + \frac{\mu\hbar e}{2mc} \vec{\sigma}_{\alpha\beta} \cdot \vec{B}^A(\vec{r}_1, t)$$

$$- \delta_{\alpha\beta} \, e U^A(\vec{r}_1, t) \Big\} \psi_\beta(\vec{r}_1, t) ,$$

$$\hat{H}_2(t) \equiv \frac{1}{2} \int\int d^3\vec{r}_1 d^3\vec{r}_2 \, \psi^\dagger_\alpha(\vec{r}_1, t) \Big\{ \psi^\dagger_\lambda(\vec{r}_2, t) \frac{e^2}{r_{12}} \psi_\lambda(\vec{r}_2, t)$$

$$- 2 \frac{e^2 n(\vec{r}_2)}{r_{12}} \Big\} \psi_\alpha(\vec{r}_1, t) ,$$

and

$$\hat{H}_3(t) \equiv \frac{1}{2} \left(\frac{e}{2mc} \right) \int d^3\vec{r}_1 \, \psi^\dagger_\alpha(\vec{r}_1, t) \Big\{ \delta_{\alpha\beta} \vec{P}(\vec{r}_1, t) \cdot \hat{\vec{A}}(\vec{r}_1, t)$$

$$+ \delta_{\alpha\beta} \hat{\vec{A}}(\vec{r}_1, t) \cdot \vec{P}(\vec{r}_1, t) + \mu\hbar \, \vec{\sigma}_{\alpha\beta} \cdot \left[\nabla_1 \times \hat{\vec{A}}(\vec{r}_1, t) \right] \Big\} \psi_\beta(\vec{r}_1, t) .$$

The operator $\hat{\vec{A}}(\vec{r}, t)$ is given by the definition

$$\hat{\vec{A}}(\vec{r}_1, t) \equiv -\frac{e}{2mc} \int d^3\vec{r}_2\, \psi^\dagger_\gamma(\vec{r}_2, t) \left\{ \frac{\delta_{\gamma\lambda}}{r_{12}} \vec{P}(\vec{r}_2, t) + \vec{P}(\vec{r}_2, t) \frac{\delta_{\gamma\lambda}}{r_{12}} + \mu\hbar\, \vec{\sigma}_{\gamma\lambda} \times \frac{\vec{r}_{12}}{r_{12}^3} \right\} \psi_\lambda(\vec{r}_2, t). \tag{III-3}$$

It is also convenient to define another operator $\hat{U}$ such that

$$\hat{U}(\vec{r}_1, t) \equiv \int d^3\vec{r}_2 \left\{ -\psi^\dagger_\lambda(\vec{r}_2, t) \frac{e}{r_{12}} \psi_\lambda(\vec{r}_2, t) + \frac{e\, n(\vec{r}_2)}{r_{12}} \right\}. \tag{III-4}$$

The operators $\hat{U}$ and $\hat{\vec{A}}$ are so defined for reasons which will become clear in retrospect. The "radiative correction" 2g is included in $\hat{H}(t)$ even though it is small because of some interesting results arising from its presence. The one-particle operator $\hat{H}_1$ is the usual form of Hamiltonian for electrons acted upon by an "externally" applied electromagnetic field, the operator $\hat{H}_2$ combines the two-particle electrostatic Coulomb interactions of the electrons and the Coulomb interactions between the electrons and the positive charges, while $\hat{H}_3$ is an operator containing the "magnetic interactions" between the electrons in the lowest order non-time retarded form (reference (36)). The "magnetic interactions" probably make only a slight contribution, but they are quite essential to the formulation of the charge density ρ and the current density $\vec{j}$ as functionals of the total field potentials U^T and $\vec{A}^T$.

The particle number operator $\hat{N}$ for the electrons is given by the equation:

$$\hat{N} = \int d^3\vec{r}_1 \, \psi_a^\dagger(\vec{r}_1, t) \, \psi_a(\vec{r}_1, t). \qquad \text{(III-5)}$$

We shall be considering Grand Canonical Ensemble averages and it is convenient to define a new Hamiltonian $\hat{\mathscr{H}}(t)$ for this purpose. The operator $\zeta\hat{N}$ represents an arbitrary shift in the origin of the energy for eigenstates of both the operators $\hat{H}(t)$ and $\hat{N}$ where ζ is a constant taken to be equal to the chemical potential for the electrons when the system is initially in a state of thermodynamic equilibrium before the time t_o. The Hamiltonian for the system may therefore be redefined as follows:

$$\hat{\mathscr{H}}(t) = \hat{H}(t) - \zeta\hat{N}. \qquad \text{(III-6)}$$

An electron number density operator may be defined from (III-5) by the prescription

$$\hat{N} \equiv \int d^3\vec{r}_1 \, \hat{\rho}_N(\vec{r}_1, t_1), \qquad \text{(III-7)}$$

so that

$$\hat{\rho}_N(\vec{r}_1, t_1) \equiv \hat{\rho}_N(1) = \psi_a^\dagger(\vec{r}_1, t_1) \, \psi_a(\vec{r}_1, t_1).$$

Similarly, an electron energy density operator may be defined from (III-2) by the prescription

$$\hat{H}(t_1) \equiv \int d^3\vec{r}_1 \, \hat{h}_v(\vec{r}_1, t_1) \qquad \text{(III-8)}$$

so that

$$\hat{h}_v(\vec{r}_1, t_1) \equiv \hat{h}_v(1) = \psi^\dagger_\alpha(\vec{r}_1, t_1)\Big\{\delta_{\alpha\beta}\,\frac{1}{2m}\left[\vec{P}(\vec{r}_1, t_1)\right]^2 + \frac{\mu\hbar e}{2mc}\,\vec{\sigma}_{\alpha\beta}\cdot\vec{B}^A(\vec{r}_1,t_1)$$

$$- \delta_{\alpha\beta}\, e\, U^A(\vec{r}_1, t_1) + \frac{1}{2}\,\delta_{\alpha\beta}\int d^3\vec{r}_2\left[\psi^\dagger_\lambda(\vec{r}_2, t)\,\frac{e^2}{r_{12}}\,\psi_\lambda(\vec{r}_2, t)^2 - \frac{e^2 n(\vec{r}_2)}{r_{12}}\right]$$

$$+ \delta_{\alpha\beta}\,\frac{e}{4mc}\,\vec{P}(\vec{r}_1, t_1)\cdot\hat{\vec{A}}(\vec{r}_1, t_1) + \delta_{\alpha\beta}\,\frac{e}{4mc}\,\hat{\vec{A}}(\vec{r}_1, t_1)\cdot\vec{P}(\vec{r}_1, t_1)$$

$$+ \frac{\mu\hbar e}{4mc}\,\vec{\sigma}_{\alpha\beta}\cdot\left[\nabla_1\times\hat{\vec{A}}(\vec{r}_1, t_1)\right]\Big\}\,\psi_\beta(\vec{r}_1, t_1)\,.$$

The field operators $\psi^\dagger$ and ψ obey the following anti-commutation relations (Fermion field):

$$\begin{aligned}
\psi_\alpha(\vec{r}_1, t)\,\psi^\dagger_\beta(\vec{r}_2, t) + \psi^\dagger_\beta(\vec{r}_2, t)\,\psi_\alpha(\vec{r}_1, t) &= \delta_{\alpha\beta}\,\delta^3(\vec{r}_1 - \vec{r}_2)\\
\psi_\alpha(\vec{r}_1, t)\,\psi_\beta(\vec{r}_2, t) + \psi_\beta(\vec{r}_2, t)\,\psi_\alpha(\vec{r}_1, t) &= 0 \qquad\qquad \text{(III-9)}\\
\psi^\dagger_\alpha(\vec{r}_1, t)\,\psi^\dagger_\beta(\vec{r}_2, t) + \psi^\dagger_\beta(\vec{r}_2, t)\,\psi^\dagger_\alpha(\vec{r}_1, t) &= 0\,.
\end{aligned}$$

Time evolution equations for the field operators $\psi^\dagger$ and ψ are given by (Heisenberg representation)

$$\begin{aligned}
i\hbar\,\frac{\partial\psi_\alpha(\vec{r}_1, t_1)}{\partial t_1} &= \left[\psi_\alpha(\vec{r}_1, t_1), \mathcal{H}(t_1)\right]\\
& \qquad\qquad\qquad\qquad \text{(III-10)}\\
i\hbar\,\frac{\partial\psi^\dagger_\alpha(\vec{r}_1, t_1)}{\partial t_1} &= \left[\psi^\dagger_\alpha(\vec{r}_1, t_1), \mathcal{H}(t_1)\right],
\end{aligned}$$

where $[F, H]$ is the commutator:

$$[F,H] = FH - HF. \tag{III-11}$$

Using the Hamiltonian given by (III-6) in the first of equations (III-10), one obtains the equation of motion for ψ:

$$i\hbar \frac{\partial\psi_\alpha(1)}{\partial t_1} = \Big\{ \delta_{\alpha\beta} \frac{1}{2m} [\vec{P}(1)]^2 + \frac{\mu\hbar e}{2mc} \vec{\sigma}_{\alpha\beta} \cdot \vec{B}^A(1) - \delta_{\alpha\beta}\zeta - \delta_{\alpha\beta}\, e U^A(1)$$

$$- \delta_{\alpha\beta}\, e\hat{U}(1) + \delta_{\alpha\beta} \frac{e}{2mc} \vec{P}(1) \cdot \hat{\vec{A}}(1) + \delta_{\alpha\beta} \frac{e}{2mc} \hat{\vec{A}}(1) \cdot \vec{P}(1)$$

$$+ \frac{\mu\hbar e}{2mc} \vec{\sigma}_{\alpha\beta} \cdot [\nabla_1 \times \hat{\vec{A}}(1)] \Big\} \psi_\beta(1) . \tag{III-12}$$

The identity

$$\nabla_2 \frac{1}{r_{12}} = \frac{\vec{r}_{12}}{r_{12}^3} = -\nabla_1 \frac{1}{r_{12}} = -\frac{\vec{r}_{21}}{r_{12}^3}, \tag{III-13}$$

and the anti-commutation relations (III-9) were used in deriving the above expression. The notation defined in (II- 8), $Q(\vec{r}_I, t_1) \equiv Q(1)$, is applied here to operators and functions alike.

To obtain the time evolution equation for $\psi^\dagger$, we assume that $\psi^\dagger$ and ψ both vanish at the limits of the spatial region for which these operators are defined:

$$\psi^\dagger_\alpha(\vec{r}, t) = \psi_\beta(\vec{r}, t) = 0 \text{ for } |\vec{r}| = \infty . \tag{III-14}$$

In the case of periodic systems periodic boundary conditions are applicable.

The Hermitian conjugate of equation (III-12) is the following time evolution equation for $\psi^\dagger$:

$$i\hbar \frac{\partial \psi_\alpha^\dagger(1)}{\partial t_1} = -\Big\{\delta_{\beta\alpha}\frac{1}{2m}\left[\vec{P}^*(1)\right]^2 + \frac{\mu\hbar e}{2mc}\vec{\sigma}_{\beta\alpha}\cdot\vec{B}^A(1) - \delta_{\beta\alpha}\zeta$$

$$- \delta_{\beta\alpha}\, e\, U^A(1)\Big\}\psi_\beta^\dagger(1) + \psi_\alpha^\dagger(1)\, e\, \hat{U}(1) - \frac{e}{2mc}\vec{P}^*(1)\cdot\left[\psi_\alpha^\dagger(1)\hat{\vec{A}}(1)\right] \tag{III-15}$$

$$- \frac{e}{2mc}\left[\vec{P}^*(1)\psi_\alpha^\dagger(1)\right]\cdot\hat{\vec{A}}(1) - \frac{\mu\hbar e}{2mc}\psi_\beta^\dagger(1)\vec{\sigma}_{\beta\alpha}\cdot\left[\nabla_1 \times \hat{\vec{A}}(1)\right].$$

At this point it can be seen that the operator $\hat{\vec{A}}$ may be written in the following form:

$$\hat{\vec{A}}(\vec{r}_1, t_1) = -\int d^3\vec{r}_2 \frac{e}{2mcr_{12}}\Big\{\psi_\lambda^\dagger(\vec{r}_2, t_1)\left[\vec{P}(\vec{r}_2, t_1)\psi_\lambda(\vec{r}_2, t_1)\right]$$

$$+ \left[\vec{P}^*(\vec{r}_2, t_1)\psi_\lambda^\dagger(\vec{r}_2, t_1)\right]\psi_\lambda(\vec{r}_2, t_1) \tag{III-16}$$

$$+ \mu\hbar\left[\nabla_2 \times \psi_\gamma^\dagger(\vec{r}_2, t_1)\vec{\sigma}_{\gamma\lambda}\psi_\lambda(\vec{r}_2, t_1)\right]\Big\}.$$

The charge density operator $\hat{\rho}_e$ for the electrons is given by

$$\hat{\rho}_e(1) = -e\,\psi_\alpha^\dagger(1)\psi_\alpha(1) \tag{III-17}$$

and the total charge density operator $\hat{\rho}$ for the system is given by

$$\hat{\rho}(1) = \hat{\rho}_e(1) + en(\vec{r}_1). \tag{III-18}$$

A current density operator $\hat{\vec{j}}$ may be defined by an equation of continuity:

$$\frac{\partial \hat{\rho}(1)}{\partial t_1} + \nabla_1 \cdot \hat{\vec{j}}(1) = 0 \; . \tag{III-19}$$

Since the positive charge background is stationary,

$$\frac{\partial \hat{\rho}(1)}{\partial t} = \frac{\partial \hat{\rho}_e(1)}{\partial t} \; .$$

Also, $\hat{\vec{j}}$ will represent the electron current density. By combining the equations (III-12) and (III-15) with (III-19) one obtains:

$$i\hbar \frac{\partial \hat{\rho}(1)}{\partial t_1} = -i\hbar \nabla_1 \cdot \left(-\frac{e}{2m}\right)\left\{\psi_a^\dagger(1)\left[\vec{P}(1)\psi_a(1)\right] + \left[\vec{P}^*(1)\psi_a^\dagger(1)\right]\psi_a(1) + 2\frac{e}{c}\psi_a^\dagger(1)\,\hat{\vec{A}}(1)\,\psi_a(1)\right\} \; , \tag{III-20}$$

so by comparing equation (III-20) with equation (III-19), one finds that $\hat{\vec{j}}$ may be identified as

$$\hat{\vec{j}}(1) = -\frac{e}{2m}\left\{\psi_a^\dagger(1)\left[\vec{P}(1)\,\psi_a(1)\right] + \left[\vec{P}^*(1)\,\psi_a^\dagger(1)\right]\psi_a(1) + 2\frac{e}{c}\psi_a^\dagger(1)\,\hat{\vec{A}}(1)\,\psi_a(1)\right\} \; . \tag{III-21}$$

As a matter of fact, equation (III-21) is incomplete; the missing term may be calculated from a variational principle relating a variation in the field potentials to a variation in the Hamiltonian:

$$\delta \hat{\mathscr{H}}(t_1) = \int d^3\vec{r}_1 \left[\delta U^A(1)\, \hat{\rho}_e(1) - \frac{1}{c}\, \delta \vec{A}^A(1) \cdot \vec{j}(1) \right] . \qquad \text{(III-22)}$$

(See reference (39), Section 128, concerning equation (III-22).) The first variation of the Hamiltonian (III-6) for a variation in the potentials U^A and $\vec{A}^A$ is given by:

$$\delta \hat{\mathscr{H}}'(t_1) = \int d^3\vec{r}_1 \psi_a^\dagger(1) \Big\{ \delta_{a\beta} \frac{1}{2m} \vec{P}(1) \cdot \frac{e}{c}\, \delta \vec{A}^A(1) + \delta_{a\beta} \frac{1}{2m} \frac{e}{c}\, \delta \vec{A}^A(1) \cdot \vec{P}(1)$$

$$+ \frac{\mu \hbar e}{2mc} \vec{\sigma}_{a\beta} \cdot \left[\nabla_1 \times \delta \vec{A}^A(1) \right] - \delta_{a\beta}\, e\, \delta U^A(1) \qquad \text{(III-23)}$$

$$+ \delta_{a\beta} \frac{e^2}{mc^2}\, \delta \vec{A}^A(1) \cdot \hat{\vec{A}}(1) \Big\} \psi_\beta(1) .$$

Equation (III-23) may be rearranged by use of the vector identity $\vec{C} \cdot (\nabla \times \vec{D}) = \nabla \cdot (\vec{D} \times \vec{C}) + \vec{D} \cdot (\nabla \times \vec{C})$, by several integrations by parts, and by the use of the boundary conditions (III-14). The result is the following equation:

$$\delta \hat{\mathscr{H}}(t_1) = \int d^3\vec{r}_1\, \delta U^A(1) \Big\{ - e\, \psi_a^\dagger(1) \psi_a(1) \Big\}$$

$$- \frac{1}{c} \int d^3\vec{r}_1\, \delta \vec{A}^A(1) \cdot \left(-\frac{e}{2m} \right) \Big\{ \psi_a^\dagger(1) \left[\vec{P}(1) \psi_a(1) \right] + \left[\vec{P}^*(1) \psi_a^\dagger(1) \right] \psi_a(1)$$

$$\text{(III-24)}$$

$$+ \mu \hbar\, \nabla_1 \times \psi_a^\dagger(1)\, \vec{\sigma}_{a\beta}\, \psi_\beta(1) + 2 \frac{e}{c}\, \psi_a^\dagger(1)\, \hat{\vec{A}}(1)\, \psi_a(1) \Big\} .$$

From a comparison of (III-24) with (III-22) one finds that it is possible to make the identifications

$$\hat{\rho}_e(1) = -e\psi_a^\dagger(1)\psi_a(1) ,$$

and

$$\hat{\vec{j}}(1) = -\frac{e}{2m}\left\{\psi_a^\dagger(1)\left[\vec{P}(1)\psi_a(1)\right] + \left[\vec{P}^*(1)\psi_a^\dagger(1)\right]\psi_a(1)\right.$$

$$\left. + 2\frac{e}{c}\psi_a^\dagger(1)\hat{\vec{A}}(1)\psi_a(1) + \mu\hbar\nabla_1 \times \psi_a^\dagger(1)\vec{\sigma}_{a\beta}\psi_\beta(1)\right\} . \qquad \text{(III-25)}$$

If we compare (III-25) with (III-21) we find that in (III-25) there is the extra term

$$\hat{\vec{j}}_M(1) \equiv -\frac{\mu\hbar e}{2m}\nabla_1 \times \left\{\psi_a^\dagger(1)\vec{\sigma}_{a\beta}\psi_\beta(1)\right\}. \qquad \text{(III-26)}$$

This term is related to the spin magnetism of the electrons, since

$$\hat{\vec{M}}_s(1) \equiv -\frac{\mu\hbar e}{2mc}\psi_a^\dagger(1)\vec{\sigma}_{a\beta}\psi_\beta(1) \qquad \text{(III-27)}$$

is identified as the spin magnetic polarization density operator (see (I-13)). This term (III-26) apparently failed to appear in (III-21) because $\hat{\vec{j}}$ was identified there from terms of the form $\nabla \cdot \hat{\vec{j}}$ (III-19), and because of the vector identity $\nabla \cdot (\nabla \times \vec{C}) \equiv 0$. Thus, the operator $\hat{\vec{j}}_M(1)$ defined in (III-26) is the spin magnetic polarization current density operator for the electron gas. By combining the definitions (III-26) and (III-27) one obtains

$$\hat{\vec{j}}_M(1) = c\, \nabla_1 \times \hat{\vec{M}}_s(1)\,. \qquad \text{(III-28)}$$

It is possible to simplify the forms of the equations (III-17, 25, 26, and 27) by the introduction of the spinor notation for the field operators (column and row matrices):

$$\psi^\dagger(\vec{r}, t) \equiv \begin{bmatrix} \psi_1^\dagger(\vec{r}, t) \\ \psi_2^\dagger(\vec{r}, t) \end{bmatrix}, \qquad \psi(\vec{r}, t) = \begin{bmatrix} \psi_1(\vec{r}, t) \\ \psi_2(\vec{r}, t) \end{bmatrix}$$

and (III-29)

$$\tilde{\psi}^\dagger(\vec{r}, t) \equiv \begin{bmatrix} \psi_1^\dagger(\vec{r}, t) & \psi_2^\dagger(\vec{r}, t) \end{bmatrix},$$

$$\tilde{\psi}^\dagger(\vec{r}, t) \equiv \begin{bmatrix} \psi_1^\dagger(\vec{r}, t) & \psi_2^\dagger(\vec{r}, t) \end{bmatrix}.$$

The unit matrix is designated by the term I :

$$I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (I)_{\alpha\beta} = \delta_{\alpha\beta}\,. \qquad \text{(III-30)}$$

The transpose of any 2×2 square matrix γ is designated by

$$\text{Transpose of } \gamma \equiv \tilde{\gamma}$$
$$\text{where } (\tilde{\gamma})_{\alpha\beta} = (\gamma)_{\beta\alpha}\,. \qquad \text{(III-31)}$$

Therefore, we may write the equations (III-17, 25, 26, and 27) as follows:

$$\rho_e(1) = -e\,\tilde{\psi}^{\dagger}(1)\,\psi(1)\,,$$

$$\hat{\vec{j}}(1) = -\frac{e}{2m}\left\{\tilde{\psi}^{\dagger}(1)\left[\vec{P}(1)\,\psi(1)\right] + \left[\vec{P}^{*}(1)\,\tilde{\psi}^{\dagger}(1)\right]\psi(1)\right.$$

$$\left. + \mu\hbar\,\nabla_1 \times \tilde{\psi}^{\dagger}(1)\,\vec{\sigma}\,\psi(1) + 2\frac{e}{c}\,\tilde{\psi}^{\dagger}(1)\,\hat{\vec{A}}(1)\,\psi(1)\right\}, \qquad \text{(III-32)}$$

$$\hat{\vec{M}}_s(1) = -\frac{\mu\hbar e}{2mc}\,\tilde{\psi}^{\dagger}(1)\,\vec{\sigma}\,\psi(1)\,,$$

and

$$\hat{\vec{j}}_M(1) = c\,\nabla_1 \times \hat{\vec{M}}_s(1)\,.$$

CHAPTER IV

"EXPECTATION VALUES" IN THE DENSITY MATRIX FORMALISM: ENSEMBLE AVERAGES

This chapter is devoted to a brief discussion of "macroscopic expectation values" or "probabilistic averages" of the quantum field theoretic operators. The density matrix formalism is utilized here using the results of equilibrium statistical mechanics. Only a brief sketch is given here of this important and essential aspect of the overall formalism; more complete discussions may be found in the references (1), (2), (3), and (9). The following notation is adopted in general:

$$\text{"Expectation Value" of } \hat{F} \equiv \langle \hat{F} \rangle \, , \qquad \text{(IV-1)}$$

where $\hat{F}$ is an arbitrary field theoretic operator.

The "expectation values" of the field theoretic density operators $\hat{\rho}$ and $\hat{\vec{j}}$ (III-18, 32) may be interpreted in a probabilistic sense similarly to the probability density (the square of the amplitude of the wave function) in ordinary Quantum Mechanics.

The field theoretic "expectation values" are defined in terms of the density matrix formalism:

$$\langle \hat{F} \rangle = \mathrm{Sp}\,(\hat{f}\hat{F}) \, , \qquad \text{(IV-2)}$$

where the "Sp" refers to the spur, or trace of the products of the operators $\hat{f}$ and $\hat{F}$, and $\hat{f}$ is the density matrix operator (this operator is usually designated by the symbol $\hat{\rho}$ in the literature, however, since we have already introduced the symbol ρ for the charge density operator, we thus introduce the notation $\hat{f}$ for the density matrix to avoid confusion). The meaning of the trace in equation (IV-2) is simply that, given an arbitrary complete set of orthonormal state vectors $|a\rangle$, one may write equation (IV-2) as sums of ordinary expectation values over the set $|a\rangle$:

$$\langle\hat{F}\rangle = \sum_{a} \langle a|\hat{f}\hat{F}|a\rangle ,$$

or (IV-3)

$$\langle\hat{F}\rangle = \sum_{a}\sum_{a'} \langle a|\hat{f}|a'\rangle\langle a'|\hat{F}|a\rangle$$

by the "chain rule" for operator products. The single parameter a actually represents all possible parameters associated with the state vector $|a\rangle$.

For operators in the Heisenberg representation the density matrix is time independent, (see reference (9)), hence one may use the operator $\hat{f}(t < t_o)$ for the system in the initial state of thermodynamic equilibrium. From equilibrium statistical mechanics, we have the result that for systems in a state of definite temperature T with chemical potential ζ the density matrix is given by the Grand Canonical Ensemble (see reference (40)):

$$\hat{f}(t < t_o) \equiv \hat{f}_o \equiv Z^{-1} e^{-\beta(\hat{H}_o - \zeta\hat{N}_o)} , \tag{IV-4}$$

where Z is the Grand Partition Function

$$Z \equiv \mathrm{Sp}\, e^{-\beta(\hat{H}_o - \zeta\hat{N}_o)} ,$$

and $\beta = \frac{1}{kT}$ where T is the temperature in degrees absolute, and $k = 1.38 \times 10^{-16}$ erg per degree absolute is the Boltzmann constant. The operators $\hat{H}_o$ and $\hat{N}_o$ are the time-independent Hamiltonian and the electron number operator given by (III-2) and (III-5) prior to the application of an electromagnetic perturbation at the time t_o. We let

$$\hat{\mathcal{H}}_o = \hat{H}_o - \zeta \hat{N}_o \qquad \text{(IV-5)}$$

after (III-6), then from equation (IV-4)

$$\hat{f}_o = Z^{-1} e^{-\beta \hat{\mathcal{H}}_o}$$

and (IV-6)

$$Z = \mathrm{Sp}\, e^{-\beta \hat{\mathcal{H}}_o}$$

It has been shown by Martin and Schwinger (reference (1))(see also reference (3)) that for systems in a state of definite energy and having a definite number of particles, the expectation values of operators may be approximated by averaging over the appropriate Grand Canonical Ensemble (equivalence of a microcanonical ensemble and a grand canonical ensemble) if the number of particles is very large. Thus, in either case, the operator $\hat{f}_o$ given by (IV-4, 6) is the density matrix we require and

$$\langle \hat{F} \rangle = \mathrm{Sp}\,(\hat{f}_o \hat{F}) \qquad \text{(IV-7)}$$

for operators $\hat{F}$ in the Heisenberg representation. This result (IV-7) is not applicable to isolated systems originating in nonequilibrium states.

The "expectation values" for $\hat{\rho}$ and $\hat{\vec{j}}$ are found with the definition (IV-7) from the equations (III-18, 32)

$$\begin{aligned}
\rho(1) &= \langle \hat{\rho}(1) \rangle = \langle \hat{\rho}_e(1) \rangle + \langle e n(\vec{r}_1) \rangle \\
&= \langle \hat{\rho}_e(1) \rangle + e n(\vec{r}_1) \\
&= -e \langle \widetilde{\psi}^{\dagger}(1)\psi(1) \rangle + e n(\vec{r}_1) \\
&= -e \langle \psi_a^{\dagger}(1)\psi_a(1) \rangle + e n(r_1) ,
\end{aligned} \tag{IV-8}$$

and

$$\begin{aligned}
\vec{j}(1) &= \langle \hat{\vec{j}}(1) \rangle \\
&= -\frac{e}{2m}\Big\{ \vec{P}(1) \langle \widetilde{\psi}^{\dagger}(2)\psi(1) \rangle + \vec{P}^*(2) \langle \widetilde{\psi}^{\dagger}(1)\psi(2) \rangle \\
&\quad + 2\frac{e}{c} \langle \widetilde{\psi}^{\dagger}(1)\hat{\vec{A}}(1)\psi(1) \rangle \Big\}_{2\to 1} + \vec{j}_M(1) \\
&= -\frac{e}{2m}\Big\{ \vec{P}(1) \langle \psi_a^{\dagger}(2)\psi_a(1) \rangle + \vec{P}^*(2) \langle \psi_a^{\dagger}(1)\psi_a(2) \rangle \\
&\quad + 2\frac{e}{c} \langle \psi_a^{\dagger}(1)\hat{\vec{A}}(1)\psi_a(1) \rangle \Big\}_{2\to 1} + \vec{j}_M(1)
\end{aligned} \tag{IV-9}$$

where

$$\begin{aligned}
\vec{j}_M(1) &= \langle \hat{\vec{j}}_M(1) \rangle \\
&= c\, \nabla_1 \times \vec{M}_s(1)
\end{aligned} \tag{IV-10}$$

with

$$\hat{\vec{M}}_s(1) = \langle \hat{\vec{M}}_s(1) \rangle$$

$$= - \frac{\mu \hbar e}{2mc} \langle \tilde{\psi}^{\dagger}(1)\, \vec{\sigma}\, \psi(1) \rangle \qquad \text{(IV-11)}$$

$$= - \frac{\mu \hbar e}{2mc} \langle \psi_{\alpha}^{\dagger}(1)\, \vec{\sigma}_{\alpha\beta}\, \psi_{\beta}(1) \rangle$$

The "subscript" notation introduced here with the brackets,

$$\left\{ \cdot \cdot \cdot \right\}_{2 \to 1} ,$$

indicates the designated change of variable after the differential operation within the brackets is performed.

We introduce here the following notation for a "trace" over quantities with spin index:

$$\mathrm{Tr}\,(AB) \equiv \sum_{\alpha=1}^{2} (AB)_{\alpha\alpha} \equiv (AB)_{\alpha\alpha}$$

$$= \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} A_{\alpha\beta}\, B_{\beta\alpha} \equiv A_{\alpha\beta}\, B_{\beta\alpha} . \qquad \text{(IV-12)}$$

Notice the use of the symbol Tr here as contrasted with the symbol Sp as used in conjunction with the density matrix (IV-4); this distinction in the designation of the two traces is observed throughout this paper.

CHAPTER V

TEMPERATURE DEPENDENT GREEN'S FUNCTIONS FOR THE ELECTRON GAS AND THEIR RELATION TO THE CHARGE AND CURRENT DENSITIES

Green's Functions and Their Time Evolution Equations

In this chapter we consider the Green's functions as defined by Martin and Schwinger (reference (1)). These functions are temperature dependent space-time correlation functions formed from the "expectation values" of certain combinations (Wick products) of the field operators $\psi^\dagger$ and ψ.

The one-particle Green's function for the electron field is given by (according to the definition for a Fermion field):

$$G_{1\alpha\beta}(\vec{r}_1, t_1; \vec{r}_2, t_2) \equiv G_{1\alpha\beta}(1,2) = -i\varepsilon_1(1,2)\left\langle \left[\psi_\alpha(1)\psi^\dagger_\beta(2)\right]_+ \right\rangle \tag{V-1}$$

where $\varepsilon_1(1,2)$ is the "time-ordered" function

$$\varepsilon_1(1,2) = \begin{cases} +1 \text{ for } t_1 > t_2 \\ -1 \text{ for } t_1 < t_2 \end{cases}$$

and $\left[\ldots\right]_+$ is a time-ordering symbol indicating that the operators within the square brackets are ordered from right to left in the order of increasing times. That is

$$\left[\psi_\alpha(1)\psi^\dagger_\beta(2)\right]_+ = \begin{cases} \psi_\alpha(1)\psi^\dagger_\beta(2) & \text{for } t_1 > t_2 \\ \psi^\dagger_\beta(2)\psi_\alpha(1) & \text{for } t_1 < t_2\ . \end{cases} \tag{V-2}$$

Thus

$$G_{1\alpha\beta}(1,2) = \begin{cases} -i\langle\psi_\alpha(1)\psi^\dagger_\beta(2)\rangle, & t_1 > t_2 \\ +i\langle\psi^\dagger_\beta(2)\psi_\alpha(1)\rangle, & t_1 < t_2\ . \end{cases} \tag{V-3}$$

Similarly, the "two-particle" Green's function is defined by

$$G_{2\alpha\beta\gamma\delta}(12;34) = (-i)^2\,\varepsilon_2(1,2;4,3)\left\langle\left[\psi_\alpha(1)\psi_\beta(2)\psi^\dagger_\gamma(3)\psi^\dagger_\delta(4)\right]_+\right\rangle, \tag{V-4}$$

where

$$\varepsilon_2(1,2;4,3) = \begin{cases} +1 & \text{if an \underline{even} number of permutations is required to rearrange the ordered set of times } (t_1, t_2, t_4, t_3) \text{ in the order of increasing times from right to left.} \\ -1 & \text{if an \underline{odd} number of permutations is required.} \end{cases}$$

In general, the "n-particle" Green's functions are designated by

$$G_{n\alpha_1\alpha_2\cdots\alpha_n,\alpha_{n+1}\cdots\alpha_{2n}}(1,2,\cdots n;\,n+1,\cdots 2n)$$

$$= (-i)^n\varepsilon_n(1,2,\cdots n;2n,\cdots n+1)\left\langle\left[\psi_{\alpha_1}(1)\psi_{\alpha_2}(2)\cdots\psi_{\alpha_n}(n)\psi^\dagger_{\alpha_{n+1}}(n+1)\cdots\psi^\dagger_{\alpha_{2n}}(2n)\right]_+\right\rangle \tag{V-5}$$

where ε_n is defined similarly to ε_1 and ε_2. Notice that in all cases, the Green's functions have an equal number of creation and annihilation operators.

Equations of motion can be obtained for the Green's functions. In the case of G_1:

$$i\hbar \frac{\partial}{\partial t_1} G_{1\alpha\beta}(1,2) = -i\varepsilon_1(1,2)\left\langle\left[i\hbar \frac{\partial\psi_\alpha(1)}{\partial t_1}\psi_\beta^\dagger(2)\right]_+\right\rangle$$

$$+ i\hbar\,\delta(t_1 - t_2)\lim_{\delta\to 0^+}\int_{t_2-\delta}^{t_2+\delta} dt_1 \frac{\partial}{\partial t_1}\left\langle -i\varepsilon_1(1,2)\left[\psi_\alpha(1)\psi_\beta^\dagger(2)\right]_+\right\rangle . \tag{V-6}$$

The last term on the right hand side of equation (V-6) arises because of the discontinuity in G_1 at $t_1 = t_2$ (see Mathematical Appendix IV). The integral in (V-6) may be evaluated as follows:

$$\lim_{\delta\to 0^+}\int_{t_2-\delta}^{t_2+\delta} dt_1 \frac{\partial}{\partial t_1}\left\langle -i\varepsilon_1(1,2)\left[\psi_\alpha(1)\psi_\beta^\dagger(2)\right]_+\right\rangle$$

$$= \lim_{\delta\to 0^+}\left\langle -i\varepsilon_1(2_+,2)\,\psi(\vec{r}_1, t_2+\delta)\,\psi_\beta^\dagger(2) + i\varepsilon_1(2_-,2)\,\psi_\beta^\dagger(2)\,\psi(\vec{r}_1, t_2-\delta)\right\rangle \tag{V-7}$$

$$= -i\left\langle\psi_\alpha(\vec{r}_1, t_2)\,\psi_\beta^\dagger(2) + \psi_\beta^\dagger(2)\,\psi_\alpha(\vec{r}_1, t_2)\right\rangle$$

$$= -i\,\delta^3(\vec{r}_1 - \vec{r}_2)$$

from (III-9) and since

$$\varepsilon_1(2_+,2) = -\varepsilon_1(2_-,2) = 1 .$$

(The subscripts (+) and (-) refer to a time ordering, i. e., $1_\pm$ implies $t_1 \pm \delta$ where δ is an "infinitesimal.") Therefore, from (V-6) and (V-7) one obtains

$$i\hbar \frac{\partial}{\partial t_1} G_{1\alpha\beta}(1,2) - (-i)\varepsilon_1(1,2)\left\langle \left[i\hbar \frac{\partial \psi_\alpha(1)}{\partial t_1} \psi^\dagger_\beta(2) \right]_+ \right\rangle$$

$$= \delta_{\alpha\beta}\, \delta(t_1 - t_2)\, \delta^3(\vec{r}_1 - \vec{r}_2) \equiv \delta_{\alpha\beta}\, \delta^4(1,2) . \qquad \text{(V-8)}$$

With the aid of equation (III-12) and the definitions (V-1) and (V-4) for G_1 and G_2, equation (V-8) may be written in the form

$$\left[\delta_{\alpha\lambda}\, i\hbar \frac{\partial}{\partial t_1} - h_{\alpha\lambda}(1)\right] G_{1\lambda\beta}(1,2) = \hbar\, \delta_{\alpha\beta}\, \delta^4(1,2)$$

$$- e\int d^3\vec{r}_3 \frac{1}{r_{13}} \left\{ i\, e\, G_{2\gamma\alpha\gamma\beta}(\vec{r}_3, t_1+; \vec{r}_1, t_1; \vec{r}_3, t_1++; \vec{r}_2, t_2) + e n(\vec{r}_3)\, G_{1\alpha\beta}(1,2) \right\}$$

$$+ \frac{\vec{P}(1)}{2m} \cdot \frac{e}{c} \int d^3\vec{r}_3 \frac{(i\,e)}{2mcr_{13}} \left\{ \vec{P}(\vec{r}_3, t_1)\, G_{2\gamma\alpha\gamma\beta}(\vec{r}_3, t_1+; \vec{r}_1, t_1; \vec{r}_4, t_1++; \vec{r}_2, t_2) \right.$$

$$+ \vec{P}^*(\vec{r}_3, t_1)\, G_{2\gamma\alpha\gamma\beta}(\vec{r}_4, t_1+; \vec{r}_1, t_1; \vec{r}_3, t_1++; \vec{r}_2, t_2)$$

$$\left. + \mu\hbar \nabla_3 \times \vec{\sigma}_{\gamma\lambda}\, G_{2\lambda\alpha\gamma\beta}(\vec{r}_3, t_1+; \vec{r}_1, t_1; \vec{r}_3, t_1++; \vec{r}_2, t_2) \right\}_{\vec{r}_4 \to \vec{r}_3} \qquad \text{(V-9)}$$

$$+ \frac{e}{c} \int d^3\vec{r}_3 \frac{(i\,e)}{2mcr_{13}} \left\{ \vec{P}(\vec{r}_3, t_1) \cdot \frac{\vec{P}(1)}{2m}\, G_{2\gamma\alpha\gamma\beta}(\vec{r}_3, t_1+; \vec{r}_1, t_1; \vec{r}_4, t_1++; \vec{r}_2, t_2) \right.$$

$$+ \vec{P}^*(\vec{r}_3, t_1) \cdot \frac{\vec{P}(1)}{2m}\, G_{2\gamma\alpha\gamma\beta}(\vec{r}_4, t_1+; \vec{r}_1, t_1; \vec{r}_3, t_1++; \vec{r}_2, t_2)$$

$$\left. + \mu\hbar (\nabla_3 \times \vec{\sigma}_{\gamma\lambda}) \cdot \frac{\vec{P}(1)}{2m}\, G_{2\lambda\alpha\gamma\beta}(\vec{r}_3, t_1+; \vec{r}_1, t_1; \vec{r}_3, t_1++; \vec{r}_2, t_2) \right\}_{\vec{r}_4 \to \vec{r}_3}$$

$$+ \frac{\mu\hbar e}{2mc} \vec{\sigma}_{\alpha\eta} \cdot \left\{ \nabla_1 \times \int d^3\vec{r}_3 \frac{(i\,e)}{2mcr_{13}} \left[\vec{P}(\vec{r}_3, t_1)\, G_{2\gamma\eta\gamma\beta}(\vec{r}_3, t_1+; \vec{r}_4, t_1; \vec{r}_5, t_1++; \vec{r}_2, t_2) \right. \right.$$

$$+ \vec{P}^*(\vec{r}_3, t_1) G_{2\gamma\eta\gamma\beta}(\vec{r}_5, t_1+; \vec{r}_4, t_1; \vec{r}_3, t_1++; \vec{r}_2, t_2)$$

$$+ \mu\hbar \nabla_3 \times \vec{\sigma}_{\gamma\lambda} G_{2\lambda\eta\gamma\beta}(\vec{r}_3, t_1+; \vec{r}_4, t_1; \vec{r}_3, t_1++; \vec{r}_2, t_2) \Big] \Big\}_{\substack{\vec{r}_4 \to \vec{r}_1 \\ \vec{r}_5 \to \vec{r}_3}} ,$$

where

$$h_{\alpha\beta}(1) \equiv \delta_{\alpha\beta} \frac{1}{2m} \left[\vec{P}(1) \right]^2 + \frac{\mu\hbar e}{2mc} \vec{\sigma}_{\alpha\beta} \cdot \vec{B}^A(1) - \delta_{\alpha\beta} \zeta - \delta_{\alpha\beta}\, e\, U^A(1) .$$

The + signs following the time co-ordinates t in (V-9) indicate the proper time ordering arising from the original ordering of the field operators $\psi^\dagger$ and ψ:

$$t_1+ \equiv t_1 + \delta$$
$$t_1++ \equiv t_1 + 2\delta , \qquad \text{(V-10)}$$

where the δ is an "infinitesimal" quantity. For the Green's function

$$G_{1\beta\alpha}(2,1) = -i\varepsilon_1(2,1) \left\langle \left[\psi_\beta(2) \psi_\alpha^\dagger(1) \right]_+ \right\rangle , \qquad \text{(V-11)}$$

an equation of motion can be obtained which is "conjugate" to equation (V-9):

$$\left[-\delta_{\lambda\alpha}\, i\hbar \frac{\partial}{\partial t_1} - h_{\lambda\alpha}^\dagger(1) \right] G_{1\beta\lambda}(2,1) = \hbar\, \delta_{\beta\alpha}\, \delta^4(2,1)$$

$$- e \int d^3\vec{r}_3 \frac{1}{r_{13}} \left\{ i e\, G_{2\beta\gamma\alpha\gamma}(\vec{r}_2, t_2; \vec{r}_3, t_1; \vec{r}_1, t_1++; \vec{r}_3, t_1+) + en(\vec{r}_3)\, G_{1\beta\alpha}(2,1) \right\}$$

$$+ \frac{\vec{P}^*(1)}{2m} \cdot \frac{e}{c} \int d^3\vec{r}_3 \frac{(i e)}{2mcr_{13}} \left\{ \vec{P}(\vec{r}_3, t_1)\, G_{2\beta\gamma\alpha\gamma}(\vec{r}_2, t_2; \vec{r}_3, t_1; \vec{r}_1, t_1++; \vec{r}_4, t_1+) \right.$$

$$+ \vec{P}^*(\vec{r}_3, t_1) G_{2\beta\gamma\alpha\gamma}(\vec{r}_2, t_2; \vec{r}_4, t_1; \vec{r}_1, t_1++; \vec{r}_3, t_1+) \tag{V-12}$$

$$+ \mu\hbar \nabla_3 \times \vec{\sigma}_{\gamma\lambda} G_{2\beta\lambda\alpha\gamma}(\vec{r}_2, t_2; \vec{r}_3, t_1; \vec{r}_1, t_1++; \vec{r}_3, t_1+)\Big\}_{\vec{r}_4 \to \vec{r}_3}$$

$$+\frac{e}{c}\int d^3\vec{r}_3 \frac{(ie)}{2mcr_{13}}\Big\{\vec{P}(\vec{r}_3, t_1) \cdot \frac{\vec{P}^*(1)}{2m} G_{2\beta\gamma\alpha\gamma}(\vec{r}_2, t_2; \vec{r}_3, t_1; \vec{r}_1, t_1++; \vec{r}_4, t_1+)$$

$$+ \vec{P}^*(\vec{r}_3, t_1) \cdot \frac{\vec{P}^*(1)}{2m} G_{2\beta\gamma\alpha\gamma}(\vec{r}_2, t_2; \vec{r}_4, t_1; \vec{r}_1, t_1++; \vec{r}_3, t_1+)$$

$$+ \mu\hbar (\nabla_3 \times \vec{\sigma}_{\gamma\lambda}) \cdot \frac{\vec{P}^*(1)}{2m} G_{2\beta\lambda\alpha\gamma}(\vec{r}_2, t_2; \vec{r}_3, t_1; \vec{r}_1, t_1++; \vec{r}_3, t_1+)\Big\}_{\vec{r}_4 \to \vec{r}_3}$$

$$+\frac{\mu\hbar e}{2mc} \vec{\sigma}_{\eta\alpha} \cdot \Big\{\nabla_1 \times \int d^3\vec{r}_3 \frac{(ie)}{2mcr_{13}}\Big[\vec{P}(\vec{r}_3, t_1) G_{2\beta\gamma\eta\gamma}(\vec{r}_2, t_2; \vec{r}_3, t_1; \vec{r}_4, t_1++; \vec{r}_5, t_1+)$$

$$+ \vec{P}^*(\vec{r}_3, t_1) G_{2\beta\gamma\eta\gamma}(\vec{r}_2, t_2; \vec{r}_5, t_1; \vec{r}_4, t_1++; \vec{r}_3, t_1+)$$

$$+ \mu\hbar \nabla_3 \times \vec{\sigma}_{\gamma\lambda} G_{2\beta\lambda\eta\gamma}(\vec{r}_2, t_2; \vec{r}_3, t_1; \vec{r}_4, t_1++; \vec{r}_3, t_1+)\Big]\Big\}_{\substack{\vec{r}_4 \to \vec{r}_1 \\ \vec{r}_5 \to \vec{r}_3}} ,$$

where

$$h^{\dagger}_{\beta\alpha}(1) = \delta_{\beta\alpha} \frac{1}{2m}\Big[\vec{P}^*(1)\Big]^2 + \frac{\mu\hbar c}{2mc} \vec{\sigma}_{\beta\alpha} \cdot \vec{B}^A(1) - \delta_{\beta\alpha}\zeta - \delta_{\beta\alpha} e U^A(1) .$$

There is an equation of motion for G_2 similar to equation (V-9) for G_1:

$$\Big[\delta_{\alpha\eta} i\hbar \frac{\partial}{\partial t_1} - h_{\alpha\eta}(1)\Big] G_{2\eta\beta\gamma\lambda}(12;34) = \hbar \delta_{\alpha\gamma} \delta^4(1,3) G_{1\beta\lambda}(2,4)$$

$$- \hbar \delta_{\alpha\lambda} \delta^4(1,4) G_{1\beta\gamma}(2,3)$$

$$
- e\int d^3\vec{r}_5 \frac{1}{r_{15}} \Big\{ i e G_{3\eta\alpha\beta\eta\gamma\lambda}(\vec{r}_5, t_1+; \vec{r}_1, t_1; \vec{r}_2, t_2; \vec{r}_5, t_1++; \vec{r}_3, t_3; \vec{r}_4, t_4) \tag{V-13}
$$

$$
+ e\, n(\vec{r}_5)\, G_{2\alpha\beta\gamma\lambda}(12;34) \Big\}
$$

+ (integral terms involving G_3 which arise from the "magnetic interactions" between electrons).

The equation (V-13) can be converted from the differential form in G_2 to a direct equation for G_2 if we first pre-multiply both sides of the equation by $G_{1\nu\alpha}(5, 1)$, then sum over α, and finally integrate over the co-ordinates $(\vec{r}_1, t_1)$:

$$
\int d(1)\, G_{1\nu\alpha}(5, 1) \left[\delta_{\alpha\eta}\, i\hbar \frac{\partial}{\partial t_1} - h_{\alpha\eta}(1) \right] G_{2\eta\beta\gamma\lambda}(12;34)
$$

$$
= \hbar\, \delta_{\alpha\gamma} \int d(1)\, \delta^4(1, 3)\, G_{1\nu\alpha}(5, 1)\, G_{1\beta\lambda}(2, 4) \tag{V-14}
$$

$$
- \hbar\, \delta_{\alpha\lambda} \int d(1)\, \delta^4(1, 4)\, G_{1\nu\alpha}(5, 1)\, G_{1\beta\gamma}(2, 3)
$$

$$
- e \iint d(1)\, d^3\vec{r}_5 \frac{1}{r_{15}} G_{1\nu\alpha}(5, 1) \Big\{ i e G_{3\eta\alpha\beta\eta\gamma\lambda}(\vec{r}_5, t_1+; \vec{r}_1, t_1; \vec{r}_2, t_2; \vec{r}_5, t_1++; \vec{r}_3, t_3; \vec{r}_4, t_4)
$$

$$
+ e\, n(\vec{r}_5)\, G_{2\alpha\beta\gamma\lambda}(12;34) \Big\}
$$

+ (integral terms containing G_1 and G_3 which arise directly from the "magnetic interactions" between electrons).

After performing some integrations by parts on (V-14), and making use of the boundary conditions (III-14) (we assume that $G_1(1, 2)$ vanishes for infinite time separations and infinite spatial separations, $|t_1 - t_2| \to \infty$ and $|\vec{r}_1 - \vec{r}_2| \to \infty$), one obtains

$$\int d(1)\, G_{2\eta\beta\gamma\lambda}(12;34)\left[-\delta_{\alpha\eta}\, i\hbar \frac{\partial}{\partial t_1} - h^{\dagger}_{\alpha\eta}(1)\right] G_{1\nu\alpha}(5,1) \tag{V-15}$$

$$= \hbar G_{1\nu\gamma}(5,3)\, G_{1\beta\lambda}(2,4) - \hbar G_{1\nu\lambda}(5,4)\, G_{1\beta\gamma}(2,3)$$

+ (integral terms involving G_1, G_2, and G_3 which arise directly from the interactions between electrons).

By comparison of (V-12) with (V-15) we see that equation (V-15) further reduces to

$$\int d(1)\, G_{2\eta\beta\gamma\lambda}(12;34)\hbar\,\delta_{\nu\eta}\,\delta^4(5,1) = \hbar G_{2\nu\beta\gamma\lambda}(52;34) \tag{V-16}$$

$$= \hbar G_{1\nu\gamma}(5,3)\, G_{1\beta\lambda}(2,4) - \hbar G_{1\nu\lambda}(5,4)\, G_{1\beta\gamma}(2,3)$$

+ (integral interaction terms which contain G_1, G_2, and G_3).

The equation (V-16) expresses G_2 as a Hartree-Fock type of combinations of G_1 as shown explicitly, plus some terms arising directly from the interactions between particles which are designated as "correlation" terms. The first term on the right hand side of equation (V-16) is the "direct term" of the Hartree-Fock expansion, while the second term is the "exchange" contribution.

If we substitute for G_2 from (V-16) into equation (V-9) and show explicitly the results of the "Hartree-Fock" contributions from G_2, we obtain the following equation:

$$\left[\delta_{\alpha\lambda}\, i\hbar \frac{\partial}{\partial t_1} - h_{\alpha\lambda}(1)\right] G_{1\lambda\beta}(1,2) = \hbar\,\delta_{\alpha\beta}\,\delta^4(1,2)$$

$$- e\langle \hat{U}(1)\rangle\, G_{1\alpha\beta}(1,2) + \frac{\vec{P}(1)}{2m}\cdot\frac{e}{c}\langle \hat{\vec{A}}(1)\rangle\, G_{1\alpha\beta}(1,2)$$

$$+\frac{e}{c}\langle\hat{\vec{A}}(1)\rangle\cdot\frac{\vec{P}(1)}{2m}G_{1\alpha\beta}(1,2)+\frac{\mu\hbar e}{2mc}\vec{\sigma}_{\alpha\lambda}\cdot\left[\nabla_1\times\langle\hat{\vec{A}}(1)\rangle\right]G_{1\lambda\beta}(1,2)$$

$$+\int d^3\vec{r}_3\,\frac{(ie^2)}{r_{13}}\,G_{1\alpha\gamma}(\vec{r}_1,t_1;\vec{r}_3,t_1+)\,G_{1\gamma\beta}(\vec{r}_3,t_1;\vec{r}_2,t_2) \qquad \text{(V-17)}$$

+ ("exchange" contributions of order $\left(\frac{v}{c}\right)^2$ and all other "correlation" interaction contributions) ,

where $\langle\hat{U}(1)\rangle$ and $\langle\hat{\vec{A}}(1)\rangle$ are obtained from the definitions (III-4, 16) and (V-1) and are given by

$$\langle\hat{U}(1)\rangle=\int d^3\vec{r}_3\,\frac{1}{r_{13}}\left\{-e\langle\psi^\dagger_\gamma(\vec{r}_3,t_1)\psi_\gamma(\vec{r}_3,t_1)\rangle+en(\vec{r}_3)\right\}$$

$$=\int d^3\vec{r}_3\,\frac{1}{r_{13}}\left\{ieG_{1\gamma\gamma}(\vec{r}_3,t_1;\vec{r}_3,t_1+)+en(\vec{r}_3)\right\}$$

$$\langle\hat{\vec{A}}(1)\rangle=-\int d^3\vec{r}_3\,\frac{e}{2mcr_{13}}\left\{\vec{P}(\vec{r}_3,t_1)\langle\psi^\dagger_\gamma(\vec{r}_4,t_1)\psi_\gamma(\vec{r}_3,t_1)\rangle\right. \qquad \text{(V-18)}$$

$$+\vec{P}^*(\vec{r}_3,t_1)\langle\psi^\dagger_\gamma(\vec{r}_3,t_1)\psi_\gamma(\vec{r}_3,t_1)\rangle$$

$$\left.+\mu\hbar\nabla_3\times\langle\psi^\dagger_\gamma(\vec{r}_3,t_1)\vec{\sigma}_{\gamma\lambda}\psi_\lambda(\vec{r}_3,t_1)\rangle\right\}_{\vec{r}_4\to\vec{r}_3}$$

$$=\int d^3\vec{r}_3\,\frac{(ie)}{2mcr_{13}}\left\{\vec{P}(\vec{r}_3,t_1)G_{1\gamma\gamma}(\vec{r}_3,t_1;\vec{r}_4,t_1+)\right.$$

$$+\vec{P}^*(\vec{r}_3,t_1)G_{1\gamma\gamma}(\vec{r}_4,t_1;\vec{r}_3,t_1+)$$

$$\left.+\mu\hbar\nabla_3\times\vec{\sigma}_{\gamma\eta}G_{1\eta\gamma}(\vec{r}_3,t_1;\vec{r}_3,t_1+)\right\}_{\vec{r}_4\to\vec{r}_3}.$$

In a similar manner, equation (V-12) reduces to

$$\left[-\delta_{\lambda\alpha} i\hbar \frac{\partial}{\partial t_1} - h^{\dagger}_{\lambda\alpha}(1)\right] G_{1\beta\lambda}(2,1) = \hbar\delta_{\beta\alpha}\,\delta^4(2,1) - e\langle\hat{U}(1)\rangle G_{1\beta\alpha}(2,1)$$

$$+ \frac{\vec{P}^*(1)}{2m}\cdot\frac{e}{c}\langle\hat{\vec{A}}(1)\rangle G_{1\beta\alpha}(2,1) + \frac{e}{c}\langle\hat{\vec{A}}(1)\rangle\cdot\frac{\vec{P}^*(1)}{2m} G_{1\beta\alpha}(2,1)$$

$$+ \frac{\mu\hbar e}{2mc}\vec{\sigma}_{\lambda\alpha}\cdot\left[\nabla_1 \times \langle\hat{\vec{A}}(1)\rangle\right] G_{1\beta\lambda}(2,1) \qquad \text{(V-19)}$$

$$+\int d^3\vec{r}_3 \frac{(ie^2)}{r_{13}} G_{1\beta\gamma}(\vec{r}_2,t_2;\vec{r}_3,t_1) G_{1\gamma\alpha}(\vec{r}_3,t_1;\vec{r}_1,t_1+)$$

+ ("exchange" terms of order $\left(\frac{v}{c}\right)^2$ and all other "correlation" interaction contributions).

Charge and Current Densities

The charge and current densities ρ and $\vec{j}$ are given from equations (IV-8) through (IV-11) with the aid of the definitions of G_1 and G_2 (V-1) and (V-4) as follows:

$$\rho(1) = ie\,G_{1\gamma\gamma}(1,1_+) + en(\vec{r}_1)$$

$$= ie\,\mathrm{Tr}\,G_1(1,1_+) + en(\vec{r}_1), \qquad \text{(V-20)}$$

and

$$\vec{j}(1) = \frac{ie}{2m}\left\{\vec{P}(1)G_{1\gamma\gamma}(1,2) + \vec{P}^*(2)G_{1\gamma\gamma}(1,2)\right\}_{2\to 1_+} + \vec{j}_M(1)$$

$$+\frac{ie}{m}\frac{e}{c}\int d^3\vec{r}_3 \frac{(ie)}{2mcr_{13}}\left\{\vec{P}(\vec{r}_3,t_1)G_{2\gamma\alpha\alpha\gamma}(\vec{r}_3,t_1+;\vec{r}_1,t_1;\vec{r}_1,t_1+++;\vec{r}_4,t_1++)\right.$$

$$\text{(V-21)}$$

$$+ \vec{P}^*(\vec{r}_4, t_1) G_{2\lambda\alpha\alpha\gamma}(\vec{r}_3, t_1+; \vec{r}_1, t_1; \vec{r}_1, t_1+++; \vec{r}_4, t_1++)$$

$$+ \mu\hbar \nabla_3 \times \vec{\sigma}_{\gamma\lambda} G_{2\lambda\alpha\alpha\gamma}(\vec{r}_3, t_1+; \vec{r}_1, t_1; \vec{r}_1, t_1+++; \vec{r}_3, t_1++) \Big\}_{\vec{r}_4 \to \vec{r}_3} ,$$

with

$$\vec{j}_M(1) = c \nabla_1 \times \vec{M}_s(1)$$

$$\vec{M}_s(1) = \frac{i\mu\hbar e}{2mc} \vec{\sigma}_{\gamma\lambda} G_{1\lambda\gamma}(1, 1_+) \qquad \text{(V-22)}$$

$$= \frac{i\mu\hbar e}{2mc} \operatorname{Tr} \vec{\sigma} G_1(1, 1_+) .$$

The symbol (n_+) represents the set of co-ordinates $(\vec{r}_n, t_n+)$. If we substitute for G_2 from (V-16) into (V-21), and show explicitly only the terms derived from the "Hartree-Fock" combinations of G_1, we obtain

$$\vec{j}(1) = \frac{ie}{2m} \Big\{ \vec{P}(1) G_{1\gamma\gamma}(1, 2) + \vec{P}^*(2) G_{1\gamma\gamma}(1, 2)$$

$$+ 2\frac{e}{c} \langle \hat{\vec{A}}(1) \rangle G_{1\gamma\gamma}(1, 1_+) \Big\}_{2 \to 1_+} + \vec{j}_M(1) \qquad \text{(V-23)}$$

+ ("exchange" contributions of order $\left(\frac{v}{c}\right)^2$ and all other "interaction" terms).

From a comparison of (V-18) and (V-23) one finds a similarity between the form of (V-23) and the integrand in (V-18). That is, except for the term $2\frac{e}{c} \langle \hat{\vec{A}}(1) \rangle G_{1\gamma\gamma}(1, 1_+)$, the integrand in (V-18) appears to be basically of the form

$$\frac{1}{cr_{13}} \vec{j}(\vec{r}_3, t_1) .$$

Specifically, if the interaction term

$$\left(2 \frac{e}{c} \langle \hat{\vec{A}}(1) \rangle \, G_{1\gamma\gamma}(1, 1_+)\right),$$

of order $\left(\frac{1}{c}\right)$ is neglected, then to the lowest order in $\left(\frac{1}{c}\right)$ equation (V-23) is given by

$$\vec{j}(1) \cong \frac{ie}{2m} \left\{ \vec{P}(1) G_{1\gamma\gamma}(1, 2) + \vec{P}^*(2) G_{1\gamma\gamma}(1, 2) \right\}_{2 \to 1_+} + \vec{j}_M(1)$$

$$= \frac{ie}{2m} \operatorname{Tr} \left\{ \vec{P}(1) G_1(1, 2) + \vec{P}^*(2) G_1(1, 2) \right\}_{2 \to 1_+} + \vec{j}_M(1) \,. \qquad \text{(V-24)}$$

This "basic" form of $\vec{j}$ (V-24) recurs because of the electron interaction terms in $\langle \hat{\vec{A}} \rangle$.

From (V-18) and (V-24) one has the result that

$$\langle \hat{\vec{A}}(1) \rangle \cong \int d^3 \vec{r}_3 \, \frac{\vec{j}(\vec{r}_3, t_1)}{c r_{13}} \qquad \text{(V-25)}$$

Moreover, from (V-18) and (V-20) one also finds that $\langle \hat{U} \rangle$ takes the form

$$\langle \hat{U}(1) \rangle = \int d^3 \vec{r}_3 \, \frac{\rho(\vec{r}_3, t_1)}{r_{13}} \,. \qquad \text{(V-26)}$$

Now, in the equations (I-12) U and $\vec{A}$ are given in the "time-retarded" form. If one makes infinite series expansions for U and $\vec{A}$ with the "retardation" term

$$\left(- \frac{r_{12}}{c}\right)$$

in the integrands as the "expansion parameter," the results take the form

$$U(\vec{r}_1, t_1) = \int d^3\vec{r}_2 \frac{\rho(\vec{r}_2, t_1)}{r_{12}} - \frac{1}{c}\frac{\partial}{\partial t_1}\int d^3\vec{r}_2\, \rho(\vec{r}_2, t_1) + \cdots$$

$$\vec{A}(\vec{r}_1, t_1) = \int d^3\vec{r}_2 \frac{\vec{j}(\vec{r}_2, t_1)}{c r_{12}} - \frac{1}{c^2}\frac{\partial}{\partial t_1}\int d^3\vec{r}_2\, \vec{j}(\vec{r}_2, t_1) + \cdots . \tag{V-27}$$

Comparing the equations (V-27) with (V-25) and (V-26) we find that

$$U(\vec{r}_1, t_1) = \langle \hat{U}(1) \rangle$$

and (V-28)

$$\vec{A}(\vec{r}_1, t_1) = \langle \hat{\vec{A}}(1) \rangle$$

to the lowest order of $\left(\frac{1}{c}\right)$. Thus, from (V-28) one may state (V-21) in the form

$$\vec{j}(1) \cong \frac{ie}{2m}\left\{\vec{\pi}^T(1)\, G_{1\gamma\gamma}(1,2) + \vec{\pi}^T{*}(2)\, G_{1\gamma\gamma}(1,2)\right\}_{2\to 1_+} + \vec{j}_M(1)$$

$$= \frac{ie}{2m}\, \mathrm{Tr}\left\{\vec{\pi}^T(1)\, G_1(1,2) + \vec{\pi}^T{*}(2)\, G_1(1,2)\right\}_{2\to 1_+} + \vec{j}_M(1), \tag{V-29}$$

where

$$\vec{\pi}^T(1) \equiv \vec{P}(1) + \frac{e}{c}\vec{A}(1) = \frac{\hbar}{i}\nabla_1 + \frac{e}{c}\vec{A}^T(1)$$

$$\vec{\pi}^T{*}(1) \equiv \vec{P}{*}(1) + \frac{e}{c}\vec{A}(1) = -\frac{\hbar}{i}\nabla_1 + \frac{e}{c}\vec{A}^T(1). \tag{V-30}$$

The "exchange" terms of order $\left(\frac{v}{c}\right)^2$ as well as all "correlation" terms from G_2 are omitted from this final equation for the current density $\vec{j}$, since we are interested here in only the lowest order contributions. In any case, the terms in $\left(\frac{v}{c}\right)^2$ are incomplete, as we have seen from the discussion of the Hamiltonian in Chapter III (time-retardation terms and other relativistic contributions, both of order $\left(\frac{v}{c}\right)^2$, were omitted).

We return now to a consideration of equation (V-17). From a comparison of (V-17) with (V-28) we see that in the lowest order equation (V-17) may be written

$$\left[\delta_{\alpha\lambda}\, i\hbar \frac{\partial}{\partial t_1} - h^T_{\alpha\lambda}(1)\right] G_{1\lambda\beta}(1,2) = \hbar\,\delta_{\alpha\beta}\,\delta^4(1,2)$$
$$+\int d^3\vec{r}_3\, \frac{(ie^2)}{r_{13}} G_{1\alpha\gamma}(\vec{r}_1, t_1; \vec{r}_3, t_1+)\, G_{1\gamma\beta}(\vec{r}_3, t_1; \vec{r}_2, t_2)\,, \qquad \text{(V-31)}$$

where

$$h^T_{\alpha\beta}(1) \equiv \delta_{\alpha\beta}\frac{1}{2m}\left[\vec{\pi}^T(1)\right]^2 + \frac{\mu\hbar e}{2mc}\,\vec{\sigma}_{\alpha\beta}\cdot \vec{B}^T(1) - \delta_{\alpha\beta}\,\zeta - \delta_{\alpha\beta}\, e\,U^T(1)\,.$$

The relations

$$\vec{B}^A(1) = \nabla_1 \times \vec{A}^A(1)$$
$$\vec{B}(1) = \nabla_1 \times \vec{A}(1) \qquad \text{(V-32)}$$

and

$$\vec{B}^T(1) = \nabla_1 \times \vec{A}^T(1)$$

were used in (V-31). Similarly, the "conjugate" equation (V-19) reduces to the form

$$\left[-\delta_{\lambda\alpha}\, i\hbar \frac{\partial}{\partial t_1} - h^{\dagger T}_{\lambda\alpha}(1)\right] G_{1\beta\lambda}(2,1) = \hbar\,\delta_{\beta\alpha}\,\delta^4)2,1)$$
$$+\int d^3\vec{r}_3\, \frac{(ie^2)}{r_{13}} G_{1\beta\gamma}(\vec{r}_2, t_2; \vec{r}_3, t_1)\, G_{1\gamma\alpha}(\vec{r}_3, t_1; \vec{r}_1, t_1+)\,, \qquad \text{(V-33)}$$

where

$$h^{\dagger T}_{\beta\alpha}(1) \equiv \delta_{\beta\alpha} \frac{1}{2m} \left[\vec{\pi}^{T} *(1)\right]^2 + \frac{\mu\hbar e}{2mc} \vec{\sigma}_{\beta\alpha} \cdot \vec{B}^{T}(1) - \delta_{\beta\alpha} \zeta - \delta_{\beta\alpha} e U^{T}(1) .$$

It should be pointed out that in equation (V-31) a term

$$\frac{1}{2m} \frac{e}{c} \vec{A}(1) \cdot \frac{e}{c} \vec{A}(1) G_{1\alpha\beta}(1, 2)$$

was inserted into the operator $h^{T}_{\alpha\beta}(1)$; while in equation (V-33) the quantity

$$\frac{1}{2m} \frac{e}{c} \vec{A}(1) \cdot \frac{e}{c} \vec{A}(1) G_{1\beta\alpha}(2, 1)$$

was similarly incorporated with $h^{\dagger T}_{\lambda\alpha} G_{1\beta\lambda}(2, 1)$. This insertion is justified partly by the contention that such a term would have appeared if the Hamiltonian had included an equivalent operator of order $\left(\frac{v}{c}\right)^4$, and by the fact that the inclusion of the terms "round off" equations (V-31, 33) up to terms of order $\left(\frac{v}{c}\right)$.

In retrospect, we see now that while the equations (V-9), (V-12), (V-20), and (V-21) represent G_1, ρ, and $\vec{j}$ correctly up to order $\left(\frac{v}{c}\right)$ as functionals of U^A and $\vec{A}^A$, the equations (V-20), (V-29), (V-31), and (V-33) represent G_1, ρ, and $\vec{j}$ correctly up to order $\left(\frac{v}{c}\right)$ as functionals of U^T and $\vec{A}^T$. Thus a "renormalization" of G_1, ρ, and $\vec{j}$ from functionals of the "externally" applied field potentials to functionals of the total electromagnetic field potentials has been accomplished. The interactions between particles are accounted for up to the lowest order of the exchange contribution; the most significant contribution from the interactions appears now in the formulation in terms of the "self-consistent fields" (total field potentials) of the system. One may consider the "renormalized" values of G_1, ρ, and $\vec{j}$ represented by equations (V-20, 29, 31, 33) to constitute a new theory in which $\vec{A}^T$ and U^T satisfy Maxwell's equations (Chapter I).

The "renormalization" of G_1, ρ, and $\vec{j}$ from functionals of the applied potentials to functionals of the total potentials was established in the lowest order for U and $\vec{A}$ (V-28). In actuality the renormalization is valid at least to the second order for U, as can easily be seen since

$$\begin{aligned} \int d^3\vec{r}_2\, \rho(\vec{r}_2, t_1) &= \text{total charge of the system} \\ &= \text{constant} \\ &= 0 \quad \text{(by definition the system is electrically neutral).} \end{aligned} \tag{V-34}$$

Functional Derivatives of the Charge and Current Densities

Now that we have ρ and $\vec{j}$ as functionals of $\vec{A}^T$ and U^T given by equations (V-20) and (V-29), we can in principle solve explicitly for ρ and $\vec{j}$ if $\vec{A}^T$ and U^T are independently known. If $\vec{A}^T$ and U^T are time dependent, this problem may be quite difficult to solve directly. In this case the utility of the perturbation theory developed in Chapter II becomes quite apparent. One may first consider ρ and $\vec{j}$ for arbitrary perturbing fields, and finally fix the values of the fields through Maxwell's equations.

The initial equilibrium state of the system is assumed to be known in terms of the Green's functions and their functional derivatives.

From (V-20) and (V-29), one obtains the functional derivatives

$$\frac{\delta \rho(1)}{\delta U^T(2)_o} = ie \, \mathrm{Tr} \, \frac{\delta G_1(1, 1_+)}{\delta U^T(2)_o}$$

$$\frac{\delta \rho(1)}{\delta A_n^T(2)_o} = ie \, \mathrm{Tr} \, \frac{\delta G_1(1, 1_+)}{\delta A_n^T(2)_o}$$

$$\frac{\delta \vec{j}(1)}{\delta U^T(2)_o} = \frac{ie}{2m} \mathrm{Tr} \left\{ \vec{\pi}^T(1)_o \frac{\delta G_1(1, 3)}{\delta U^T(2)_o} + \vec{\pi}^T{*}(3)_o \frac{\delta G_1(1, 3)}{\delta U^T(2)_o} \right\}_{3 \to 1_+} + \frac{\delta \vec{j}_M(1)}{\delta U^T(2)_o} \qquad \text{(V-35)}$$

$$\frac{\delta \vec{j}(1)}{\delta A_n^T(2)_o} = \frac{ie}{2m} \mathrm{Tr} \left\{ \vec{i}_n \frac{2e}{c} \delta^4(1, 2) G_1(1, 1_+)_o + \vec{\pi}^T(1)_o \frac{\delta G_1(1, 3)}{\delta A_n^T(2)_o} + \vec{\pi}^T{*}(3)_o \frac{\delta G_1(1, 3)}{\delta A_n^T(2)_o} \right\}_{3 \to 1_+} + \frac{\delta \vec{j}_M(1)}{\delta A_n^T(2)_o} ,$$

with

$$\frac{\delta \vec{j}_M(1)}{\delta U^T(2)_o} = c \nabla_1 \times \frac{\delta \vec{M}_s(1)}{\delta U^T(2)_o}$$

$$\frac{\delta \vec{j}_M(1)}{\delta A_n^T(2)_o} = c \nabla_1 \times \frac{\delta \vec{M}_s(1)}{\delta A_n^T(2)_o} \qquad \text{(V-36)}$$

$$\frac{\delta \vec{M}_s(1)}{\delta U^T(2)_o} = \frac{i\mu\hbar e}{2mc} \mathrm{Tr} \, \vec{\sigma} \, \frac{\delta G_1(1, 1_+)}{\delta U^T(2)_o}$$

$$\frac{\delta \vec{M}_s(1)}{\delta A_n^T(2)_o} = \frac{i\mu\hbar e}{2mc} \mathrm{Tr}\, \vec{\sigma} \frac{\delta G_1(1,1_+)}{\delta A_n^T(2)_o} \quad .$$

It should be understood that all expressions displayed in equations (V-35, 36) represent the first variations of the expressions (V-20, 29) given in the limit of no perturbations (e. g., the initial state of the system at times $t < t_o$):

$$\delta U^T \to 0$$
$$\delta \vec{A}^T \to 0 \;. \tag{V-37}$$

The applied magnetic field is taken to be aligned along the z-axis making the z-axis a preferred axis of symmetry. Accordingly, one may define several useful combinations of operators and combinations of the current density components:

$$\left.\begin{aligned} \vec{\pi}^T(1)_o &\equiv \vec{\pi}(1) \\ \vec{\pi}^T{}^*(1)_o &\equiv \vec{\pi}^*(1) \end{aligned}\right\} \text{with "equilibrium" value } \vec{A}_o^T \text{ for } \vec{A}^T$$

$$\pi_+(1) \equiv \pi_x(1) + i\pi_y(1)$$

$$\pi_-(1) \equiv \pi_x(1) - i\pi_y(1)$$

$$\pi_+^*(1) \equiv \pi_x^*(1) - i\pi_y^*(1)$$

$$\pi_-^*(1) \equiv \pi_x^*(1) + i\pi_y^*(1)$$

$$x_+ \equiv x + iy$$

$$x_- \equiv x - iy \tag{V-38}$$

$$\frac{\partial}{\partial x_+} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial x_-} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\sigma_+ \equiv \sigma_x + i\sigma_y$$

$$\sigma_- \equiv \sigma_x + i\sigma_y$$

$$A_+^T(1) \equiv A_x^T(1) + iA_y^T(1)$$

$$A_-^T(1) \equiv A_x^T(1) - iA_y^T(1)$$

$$\frac{\delta}{\delta A_+^T(2)} \equiv \frac{1}{2}\left[\frac{\delta}{\delta A_x^T(2)} - i\frac{\delta}{\delta A_y^T(2)}\right]$$

$$\frac{\delta}{\delta A_-^T(2)} \equiv \frac{1}{2}\left[\frac{\delta}{\delta A_x^T(2)} + i\frac{\delta}{\delta A_y^T(2)}\right]$$

$$j_+(1) \equiv j_x(1) + ij_y(1)$$

$$j_-(1) \equiv j_x(1) - ij_y(1)$$

$$M_s(1)_+ \equiv M_s(1)_x + iM_s(1)_y$$

$$M_s(1)_- \equiv M_s(1)_x - iM_s(1)_y \ .$$

From (V-30) and (V-38), the following combinations are possible

$$\begin{aligned}
\frac{h}{i}\frac{\partial}{\partial x_+} &= \frac{1}{4}\left[\pi_- - \pi_+^*\right] \\
\frac{h}{i}\frac{\partial}{\partial x_-} &= \frac{1}{4}\left[\pi_+ - \pi_-^*\right] \\
\frac{h}{i}\frac{\partial}{\partial z} &= \frac{1}{2}\left[\pi_z - \pi_z^*\right] \\
\frac{e}{c}A_+^T(1)_o &= \frac{1}{2}\left[\pi_+ + \pi_-^*\right] \\
\frac{e}{c}A_-^T(1)_o &= \frac{1}{2}\left[\pi_- + \pi_+^*\right] \\
\frac{e}{c}A_z^T(1)_o &= \frac{1}{2}\left[\pi_z + \pi_z^*\right].
\end{aligned} \tag{V-39}$$

With the above definitions (V-38), one may write equations (V-35, 36) as follows:

$$\frac{\delta\rho(1)}{\delta U^T(2)_o} = i\,e\,\mathrm{Tr}\,\frac{\delta G_1(1,1_+)}{\delta U^T(2)_o}$$

$$\frac{\delta\rho(1)}{\delta A_+^T(2)_o} = \frac{1}{2}\left[\frac{\delta\rho(1)}{\delta A_x^T(2)_o} - i\,\frac{\delta\rho(1)}{\delta A_y^T(2)_o}\right] = i\,e\,\mathrm{Tr}\,\frac{\delta G_1(1,1_+)}{\delta A_+^T(2)_o}$$

$$\frac{\delta\rho(1)}{\delta A_-^T(2)_o} = \frac{1}{2}\left[\frac{\delta\rho(1)}{\delta A_x^T(2)_o} + i\,\frac{\delta\rho(1)}{\delta A_y^T(2)_o}\right] = i\,e\,\mathrm{Tr}\,\frac{\delta G_I(1,1_+)}{\delta A_-^T(2)_o}$$

$$\frac{\delta \rho(1)}{\delta A_z^T(2)_o} = i e \operatorname{Tr} \frac{\delta G_1(1, 1_+)}{\delta A_z^T(2)_o}$$

$$\frac{\delta j_+(1)}{\delta U^T(2)_o} = \left[\frac{\delta j_x(1)}{\delta U^T(2)_o} + i \frac{\delta j_y(1)}{\delta U^T(2)_o}\right]$$

$$= \frac{ie}{2m} \operatorname{Tr} \left\{\left[\pi_+(1) + \pi_-^*(3)\right] \frac{\delta G_1(1, 3)}{\delta U^T(2)_o}\right\}_{3 \to 1_+} + \frac{\delta j_M(1)_+}{\delta U^T(2)_o}$$

$$\frac{\delta j_+(1)}{\delta A_+^T(2)_o} = \frac{1}{2}\left[\frac{\delta j_x(1)}{\delta A_x^T(2)_o} + i \frac{\delta j_y(1)}{\delta A_x^T(2)_o} - i \frac{\delta j_x(1)}{\delta A_y^T(2)_o} + \frac{\delta j_y(1)}{\delta A_y^T(2)_o}\right]$$

$$= \frac{ie}{2m} \operatorname{Tr} \left\{\frac{2e}{c} \delta^4(1, 2) G_1(1, 1_+)_o + \left[\pi_+(1) + \pi_-^*(3)\right] \frac{\delta G_1(1,3)}{\delta A_+^T(2)_o}\right\}_{3 \to 1_+}$$

$$+ \frac{\delta j_M(1)_+}{\delta A_+^T(2)_o} . \qquad \text{(V-40)}$$

$$\frac{\delta j_+(1)}{\delta A_-^T(2)_o} = \frac{1}{2}\left[\frac{\delta j_x(1)}{\delta A_x^T(2)_o} + i \frac{\delta j_y(1)}{\delta A_x^T(2)_o} + i \frac{\delta j_x(1)}{\delta A_y^T(2)_o} - \frac{\delta j_y(1)}{\delta A_y^T(2)_o}\right]$$

$$= \frac{ie}{2m} \operatorname{Tr} \left\{\left[\pi_+(1) + \pi_-^*(3)\right] \frac{\delta G_1(1, 3)}{\delta A_-^T(2)_o}\right\}_{3 \to 1_+} + \frac{\delta j_M(1)_+}{\delta A_-^T(2)_o}$$

$$\frac{\delta j_+(1)}{\delta A_z^T(2)_o} = \left[\frac{\delta j_x(1)}{\delta A_z^T(2)_o} + i\,\frac{\delta j_y(1)}{\delta A_z^T(2)_o}\right]$$

$$= \frac{ie}{2m}\,\mathrm{Tr}\left\{\left[\pi_+(1) + \pi_-^*(3)\right]\frac{\delta G_1(1,3)}{\delta A_z^T(2)_o}\right\}_{3\to 1_+} + \frac{\delta j_M(1)_+}{\delta A_z^T(2)_o}$$

$$\frac{\delta j_-(1)}{\delta U^T(2)_o} = \left[\frac{\delta j_x(1)}{\delta U^T(2)_o} - i\,\frac{\delta j_y(1)}{\delta U^T(2)_o}\right]$$

$$= \frac{ie}{2m}\,\mathrm{Tr}\left\{\left[\pi_-(1) + \pi_+^*(3)\right]\frac{\delta G_1(1,3)}{\delta U^T(2)_o}\right\}_{3\to 1_+} + \frac{\delta j_M(1)_-}{\delta U^T(2)_o}$$

$$= \frac{1}{2}\left[\frac{\delta j_x(1)}{\delta A_x^T(2)_o} - i\,\frac{\delta j_y(1)}{\delta A_x^T(2)_o} - i\,\frac{\delta j_x(1)}{\delta A_y^T(2)_o} - \frac{\delta j_x(1)}{\delta A_y^T(2)_o}\right]$$

$$\frac{\delta j_-(1)}{\delta A_+^T(2)_o} = \frac{ie}{2m}\,\mathrm{Tr}\left\{\left[\pi_-(1) + \pi_+^*(3)\right]\frac{\delta G_1(1,3)}{\delta A_+^T(2)_o}\right\}_{3\to 1_+} + \frac{\delta j_M(1)_-}{\delta A_+^T(2)_o}$$

$$\frac{\delta j_-(1)}{\delta A_-^T(2)_o} = \frac{1}{2}\left[\frac{\delta j_x(1)}{\delta A_x^T(2)_o} - i\,\frac{\delta j_y(1)}{\delta A_x^T(2)_o} + i\,\frac{\delta j_x(1)}{\delta A_y^T(2)_o} + \frac{\delta j_y(1)}{\delta A_y^T(2)_o}\right]$$

$$= \frac{ie}{2m}\,\mathrm{Tr}\left\{\frac{2e}{c}\,\delta^4(1,2)\,G_1(1,1_+)_o + \left[\pi_-(1) + \pi_+^*(3)\right]\frac{\delta G_1(1,3)}{\delta A_-^T(2)_o}\right\}_{3\to 1_+}$$

$$+ \frac{\delta j_M(1)_-}{\delta A_-^T(2)_o}$$

$$\frac{\delta j_-(1)}{\delta A_z^T(2)_o} = \left[\frac{\delta j_x(1)}{\delta A_z^T(2)_o} - i\,\frac{\delta j_y(1)}{\delta A_z^T(2)_o}\right]$$

$$= \frac{ie}{2m}\,\mathrm{Tr}\left\{\left[\pi_-(1) + \pi_+^*(3)\right]\frac{\delta G_1(1,3)}{\delta A_z^T(2)_o}\right\}_{3\to 1_+} + \frac{\delta j_M(1)_-}{\delta A_z^T(2)_o}$$

$$\frac{\delta j_z(1)}{\delta U^T(2)_o} = \frac{ie}{2m}\,\mathrm{Tr}\left\{\left[\pi_z(1) + \pi_z^*(3)\right]\frac{\delta G_1(1,3)}{\delta U^T(2)_o}\right\}_{3\to 1_+} + \frac{\delta j_M(1)_z}{\delta U^T(2)_o}$$

$$\frac{\delta j_z(1)}{\delta A_+^T(2)_o} = \frac{1}{2}\left[\frac{\delta j_z(1)}{\delta A_x^T(2)_o} - i\,\frac{\delta j_z(1)}{\delta A_y^T(2)_o}\right]$$

$$= \frac{ie}{2m}\,\mathrm{Tr}\left\{\left[\pi_z(1) + \pi_z^*(3)\right]\frac{\delta G_1(1,3)}{\delta A_+^T(2)_o}\right\}_{3\to 1_+} + \frac{\delta j_M(1)_z}{\delta A_+^T(2)_o}$$

$$\frac{\delta j_z(1)}{\delta A_-^T(2)_o} = \frac{1}{2}\left[\frac{\delta j_z(1)}{\delta A_x^T(2)_o} + i\,\frac{\delta j_z(1)}{\delta A_y^T(2)_o}\right]$$

$$= \frac{ie}{2m}\,\mathrm{Tr}\left\{\left[\pi_z(1) + \pi_z^*(3)\right]\frac{\delta G_1(1,3)}{\delta A_-^T(2)_o}\right\}_{3\to 1_+} + \frac{\delta j_M(1)_z}{\delta A_-^T(2)_o}$$

$$\frac{\delta j_z(1)}{\delta A_z^T(2)_o} = \frac{ie}{2m}\,\mathrm{Tr}\left\{\frac{2e}{c}\,\delta^4(1,2)\,G_1(1,1_+)_o\right.$$

$$\left. + \left[\pi_z(1) + \pi_z^*(3)\right]\frac{\delta G_1(1,3)}{\delta A_z^T(2)_o}\right\}_{3\to 1_+} + \frac{\delta j_M(1)_z}{\delta A_z^T(2)_o}$$

with

$$\frac{\delta j_M(1)_+}{\delta U^T(2)_o} = -ic\,2\,\frac{\partial}{\partial x_{1-}}\frac{\delta M_s(1)_z}{\delta U^T(2)_o} + ic\,\frac{\partial}{\partial z_1}\frac{\delta M_s(1)_+}{\delta U^T(2)_o}$$

$$\frac{\delta j_M(1)_+}{\delta A^T_{\left\{ {\pm \atop z} \right\}}(2)_o} = -ic\,2\,\frac{\partial}{\partial x_{1-}}\frac{\delta M_s(1)_z}{\delta A^T_{\left\{ {\pm \atop z} \right\}}(2)_o} + ic\,\frac{\partial}{\partial z_1}\frac{\delta M_s(1)_+}{\delta A^T_{\left\{ {\pm \atop z} \right\}}(2)_o}$$

$$\frac{\delta j_M(1)_-}{\delta U^T(2)_o} = ic\,2\,\frac{\partial}{\partial x_{1+}}\frac{\delta M_s(1)_z}{\delta U^T(2)_o} - ic\,\frac{\partial}{\partial z_1}\frac{\delta M_s(1)_-}{\delta U^T(2)_o}$$

$$\frac{\delta j_M(1)_-}{\delta A^T_{\left\{ {\pm \atop z} \right\}}(2)_o} = ic\,2\,\frac{\partial}{\partial x_{1+}}\frac{\delta M_s(1)_z}{\delta A^T_{\left\{ {\pm \atop z} \right\}}(2)_o} - ic\,\frac{\partial}{\partial z_1}\frac{\delta M_s(1)_-}{\delta A^T_{\left\{ {\pm \atop z} \right\}}(2)_o}$$

$$\frac{\delta j_M(1)_z}{\delta U^T(2)_o} = ic\,\frac{\partial}{\partial x_{1-}}\frac{\delta M_s(1)_-}{\delta U^T(2)_o} - ic\,\frac{\partial}{\partial x_{1+}}\frac{\delta M_s(1)_+}{\delta U^T(2)_o}$$

$$\frac{\delta j_M(1)_z}{\delta A^T_{\left\{ {\pm \atop z} \right\}}(2)_o} = ic\,\frac{\partial}{\partial x_{1-}}\frac{\delta M_s(1)_-}{\delta A^T_{\left\{ {\pm \atop z} \right\}}(2)_o} - ic\,\frac{\partial}{\partial x_{1+}}\frac{\delta M_s(1)_+}{\delta A^T_{\left\{ {\pm \atop z} \right\}}(2)_o} \qquad \text{(V-41)}$$

and

$$\frac{\delta M_s(1)_+}{\delta U^T(2)_o} = \frac{i\mu\hbar e}{2mc}\,\mathrm{Tr}\,\sigma_+\,\frac{\delta G_1(1,1_+)}{\delta U^T(2)_o}$$

$$\frac{\delta M_s(1)_+}{\delta A^T_{\left\{ {\pm \atop z} \right\}}(2)_o} = \frac{i\mu\hbar e}{2mc}\,\mathrm{Tr}\,\sigma_+\,\frac{\delta G_1(1,1_+)}{\delta A^T_{\left\{ {\pm \atop z} \right\}}(2)_o}$$

$$\frac{\delta M_s(1)_-}{\delta U^T(2)_o} = \frac{i\mu\hbar e}{2mc}\,\mathrm{Tr}\,\sigma_-\,\frac{\delta G_1(1,1_+)}{\delta U^T(2)_o}$$

$$\frac{\delta M_s(1)_-}{\delta A^T_{\{\pm z\}}(2)_o} = \frac{i\mu\hbar e}{2mc} \operatorname{Tr} \sigma_- \frac{\delta G_1(1, 1_+)}{\delta A^T_{\{\pm z\}}(2)_o}$$

$$\frac{\delta M_s(1)_z}{\delta U^T(2)_o} = \frac{i\mu\hbar e}{2mc} \operatorname{Tr} \sigma_z \frac{\delta G_1(1, 1_+)}{\delta U^T(2)_o}$$

$$\frac{\delta M_s(1)_z}{\delta A^T_{\{\pm z\}}(2)_o} = \frac{i\mu\hbar e}{2mc} \operatorname{Tr} \sigma_z \frac{\delta G_1(1, 1_+)}{\delta A^T_{\{\pm z\}}(2)_o}$$

The functional derivatives

$$\frac{\delta G_1(1, 3)}{\delta U^T(2)_o} \quad \text{and} \quad \frac{\delta G_1(1, 3)}{\delta A_q^T(2)_o}$$

are found in Cartesian co-ordinate form from the equations (V-31) and (V-33). The first variation of equation (V-31) with respect to $U^T(2)$ results in the equation

$$\left\{\delta_{\alpha\lambda} i\hbar \frac{\partial}{\partial t_1} - h^T_{\alpha\lambda}(1)_o\right\} \frac{\delta G_{1\lambda\beta}(1, 3)}{\delta U^T(2)_o} + e\, \delta^4(1, 2) G_{1\alpha B}(1, 3)_o$$

$$= \int d^3\vec{r}_4 \frac{(ie^2)}{r_{14}} \left\{ \frac{\delta G_{1\alpha\gamma}(\vec{r}_1, t_1; \vec{r}_4, t_1+)}{\delta U^T(2)_o} G_{1\gamma\beta}(\vec{r}_4, t_1; \vec{r}_3, t_3)_o \right. \qquad \text{(V-42)}$$

$$\left. + G_{1\alpha\gamma}(\vec{r}_1, t_1; \vec{r}_4; t_1+)_o \frac{\delta G_{1\gamma\beta}(\vec{r}_4, t_1; \vec{r}_3, t_3)}{\delta U^T(2)_o} \right\} .$$

To obtain

$$\frac{\delta G_1(1,3)}{\delta U^T(2)_o},$$

both sides of equation (V-42) are pre-multiplied by $G_{1\eta\alpha}(5,1)_o$, then summed over the spin index α, and integrated over the set of co-ordinates $(\vec{r}_1, t_1)$, i.e.,

$$\int d(1)\, G_{1\eta\alpha}(5,1)_o \left\{ \delta_{\alpha\lambda}\, i\hbar \frac{\partial}{\partial t_1} - h^T_{\alpha\lambda}(1)_o \right\} \frac{\delta G_{1\lambda\beta}(1,3)}{\delta U^T(2)_o}$$

$$+ e \int d(1)\, G_{1\eta\alpha}(5,1)_o\, \delta^4(1,2)\, G_{1\alpha\beta}(1,3)_o =$$

$$\iint d(1) d^3\vec{r}_4 \frac{(ie^2)}{r_{14}} \left\{ G_{1\eta\alpha}(5,1)_o \frac{\delta G_{1\alpha\gamma}(\vec{r}_1, t_1; \vec{r}_4, t_1+)}{\delta U^T(2)_o} G_{1\gamma\beta}(\vec{r}_4, t_1; \vec{r}_3, t_3)_o \right.$$

$$\left. + G_{1\eta\alpha}(5,1)_o\, G_{1\alpha\gamma}(\vec{r}_1, t_1; \vec{r}_4, t_1+)_o \frac{\delta G_{1\gamma\beta}(\vec{r}_4, t_1; \vec{r}_3, t_3)}{\delta U^T(2)_o} \right\} \tag{V-43}$$

After several integrations by parts in the first term on the left hand side of equation (V-43) and with the use of equation (V-33) one obtains the following result:

$$\frac{\delta G_{1\eta\beta}(5,3)}{\delta U^T(2)_o} = -\frac{e}{\hbar} G_{1\eta\lambda}(5,2)_o\, G_{1\lambda\beta}(2,3)_o$$

$$+ \frac{ie^2}{\hbar} \iint d(1) d^3\vec{r}_4 \frac{1}{r_{14}} G_{1\eta\alpha}(5,1)_o \frac{\delta G_{1\alpha\gamma}(\vec{r}_1, t_1; \vec{r}_4, t_1+)}{\delta U^T(2)_o} G_{1\gamma\beta}(\vec{r}_4, t_1; \vec{r}_3, t_3)_o. \tag{V-44}$$

If the exchange contribution to (V-44) is considered negligible, then

$$\frac{\delta G_{1\alpha\beta}(1,3)}{\delta U^T(2)_o} \cong -\frac{e}{\hbar} G_{1\alpha\lambda}(1,2)_o G_{1\lambda\beta}(2,3)_o$$

or, in matrix form, (V-45)

$$\frac{\delta G_1(1,3)}{\delta U^T(2)_o} \cong -\frac{e}{\hbar} G_1(1,2)_o G_1(2,3)_o .$$

The first variation of equation (V-31) with respect to $A_q^T(2)$ is

$$\left\{\delta_{\alpha\lambda} i\hbar \frac{\partial}{\partial t_1} - h^T_{\alpha\lambda}(1)_o\right\} \frac{\delta G_{1\lambda\beta}(1,3)}{\delta A_q^T(2)_o} - \left\{\delta_{\alpha\lambda} \frac{e}{2mc} \delta^4(1,2)\pi_q(1)\right.$$

$$\left. + \delta_{\alpha\lambda} \frac{e}{2mc} \pi_q(1)\delta^4(1,2) + \frac{\mu\hbar e}{2mc}\left[\vec{\sigma}_{\alpha\lambda} \times \nabla_1 \delta^4(1,2)\right]_q\right\} G_{1\lambda\beta}(1,3)_o$$

(V-46)

$$= \int d^3\vec{r}_4 \frac{(i e^2)}{r_{14}} \left\{\frac{\delta G_{1\alpha\gamma}(\vec{r}_1, t_1; \vec{r}_4, t_1+)}{\delta A_q^T(2)_o} G_{1\gamma\beta}(\vec{r}_4, t_1; \vec{r}_3, t_3)_o\right.$$

$$\left. + G_{1\alpha\gamma}(\vec{r}_1, t_1; \vec{r}_4, t_1+)_o \frac{\delta G_{1\gamma\beta}(\vec{r}_4, t_1; \vec{r}_3, t_3)}{\delta A_q^T(2)_o}\right\} .$$

By the same procedure that led to equations (V-44, 45) we find from equation (V-46) that

$$\frac{\delta G_{1\eta\beta}(5,3)}{\delta A_q^T(2)_o} = \frac{e}{2mc\hbar}\left\{\left[\pi_q(2) + \pi_q{}^*(6)\right] G_{1\eta\lambda}(5,6)_o G_{1\lambda\beta}(2,3)_o\right\}_{6\to 2}$$

$$+\frac{\mu e}{2mc}\left[\nabla_2 \times G_{1\eta\alpha}(5,2)_o\, \vec{\sigma}_{\alpha\lambda}\, G_{1\lambda\beta}(2,3)_o\right]_q \tag{V-47}$$

$$+\frac{ie^2}{\hbar}\iint d(1)d^3\vec{r}_4 \frac{1}{r_{14}}\, G_{1\eta\alpha}(5,1)_o\, \frac{\delta G_{1\alpha\lambda}(\vec{r}_1,t_1;\vec{r}_4,t_1+)}{\delta A_q^T(2)_o} G_{1\lambda\beta}(\vec{r}_4,t_1;\vec{r}_3,t_3)_o.$$

If the exchange contribution in (V-47) is considered negligible, then

$$\frac{\delta G_{1\alpha\beta}(1,3)}{\delta A_q^T(2)_o} \cong \frac{e}{2mc\hbar}\left\{\left[\pi_q(2)+\pi_q^*(4)\right] G_{1\alpha\gamma}(1,4)_o\, G_{1\gamma\beta}(2,3)_o\right.$$

$$\left.+\mu\hbar\left[\nabla_2 \times G_{1\alpha\gamma}(1,2)_o\, \vec{\sigma}_{\gamma\lambda}\, G_{1\lambda\beta}(2,3)_o\right]_{-q}\right\}_{4\to 2}$$

or, in matrix form, (V-48)

$$\frac{\delta G_1(1,3)}{\delta A_q^T(2)_o} \cong \frac{e}{2mc\hbar}\left\{\left[\pi_q(2)+\pi_q^*(4)\right] G_1(1,4)_o\, G_1(2,3)_o\right.$$

$$\left.+\mu\hbar\left[\nabla_2 \times G_1(1,2)_o\, \vec{\sigma}\, G_1(2,3)_o\right]_{-q}\right\}_{4\to 2}$$

With the aid of the definitions (V-38), one obtains from (V-48) the equations

$$\frac{\delta G_1(1,3)}{\delta A_+^T(2)_o} = \frac{e}{4mc\hbar}\left\{\left[\pi_-(2)+\pi_+^*(4)\right] G_1(1,4)_o\, G_1(2,3)_o\right.$$

$$\left.+\mu\hbar\left[i\,2\frac{\partial}{\partial x_{2+}} G_1(1,2)_o\, \sigma_z\, G_1(2,3)_o - i\frac{\partial}{\partial z_2} G_1(1,2)_o\, \sigma_-\, G_1(2,3)_o\right]\right\}_{4\to 2}$$

$$\frac{\delta G_1(1,3)}{\delta A_-^T(2)_o} = \frac{e}{4mc\hbar}\left\{\left[\pi_+(2)+\pi_-^*(4)\right] G_1(1,4)_o\, G_1(2,3)_o\right.$$

$$\left. + \mu\hbar\left[-i\,2\frac{\partial}{\partial x_{2-}} G_1(1,2)_o\,\sigma_z\, G_1(2,3)_o + i\frac{\partial}{\partial z_2} G_1(1,2)_o\,\sigma_+\, G_1(2,3)_o\right]\right\}_{4\to 2}$$

(V-49)

$$\frac{\delta G_1(1,3)}{\delta A_z^T(2)_o} = \frac{e}{2mc\hbar}\left\{\left[\pi_z(2)+\pi_z^*(4)\right] G_1(1,4)_o\, G_1(2,3)_o\right.$$

$$\left. + i\mu\hbar\left[\frac{\partial}{\partial x_{2-}} G_1(1,2)_o\,\sigma_-\, G_1(2,3)_o - \frac{\partial}{\partial x_{2+}} G_1(1,2)_o\,\sigma_+\, G_1(2,3)_o\right]\right\}_{4\to 2} .$$

With the help of the equations (V-45) and (V-49), equations (V-40, 41) may be written in the form

$$\frac{\delta \rho(1)}{\delta U^T(2)_o} = -\frac{i e^2}{\hbar}\,\mathrm{Tr}\, G_1(1,2)_o\, G_1(2,1)_o$$

$$\frac{\delta \rho(1)}{\delta A_+^T(2)_o} = \frac{i e^2}{4mc\hbar}\,\mathrm{Tr}\left\{\left[\pi_-(2)+\pi_+^*(4)\right] G_1(1,4)_o\, G_1(2,1)_o\right.$$

$$\left. + i\mu\hbar\left[2\frac{\partial}{\partial x_{2+}} G_1(1,2)_o\sigma_z\, G_1(2,1)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o\sigma_-\, G_1(2,1)_o\right]\right\}_{4\to 2}$$

$$\frac{\delta \rho(1)}{\delta A_-^T(2)_o} = \frac{i e^2}{4mc\hbar}\,\mathrm{Tr}\left\{\left[\pi_+(2)+\pi_-^*(4)\right] G_1(1,4)_o\, G_1(2,1)_o\right.$$

$$\left. - i\mu\hbar\left[2\frac{\partial}{\partial x_{2-}} G_1(1,2)_o\sigma_z\, G_1(2,1)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o\sigma_+\, G_1(2,1)_o\right]\right\}_{4\to 2}$$

$$\frac{\delta \rho(1)}{\delta A_z^T(2)_o} = \frac{ie^2}{2mc\hbar} \mathrm{Tr} \left\{ \left[\pi_z(2) + \pi_z^*(4) \right] G_1(1,4)_o \, G_1(2,1)_o \right.$$

$$\left. + i\mu\hbar \left[\frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_- G_1(2,1)_o - \frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_+ G(2,1)_o \right] \right\}_{4 \to 2}$$

$$\frac{\delta j_+(1)}{\delta U^T(2)_o} = \frac{ie^2}{2m\hbar} \mathrm{Tr} \left\{ \left[\pi_+(1) + \pi_-^*(3) \right] G_1(1,2)_o \, G_1(2,3)_o \right\}_{3 \to 1} + \frac{\delta j_M(1)_+}{\delta U^T(2)_o}$$

$$\frac{\delta j_+(1)}{\delta A_+^T(2)_o} = \frac{ie^2}{8m^2 c\hbar} \mathrm{Tr} \left\{ 8\hbar m \, \delta^4(1,2) \, G_1(1,1_+)_o \right.$$

(V-50)

$$+ \left[\pi_+(1) + \pi_-^*(3) \right] \left[\pi_-(2) + \pi_+^*(4) \right] G_1(1,4)_o \, G_1(2,3)_o$$

$$\left. + i\mu\hbar \left[\pi_+(1) + \pi_-^*(3) \right] \left[2 \frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_z G_1(2,3)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o \sigma_- G_1(2,3)_o \right] \right\}_{\substack{3 \to 1 \\ 4 \to 2}}$$

$$+ \frac{\delta j_M(1)_+}{\delta A_+^T(2)_o}$$

$$\frac{\delta j_+(1)}{\delta A_-^T(2)_o} = \frac{ie^2}{8m^2 c\hbar} \mathrm{Tr} \left\{ \left[\pi_+(1) + \pi_-^*(3) \right] \left[\pi_+(2) + \pi_-^*(4) \right] G_1(1,4)_o \, G_1(2,3)_o \right.$$

$$\left. - i\mu\hbar \left[\pi_+(1) + \pi_-^*(3) \right] \left[2 \frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_z G_1(2,3)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o \sigma_+ G_1(2,3)_o \right] \right\}$$

$$+ \frac{\delta j_M(1)_+}{\delta A_-^T(2)_o}$$

$$\frac{\delta j_+(1)}{\delta A_z^T(2)_o} = \frac{ie^2}{4m^2 c\hbar} \mathrm{Tr} \left\{ \left[\pi_+(1) + \pi_-^*(3)\right]\left[\pi_z(2) + \pi_z^*(4)\right] G_1(1,4)_o \, G_1(2,3)_o \right.$$

$$\left. + i\mu\hbar \left[\pi_+(1) + \pi_-^*(3)\right]\left[\frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_- G_1(2,3)_o - \frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_+ G_1(2,3)_o\right]\right\}_{\substack{3\to 1\\ 4\to 2}}$$

$$+ \frac{\delta j_M(1)_+}{\delta A_z^T(2)_o}$$

$$\frac{\delta j_-(1)}{\delta U^T(2)_o} = -\frac{ie^2}{2m\hbar} \mathrm{Tr} \left\{ \left[\pi_-(1) + \pi_+^*(3)\right] G_1(1,2)_o \, G_1(2,3)_o \right\}_{3\to 1} + \frac{\delta j_M(1)_-}{\delta U^T(2)_o}$$

$$\frac{\delta j_-(1)}{\delta A_+^T(2)_o} = \frac{ie^2}{8m^2 c\hbar} \mathrm{Tr} \left\{ \left[\pi_-(1) + \pi_+^*(3)\right]\left[\pi_-(2) + \pi_+^*(4)\right] G_1(1,4)_o \, G_1(2,3)_o \right.$$

$$\left. + i\mu\hbar \left[\pi_-(1) + \pi_+^*(3)\right]\left[2\frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_z G_1(2,3)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o \sigma_- G_1(2,3)_o\right]\right\}_{\substack{3\to 1\\ 4\to 2}}$$

$$+ \frac{\delta j_M(1)_-}{\delta A_+^T(2)_o}$$

$$\frac{\delta j_-(1)}{\delta A_-^T(2)_o} = \frac{ie^2}{8m^2 c\hbar} \mathrm{Tr} \left\{ 8\hbar m \, \delta^4(1,2) \, G_1(1,1_+)_o \right.$$

$$+ \left[\pi_-(1) + \pi_+^*(3)\right]\left[\pi_+(2) + \pi_-^*(4)\right] G_1(1,4)_o \, G_1(2,3)_o$$

$$\left. - i\mu\hbar \left[\pi_-(1) + \pi_+^*(3)\right]\left[2\frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_z G_1(2,3)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o \sigma_+ G_1(2,3)_o\right]\right\}_{\substack{3\to 1\\ 4\to 2}}$$

$$+ \frac{\delta j_M(1)_-}{\delta A_-^T(2)_o}$$

$$\frac{\delta j_-(1)}{\delta A_z^T(2)_o} = \frac{ie^2}{4m^2 c\hbar} \mathrm{Tr} \Big\{ \Big[\pi_-(1) + \pi_+^*(3)\Big]\Big[\pi_z(2) + \pi_z^*(4)\Big] G_1(1,4)_o G_1(2,3)_o$$

$$+ i\mu\hbar \Big[\pi_-(1) + \pi_+^*(3)\Big]\Big[\frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_- G_1(2,3)_o - \frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_+ G_1(2,3)_o\Big]\Big\}_{\substack{3\to 1\\ 4\to 2}}$$

$$+ \frac{\delta j_M(1)_-}{\delta A_z^T(2)_o}$$

$$\frac{\delta j_z(1)}{\delta U^T(2)_o} = - \frac{ie^2}{2m\hbar} \mathrm{Tr} \Big\{ \Big[\pi_z(1) + \pi_z^*(3)\Big] G_1(1,2)_o G_1(2,3)_o \Big\}_{3\to 1} + \frac{\delta j_M(1)_z}{\delta U^T(2)_o}$$

$$\frac{\delta j_z(1)}{\delta A_+^T(2)_o} = \frac{ie^2}{8m^2 c\hbar} \mathrm{Tr} \Big\{ \Big[\pi_z(1) + \pi_z^*(3)\Big]\Big[\pi_-(2) + \pi_+^*(4)\Big] G_1(1,4)_o G_1(2,3)_o$$

$$+ i\mu\hbar \Big[\pi_z(1) + \pi_z^*(3)\Big]\Big[2\frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_z G_1(2,3)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o \sigma_- G_1(2,3)_o\Big]\Big\}_{\substack{3\to 1\\ 4\to 2}}$$

$$+ \frac{\delta j_M(1)_z}{\delta A_+^T(2)_o}$$

$$\frac{\delta j_z(1)}{\delta A_-^T(2)_o} = \frac{ie^2}{8m^2 c\hbar} \mathrm{Tr} \Big\{ \Big[\pi_z(1) + \pi_z^*(3)\Big]\Big[\pi_+(2) + \pi_-^*(4)\Big] G_1(1,4)_o G_1(2,3)_o$$

$$- i\mu\hbar \Big[\pi_z(1) + \pi_z^*(3)\Big]\Big[2\frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_z G_1(2,3)_o - G_1(1,2)_o \sigma_+ G_1(2,3)_o\Big]\Big\}_{\substack{3\to 1\\ 4\to 2}}$$

$$+ \frac{\delta j_M(1)_z}{\delta A_-^T(2)_o}$$

$$\frac{\delta j_z(1)}{\delta A_z^T(2)_o} = \frac{ie^2}{4m^2 c\hbar} \mathrm{Tr} \Big\{ 4\hbar m\, \delta^4(1,2)\, G_1(1,1_+)_o$$

$$+ \Big[\pi_z(1) + \pi_z{*}(3)\Big]\Big[\pi_z(2) + \pi_z{*}(4)\Big] G_1(1,4)_o\, G_1(2,3)_o$$

$$+ i\mu\hbar\Big[\pi_z(1) + \pi_z{*}(3)\Big]\Big[\frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_- G_1(2,3)_o - \frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_+ G_1(2,3)_o\Big]\Big\}_{\substack{3\to 1\\ 4\to 2}}$$

$$+ \frac{\delta j_M(1)_z}{\delta A_z^T(2)_o}$$

The variational derivatives of $\vec{j}_M$ are again given by equations (V-41) and the variational derivatives of $\vec{M}_s$ are given by

$$\frac{\delta M_s(1)_+}{\delta U^T(2)_o} = -\frac{i\mu e^2}{2mc} \mathrm{Tr}\, \sigma_+ G_1(1,2)_o\, G_1(2,1)_o$$

$$\frac{\delta M_s(1)_+}{\delta A_+^T(2)_o} = \frac{i\mu e^2}{8m^2 c^2} \mathrm{Tr}\, \sigma_+ \Big\{\Big[\pi_-(2) + \pi_+{*}(4)\Big] G_1(1,4)_o\, G_1(2,1)_o$$

$$+ i\mu\hbar\Big[2\frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_z G_1(2,1)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o \sigma_- G_1(2,1)_o\Big]\Big\}_{4\to 2}$$

$$\frac{\delta M_s(1)_+}{\delta A_-^T(2)_o} = \frac{i\mu e^2}{8m^2c^2} \operatorname{Tr} \sigma_+ \Big\{ \Big[\pi_+(2) + \pi_1^*(4)\Big] G_1(1,4)_o\, G_1(2,1)_o$$

$$- i\mu\hbar \Big[2 \frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_z G_1(2,1)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o \sigma_+ G_1(2,1)_o \Big]\Big\}_{4\to 2}$$

$$\frac{\delta M_s(1)_+}{\delta A_z^T(2)_o} = \frac{i\mu e^2}{4m^2c^2} \operatorname{Tr} \sigma_+ \Big\{ \Big[\pi_z(2) + \pi_z^*(4)\Big] G_1(1,4)_o\, G_1(2,1)_o$$

$$+ i\mu\hbar \Big[\frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_- G_1(2,1)_o - \frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_+ G_1(2,1)_o \Big]\Big\}_{4\to 2}$$

$$\frac{\delta M_s(1)_-}{\delta U^T(2)_o} = - \frac{i\mu e^2}{2mc} \operatorname{Tr} \sigma_- \, G_1(1,2)_o\, G_1(2,1)_o \qquad \text{(V-51)}$$

$$\frac{\delta M_s(1)_-}{\delta A_+^T(2)_o} = \frac{i\mu e^2}{8m^2c^2} \operatorname{Tr} \sigma_- \Big\{ \Big[\pi_-(2) + \pi_+^*(4)\Big] G_1(1,4)_o\, G_1(2,1)_o$$

$$+ i\mu\hbar \Big[2 \frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_z G_1(2,1)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o \sigma_- G_1(2,1)_o \Big]\Big\}_{4\to 2}$$

$$\frac{\delta M_s(1)_-}{\delta A_-^T(2)_o} = \frac{i\mu e^2}{8m^2c^2} \operatorname{Tr} \sigma_- \Big\{ \Big[\pi_+(2) + \pi_-^*(4)\Big] G_1(1,4)_o\, G_1(2,1)_o$$

$$- i\mu\hbar \Big[2 \frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_z G_1(2,1)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o \sigma_+ G_1(2,1)_o \Big]\Big\}_{4\to 2}$$

$$\frac{\delta M_s(1)_-}{\delta A_z^T(2)_o} = \frac{i\mu e^2}{4m^2c^2} \operatorname{Tr} \sigma_- \left\{ \left[\pi_z(2) + \pi_z^*(4) \right] G_1(1,4)_o \, G_1(2,1)_o \right.$$

$$\left. + \, i\mu\hbar \left[\frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_- G_1(2,1)_o - \frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_+ G_1(2,1)_o \right] \right\}_{4\to 2}$$

$$\frac{\delta M_s(1)_z}{\delta U^T(2)_o} = -\frac{i\mu e^2}{2mc} \operatorname{Tr} \sigma_z \, G_1(1,2)_o \, G_1(2,1)_o$$

$$\frac{\delta M_s(1)_z}{\delta A_+^T(2)_o} = \frac{i\mu e^2}{8m^2c^2} \operatorname{Tr} \sigma_z \left\{ \left[\pi_-(2) + \pi_+^*(4) \right] G_1(1,4)_o \, G_1(2,1)_o \right.$$

$$\left. - \, i\mu\hbar \left[2 \frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_z G_1(2,1)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o \sigma_- G_1(2,1)_o \right] \right\}_{4\to 2}$$

$$\frac{\delta M_s(1)_z}{\delta A_-^T(2)_o} = \frac{i\mu e^2}{8m^2c^2} \operatorname{Tr} \sigma_z \left\{ \left[\pi_+(2) + \pi_-^*(4) \right] G_1(1,4)_o \, G_1(2,1)_o \right.$$

$$\left. - \, i\mu\hbar \left[2 \frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_z G_1(2,1)_o - \frac{\partial}{\partial z_2} G_1(1,2)_o \sigma_+ G_1(2,1)_o \right] \right\}_{4\to 2}$$

$$\frac{\delta M_s(1)_z}{\delta A_z^T(2)_o} = \frac{i\mu e^2}{4m^2c^2} \operatorname{Tr} \sigma_z \left\{ \left[\pi_z(2) + \pi_z^*(4) \right] G_1(1,4)_o \, G_1(2,1)_o \right.$$

$$\left. + \, i\mu\hbar \left[\frac{\partial}{\partial x_{2-}} G_1(1,2)_o \sigma_- G_1(2,1)_o - \frac{\partial}{\partial x_{2+}} G_1(1,2)_o \sigma_+ G_1(2,1)_o \right] \right\}_{4\to 2} .$$

The equations (V-50, 51) are given in the "self-consistent field approximation" (all exchange and higher order correlation contributions are neglected); the appropriate Green's function to be used in this approximation will be the solution of equation (V-31) with the exchange term omitted.

There are a great number of terms in equations (V-50, 51) involving the Pauli spin matrices. Fortunately, if G_1 is diagonal a number of the spin terms vanish immediately due to the trace operations involved in the equations. The spin matrices are given by (see reference (35), section 10):

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{(V-52)}$$

and thus

$$\sigma_+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},$$

$$\sigma_- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

The following combinations appear in the equations (V-50, 51) which vanish identically for diagonal G_1:

$$\mathrm{Tr}\, G_1(1,4)_o \sigma_- G_1(2,3)_o = \mathrm{Tr}\, G_1(1,4)_o \sigma_+ G_1(2,3)_o$$

$$= \mathrm{Tr}\, \sigma_- G_1(1,4)_o G_1(2,3)_o = \mathrm{Tr}\, \sigma_+ G_1(1,4)_o G_1(2,3)_o$$

$$= \mathrm{Tr}\, \sigma_z G_1(1,4)_o \sigma_- G_1(2,3)_o = \mathrm{Tr}\, \sigma_z G_1(1,4)_o \sigma_+ G_1(2,3)_o$$

$$= \mathrm{Tr}\, \sigma_- G_1(1,4)_o \sigma_z G_1(2,3)_o = \mathrm{Tr}\, \sigma_+ G_1(1,4)_o \sigma_z G_1(2,3)_o \qquad \text{(V-53)}$$

$$= \mathrm{Tr}\, \sigma_- G_1(1,4)_o \sigma_- G_1(2,3)_o = \mathrm{Tr}\, \sigma_+ G_1(1,4)_o \sigma_+ G_1(2,3)_o$$

$$\equiv 0, \text{ for diagonal } G_1.$$

Also for diagonal G_1 we have the relations

$$\sigma_z G_1(1,4)_o \sigma_z G_1(2,3)_o = G_1(1,4)_o G_1(2,3)_o$$

$$\sigma_z G_1(1,4)_o G_1(2,3)_o = G_1(1,4)_o \sigma_z G_1(2,3)_o . \qquad \text{(V-54)}$$

The use of equations (V-53) and (V-54) with equations (V-50, 51) simplifies and shortens the work of calculation a great deal.

The term $\mathrm{Tr}\, G_1(1, 1_+)_o$ appears in the equations (V-50) a number of times; this term may be replaced by a constant factor wherever it appears. The positive charge background is taken here to be uniform, thus

$$n(\vec{r}_1) = \text{constant}. \qquad \text{(V-55)}$$

If the equilibrium value of the electron number density operator $\hat{\rho}_N$ (III-7) is defined as

$$n_e = \langle \hat{\rho}_N(1) \rangle_o , \qquad \text{(V-56)}$$

we see by comparison of (III-7) with the definition of G_1 (V-1) that

$$n_e = -i \, Tr \, G_1(1, 1_+)_o,$$

or (V-57)

$$Tr \, G_1(1, 1_+)_o = i n_e,$$

where n_e is the equilibrium value of the electron number density. Because of the imposed overall charge neutrality of the system one has the result

$$n_e = n(\vec{r}) = \text{constant}. \quad \text{(V-58)}$$

CHAPTER VI

SOLUTION FOR THE ONE-PARTICLE GREEN'S FUNCTION

Methods of solution for Green's functions have been outlined in the references (1) through (8). An approach similar to that given by Ashby (see references (2) and (3)) is utilized here.

A uniform magnetic field is applied to the electron gas both in the equilibrium state and after the action of an electromagnetic perturbation. The magnetic field $\vec{B}_o^T(1)(t_1 \leq t_o)$ is then constant, given by

$$\vec{B}_o^T(1) = \nabla_1 \times \vec{A}_o^T(1), \tag{VI-1}$$

where $\vec{A}_o^T$ is time-independent. The direction of $\vec{B}_o^T$ is taken along the z-axis:

$$\vec{B}_o^T(1) = \nabla_1 \times \vec{A}_o^T(1) = \vec{i}_z B_o, \quad B_o \geq 0 . \tag{VI-2}$$

The potential $U_o^T(1)$ is taken to be zero for $t \leq t_o$ (equilibrium state); thus there is no applied electric field. The choice $\vec{B}_o^T$ = constant ($\vec{A}_o^T$ time-independent) and $U_o^T(1) = 0$ is consistent with Maxwell's equations (Chapter I) and with the equations (V-20) and (V-29) for ρ and $\vec{j}$. The relations

$$\begin{aligned} \rho_o\left[\vec{A}_o^T, U_o^T\right] &= 0 \\ \vec{j}_o\left[\vec{A}_o^T, U_o^T\right] &= 0 , \end{aligned} \tag{VI-3}$$

can be verified in the "self-consistent field approximation" and thus the choice made for $\vec{A}_o^T$ and U_o^T is certainly proper to this order.

No attempt is made here to obtain a complete solution of the equations (V-31) and (V-33); the exchange contributions are neglected. In order to better visualize the nature of this approximation, we introduce the function $G_{o\alpha\beta}(1, 2)$ such that

$$\left\{\delta_{\alpha\lambda}\, i\hbar \frac{\partial}{\partial t_1} - h^T_{\alpha\lambda}(1)_o\right\} G_{o\lambda\beta}(1, 2) = \hbar\, \delta_{\alpha\beta}\, \delta^4(1, 2)$$

with (VI-4)

$$h^T_{\alpha\lambda}(1)_o = \delta_{\alpha\lambda} \frac{1}{2m} \vec{\pi}(1)^2 + \frac{\mu\hbar\omega_B}{2} \sigma_{z\alpha\lambda} - \delta_{\alpha\lambda}\zeta ,$$

and

$$\left\{- \delta_{\lambda\alpha}\, i\hbar \frac{\partial}{\partial t_1} - h^{\dagger T}_{\lambda\alpha}(1)_o\right\} G_{o\beta\lambda}(2, 1) = \hbar\, \delta_{\beta\alpha}\, \delta^4(2, 1)$$

with (VI-5)

$$h^{\dagger T}_{\lambda\alpha}(1)_o = \delta_{\lambda\alpha} \frac{1}{2m} \vec{\pi}^*(1)^2 + \frac{\mu\hbar\omega_B}{2} \sigma_{z\lambda\alpha} - \delta_{\lambda\alpha}\zeta ,$$

and where

$$\omega_B \equiv \frac{e B_o}{mc} . \qquad \text{(VI-6)}$$

If both sides of equation (V-31) are pre-multiplied by $G_{o\eta\alpha}(5, 1)$, then summed over the spin index α, and integrated over the set of co-ordinates $(\vec{r}_1, t_1)$, one obtains

$$\int d(1)\, G_{o\eta\alpha}(5, 1) \left\{\delta_{\alpha\lambda}\, i\hbar \frac{\partial}{\partial t_1} - h^T_{\alpha\lambda}(1)_o\right\} G_{1\lambda\beta}(1, 2)_o = \hbar G_{o\eta\beta}(5, 2)$$

(VI-7)

$$+ i e^2 \int d(1)\, d^3\vec{r}_3 \frac{1}{r_{13}} G_{o\eta\alpha}(5, 1)\, G_{1\alpha\gamma}(\vec{r}_1, t_1; \vec{r}_3, t_1+)_o\, G_{1\gamma\beta}(\vec{r}_3, t_1; \vec{r}_2, t_2)_o .$$

After several integrations by parts and with the use of the assumption that both $G_1(1,2)$ and $G_o(1,2)$ vanish for infinite time and spatial separations, one obtains the result

$$G_{1\eta\beta}(5,2)_o = G_{o\eta\beta}(5,2)$$

$$+ \frac{ie^2}{\hbar}\int d(1) d^3\vec{r}_3 \frac{1}{r_{13}} G_{o\eta\alpha}(5,1) G_{1\alpha\gamma}(\vec{r}_1, t_1; \vec{r}_3, t_1+)_o G_{1\gamma\beta}(\vec{r}_3, t_1; \vec{r}_2, t_2)_o ,$$

or (VI-8)

$$G(5,2)_o = G_o(5,2)$$

$$+ \frac{ie^2}{\hbar}\int d(1) d^3\vec{r}_3 \frac{1}{r_{13}} G_o(5,1) G_1(\vec{r}_1, t_1; \vec{r}_3, t_1+)_o G_1(\vec{r}_3, t_1; \vec{r}_2, t_2)_o .$$

Similarly, if one pre-multiplies both sides of equations (V-33) by $G_{o\alpha\eta}(1,5)$, sums over the spin index α, and integrates over the set of co-ordinates $(\vec{r}_1, t_1)$, one obtains

$$\int d(1) G_{o\alpha\eta}(1,5) \left\{ -\delta_{\lambda\alpha} i\hbar \frac{\partial}{\partial t_1} - h^{\dagger T}_{\lambda\alpha}(1)_o \right\} G_{1\beta\lambda}(2,1)_o = \hbar G_{o\beta\eta}(2,5) \qquad \text{(VI-9)}$$

$$+ ie^2 \int d(1) d^3\vec{r}_3 \frac{1}{r_{13}} G_{1\beta\gamma}(\vec{r}_2, t_2; \vec{r}_3, t_1)_o G_{1\gamma\alpha}(\vec{r}_3, t_1; \vec{r}_1, t_1+)_o G_{o\alpha\eta}(1,5) .$$

The resulting integral equation is

$$G_{1\beta\eta}(2,5)_o = G_{o\beta\eta}(2,5)$$

$$+ \frac{ie^2}{\hbar}\int d(1) d^3\vec{r}_3 \frac{1}{r_{13}} G_{1\beta\gamma}(\vec{r}_2, t_2; \vec{r}_3, t_1)_o G_{1\gamma\alpha}(\vec{r}_3, t_1; \vec{r}_1, t_1+)_o G_{o\alpha\eta}(1,5)$$

or (VI-10)

$$G_1(2,1) = G_o(2,1)$$

$$+ \frac{ie^2}{\hbar}\int d(4)d^3\vec{r}_3 \frac{1}{r_{34}} G_1(\vec{r}_2, t_2; \vec{r}_3, t_4)_o G_1(\vec{r}_3, t_4; \vec{r}_4, t_4+) G_o(4,1) .$$

Equations (VI-8) and (VI-10) are equivalent; they are integral equations for G_1 in terms of the function G_o. We see that if the exchange interactions were absent, then the result would be $G_1(1,2)_o = G_o(1,2)$. Thus, G_o is clearly the Green's function for an equilibrium system of electrons interacting only with the applied fields

$$\vec{E}_o^T(1) = -\nabla_1 U_o^T(1) - \frac{1}{c}\frac{\partial \vec{A}_o^T(1)}{\partial t_1} = 0$$

$$\vec{B}_o^T(1) = \nabla_1 \times \vec{A}_o^T(1) = \vec{i}_z B_o = \text{constant.} \quad \text{(VI-11)}$$

In the "self-consistent field approximation"

$$G_1(1,2)_o \cong G_o(1,2) . \quad \text{(VI-12)}$$

For sake of completeness, the essential ideas in the solution of equation (VI-4) are now discussed. The treatment employed here is that of Martin and Schwinger (see reference (1)) and of Ashby (references (2) and (3)).

We shall use subscripts (f) with quantities and operators relating only to the interaction of the electrons with the applied field $\vec{B}_o^T$. The time-independent Hamiltonian leading to the equation (VI-4) is given by

$$\hat{H}_f = \int d^3\vec{r}_1 \, \psi_\alpha^\dagger(1) \left\{ \delta_{\alpha\beta} \frac{1}{2m} \vec{\pi}(1)^2 + \frac{\mu\hbar\omega_B}{2} \sigma_{z\alpha\beta} \right\} \psi_\beta(1), \quad \text{(VI-13)}$$

where $\psi^\dagger$ and ψ are the creation and annihilation operators, respectively, for the electron. The number operator is

$$\hat{N}_f = \int d^3\vec{r}_1\ \psi_\alpha^\dagger(1)\,\psi_\alpha(1)\ . \qquad \text{(VI-14)}$$

In the same manner as before a new Hamiltonian directly related to the Grand Canonical Ensemble may be constructed with the origin of the energy redefined (see equation (III-6)):

$$\hat{\mathcal{H}}_f = \hat{H}_f - \zeta\,\hat{N}_f\ . \qquad \text{(VI-15)}$$

The field operators $\psi^\dagger$ and ψ corresponding to the Hamiltonian $\hat{\mathcal{H}}_f$ obey the usual equations of motion for Heisenberg field operators:

$$i\hbar\frac{\partial\psi_\alpha(1)}{\partial t_1} = \left[\psi_\alpha(1), \hat{\mathcal{H}}_f\right],$$
$$i\hbar\frac{\partial\psi_\alpha^\dagger(1)}{\partial t_1} = \left[\psi_\alpha^\dagger(1), \hat{\mathcal{H}}_f\right]. \qquad \text{(VI-16)}$$

The Green's functions are also defined as before:

$$G_{o\alpha\beta}(1,2) \equiv -i\varepsilon_1(1,2)\ \mathrm{Sp}\left\{\hat{f}_f\left[\psi_\alpha(1)\,\psi_\beta^\dagger(2)\right]_+\right\}$$
$$= -i\varepsilon_1(1,2)\left\langle\left[\psi_\alpha(1)\,\psi_\beta^\dagger(2)\right]_+\right\rangle , \qquad \text{(VI-17)}$$

where

$$\hat{f}_f \equiv Z_f^{-1}\,e^{-\beta\hat{\mathcal{H}}_f}$$

and

$$Z_f \equiv \mathrm{Sp}\left\{e^{-\beta\hat{\mathcal{H}}_f}\right\}\ . \qquad \text{(VI-18)}$$

The use of the equations (VI-16), together with the definition for G_o (VI-17), leads to the equation of motion (VI-4) for G_o.

A generating function $\mathcal{G}_{\gamma\lambda}^{(\tau-i\beta\hbar)}(1,2)$ is defined as follows:

$$\mathcal{G}_{\gamma\lambda}^{(\tau-i\beta\hbar)}(1,2) \equiv \frac{Sp\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\left(-i\varepsilon_1(1,2)\left[\psi_\gamma(1)\psi_\lambda^\dagger(2)\right]_+\right)\right\}}{Sp\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\right\}}, \quad \text{(VI-19)}$$

where $\tau = t_f - t_o \geq 0$ is a real parameter with dimensions of time. The generating function $\mathcal{G}$ differs from the function G_o by the replacement $\beta \to \beta + \frac{i}{\hbar}\tau$ in the "exponential" operator

$$e^{-\beta\hat{\mathcal{H}}_f} .$$

The function $\mathcal{G}$ satisfied the same equation of motion as does G_o:

$$\left\{\delta_{\alpha\lambda}\, i\hbar\frac{\partial}{\partial t_1} - h_{\alpha\lambda}^T(1)_o\right\}\mathcal{G}_{\lambda\beta}^{(\tau-i\beta\hbar)}(1,2) = \hbar\,\delta_{\alpha\beta}\,\delta^4(1,2) . \quad \text{(VI-20)}$$

The difference between G_o and $\mathcal{G}$ is manifested in different sets of subsidiary conditions for the two functions; these subsidiary conditions are to be developed next. Consider first the auxiliary functions

$$\mathcal{G}_{>\gamma\lambda}^{(\tau-i\beta\hbar)}(1,2) = -i\,\frac{Sp\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\psi_\gamma(1)\,\psi_\lambda^\dagger(2)\right\}}{Sp\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\right\}},$$

$$\mathcal{G}_{<\gamma\lambda}^{(\tau-i\beta\hbar)}(1,2) = +i\,\frac{Sp\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\psi_\lambda^\dagger(2)\,\psi_\gamma(1)\right\}}{Sp\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\right\}}, \quad \text{(VI-21)}$$

and

$$G_{>\gamma\lambda}(1,2) = -i \frac{Sp\left\{e^{-\beta\hat{\mathcal{H}}_f} \psi_\gamma(1)\psi_\lambda^\dagger(2)\right\}}{Sp\left\{e^{-\beta\hat{\mathcal{H}}_f}\right\}}$$

$$G_{<\gamma\lambda}(1,2) = +i \frac{Sp\left\{e^{-\beta\hat{\mathcal{H}}_f} \psi_\lambda^\dagger(2)\psi_\gamma(1)\right\}}{Sp\left\{e^{-\beta\hat{\mathcal{H}}_f}\right\}} . \tag{VI-22}$$

From the definitions (VI-17) and (VI-19) we know that

$$\mathcal{G}_{\gamma\lambda}^{(\tau-i\beta\hbar)}(1,2) = \begin{cases} \mathcal{G}_{>\gamma\lambda}^{(\tau-i\beta\hbar)}(1,2), & t_1 > t_2 \\ \mathcal{G}_{<\gamma\lambda}^{(\tau-i\beta\hbar)}(1,2), & t_1 < t_2 \end{cases} \tag{VI-23}$$

and

$$G_{o\gamma\lambda}(1,2) = \begin{cases} G_{>\gamma\lambda}(1,2), & t_1 > t_2 \\ G_{<\gamma\lambda}(1,2), & t_1 < t_2 \end{cases} \tag{VI-24}$$

The Heisenberg operators $\psi^\dagger$ and ψ corresponding to the Hamiltonian $\hat{\mathcal{H}}_f$ are given by

$$\psi_a^\dagger(1) = e^{+\frac{i}{\hbar}(t_1-t_o)\hat{\mathcal{H}}_f} \psi_a^\dagger(\vec{r}_1, t_o)\, e^{-\frac{i}{\hbar}(t_1-t_o)\hat{\mathcal{H}}_f}$$

$$\psi_a(1) = e^{+\frac{i}{\hbar}(t_1-t_o)\hat{\mathcal{H}}_f} \psi_a(\vec{r}_1, t_o)\, e^{-\frac{i}{\hbar}(t_1-t_o)\hat{\mathcal{H}}_f} , \tag{VI-25}$$

as may be verified with the aid of the equations (VI-16). We shall consider the equations (VI-25) to be valid for complex times as well as for real times. Now, from (VI-21) and (VI-25)

$$\mathcal{G}^{(\tau-i\beta\hbar)}_{>\gamma\lambda}(\vec{r}_1,t_f;\vec{r}_2,t_2) = -i\,\frac{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\,e^{\frac{i}{\hbar}\tau\hat{\mathcal{H}}_f}\,\psi_\gamma(\vec{r}_1,t_o)\,e^{-\frac{i}{\hbar}\tau\hat{\mathcal{H}}_f}\,\psi^\dagger_\lambda(2)\right\}}{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\right\}}$$

$$= -i\,\frac{\mathrm{Sp}\left\{e^{-\beta\hat{\mathcal{H}}_f}\,e^{-\frac{i}{\hbar}\tau\hat{\mathcal{H}}_f}\,\psi^\dagger_\lambda(2)\,e^{-\beta\hat{\mathcal{H}}_f}\,\psi_\gamma(\vec{r}_1,t_o)\,e^{\beta\hat{\mathcal{H}}_f}\right\}}{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\right\}} \qquad \text{(VI-26)}$$

$$= -i\,\frac{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\,\psi^\dagger_\lambda(2)\,\psi_\gamma(\vec{r}_1,t_o+i\beta\hbar)\right\}}{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\right\}}$$

in which the cyclic invariance of the trace has been utilized; recall also that $\tau = t_f - t_o \geq 0$. From (VI-21) one also has

$$\mathcal{G}^{(\tau-i\beta\hbar)}_{<\gamma\lambda}(\vec{r}_1,t_o+i\beta\hbar;\vec{r}_2,t_2) = +i\,\frac{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\,\psi^\dagger_\lambda(2)\,\psi_\gamma(\vec{r}_1,t_o+i\beta\hbar)\right\}}{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\right\}} \qquad \text{(VI-27)}$$

Thus we see from a comparison of (VI-26) and (VI-27) that

$$\mathcal{G}^{(\tau-i\beta\hbar)}_{>\gamma\lambda}(\vec{r}_1,t_f;\vec{r}_2,t_2) = -\mathcal{G}^{(\tau-i\beta\hbar)}_{<\gamma\lambda}(\vec{r}_1,t_o+i\beta\hbar;\vec{r}_2,t_2). \qquad \text{(VI-28)}$$

Similarly

$$\mathcal{S}^{(\tau-i\beta\hbar)}_{<\gamma\lambda}(\vec{r}_1,t_1;\vec{r}_2,t_f) = -\mathcal{S}^{(\tau-i\beta\hbar)}_{>\gamma\lambda}(\vec{r}_1,t_1;\vec{r}_2,t_o+i\beta\hbar). \qquad \text{(VI-29)}$$

By the same procedures that led to (VI-28, 29) one can obtain subsidiary conditions for the Green's function G_o, which are

$$G_{>\gamma\lambda}(\vec{r}_1,t_o;\vec{r}_2,t_2) = -G_{<\gamma\lambda}(\vec{r}_1,t_o+i\beta\hbar;\vec{r}_2,t_2)$$

and

$$G_{<\gamma\lambda}(\vec{r}_1,t_1;\vec{r}_2,t_o-i\beta\hbar) = -G_{>\gamma\lambda}(\vec{r}_1,t_1;\vec{r}_2,t_o) \qquad \text{(VI-30)}$$

In the case that $\beta \to 0$, the generating function $\mathcal{S}^{(\tau-i\beta\hbar)}_{\gamma\lambda}(1,2) \to \mathcal{S}^{(\tau)}_{\gamma\lambda}(1,2)$ and the subsidiary conditions (VI-28, 29) reduce to the boundary conditions

$$\mathcal{S}^{(\tau)}_{>\gamma\lambda}(\vec{r}_1,t_f;\vec{r}_2,t_2) = -\mathcal{S}^{(\tau)}_{<\gamma\lambda}(\vec{r}_1,t_o;\vec{r}_2,t_2)$$

$$\mathcal{S}^{(\tau)}_{>\gamma\lambda}(\vec{r}_1,t_1;\vec{r}_2,t_o) = -\mathcal{S}^{(\tau)}_{<\gamma\lambda}(\vec{r}_1,t_1;\vec{r}_2,t_f) \qquad \text{(VI-31)}$$

If the solution of $\mathcal{S}^{(\tau)}_{\gamma\lambda}(1,2)$ is known, the Green's function $G_{o\gamma\lambda}(1,2)$ is obtained from $\mathcal{S}^{(\tau)}_{\gamma\lambda}(1,2)$ by the replacement $\tau \to -i\beta\hbar$.

The boundary conditions (VI-31) define $\mathcal{S}^{(\tau)}_{\gamma\lambda}(1,2)$ for the restricted intervals of the time $t_o \leq t_1 \leq t_f$, and $t_o \leq t_2 \leq t_f$, however, the final results are functions of the difference $\tau = t_f - t_o$ and by analytic continuation one obtains $\mathcal{S}^{(\tau)}_{\gamma\lambda}(1,2)$ for unrestricted values of t_1 and t_2.

The function $\mathcal{S}^{(\tau)}_{\gamma\lambda}(1,2)$ depends explicitly upon the difference $(t_1 - t_2)$ in times t_1 and t_2 as can be seen from a substitution of the equation (VI-25) into the equations (VI-21):

$$\mathcal{G}^{(\tau-i\beta\hbar)}_{>\gamma\lambda}(1,2)$$

$$= -i\,\frac{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\, e^{+\frac{i}{\hbar}(t_1-t_o)\hat{\mathcal{H}}_f}\,\psi_\gamma(\vec{r}_1,t_o)\,e^{-\frac{i}{\hbar}(t_1-t_o)\hat{\mathcal{H}}_f}\, e^{+\frac{i}{\hbar}(t_2-t_o)\hat{\mathcal{H}}_f}\,\psi^\dagger_\lambda(\vec{r}_2,t_o)\,e^{-\frac{i}{\hbar}(t_2-t_o)\hat{\mathcal{H}}_f}\right\}}{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\right\}}$$

$$= -i\,\frac{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\, e^{\frac{i}{\hbar}(t_1-t_2)\hat{\mathcal{H}}_f}\,\psi_\gamma(\vec{r}_2,t_o)\,e^{-\frac{i}{\hbar}(t_1-t_2)\hat{\mathcal{H}}_f}\,\psi^\dagger_\lambda(\vec{r}_2,t_o)\right\}}{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\right\}}$$

and (VI-32)

$$\mathcal{G}^{(\tau-i\beta\hbar)}_{<\gamma\lambda}(1,2)$$

$$= +i\,\frac{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\, e^{+\frac{i}{\hbar}(t_2-t_o)\hat{\mathcal{H}}_f}\,\psi^\dagger_\lambda(\vec{r}_2,t_o)\,e^{-\frac{i}{\hbar}(t_2-t_o)\hat{\mathcal{H}}_f}\, e^{+\frac{i}{\hbar}(t_1-t_o)\hat{\mathcal{H}}_f}\,\psi_\gamma(\vec{r}_1,t_o)\,e^{-\frac{i}{\hbar}(t_1-t_o)\hat{\mathcal{H}}_f}\right\}}{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\right\}}$$

$$= +i\,\frac{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\, e^{-\frac{i}{\hbar}(t_1-t_2)\hat{\mathcal{H}}_f}\,\psi^\dagger_\lambda(\vec{r}_2,t_o)\,e^{+\frac{i}{\hbar}(t_1-t_2)\hat{\mathcal{H}}_f}\,\psi_\gamma(\vec{r}_1,t_o)\right\}}{\mathrm{Sp}\left\{e^{-\frac{i}{\hbar}(\tau-i\beta\hbar)\hat{\mathcal{H}}_f}\right\}}$$

by the cyclic property of the trace.

The time dependence of $\mathcal{G}^{(\tau)}_{\gamma\lambda}(1,2)$ may be expressed by the Fourier expansions

$$\mathcal{G}^{(\tau)}_{\gamma\lambda}(1,2) \equiv \sum_q e^{-\frac{iq\pi}{\tau}(t_1-t_2)} \mathcal{G}^{(\tau)}_{\gamma\lambda}(q;\vec{r}_1,\vec{r}_2)$$

and (VI-33)

$$\mathcal{G}^{(\tau)}_{\gtrless\gamma\lambda}(1,2) \equiv \sum_q e^{-\frac{iq\pi}{\tau}(t_1-t_2)} \mathcal{G}^{(\tau)}_{\gtrless\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) ,$$

where the index q represents the set of all integers. The boundary conditions (VI-31) (also known as "antiperiodicity" conditions) restrict the range of integers q to the odd integers alone. This is shown as follows. From equations (VI-23, 33) one has the results

$$\mathcal{G}^{(\tau)}_{\gamma\lambda}(\vec{r}_1,t_f;\vec{r}_2,t_2) = \mathcal{G}^{(\tau)}_{>\gamma\lambda}(\vec{r}_1,t_f;\vec{r}_2,t_2)$$

or (VI-34)

$$\sum_q e^{-\frac{iq\pi}{\tau}(t_f-t_2)} \mathcal{G}^{(\tau)}_{\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) = \sum_q e^{-\frac{iq\pi}{\tau}(t_f-t_2)} \mathcal{G}^{(\tau)}_{>\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) ,$$

and

$$\mathcal{G}^{(\tau)}_{\gamma\lambda}(\vec{r}_1,t_o;\vec{r}_2,t_2) = \mathcal{G}^{(\tau)}_{<\gamma\lambda}(\vec{r}_1,t_o;\vec{r}_2,t_2)$$

or (VI-35)

$$\sum_q e^{-\frac{iq\pi}{\tau}(t_o-t_2)} \mathcal{G}^{(\tau)}_{\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) = \sum_q e^{-\frac{iq\pi}{\tau}(t_o-t_2)} \mathcal{G}^{(\tau)}_{<\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) .$$

Since equations (VI-34, 35) are valid for arbitrary t_2 in the range $t_o < t_2 < t_f$, one may conclude that

$$\mathcal{G}^{(\tau)}_{\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) = \mathcal{G}^{(\tau)}_{>\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) = \mathcal{G}^{(\tau)}_{<\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) . \qquad \text{(VI-36)}$$

Since $\tau = t_f - t_o$, one has from equations (VI-31) the results

$$\sum_q e^{-\frac{iq\pi}{\tau}(t_f-t_2)} \mathcal{G}^{(\tau)}_{>\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) = -\sum_q e^{-\frac{iq\pi}{\tau}(t_o-t_2)} \mathcal{G}^{(\tau)}_{<\gamma\lambda}(q;\vec{r}_1,\vec{r}_2)$$

$$= -\sum_q e^{iq\pi} e^{-\frac{iq\pi}{\tau}(t_f-t_2)} \mathcal{G}^{(\tau)}_{<\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) \qquad \text{(VI-37)}$$

$$= \sum_q (-1)^{q+1} e^{-\frac{iq\pi}{\tau}(t_f-t_2)} \mathcal{G}^{(\tau)}_{<\gamma\lambda}(q;\vec{r}_1,\vec{r}_2),$$

and

$$\sum_q e^{-\frac{iq\pi}{\tau}(t_1-t_o)} \mathcal{G}^{(\tau)}_{>\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) = -\sum_q e^{-\frac{iq\pi}{\tau}(t_1-t_f)} \mathcal{G}^{(\tau)}_{<\gamma\lambda}(q;\vec{r}_1,\vec{r}_2)$$

$$= -\sum_q e^{iq\pi} e^{-\frac{iq\pi}{\tau}(t_1-t_o)} \mathcal{G}^{(\tau)}_{<\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) \qquad \text{(VI-38)}$$

$$= \sum_q (-1)^{q+1} \mathcal{G}^{(\tau)}_{<\gamma\lambda}(q;\vec{r}_1,\vec{r}_2)\, e^{-\frac{iq\pi}{\tau}(t_1-t_o)}.$$

Since t_1 and t_2 are arbitrary in the range $t_o < t_1,\ t_2 < t_f$, we have the result

$$\mathcal{G}^{(\tau)}_{>\gamma\lambda}(q;\vec{r}_1,\vec{r}_2) = (-1)^{q+1} \mathcal{G}^{(\tau)}_{<\gamma\lambda}(q;\vec{r}_1,\vec{r}_2). \qquad \text{(VI-39)}$$

Comparison of equations (VI-36) and (VI-39) shows that the choice of q even is contradictory, while all the conditions are mutually satisfied for q odd. This result is related to the fact that the electrons obey

Fermi-Dirac statistics. In the case of particles obeying Bose-Einstein statistics, the boundary conditions would be periodic and the choice of the even integers rather than the odd integers would then be the correct one (see references (2) and (3)). The orthonormality relation for the set $e^{\frac{iq\pi}{\tau}t}$ is

$$\frac{1}{\tau}\int_{t_o}^{t_f} e^{-\frac{i(q-q')\pi}{\tau}t}\, dt = \delta_{q,q'} \text{ (odd integers)} . \tag{VI-40}$$

(Recall that, by definition, $0 \leq |t_1 - t_2| \leq \tau$.)

Thus far the choice of the gauge of the vector potential $\vec{A}_o^T(1)$ has been left arbitrary, but before a solution of equation (VI-20) can be obtained, the gauge of $\vec{A}_o^T(1)$ must be specified. Since

$$\nabla_1 \times \left[\vec{i}_x B_o y_1 + \vec{i}_y B_o x_1\right] = 0,$$

one can take

$$\begin{aligned}\vec{A}_o^T(1) &= \frac{1}{2}\vec{B}_o^T(1) \times \vec{r}_1 + \lambda\left[\vec{i}_x B_o y_1 + \vec{i}_y B_o x_1\right] \\ &= \frac{1}{2}\left[-\vec{i}_x B_o y_1 + \vec{i}_y B_o x_1\right] + \lambda\left[\vec{i}_x B_o y_1 + \vec{i}_y B_o x_1\right],\end{aligned} \tag{VI-41}$$

where λ is any real constant. The representation chosen here is the "symmetric" gauge ($\lambda = 0$):

$$\vec{A}_o^T(1) = \frac{1}{2}\vec{B}_o^T(1) \times \vec{r}_1 . \tag{VI-42}$$

The solution to the Schrodinger equation for an electron in the field $\vec{B}_o^T$ with the gauge (VI-42) is developed in Mathematical Appendix V, together with a number of useful identities. The eigenfunctions are designated by

$$v_{n,\ell,k}(\vec{r}_1) = \begin{cases} e^{ikz_1} u_{n,\ell}(\rho_1, \theta_1) & \text{(polar cylindrical co-ordinates)} \\ e^{ikz_1} w_{n,\ell}(x_1, y_1) & \text{(Cartesian co-ordinates)} \ , \end{cases} \qquad \text{(VI-43)}$$

and form a complete orthonormal set. The quantum numbers n and ℓ form a discrete set (all the positive integers 0, 1, 2, 3, ... ∞), while k has a continuous range, $-\infty \leqslant k \leqslant +\infty$. The energy eigenvalues of the Hamiltonian $\hat{H}_f$ are characterized by the symbol $E_{\alpha,n,k}$, where α is the spin index. The energy eigenvalues of the Hamiltonian $\hat{\mathscr{H}}_f$ are given by $\xi_{\alpha,n,k} = E_{\alpha,n,k} - \zeta$; the eigenfunctions are the same as for $\hat{H}_f$. The orthonormality relations are

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; w^*_{n_1,\ell_1}(x,y)\, w_{n_2,\ell_2}(x,y)$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} \rho d\rho \; u^*_{n_1,\ell_1}(\rho,\theta)\, u_{n_2,\ell_2}(\rho,\theta) = \delta_{n_1,n_2}\, \delta_{\ell_1,\ell_2}$$

and (VI-44)

$$\int_{-\infty}^{\infty} dz \; e^{i(k_1 - k_2)z} = 2\pi\, \delta(k_1 - k_2) \, .$$

The generating function $\mathscr{G}^{(\tau)}_{\alpha\beta}(1,2)$ may be constructed as an expansion of the functions (VI-43) and of the set $e^{-\frac{iq\pi}{\tau}t}$:

$$\mathcal{G}^{(\tau)}_{\alpha\beta}(1,2)$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \sum_{q \text{ odd}} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \tag{VI-45}$$

$$\times C_{\alpha\beta}(n_1, \ell_1, k_1; n_2, \ell_2, k_2; q)\, v_{n_1, \ell_1, k_1}(\vec{r}_1)\, v^*_{n_2, \ell_2, k_2}(\vec{r}_2)\, e^{-\frac{iq\pi}{\tau}(t_1 - t_2)}$$

If the equation (VI-45) is substituted into equation (VI-20), one obtains by suitable integrations and by the use of the orthonormality relations (VI-40) and (VI-44) the expansion coefficient $C_{\alpha\beta}$:

$$C_{\alpha\beta}(n_1, \ell_1, k_1; n_2, \ell_2, k_2; q) = \frac{\delta_{\alpha\beta}\, \delta_{n_1, n_2}\, \delta_{\ell_1, \ell_2}\, \frac{2\pi\hbar}{\tau}\, \delta(k_1 - k_2)}{\left[\frac{\pi\hbar}{\tau} q - E_{\alpha, n, k} + \zeta\right]}, \tag{VI-46}$$

and therefore

$$\mathcal{G}^{(\tau)}_{\alpha\beta}(1,2)$$

$$= \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{q \text{ odd}} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{\delta_{\alpha\beta}\left(\frac{\hbar}{\tau}\right) v_{n,\ell,k}(\vec{r}_1)\, v^*_{n,\ell,k}(\vec{r}_2)\, e^{-\frac{i\pi q}{\tau}(t_1 - t_2)}}{\left[\frac{\pi\hbar}{\tau} q - E_{\alpha, n, k} + \zeta\right]}. \tag{VI-47}$$

The delta function $\delta^3(\vec{r}_1 - \vec{r}_2)$ in the representation of the functions $v_{n,\ell,k}(\vec{r})$ (VI-43) is

$$\delta^3(\vec{r}_1 - \vec{r}_2) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, v_{n,\ell,k}(\vec{r}_1)\, v^*_{n,\ell,k}(\vec{r}_2)$$

$$= \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} u_{n,\ell}(\rho_1, \theta_1) u^*_{n,\ell}(\rho_2, \theta_2) e^{ik(z_1 - z_2)} \qquad \text{(VI-48)}$$

$$= \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} w_{n,\ell}(x_1, y_1) w^*_{n,\ell}(x_2, y_2) e^{ik(z_1 - z_2)} .$$

The expression (VI-47) for the function $\mathcal{G}^{(\tau)}_{\alpha\beta}(1, 2)$ can be reduced to a more useful form with the aid of the well-known identities

$$\frac{1}{\tau} \sum_{q \text{ odd}} \frac{e^{-\frac{iq\pi}{\tau}t}}{\left[\frac{q\pi}{\tau} - \Omega\right]} = \begin{cases} -i \dfrac{e^{-i\Omega t}}{1 + e^{-i\Omega\tau}}, & 0 < t < \tau \\[2ex] +i \dfrac{e^{-i\Omega t}}{1 + e^{i\Omega\tau}}, & -\tau < t < 0 \end{cases} \qquad \text{(VI-49)}$$

(See reference (2) for a proof of the identities (VI-49).) From a comparison of (VI-49) with (VI-47) we see that

$$\mathcal{G}^{(\tau)}_{>\alpha\beta}(1, 2) = -i\,\delta_{\alpha\beta} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{v_{n,\ell,k}(\vec{r}_1) v^*_{n,\ell,k}(\vec{r}_2) e^{-\frac{i}{\hbar}\xi_{a,n,k}(t_1 - t_2)}}{\left[1 + e^{-\frac{i}{\hbar}\tau \xi_{a,n,k}}\right]}$$

$$\mathcal{G}^{(\tau)}_{<\alpha\beta}(1, 2) = +i\,\delta_{\alpha\beta} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{v_{n,\ell,k}(\vec{r}_1) v^*_{n,\ell,k}(\vec{r}_2) e^{-\frac{i}{\hbar}\xi_{a,n,k}(t_1 - t_2)}}{\left[1 + e^{+\frac{i}{\hbar}\tau \xi_{a,n,k}}\right]} , \qquad \text{(VI-50)}$$

where

$$\xi_{a,n,k} = E_{a,n,k} - \zeta \,. \qquad \text{(VI-51)}$$

Equations (VI-50) satisfy the boundary conditions (VI-31) for any time t_1, and any time t_2; thus the restriction $0 < |t_1 - t_2| < \tau$, on the times t_1 and t_2 may be dropped. Now, $\mathcal{G}^{(\tau)}_{\alpha\beta}(1,2)$ may be written as

$$\mathcal{G}^{(\tau)}_{\alpha\beta}(1,2) = \eta_+(1,2)\,\mathcal{G}^{(\tau)}_{>\alpha\beta}(1,2) + \eta_-(1,2)\,\mathcal{G}^{(\tau)}_{<\alpha\beta}(1,2), \qquad \text{(VI-52)}$$

where η_+ and η_- are the "step functions"

$$\eta_+(1,2) = \begin{cases} 1, & t_1 > t_2 \\ 0, & t_1 < t_2 \end{cases}$$

and (VI-53)

$$\eta_-(1,2) = \begin{cases} 0, & t_1 > t_2 \\ 1, & t_1 < t_2 \end{cases}$$

With the replacement $\tau \rightarrow -i\beta\hbar$ in the function $\mathcal{G}^{(\tau)}_{\alpha\beta}(1,2)$ (VI-50, 52) we obtain the Green's function G_o:

$$G_{o\alpha\beta}(1,2) = \eta_+(1,2)\,G_{>\alpha\beta}(1,2) + \eta_-(1,2)\,G_{<\alpha\beta}(1,2) \qquad \text{(VI-54)}$$

with

$$G_{>\alpha\beta}(1,2) = -i\,\delta_{\alpha\beta} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{v_{n,\ell,k}(\vec{r}_1)\, v^*_{n,\ell,k}(\vec{r}_2)\, e^{-\frac{i}{\hbar}\xi_{a,n,k}(t_1 - t_2)}}{\left[1 + e^{-\beta\xi_{a,n,k}}\right]}$$

$$G_{<\alpha\beta}(1,2) = +i\,\delta_{\alpha\beta} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{v_{n,\ell,k}(\vec{r}_1)\, v^*_{n,\ell,k}(\vec{r}_2)\, e^{-\frac{i}{\hbar}\xi_{a,n,k}(t_1-t_2)}}{\left[1 + e^{\beta\xi_{a,n,k}}\right]} .$$

The Green's function (VI-54) satisfies the subsidiary conditions (VI-30). There is another boundary condition not specified by the conditions (VI-30); we assume the appearance of an exponential convergence factor of the form $e^{-\varepsilon|t_1-t_2|}$, with $\varepsilon > 0$ an "infinitesimal," in the generating function and Green's function. This is done to insure that the functions vanish at infinite time separations. Such an artifice is not necessary in the case of spatial separations since this is automatically satisfied. The replacement

$$G_{o\alpha\beta}(1,2) \rightarrow e^{-\varepsilon|t_1-t_2|}\, G_{o\alpha\beta}(1,2), \quad \varepsilon \rightarrow 0^+ \qquad \text{(VI-55)}$$

is consistent with the equation of motion (VI-4) for G_o. In applications of G_o, the limiting procedure will be the final operation.

The following identities involving the "step functions" η_+ and η_- are useful in calculations involving products of G_o:

$$\begin{aligned} \eta_+(1,2)\,\eta_-(1,2) &= 0 \\ \eta_+^2(1,2) + \eta_-^2(1,2) &= 1 . \end{aligned} \qquad \text{(VI-56)}$$

With the aid of the identities (6-1) in Mathematical Appendix VI, one obtains for the Green's function $G_o(1,2)$ (Cartesian co-ordinates):

$$G_{>\alpha\beta}(1,2) = e^{\frac{im\omega_B}{2\hbar}(x_1y_2 - x_2y_1)}\, \bar{G}_{>\alpha\beta}(1,2)\, e^{-\varepsilon|t_1-t_2|}$$

(VI-57)

$$G_{<\alpha\beta}(1,2) = e^{\frac{im\omega_B}{2\hbar}(x_1y_2 - x_2y_1)}\, \bar{G}_{<\alpha\beta}(1,2)\, e^{-\varepsilon|t_1 - t_2|}$$

where we have used the definitions

$$\bar{G}_{>\alpha\beta}(1,2) \equiv -i\,\delta_{\alpha\beta}\frac{m\omega_B}{2\pi\hbar}\, e^{-\frac{V_{12}}{2}} \sum_{n=0}^{\infty} \frac{L_n(V_{12})}{n!} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(z_1 - z_2) - \frac{i}{\hbar}\xi_{a,n,k}(t_1 - t_2)}}{\left[1 + e^{-\beta\,\xi_{a,n,k}}\right]}$$

$$\bar{G}_{<\alpha\beta}(1,2) \equiv +i\,\delta_{\alpha\beta}\frac{m\omega_B}{2\pi\hbar}\, e^{-\frac{V_{12}}{2}} \sum_{n=0}^{\infty} \frac{L_n(V_{12})}{n!} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(z_1 - z_2) - \frac{i}{\hbar}\xi_{a,n,k}(t_1 - t_2)}}{\left[1 + e^{\beta\,\xi_{a,n,k}}\right]}$$

$$V_{12} \equiv \frac{m\omega_B}{2\hbar}\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right], \qquad \text{(VI-58)}$$

and where $L_n(V)$ is the Laguerre polynomial of order n. With the definition

$$G_{o\alpha\beta}(1,2) \equiv e^{\frac{im\omega_B}{2\hbar}(x_1y_2 - x_2y_1)}\, \bar{G}_{o\alpha\beta}(1,2)\, e^{-\varepsilon|t_1 - t_2|}, \qquad \text{(VI-59)}$$

one has the result

$$\bar{G}_{o\alpha\beta}(1,2) \equiv \eta_+(1,2)\,\bar{G}_{>\alpha\beta}(1,2) + \eta_-(1,2)\,\bar{G}_{<\alpha\beta}(1,2). \qquad \text{(VI-60)}$$

It is apparent that the functions $\bar{G}_<$, $\bar{G}_>$, and $\bar{G}_o$ depend manifestly upon the differences $(\vec{r}_1 - \vec{r}_2)$ in the spatial co-ordinates and the time difference, $(t_1 - t_2)$.

CHAPTER VII

THE EQUILIBRIUM ENERGY DENSITY AND THE CHEMICAL POTENTIAL OF THE ELECTRON GAS

The energy density of the gas is given by the expression

$$e_v(1) \equiv \langle \hat{h}_v(1) \rangle \qquad \text{(VII-1)}$$

and in the "self-consistent field approximation" (subscripts f)

$$\hat{h}_{vf}(1) = \psi_a^\dagger(1) \left\{ \delta_{a\beta} \frac{1}{2m} \vec{\pi}(1)^2 + \frac{\mu\hbar\omega_B}{2} \sigma_{za\beta} \right\} \psi_\beta(1) \qquad \text{(VII-2)}$$

(c.f. (III-8) and (VI-13)). Thus for electrons in an applied field $\vec{B}_o^T$

$$\begin{aligned}
e_{vf}(1) &= \langle \hat{h}_{vf}(1) \rangle \\
&= \left\{ \left[\delta_{a\beta} \frac{1}{2m} \vec{\pi}(1)^2 + \frac{\mu\hbar\omega_B}{2} \sigma_{za\beta} \right] \langle \psi_a^\dagger(2) \psi_\beta(1) \rangle \right\}_{2\to 1} \\
&= -i \left\{ \left[\delta_{a\beta} \frac{1}{2m} \vec{\pi}(1)^2 + \frac{\mu\hbar\omega_B}{2} \sigma_{za\beta} \right] G_{o\beta a}(1,2) \right\}_{2\to 1_+} \qquad \text{(VII-3)} \\
&= -i \, \mathrm{Tr} \left\{ \left[\frac{1}{2m} \vec{\pi}(1)^2 + \frac{\mu\hbar\omega_B}{2} \sigma_z \right] G_o(1,2) \right\}_{2\to 1_+}
\end{aligned}$$

The electron number density n_{ef} is found from the definition (III-7) and from (VI-14) to be

$$n_{ef} = \langle \hat{\rho}_{Nf} \rangle$$

$$= \langle \psi_a^\dagger(1)\, \psi_a(1) \rangle \qquad \text{(VII-4)}$$

$$= -i\, G_{oaa}(1, 1_+) = -i \,\mathrm{Tr}\, G_o(1, 1_+) ,$$

which is analogous to the equations (V-56, 57).

Both e_{vf} and n_{ef} are equilibrium state values and are therefore independent of the co-ordinates $\vec{r}$ and t. The solution of equation (VII-3) for e_{vf} using the Green's function (VI-54) is found to be

$$e_{vf} = \sum_{a=1}^{2} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{E_{a,n,k}\, v_{n,\ell,k}(\vec{r}_1)\, v^*_{n,\ell,k}(\vec{r}_1)}{\left[1 + e^{\beta \xi_{a,n,k}}\right]}$$

$$= \frac{m\omega_B}{2\pi\hbar} \sum_{a=1}^{2} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{E_{a,n,k}}{\left[1 + e^{\beta \xi_{a,n,k}}\right]} , \qquad \text{(VII-5)}$$

where we have used the identity

$$\sum_{\ell=0}^{\infty} v_{n,\ell,k}(\vec{r}_1)\, v^*_{n,\ell,k}(\vec{r}_1) = \frac{m\omega_B}{2\pi\hbar} \qquad \text{(VII-6)}$$

(see Mathematical Appendix VI). The number density n_{ef} is similarly given by

$$n_{ef} = \sum_{a=1}^{2} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{v_{n,\ell,k}(r_1) v^*_{n,\ell,k}(r_1)}{\left[1 + e^{\beta \xi_{a,n,k}}\right]}$$

$$= \frac{m\omega_B}{2\pi\hbar} \sum_{a=1}^{\infty} \sum_{n=0}^{\infty} \int \frac{dk}{2\pi} \frac{1}{\left[1 + e^{\beta\xi_{a,n,k}}\right]} \qquad \text{(VII-7)}$$

From (VI-51) one uses the replacement $\xi_{a,n,k} = E_{a,n,k} - \zeta$ in (VII-5) and (VII-7). The energy eigenvalues $E_{a,n,k}$ for the two values of spin index are:

$$E_{1,n,k} = (n+1+g)\hbar\omega_B + \frac{\hbar^2 k^2}{2m}$$

and (VII-8)

$$E_{2,n,k} = (n - g)\hbar\omega_B + \frac{\hbar^2 k^2}{2m}$$

where $2g = \mu - 1 > 0$, $g << 1$ from (III-1) (see Mathematical Appendix V).

We consider the solutions of the equations (VII-5) and (VII-7) first for a non-degenerate gas in the classical limit $e^{\beta\zeta} << 1$:

$$e_{vf} \cong \frac{m\omega_B}{2\pi\hbar} e^{\beta\zeta} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[E_{1,n,k}\, e^{-\beta E_{1,n,k}} + E_{2,n,k}\, e^{-\beta E_{2,n,k}}\right]$$

$$= \frac{m\omega_B}{2\pi\hbar} e^{\beta(\zeta - \frac{\hbar\omega_B}{2})} \left\{ \left[2\hbar\omega_B \cosh\frac{\mu\beta\hbar\omega_B}{2}\right]\left[\sum_{n=0}^{\infty} n e^{-n\beta\hbar\omega_B}\right] \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-\frac{\beta\hbar^2 k^2}{2m}}\right.$$

$$B \sinh \frac{\quad}{2} \quad \sum_{n=0} \quad \int_{-\infty} \frac{\quad}{2\pi}$$

(VII-9)

$$+\left[2\cosh\frac{\mu\beta\hbar\omega_B}{2}\right]\left[\sum_{n=0}^{\infty} e^{-n\beta\hbar\omega_B}\right]\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{\hbar^2k^2}{2m}\,e^{-\frac{\beta\hbar^2k^2}{2m}}\Bigg\}$$

$$=\frac{\omega_B e^{\beta\zeta}}{2\hbar^2}\left(\frac{m}{2\pi\beta}\right)^{\frac{3}{2}}\frac{\cosh\frac{\mu\beta\hbar\omega_B}{2}}{\sinh\frac{\beta\hbar\omega_B}{2}}\left\{1+\beta\hbar\omega_B\, e^{-\frac{\beta\hbar\omega_B}{2}}\operatorname{csch}\frac{\beta\hbar\omega_B}{2}\right.$$

$$\left.+\;\beta\hbar\omega_B\left[1-\mu\tanh\frac{\mu\beta\hbar\omega_B}{2}\right]\right\}$$

and

$$n_{ef}\cong\frac{m\omega_B}{2\pi\hbar}\,e^{\beta\zeta}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left[e^{-\beta E_{1,n,k}}+e^{-\beta E_{2,n,k}}\right]$$

$$=\frac{m\omega_B}{2\pi\hbar}\,e^{\beta\left(\zeta-\frac{\hbar\omega_B}{2}\right)}\,2\cosh\frac{\mu\beta\hbar\omega_B}{2}\left[\sum_{n=0}^{\infty}e^{-n\beta\hbar\omega_B}\right]\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{-\frac{\beta\hbar^2k^2}{2m}}$$

(VII-10)

$$=\frac{\beta\omega_B e^{\beta\zeta}}{\hbar^2}\left(\frac{m}{2\pi\beta}\right)^{\frac{3}{2}}\frac{\cosh\frac{\mu\beta\hbar\omega_B}{2}}{\sinh\frac{\beta\hbar\omega_B}{2}}$$

In the solution of (VII-9) and (VII-10) the following definite integrals and identities were used:

$$\int_{-\infty}^{\infty}dx\, e^{-x^2}=\sqrt{\pi}$$

$$\int dx\, x^2 e^{-x^2} = \frac{\sqrt{\pi}}{2} \,, \tag{VII-11}$$

and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \,, \quad \text{for} \quad 0 \leq x < 1$$

$$\sum_{n=0}^{\infty} n x^n = x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = x \frac{d}{dx} \left(\frac{1}{1-x} \right) \tag{VII-12}$$

$$= \frac{x}{(1-x)^2} \,, \quad \text{for} \quad 0 \leq x < 1 \,.$$

Thus, from (VII-10) one obtains for $e^{\beta\zeta}$

$$e^{\beta\zeta} \cong \left(\frac{2\pi\hbar^2\beta}{m} \right)^{\frac{3}{2}} \frac{n_{ef}}{\beta\hbar\omega_B} \frac{\sinh \frac{\beta\hbar\omega_B}{2}}{\cosh \frac{\mu\beta\hbar\omega_B}{2}} \,, \tag{VII-13}$$

and we see that a number of conditions would satisfy the restriction $e^{\beta\zeta} << 1$. Equation (VII-13) may be used to eliminate the factor $e^{\beta\zeta}$ from equation (VII-9):

$$e_{vf} \cong \frac{n_{ef}}{2\beta} \left\{ 1 + \beta\hbar\omega_B e^{-\frac{\beta\hbar\omega_B}{2}} \operatorname{csch} \frac{\beta\hbar\omega_B}{2} + \beta\hbar\omega_B \left[1 - \mu \tanh \frac{\mu\beta\hbar\omega_B}{2} \right] \right\} \tag{VII-14}$$

Equations (VII-13) and (VII-14) represent the expressions for the chemical potential and the energy density of electron gas in a magnetic field as calculated with Maxwell-Boltzmann statistics. In the limit of no magnetic field ($\omega_B = 0$)

$$e^{\beta\zeta} = \frac{n_{ef}}{2}\left(\frac{2\pi\hbar^2\beta}{m}\right)^{\frac{3}{2}} = \frac{n_{ef}}{2}\left(\frac{2\pi\hbar^2}{mkT}\right)^{\frac{3}{2}}$$

and (VII-15)

$$e_{vf} = \frac{3}{2\beta} n_{ef} = \frac{3}{2} n_{ef} kT .$$

In the case of the fully degenerate gas ($\beta \rightarrow \infty$) with applied field B_o a precise formal solution of the equations (VII-5) and (VII-7) is possible, although quite cumbersome. For convenience, distribution functions W_1 and W_2, Γ_1 and Γ_2, are defined such that

$$W_1(n,k) \doteq \frac{1}{1+e^{\beta(E_{1nk}-\zeta)}} , \quad \Gamma_1(n,k) = 1 - W_1(n,k)$$

(VII-16)

$$W_2(n,k) = \frac{1}{1+e^{\beta(E_{2nk}-\zeta)}} , \quad \Gamma_2(n,k) = 1 - W_2(n,k)$$

$$\lim_{\beta\rightarrow\infty} W_1(n,k) = \begin{cases} 1, \text{ if } (n+1+g)\hbar\omega_B + \dfrac{\hbar^2k^2}{2m} - \zeta < 0 \\ 0, \text{ if } (n+1+g)\hbar\omega_B + \dfrac{\hbar^2k^2}{2m} - \zeta > 0 \end{cases}$$

(VII-17)

$$\lim_{\beta\rightarrow\infty} W_2(n,k) = \begin{cases} 1, \text{ if } (n-g)\hbar\omega_B + \dfrac{\hbar^2k^2}{2m} - \zeta < 0 \\ 0, \text{ if } (n-g)\hbar\omega_B + \dfrac{\hbar^2k^2}{2m} - \zeta > 0 . \end{cases}$$

Thus, $W_1(n, k)$ is non-zero at $T = 0$ only in the range

$$-\sqrt{\zeta - (n+1+g)\hbar\omega_B} < \frac{\hbar k}{\sqrt{2m}} < +\sqrt{\zeta - (n+1+g)\hbar\omega_B} ,$$

While $W_2(n, k)$ is non-zero only in the range (VII-18)

$$-\sqrt{\zeta - (n - g)\hbar\omega_B} < \frac{\hbar k}{\sqrt{2m}} < +\sqrt{\zeta - (n - g)\hbar\omega_B}$$

as long as the quantities under the radical (square root sign) are non-negative. We introduce a "cutoff" value of n, defined as follows:

$N_c \equiv$ "cutoff" value of n, such that

$$(N_c - g)\hbar\omega_B \leq \zeta < (N_c + 1 - g)\hbar\omega_B \qquad \text{(VII-19)}$$

with $(1 - 2g) > 0$.

Thus, for $(N_c - g)\hbar\omega_B \leq \zeta < (N_c + g)\hbar\omega_B$ the energy density is given by

$$e_{vf} = \frac{\omega_B (2m)^{\frac{3}{2}}}{(2\pi\hbar)^2} \left\{ \sum_{n=0}^{N_c - 2} \int_0^{+\sqrt{\zeta-(n+1+g)\hbar\omega_B}} d\lambda \left[(n+1+g)\hbar\omega_B + \lambda^2\right] + \sum_{n=0}^{N_c} \int_0^{+\sqrt{\zeta-(n-g)\hbar\omega_B}} d\lambda \left[(n-g)\hbar\omega_B + \lambda^2\right] \right\}$$

(VII-20)

$$= \frac{1}{3} \frac{\omega_B (2m)^{\frac{3}{2}}}{(2\pi\hbar)^2} \left\{ \sum_{n=0}^{N_c - 2} \left[\zeta + 2(n+1+g)\hbar\omega_B\right] \sqrt{\zeta - (n+1+g)\hbar\omega_B} + \sum_{n=0}^{N_c} \left[\zeta + 2(n-g)\hbar\omega_B\right] \sqrt{\zeta - (n-g)\hbar\omega_B} \right\} ,$$

and similarly $(N_c + g)\hbar\omega_B \leq \zeta < (N_c + 1 - g)\hbar\omega_B$ leads to

$$e_{vf} = \frac{\omega_B (2m)^{\frac{3}{2}}}{3(2\pi\hbar)^2} \left\{ \sum_{n=0}^{N_c - 1} \left[\zeta + 2(n+1+g)\hbar\omega_B\right] \sqrt{\zeta - (n+1+g)\hbar\omega_B} \right.$$

$$\left. + \sum_{n=0}^{N_c} \left[\zeta + 2(n-g)\hbar\omega_B\right] \sqrt{\zeta - (n-g)\hbar\omega_B} \right\} . \qquad \text{(VII-21)}$$

The change of variable $\lambda = \frac{\hbar k}{\sqrt{2m}}$ was made in the equations (VII-20, 21). We also calculate the number density n_{ef} for the same conditions: For $(N_c - g)\hbar\omega_B \leq \zeta < (N_c + g)\hbar\omega_B$

$$n_{ef} = \frac{\omega_B (2m)^{\frac{3}{2}}}{(2\pi\hbar)^2} \left\{ \sum_{n=0}^{N_c - 2} \int_o^{+\sqrt{\zeta - (n+1+g)\hbar\omega_B}} d\lambda + \sum_{n=0}^{N_c} \int_o^{+\sqrt{\zeta - (n-g)\hbar\omega_B}} d\lambda \right\}$$

or (VII-22)

$$n_{ef} = \frac{\omega_B (2m)^{\frac{3}{2}}}{(2\pi\hbar)^2} \left\{ \sum_{n=0}^{N_c - 2} \sqrt{\zeta - (n+1+g)\hbar\omega_B} + \sum_{n=0}^{N_c} \sqrt{\zeta - (n-g)\hbar\omega_B} \right\}$$

and similarly for $(N_c + g)\hbar\omega_B \leq \zeta < (N_c + 1 - g)\hbar\omega_B$

$$n_{ef} = \frac{\omega_B (2m)^{\frac{3}{2}}}{(2\pi\hbar)^2} \left\{ \sum_{n=0}^{N_c - 1} \sqrt{\zeta - (n+1+g)\hbar\omega_B} + \sum_{n=0}^{N_c} \sqrt{\zeta - (n-g)\hbar\omega_B} \right\} . \qquad \text{(VII-23)}$$

The equations (VII-22) and (VII-23) give n_{ef} as a function of ζ, or vice versa. The equations (VII-20) and (VII-21) give e_{vf} as a function of ζ, or vice versa. Thus, in principle, one has ζ as a function of n_{ef}, and thereby e_{vf} as an implicit function of n_{ef}. Both n_{ef} and e_{vf} are readily evaluated for small N_c (large value of ω_B) directly from (VII-22, 23); for large values of N_c (small ω_B) this is more difficult.

The behavior of the function ζ versus ω_B for a fixed density n_{ef} is rather interesting. The function ζ is continuous for all values of ω_B, but the slope $\frac{d\zeta}{d\omega_B}$ is discontinuous at each of the "boundary points" $\{\omega_B\}_b$ such that $\zeta_b \equiv (N_c - g)\hbar\{\omega_B\}_b$, and at each of the "interior connecting points" $\{\omega_B\}_i$ where $\zeta_i \equiv (N_c + g)\hbar\{\omega_B\}_i$ for all values of N_c except zero. The "boundary points" as defined form the upper limit for ω_B in the range defined by N_c and the lower limit in the range defined by $N_c - 1$, while the "interior connecting points" occur within the range defined by N_c. For arbitrary $N_c > 0$ the "boundary point" is given by the expression

$$\{\omega_B\}_b = \frac{\pi\hbar}{m}\left\{\frac{2\pi n_{ef}^2}{\left[\sum_{n=0}^{N_c-2}\sqrt{N_c-\mu-n}+\sum_{n=0}^{N_c}\sqrt{N_c-n}\right]^2}\right\}^{\frac{1}{3}} \qquad \text{(VII-24)}$$

from (VII-22). The slope at this point evaluated from below $(\{\omega_B\}_b-$; upper limit for the range defined by N_c), is, again from (VII-22)

$$\left.\frac{d\zeta}{d\omega_B}\right]_- = (N_c - g)\hbar\ . \qquad \text{(VII-25)}$$

The slope from above $(\{\omega_B\}_b+)$, evaluated using (VII-23) (here replacement $N_c \rightarrow N_c - 1$ must be made in the summations since this will be the lower limit of the range defined by $N_c - 1$), is found to be

$$\left.\frac{d\zeta}{d\omega_B}\right]_{+} =$$

$$\frac{\hbar\left\{\sum_{n=0}^{N_c-1}\frac{(n-g)}{\sqrt{N_c-n}}+\sum_{n=0}^{N_c-2}\frac{(n+1+g)}{\sqrt{N_c-\mu-n}}-2\left[\sum_{n=0}^{N_c-2}\sqrt{N_c-\mu-n}+\sum_{n=0}^{N_c}\sqrt{N_c-n}\right]\right\}}{\left\{\sum_{n=0}^{N_c-1}\frac{1}{\sqrt{N_c-n}}+\sum_{n=0}^{N_c-2}\frac{1}{\sqrt{N_c-\mu-n}}\right\}} \qquad \text{(VII-26)}$$

The "interior connection point" for arbitrary N_c is given by

$$\{\omega_B\}_i = \frac{\pi\hbar}{m}\left\{\frac{2\pi n_{ef}^2}{\left[\sum_{n=0}^{N_c-1}\sqrt{N_c-n-1}+\sum_{n=0}^{N_c}\sqrt{N_c+\mu-n-1}\right]^2}\right\}^{\frac{1}{3}} \qquad \text{(VII-27)}$$

where we have used (VII-23). The slope at this point evaluated from below ($\{\omega_B\}_i-$) is given by

$$\left.\frac{d\zeta}{d\omega_B}\right]_{-} = (N_c + g)\hbar \ , \qquad \text{(VII-28)}$$

from (VII-23). From above ($\{\omega_B\}_i+$) the slope at this point as evaluated from (VII-22) is

$$\left.\frac{d\zeta}{d\omega_B}\right]_{+} =$$

$$\frac{\hbar\left\{\sum_{n=0}^{N_c-2}\frac{(n+1+g)}{\sqrt{N_c-n-1}}+\sum_{n=0}^{N_c}\frac{(n-g)}{\sqrt{N_c+\mu-n-1}}-2\left[\sum_{n=0}^{N_c-1}\sqrt{N_c-n-1}+\sum_{n=0}^{N_c}\sqrt{N_c+\mu-n-1}\right]\right\}}{\left\{\sum_{n=0}^{N_c-2}\frac{1}{\sqrt{N_c-n-1}}+\sum_{n=0}^{N_c}\frac{1}{\sqrt{N_c+\mu-n-1}}\right\}}. \qquad \text{(VII-29)}$$

The discontinuities in the slope of ζ might be termed "oscillations" in the function ζ, and the "spacings" between discontinuities (values of $\Delta\omega_B$) might be called the "periods" of the "oscillations." The "periods" decrease with increasing N_c, becoming "vanishingly small" as $N_c \to \infty$, and the number of "oscillations" become infinite since the set $N_c = 0, 1, 2, \ldots \infty$ is infinite. Since for increasing N_c the quantities $\{\omega_B\}_b$ and $\{\omega_B\}_i$ decrease monotonically, we see that the number of "oscillations" increases without limit as ω_B goes from a non-zero value to zero, or, as ω_B increases, the number of "oscillations" decreases until the field reaches the point

$$\omega_B = \frac{\pi\hbar}{m}\left(2\pi n_{ef}^2\right)^{\frac{1}{3}},$$

after which the "oscillations" cease altogether. The function ζ presumably converges to the value otherwise calculated for zero field ($\omega_B = 0$):

$$\zeta_{\omega_B\to 0} = \lim_{N_c\to\infty}(N_c - g)\hbar\{\omega_B\}_b = \lim_{N_c\to\infty}(N_c + g)\hbar\{\omega_B\}_i$$

$$= \lim_{N_c\to\infty}\frac{(N_c - g)}{\left[\sum_{n=0}^{N_c-2}\sqrt{N_c-\mu-n}+\sum_{n=0}^{N_c}\sqrt{N_c-n}\right]^{\frac{2}{3}}}\frac{\pi\hbar^2}{m}\left(2\pi n_{ef}^2\right)^{\frac{1}{3}} \qquad \text{(VII-30)}$$

$$= \lim_{N_c \to \infty} \frac{(N_c + g)}{\left[\sum_{n=0}^{N_c - 2} \sqrt{N_c - n - 1} + \sum_{n=0}^{N_c} \sqrt{N_c + \mu - n - 1} \right]^{\frac{2}{3}}} \frac{\pi \hbar^2}{m} \left(2\pi n_{ef}^2 \right)^{\frac{1}{3}} ,$$

or

$$\zeta_{\omega_B \to 0} = \left(\frac{9}{4} \right)^{\frac{1}{3}} \frac{\pi \hbar^2}{m} \left(2\pi n_{ef}^2 \right)^{\frac{1}{3}} .$$

This zero field value for ζ is obtained from equation (1-95) of reference (3), in which

$$G_{011}(1, 1_+) = G_{022}(1, 1_+) = i \int \frac{d^3\vec{k}}{(2\pi)^3}, \; \frac{\hbar^2 \vec{k}^2}{2m} < \zeta$$

$$= i\, 2\pi \int_0^{\sqrt{\frac{2m\zeta}{\hbar^2}}} \frac{k^2 dk}{(2\pi)^3} = \frac{i}{3(2\pi)^2} \left(\frac{2m\zeta}{\hbar^2} \right)^{\frac{3}{2}} ,$$

and (VII-31)

$$n_{ef} = - i \, \mathrm{Tr} \, G_o(1, 1_+) = \frac{2}{3(2\pi)^2} \left(\frac{2m\zeta}{\hbar^2} \right)^{\frac{3}{2}} .$$

Thus the behavior of the chemical potential in the ground state of the system ($\beta = \infty$) is marked by an "oscillatory" character superimposed upon an overall decrease in value as ω_B increases from zero. The function ζ is also the "Fermi Energy" of the system in the ground state.

In equations (VI-8) and (VI-10) the chemical potential ζ is the same in both the expressions $G_{o\alpha\beta}(1,2)$ and $G_{1\alpha\beta}(1,2)_o$; in such cases one generally has the result

$$n_e \neq n_{ef} . \qquad \text{(VII-32)}$$

For purposes of calculation, however, we let

$$n_e = n_{ef} , \qquad \text{(VII-33)}$$

and the calculated value of ζ will be an approximation. This procedure permits a reasonable solution for ζ in any approximation of G_1 with a specified number density n_e for the gas.

CHAPTER VIII

INTERACTION OF TIME-DEPENDENT ELECTROMAGNETIC FIELDS WITH THE ELECTRON GAS

In this chapter we consider the formal solution of the linear wave equation (II-29). The "conductance tensor" (II-31) for the electron gas is discussed in terms of its Fourier transform. Plane wave propagation in the direction of the applied magnetic field is considered for a non-degenerate gas. Some discussion is devoted to the case of complex frequencies. The quantities

$$\frac{\delta j_\ell(1)}{\delta A_m^T(2)_o}$$

are calculated from the equations (V-50) and from the Green's function (VI-54). The quantities

$$\frac{\delta \rho(1)}{\delta U^T(2)_o}, \quad \frac{\delta \rho(1)}{\delta A_m^T(2)_o}, \text{ and } \frac{\delta j_\ell(1)}{\delta U^T(2)_o}$$

may be calculated for the electron gas from

$$\frac{\delta j_\ell(1)}{\delta A_m^T(2)_o}$$

through the connecting relations (II-11, 15).

Calculation of the Quantities $\frac{\delta j_\ell(1)}{\delta A_m^T(2)_o}$

From the equations (V-50) and (VI-54) one obtains the following results:

$$\frac{\delta j_+(1)}{\delta A_+^T(2)_o} = -\frac{e^2 n_e}{m c}\,\delta^4(1,2)$$

$$+\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} Q(n_1,k_1;n_2,k_2;t_1,t_2)\,D_{++}(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2)\,\cdot$$

$$-\frac{\partial}{\partial x_{2+}}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} X(n_1,k_1;n_2,k_2;t_1,t_2)\,F_+(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2)$$

$$+\frac{\delta j_M(1)_+}{\delta A_+^T(2)_o}$$

$$\frac{\delta j_+(1)}{\delta A_-^T(2)_o}$$

$$=\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} Q(n_1,k_1;n_2,k_2;t_1,t_2)\,D_{+-}(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2)$$

$$+\frac{\partial}{\partial x_{2-}}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} X(n_1,k_1;n_2,k_2;t_1,t_2)F_+(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2)$$

$$+ \frac{\delta j_M(1)_+}{\delta A_-^T(2)_o}$$

$$\frac{\delta j_+(1)}{\delta A_z^T(2)_o}$$

$$= \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} Q(n_1, k_1; n_2, k_2; t_1, t_2) D_{+z}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$+ \frac{\delta j_M(1)_+}{\delta A_z^T(2)_o}$$

$$\frac{\delta j_-(1)}{\delta A_+^T(2)_o}$$

$$= \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} Q(n_1, k_1; n_2, k_2; t_1, t_2) D_{-+}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$- \frac{\partial}{\partial x_{2+}} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} X(n_1, k_1; n_2, k_2, t_1, t_2) F_-(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$+ \frac{\delta j_M(1)_-}{\delta A_+^T(2)_o}$$

(VIII-1)

$$\frac{\delta j_-(1)}{\delta A_-^T(2)_o} = - \frac{e^2 n_e}{mc} \delta^4(1, 2)$$

$$+ \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} Q(n_1, k_1; n_2, k_2; t_1, t_2) D_{--}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$+ \frac{\partial}{\partial x_{2-}} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} X(n_1, k_1; n_2, k_2; t_1, t_2) F_-(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$+ \frac{\delta j_M(1)_-}{\delta A_-^T(2)_o}$$

$$\frac{\delta j_-(1)}{\delta A_z^T(2)_o}$$

$$= \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} Q(n_1, k_1; n_2, k_2; t_1, t_2) D_{-z}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$+ \frac{\delta j_M(1)_-}{\delta A_z^T(2)_o}$$

$$\frac{\delta j_z(1)}{\delta A_+^T(2)_o}$$

$$= \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} Q(n_1, k_1; n_2, k_2; t_1, t_2) D_{z+}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$- \frac{\partial}{\partial x_{2+}} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} X(n_1, k_1; n_2, k_2; t_1, t_2) F_z(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$+ \frac{\delta j_M(1)_z}{\delta A_+^T(2)_o}$$

$$\frac{\delta j_z(1)}{\delta A_-^T(2)_o}$$

$$= \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} Q(n_1, k_1; n_2, k_2; t_1, t_2) D_{z-}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$+ \frac{\partial}{\partial x_{2-}} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} X(n_1, k_1; n_2, k_2; t_1, t_2) F_z(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$+ \frac{\delta j_M(1)_z}{\delta A_-^T(2)_o}$$

$$\frac{\delta j_z(1)}{\delta A_z^T(2)_o} = -\frac{e^2 n_e}{mc} \delta^4(1, 2)$$

$$+ \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} Q(n_1, k_1; n_2, k_2; t_1, t_2) D_{zz}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$+ \frac{\delta j_M(1)_z}{\delta A_z^T(2)_o} ,$$

with

$$Q(n_1, k_1; n_2, k_2; t_1, t_2)$$

$$\equiv \frac{i e^2}{8\hbar m^2 c} \Big\{ \eta_+^2(1, 2) \Big[\Gamma_1(n_1, k_1) W_1(n_2, k_2) + \Gamma_2(n_1, k_1) W_2(n_2, k_2) \Big]$$

$$+ \eta_-^2(1, 2) \Big[W_1(n_1, k_1) \Gamma_1(n_2, k_2) + W_2(n_1, k_1) \Gamma_2(n_2, k_2) \Big] \Big\}$$

$$\times\ e^{-\frac{i\hbar}{2m}(k_1^2-k_2^2)(t_1-t_2)-i(n_1-n_2)\omega_B(t_1-t_2)-\varepsilon|t_1-t_2|}\ ,\ \varepsilon>0,\ \ \varepsilon\to 0^+\ ,$$

$$D_{++}(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2)$$

$$=\sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}2m\hbar\omega_B\Big[\sqrt{(n_1+1)n_2}\,v_{n_1+1,\ell_1,k_1}(\vec{r}_1)\,v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\,v_{n_2-1,\ell_2,k_2}(\vec{r}_2)\,v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+\ (n_1+1)\,v_{n_1+1,\ell_1,k_1}(\vec{r}_1)\,v^*_{n_1+1,\ell_1,k_1}(\vec{r}_2)\,v_{n_2,\ell_2,k_2}(\vec{r}_2)\,v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+\ n_2\,v_{n_1,\ell_1,k_1}(\vec{r}_1)\,v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\,v_{n_2-1,\ell_2,k_2}(\vec{r}_2)\,v^*_{n_2-1,\ell_2,k_2}(\vec{r}_1)$$

$$+\sqrt{(n_1+1)n_2}\,v_{n_1,\ell_1,k_1}(\vec{r}_1)\,v^*_{n_1+1,\ell_1,k_1}(\vec{r}_2)\,v_{n_2,\ell_2,k_2}(\vec{r}_2)\,v^*_{n_2-1,\ell_2,k_2}(\vec{r}_1)\Big]$$

$$D_{+-}(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2)$$

$$=\sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}2m\hbar\omega_B\Big[\sqrt{(n_1+1)(n_2+1)}\,v_{n_1+1,\ell_1,k_1}(\vec{r}_1)\,v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\,v_{n_2+1,\ell_2,k_2}(\vec{r}_2)\,v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+\sqrt{n_1(n_1+1)}\,v_{n_1+1,\ell_1,k_1}(\vec{r}_1)\,v^*_{n_1-1,\ell_1,k_1}(\vec{r}_2)\,v_{n_2,\ell_2,k_2}(\vec{r}_2)\,v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+\sqrt{n_2(n_2+1)}\,v_{n_2+1,\ell_2,k_2}(\vec{r}_2)\,v^*_{n_2-1,\ell_2,k_2}(\vec{r}_1)\,v_{n_1,\ell_1,k_1}(\vec{r}_1)\,v^*_{n_1,\ell_1,k_1}(\vec{r}_2)$$

$$+\sqrt{n_1\,n_2}\quad v_{n_1,\ell_1,k_1}(\vec{r}_1)\,v^*_{n_1-1,\ell_1,k_1}(\vec{r}_2)\,v_{n_2,\ell_2,k_2}(\vec{r}_2)\,v^*_{n_2-1,\ell_2,k_2}(\vec{r}_1)\Big]$$

$$D_{+z}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$= \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \hbar\sqrt{2m\hbar\omega_B}\Big[\sqrt{n_1+1}\, k_2\, v_{n_1+1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_1+1}\, k_1\, v_{n_1+1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_2}\, k_2\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2-1,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_2}\, k_1\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2-1,\ell_2,k_2}(\vec{r}_1)\Big]$$

$$D_{-+}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$= \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} 2m\hbar\omega_B\Big[\sqrt{n_1 n_2}\, v_{n_1-1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2-1,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_1(n_1+1)}\, v_{n_1-1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1+1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_2(n_2+1)}\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2-1,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2+1,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{(n_1+1)(n_2+1)}\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v_{n_1+1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v_{n_2+1,\ell_2,k_2}(\vec{r}_1)\Big]$$

$$D_{--}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$= \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} 2m\hbar\omega_B \Big[\sqrt{n_1(n_2+1)}\, v_{n_1-1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2+1,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ n_1 v_{n_1-1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1-1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ (n_2+1) v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2+1,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2+1,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_1(n_2+1)}\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1-1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2+1,\ell_2,k_2}(\vec{r}_1) \Big]$$

$$D_{-z}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$= \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \hbar\sqrt{2m\hbar\omega_B} \Big[\sqrt{n_1}\, k_2\, v_{n_1-1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_1}\, k_1\, v_{n_1-1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_2+1}\, k_2\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2+1,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_2+1}\, k_1\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2+1,\ell_2,k_2}(\vec{r}_1) \Big]$$

$D_{z+}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$

$$= \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \hbar\sqrt{2m\hbar\omega_B}\Big[\sqrt{n_2}\, k_1\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2-1,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_1+1}\, k_1\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1+1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_2}\, k_2\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2-1,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_1+1}\, k_2\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1+1,\ell_1,k_1}(\vec{r}_2)\Big]$$

$D_{z-}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$

$$= \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \hbar\sqrt{2m\hbar\omega_B}\Big[\sqrt{n_2+1}\, k_1\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2+1,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_1}\, k_1\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1-1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_2+1}\, k_2\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2+1,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_1}\, k_2\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1-1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)\Big]$$

$D_{zz}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$

$$= \hbar^2 (k_1+k_2)^2 \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1) \quad ,$$

and

$$X(n_1, k_1; n_2, k_2; t_1, t_2)$$

$$\equiv \frac{\mu e^2}{4m^2 c}\Big\{\eta_+^2(1,2)\Big[\Gamma_1(n_1, k_1)W_1(n_2, k_2) - \Gamma_2(n_1, k_1)W_2(n_2, k_2)\Big]$$

$$+ \eta_-^2(1,2)\Big[W_1(n_1, k_1)\Gamma_1(n_2, k_2) - W_2(n_1, k_1)\Gamma_2(n_2, k_2)\Big]\Big\}$$

$$\times\, e^{-\frac{i\hbar}{2m}(k_1^2 - k_2^2)(t_1 - t_2) - i(n_1 - n_2)\omega_B(t_1 - t_2) - \varepsilon|t_1 - t_2|}, \quad \varepsilon > 0,\ \varepsilon \to 0^+ ,$$

$$F_+(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$= \sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}\sqrt{2m\hbar\omega_B}\Big[\sqrt{n_1+1}\, v_{n_1+1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_2}\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2-1,\ell_2,k_2}(\vec{r}_1)\Big]$$

(VIII-3)

$$F_-(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$= \sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}\sqrt{2m\hbar\omega_B}\Big[\sqrt{n_1}\, v_{n_1-1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_2+1}\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2+1,\ell_2,k_2}(\vec{r}_1)\Big]$$

$$F_z(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$= \hbar(k_1 + k_2)\sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty} v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)\,.$$

The variational derivatives of $\vec{j}_M$ are given as before by (V-41), and the associated variational derivatives of $\vec{M}_s$ are given by

$$\frac{\delta M_s(1)_+}{\delta U^T(2)_o} = 0$$

$$\frac{\delta M_s(1)_+}{\delta A_+^T(2)_o} = \frac{\partial}{\partial z_2}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} Y(n_1, k_1; n_2 k_2; t_1, t_2) V_1(n_1, k_1; n_2 k_2; \vec{r}_1, \vec{r}_2)$$

$$\frac{\delta M_s(1)_+}{\delta A_-^T(2)_o} = 0$$

$$\frac{\delta M_s(1)_+}{\delta A_z^T(2)_o} = -2\frac{\partial}{\partial x_{2-}}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} Y(n_1, k_1; n_2 k_2; t_1, t_2) V_1(n_1, k_1; n_2 k_2; \vec{r}_1, \vec{r}_2)$$

$$\frac{\delta M_s(1)_-}{\delta U^T(2)_o} = 0 \qquad \text{(VIII-4)}$$

$$\frac{\delta M_s(1)_-}{\delta A_+^T(2)_o} = 0$$

$$\frac{\delta M_s(1)_-}{\delta A_-^T(2)_o} = -\frac{\partial}{\partial z_2}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} Z(n_1, k_1; n_2 k_2; t_1, t_2) V_1(n_1, k_1; n_2 k_2; \vec{r}_1, \vec{r}_2)$$

$$\frac{\delta M_s(1)_-}{\delta A_z^T(2)_o} = 2\frac{\partial}{\partial x_{2+}}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} Z(n_1, k_1; n_2 k_2; t_1, t_2) V_1(n_1, k_1; n_2 k_2; \vec{r}_1, \vec{r}_2)$$

$$\frac{\delta M_s(1)_z}{\delta U^T(2)_o} = -\frac{i\,2m}{c}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} X(n_1, k_1; n_2 k_2; t_1, t_2) V_1(n_1, k_1; n_2 k_2; \vec{r}_1, \vec{r}_2)$$

$$\frac{\delta M_s(1)_z}{\delta A_+^T(2)_o} = \frac{i}{2c^2}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} X(n_1,k_1;n_2,k_2;t_1,t_2)V_2(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2)$$

$$+\frac{i\,2\mu^2\hbar^2}{c^2}\frac{\partial}{\partial x_{2+}}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} Q(n_1,k_1;n_2,k_2;t_1,t_2)V_1(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2)$$

$$\frac{\delta M_s(1)_z}{\delta A_-^T(2)_o} = \frac{i}{2c^2}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} X(n_1,k_1;n_2,k_2;t_1,t_2)V_3(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2)$$

$$-\frac{i\,2\mu^2\hbar^2}{c^2}\frac{\partial}{\partial x_{2-}}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} Q(n_1,k_1;n_2,k_2;t_1,t_2)V_1(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2)$$

$$\frac{\delta M_s(1)_z}{\delta A_z^T(2)_o} = \frac{i}{c^2}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\int_{-\infty}^{\infty}\frac{dk_2}{2\pi}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} X(n_1,k_1;n_2,k_2;t_1,t_2)F_z(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2)\,,$$

with

$Y(n_1,k_1;n_2,k_2;t_1,t_2)$

$$\equiv \frac{\mu^2e^2\hbar}{2m^2c^2}\left\{\eta_+^2(1,2)\Gamma_2(n_1,k_1)W_1(n_2,k_2)+\eta_-^2(1,2)W_2(n_1,k_1)\Gamma_1(n_2,k_2)\right\}$$

$$\times\, e^{-\frac{i\hbar}{2m}(k_1^2-k_2^2)(t_1-t_2)-i(n_1-n_2-\mu)\omega_B(t_1-t_2)-\varepsilon|t_1-t_2|}\,,\ \varepsilon>0\,,\ \varepsilon\to 0^+\,,$$

$Z(n_1,k_1;n_2,k_2;t_1,t_2)$

$$\equiv \frac{\mu^2e^2\hbar}{2m^2c^2}\left\{\eta_+^2(1,2)\Gamma_1(n_1,k_1)W_2(n_2,k_2)+\eta_-^2(1,2)W_1(n_1,k_1)\Gamma_2(n_2,k_2)\right\}$$

$$\times\, e^{-\frac{i\hbar}{2m}(k_1^2-k_2^2)(t_1-t_2)-i(n_1-n_2+\mu)\omega_B(t_1-t_2)-\varepsilon|t_1-t_2|}\,,\ \varepsilon>0,\ \varepsilon\to 0^+\,,$$

$$V_1(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2) \qquad \text{(VIII-5)}$$

$$\equiv \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$V_2(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$\equiv \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \sqrt{2mh\omega_B}\Big[\sqrt{n_2}\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2-1,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_1+1}\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1+1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)\Big]$$

$$V_3(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$\equiv \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \sqrt{2mh\omega_B}\Big[\sqrt{n_2+1}\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2+1,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)$$

$$+ \sqrt{n_1}\, v_{n_1,\ell_1,k_1}(\vec{r}_1)\, v^*_{n_1-1,\ell_1,k_1}(\vec{r}_2)\, v_{n_2,\ell_2,k_2}(\vec{r}_2)\, v^*_{n_2,\ell_2,k_2}(\vec{r}_1)\Big]$$

The above expressions were derived using equations (V-54) and (VI-56).

It is seen from the equations (VIII-1) through (VIII-5) that all the quantities

$$\frac{\delta j_\ell(1)}{\delta A^T_m(2)_o}$$

depend upon the time difference $(t_1 - t_2)$, and therefore the current density equation (II-25) is applicable here. From the connecting relations (II-11, 15) and from (II-21) one sees that

$$\frac{\delta \rho(1)}{\delta U^T(2)_o}, \quad \frac{\delta \rho(1)}{\delta A^T_m(2)_o}, \text{ and } \frac{\delta j_\ell(1)}{\delta U^T(2)_o}$$

also depend upon the time difference $(t_1 - t_2)$. Thus the charge density fluctuation equation (II-25) is also applicable here. The same would be true for any system with time-independent Hamiltonian because of the structure of the field creation and annihilation operators (see equations (VI-25)).

By making use of the identities (6-1) from Mathematical Appendix VI in the equations (VIII-1) through (VIII-5), one obtains the resulting equations for the coefficients $D_{\ell m}$, F_m, and V_m:

$$D_{+z}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$= +i\hbar(k_1 + k_2)\sqrt{2m\hbar\omega_B}\left(\frac{m\omega_B}{2\pi\hbar}\right)^{\frac{5}{2}} \frac{\sqrt{\pi}\, e^{-v_{12}}}{n_1!\, n_2!} (x_{2+} - x_{1+})\, e^{i(k_1 - k_2)(z_1 - z_2)}$$

$$\times \left\{ \frac{L_{n_2}(v_{12})}{(n_1 + 1)} \left[\frac{d}{dv_{12}} L_{n_1+1}(v_{12})\right] - L_{n_1}(v_{12}) \left[\frac{d}{dv_{12}} L_{n_2}(v_{12})\right]\right\}$$

$$D_{z+}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$= -i\hbar(k_1 + k_2)\sqrt{2m\hbar\omega_B}\left(\frac{m\omega_B}{2\pi\hbar}\right)^{\frac{5}{2}} \frac{\sqrt{\pi}\, e^{-v_{12}}}{n_1!\, n_2!} (x_{2-} - x_{1-})\, e^{i(k_1 - k_2)(z_1 - z_2)}$$

$$\times \left\{ \frac{L_{n_2}(v_{12})}{(n_1 + 1)} \left[\frac{d}{dv_{12}} L_{n_1+1}(v_{12})\right] - L_{n_1}(v_{12}) \left[\frac{d}{dv_{12}} L_{n_2}(v_{12})\right]\right\} .$$

$$D_{++}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2) = 2m\hbar\omega_B \left(\frac{m\omega_B}{2\pi\hbar}\right)^2 \frac{e^{-v_{12}}}{n_1! \, n_2!} e^{i(k_1-k_2)(z_1-z_2)}$$

$$\times \left\{ -\frac{2v_{12}}{(n_1+1)} \left[\frac{d}{dv_{12}} L_{n_1+1}(v_{12})\right]\left[\frac{d}{dv_{12}} L_{n_2}(v_{12})\right] + L_{n_1+1}(v_{12})\, L_{n_2}(v_{12}) \right.$$

$$\left. + n_2^2\, L_{n_1}(v_{12})\, L_{n_2-1}(v_{12}) \right\}$$

$$D_{+-}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2) = 2m\hbar\omega_B \left(\frac{m\omega_B}{2\pi\hbar}\right)^3 \frac{\pi e^{-v_{12}}}{n_1! \, n_2!} (x_{2+} - x_{1+})^2 e^{i(k_1-k_2)(z_1-z_2)}$$

$$\times \left\{ \frac{1}{(n_1+1)} \left[\frac{d}{dv_{12}} L_{n_1+1}(v_{12})\right] \frac{1}{(n_2+1)} \left[\frac{d}{dv_{12}} L_{n_2+1}(v_{12})\right] \right.$$

$$+ \left[\frac{d}{dv_{12}} L_{n_1}(v_{12})\right]\left[\frac{d}{dv_{12}} L_{n_2}(v_{12})\right] - \frac{1}{(n_1+1)} L_{n_2}(v_{12}) \left[\frac{d^2}{dv_{12}^2} L_{n_1+1}(v_{12})\right]$$

$$\left. - \frac{1}{(n_2+1)} L_{n_1}(v_{12}) \left[\frac{d^2}{dv_{12}^2} L_{n_2+1}(v_{12})\right] \right\}$$

$$D_{z-}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2) =$$

$$+ i\hbar(k_1+k_2)\sqrt{2m\hbar\omega_B} \left(\frac{m\omega_B}{2\pi\hbar}\right)^{\frac{5}{2}} \frac{\sqrt{\pi}\, e^{-v_{12}}}{n_1! \, n_2!} (x_{1+} - x_{2+}) e^{i(k_1-k_2)(z_1-z_2)}$$

$$\times \left\{ \frac{L_{n_1}(v_{12})}{(n_2+1)} \left[\frac{d}{dv_{12}} L_{n_2+1}(v_{12})\right] - L_{n_2}(v_{12}) \left[\frac{d}{dv_{12}} L_{n_1}(v_{12})\right] \right\}$$

$$D_{zz}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2) = \hbar^2 (k_1+k_2)^2 \left(\frac{m\omega_B}{2\pi\hbar}\right)^2 \frac{e^{-v_{12}}}{n_1! \, n_2!} L_{n_1}(v_{12}) L_{n_2}(v_{12}) e^{i(k_1-k_2)(z_1-z_2)},$$

$$D_{-+}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$= 2m\hbar\omega_B \left(\frac{m\omega_B}{2\pi\hbar}\right)^3 \frac{\pi e^{-v_{12}}}{n_1! n_2!} (x_{1-} - x_{2-})^2 e^{i(k_1 - k_2)(z_1 - z_2)}$$

$$\times \left\{ -\frac{1}{(n_2+1)} \left[\frac{d}{dv_{12}} L_{n_1+1}(v_{12})\right] \frac{1}{(n_2+1)} \left[\frac{d}{dv_{12}} L_{n_2+1}(v_{12})\right] \right.$$

$$+ \left[\frac{d}{dv_{12}} L_{n_1}(v_{12})\right]\left[\frac{d}{dv_{12}} L_{n_2}(v_{12})\right] - \frac{1}{(n_1+1)} L_{n_2}(v_{12}) \left[\frac{d^2}{dv_{12}^2} L_{n_1+1}(v_{12})\right]$$

$$\left. - \frac{1}{(n_2+1)} L_{n_1}(v_{12}) \left[\frac{d^2}{dv_{12}^2} L_{n_2+1}(v_{12})\right] \right\}$$

(VIII-6)

$$D_{--}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$= 2m\hbar\omega_B \left(\frac{m\omega_B}{2\pi\hbar}\right)^2 \frac{e^{-v_{12}}}{n_1! n_2!} e^{i(k_1 - k_2)(z_1 - z_2)}$$

$$\times \left\{ -\frac{2v_{12}}{(n_2+1)} \left[\frac{d}{dv_{12}} L_{n_1}(v_{12})\right]\left[\frac{d}{dv_{12}} L_{n_2+1}(v_{12})\right] + L_{n_1}(v_{12}) L_{n_2+1}(v_{12}) \right.$$

$$\left. + n_1^2 L_{n_1-1}(v_{12}) L_{n_2}(v_{12}) \right\}$$

$$D_{-z}(n_1, k_1; n_2, k_2; \vec{r}_1, \vec{r}_2)$$

$$= -i\hbar(k_1 + k_2)\sqrt{2m\hbar\omega_B} \left(\frac{m\omega_B}{2\pi\hbar}\right)^{\frac{5}{2}} \frac{\sqrt{\pi}\, e^{-v_{12}}}{n_1! n_2!} (x_{1-} - x_{2-})\, e^{i(k_1 - k_2)(z_1 - z_2)}$$

$$\times \left\{ \frac{L_{n_1}(v_{12})}{(n_2+1)} \left[\frac{d}{dv_{12}} L_{n_2+1}(v_{12})\right] - L_{n_2}(v_{12}) \left[\frac{d}{dv_{12}} L_{n_1}(v_{12})\right] \right\}$$

$$F_+(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2) = +i\sqrt{2m\hbar\omega_B}\left(\frac{m\omega_B}{2\pi\hbar}\right)^{\frac{5}{2}} \frac{\sqrt{\pi}\, e^{-v_{12}}}{n_1!\, n_2!} (x_{2+} - x_{1+})$$

$$\times \left\{ \frac{L_{n_2}(v_{12})}{(n_1+1)} \left[\frac{d}{dv_{12}} L_{n_1+1}(v_{12})\right] + L_{n_1}(v_{12}) \left[\frac{d}{dv_{12}} L_{n_2}(v_{12})\right] e^{i(k_1-k_2)(z_1-z_2)}\right.$$

$$F_-(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2) = +i\sqrt{2m\hbar\omega_B}\left(\frac{m\omega_B}{2\pi\hbar}\right)^{\frac{5}{2}} \frac{\sqrt{\pi}\, e^{-v_{12}}}{n_1!\, n_2!} (x_{1-} - x_{2-})$$

$$\times \left\{ \frac{L_{n_1}(v_{12})}{(n_2+1)} \left[\frac{d}{dv_{12}} L_{n_2+1}(v_{12})\right] + L_{n_2}(v_{12}) \left[\frac{d}{dv_{12}} L_{n_1}(v_{12})\right] e^{i(k_1-k_2)(z_1-z_2)}\right.$$

$$F_z(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2) = \hbar(k_1+k_2)\left(\frac{m\omega_B}{2\pi\hbar}\right)^2 \frac{e^{-v_{12}}}{n_1!\,n_2!} L_{n_1}(v_{12}) L_{n_2}(v_{12}) e^{i(k_1-k_2)(z_1-z_2)}$$

$$= \hbar(k_1+k_2)\, V_1(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2)$$

$$V_2(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2) = F_+^*(n_1,k_2;n_2,k_1;\vec{r}_1,\vec{r}_2)$$

$$V_3(n_1,k_1;n_2,k_2;\vec{r}_1,\vec{r}_2) = F_-^*(n_1,k_2,n_2,k_1;\vec{r}_1,\vec{r}_2)$$

where

$$V_{12} \equiv \frac{m\omega_B}{2\hbar}(x_{1+} - x_{2+})(x_{1-} - x_{2-})$$

$$= \frac{m\omega_B}{2\hbar}\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right].$$

From the equations (VIII-1) through (VIII-6), it is obvious that the quantities

$$\frac{\delta j_\ell(1)}{\delta A_m^T(2)_o}$$

are manifestly dependent upon the differences $(\vec{r}_1 - \vec{r}_2)$ in the spatial co-ordinates as well as upon the time differences $(t_1 - t_2)$. This was to be expected since the system (electron gas) was taken to be unbounded, and since no restrictions other than an axis of symmetry (direction of applied magnetic field) are imposed upon the system, it should exhibit translational invariance which, of course, is the precise physical interpretation of a mathematical dependence upon the difference in spatial co-ordinates. Therefore, if the Fourier transform of $\delta \vec{j}(1)$ is defined by the expression

$$\delta j_\ell(1) = \int \frac{d^3\vec{k}\, d\omega}{(2\pi)^4}\, j_\ell(\vec{k}, \omega)\, e^{i(\vec{k}\cdot\vec{r}_1 - \omega t_1)}, \qquad \text{(VIII-7)}$$

then from equations (II-19, 25, 31) one obtains the result

$$j_\ell(\vec{k}, \omega) = s^T_{\ell m}(\vec{k}, \omega)\, e^T_m(\vec{k}, \omega), \qquad \text{(VIII-8)}$$

where

$$s^T_{\ell m}(\vec{k}, \omega) = -\frac{ic}{\omega}\int d(2)\, \frac{\delta j_\ell(1)}{\delta A^T_m(2)_o}\, e^{i\vec{k}\cdot(\vec{r}_2 - \vec{r}_1) - i\omega(t_2 - t_1)}. \qquad \text{(VIII-9)}$$

The fact that

$$\frac{\delta j_\ell(1)}{\delta A^T_m(2)_o}$$

depends upon the difference $(\vec{r}_1 - \vec{r}_2)$ in the spatial co-ordinates as well as upon the difference $(t_1 - t_2)$ of the time co-ordinates imposes the following condition upon the Fourier transform of

$$\frac{\delta j_\ell(1)}{\delta A^T_m(2)_o}\;:$$

$$q_{\ell m}(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2) = (2\pi)^4 \delta^3(\vec{k}_1 - \vec{k}_2)\,\delta(\omega_1 - \omega_2)\,\bar{\bar{q}}_{\ell m}(\vec{k}_1, \omega_1), \qquad \text{(VIII-10)}$$

and from a comparison of the connecting relations (II-11, 15) with (VIII-10) one finds that

$$f(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2) = (2\pi)^4 \delta^3(\vec{k}_1 - \vec{k}_2)\,\delta(\omega_1 - \omega_2)\,\bar{\bar{f}}(\vec{k}_1, \omega_1)$$

$$g_m(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2) = (2\pi)^4 \delta^3(\vec{k}_1 - \vec{k}_2)\,\delta(\omega_1 - \omega_2)\,\bar{\bar{g}}_m(\vec{k}_1, \omega_1) \qquad \text{(VIII-11)}$$

$$p_\ell(\vec{k}_1, \omega_1; \vec{k}_2, \omega_2) = (2\pi)^4 \delta^3(\vec{k}_1 - \vec{k}_2)\,\delta(\omega_1 - \omega_2)\,\bar{\bar{p}}_\ell(\vec{k}_1, \omega_1).$$

Obviously then, all the functional derivatives

$$\frac{\delta\rho(1)}{\delta U^T(2)_o}, \quad \frac{\delta\rho(1)}{\delta A_m^T(2)_o}, \quad \frac{\delta j_\ell(1)}{\delta U^T(2)_o} \quad \text{and} \quad \frac{\delta j_\ell(1)}{\delta A_m^T(2)_o}$$

depend upon the differences of the spatial and temporal co-ordinates in the case of the uniformly magnetized electron gas in thermal equilibrium. In this case the connecting relations (II-15) reduce to the following equations:

$$\begin{aligned}
-i\omega\bar{\bar{f}}(\vec{k}, \omega) + ik_\ell\bar{\bar{p}}_\ell(\vec{k}, \omega) &= 0 \\
-i\omega\bar{\bar{g}}_m(\vec{k}, \omega) + ik_\ell\bar{\bar{q}}_{\ell m}(\vec{k}, \omega) &= 0 \\
-\frac{i\omega}{c}\bar{\bar{f}}(\vec{k}, \omega) - ik_m\bar{\bar{g}}_m(\vec{k}, \omega) &= 0 \\
-\frac{i\omega}{c}\bar{\bar{p}}_\ell(\vec{k}, \omega) - ik_m\bar{\bar{q}}_{\ell m}(\vec{k}, \omega) &= 0 .
\end{aligned} \qquad \text{(VIII-12)}$$

From equations (VIII-9) and (VIII-10) one has the result

$$\bar{\bar{q}}_{\ell m}(\vec{k}, \omega) = \frac{i\omega}{c} s^T_{\ell m}(\vec{k}, \omega) . \qquad \text{(VIII-13)}$$

Some Useful Relations for the "Conductance Tensor," the Current Density, and the Wave Equation

The current density equation (II-25) represents three scalar equations which can be expressed as

$$\frac{\partial}{\partial t_1} \delta j_x(1) = -c \int d(2) \left[\frac{\delta j_x(1)}{\delta A_x^T(2)_o} \delta E_x^T(2) + \frac{\delta j_x(1)}{\delta A_y^T(2)_o} \delta E_y^T(2) + \frac{\delta j_x(1)}{\delta A_z^T(2)_o} \delta E_z^T(2) \right]$$

$$\frac{\partial}{\partial t_1} \delta j_y(1) = -c \int d(2) \left[\frac{\delta j_y(1)}{\delta A_x^T(2)_o} \delta E_x^T(2) + \frac{\delta j_y(1)}{\delta A_y^T(2)_o} \delta E_y^T(2) + \frac{\delta j_y(1)}{\delta A_z^T(2)_o} \delta E_z^T(2) \right]$$

(VIII-14)

$$\frac{\partial}{\partial t_1} \delta j_z(1) = -c \int d(2) \left[\frac{\delta j_z(1)}{\delta A_x^T(2)_o} \delta E_x^T(2) + \frac{\delta j_z(1)}{\delta A_y^T(2)_o} \delta E_y^T(2) + \frac{\delta j_z(1)}{\delta A_z^T(2)_o} \delta E_z^T(2) \right]$$

in Cartesian co-ordinates, or they may be written in the form

$$\frac{\partial}{\partial t_1} \delta j_+(1) = -c \int d(2) \left[\frac{\delta j_+(1)}{\delta A_+^T(2)_o} \delta E_+^T(2) + \frac{\delta j_+(1)}{\delta A_-^T(2)_o} \delta E_-^T(2) + \frac{\delta j_+(1)}{\delta A_z^T(2)_o} \delta E_z^T(2) \right]$$

$$\frac{\partial}{\partial t_1} \delta j_-(1) = -c \int d(2) \left[\frac{\delta j_-(1)}{\delta A_+^T(2)_o} \delta E_+^T(2) + \frac{\delta j_-(1)}{\delta A_-^T(2)_o} \delta E_-^T(2) + \frac{\delta j_-(1)}{\delta A_z^T(2)_o} \delta E_z^T(2) \right]$$

(VIII-15)

$$\frac{\partial}{\partial t_1} \delta j_z(1) = -c \int d(2) \left[\frac{\delta j_z(1)}{\delta A_+^T(2)_o} \delta E_+^T(2) + \frac{\delta j_z(1)}{\delta A_-^T(2)_o} \delta E_-^T(2) + \frac{\delta j_z(1)}{\delta A_z^T(2)_o} \delta E_z^T(2) \right]$$

where

$$E^T_+(1) \equiv E^T_x(1) + i E^T_y(1)$$

$$E^T_-(1) \equiv E^T_x(1) - i E^T_y(1) \; . \qquad \text{(VIII-16)}$$

The form given by (VIII-15) is the most useful for calculations since the functional derivatives are in a form best suited to operations on the basis functions (VI-43). Then, from the definitions (V-40), we find that

$$\frac{\delta\rho(1)}{\delta A^T_x(2)_o} = \left[\frac{\delta\rho(1)}{\delta A^T_+(2)_o} + \frac{\delta\rho(1)}{\delta A^T_-(2)_o}\right]$$

$$\frac{\delta\rho(1)}{\delta A^T_y(2)_o} = i\left[\frac{\delta\rho(1)}{\delta A^T_+(2)_o} - \frac{\delta\rho(1)}{\delta A^T_-(2)_o}\right]$$

$$\frac{\delta j_x(1)}{\delta U^T(2)_o} = \frac{1}{2}\left[\frac{\delta j_+(1)}{\delta U^T(2)_o} + \frac{\delta j_-(1)}{\delta U^T(2)_o}\right]$$

$$\frac{\delta j_x(1)}{\delta A^T_x(2)_o} = \frac{1}{2}\left[\frac{\delta j_+(1)}{\delta A^T_+(2)_o} + \frac{\delta j_+(1)}{\delta A^T_-(2)_o} + \frac{\delta j_-(1)}{\delta A^T_+(2)_o} + \frac{\delta j_-(1)}{\delta A^T_-(2)_o}\right]$$

$$\frac{\delta j_x(1)}{\delta A^T_y(2)_o} = \frac{i}{2}\left[\frac{\delta j_+(1)}{\delta A^T_+(2)_o} - \frac{\delta j_+(1)}{\delta A^T_-(2)_o} + \frac{\delta j_-(1)}{\delta A^T_+(2)_o} - \frac{\delta j_-(1)}{\delta A^T_-(2)_o}\right]$$

$$\frac{\delta j_x(1)}{\delta A^T_z(2)_o} = \frac{1}{2}\left[\frac{\delta j_+(1)}{\delta A^T_z(2)_o} + \frac{\delta j_-(1)}{\delta A^T_z(2)_o}\right]$$

(VIII-17)

$$\frac{\delta j_y(1)}{\delta U^T(2)_o} = -\frac{i}{2}\left[\frac{\delta j_+(1)}{\delta U^T(2)_o} - \frac{\delta j_-(1)}{\delta U^T(2)_o}\right]$$

$$\frac{\delta j_y(1)}{\delta A_x^T(2)_o} = -\frac{i}{2}\left[\frac{\delta j_+(1)}{\delta A_+^T(2)_o} + \frac{\delta j_+(1)}{\delta A_-^T(2)_o} - \frac{\delta j_-(1)}{\delta A_+^T(2)_o} - \frac{\delta j_-(1)}{\delta A_-^T(2)_o}\right]$$

$$\frac{\delta j_y(1)}{\delta A_y^T(2)_o} = \frac{1}{2}\left[\frac{\delta j_+(1)}{\delta A_+^T(2)_o} - \frac{\delta j_+(1)}{\delta A_-^T(2)_o} - \frac{\delta j_-(1)}{\delta A_+^T(2)_o} + \frac{\delta j_-(1)}{\delta A_-^T(2)_o}\right]$$

$$\frac{\delta j_y(1)}{\delta A_z^T(2)_o} = -\frac{i}{2}\left[\frac{\delta j_+(1)}{\delta A_z^T(2)_o} - \frac{\delta j_-(1)}{\delta A_z^T(2)_o}\right]$$

$$\frac{\delta j_z(1)}{\delta A_x^T(2)_o} = \left[\frac{\delta j_z(1)}{\delta A_+^T(2)_o} + \frac{\delta j_z(1)}{\delta A_-^T(2)_o}\right]$$

$$\frac{\delta j_z(1)}{\delta A_y^T(2)_o} = i\left[\frac{\delta j_z(1)}{\delta A_+^T(2)_o} - \frac{\delta j_z(1)}{\delta A_-^T(2)_o}\right].$$

From (VIII-13) and (VIII-17) we have

$$s_{xx}^T(\vec{k},\omega) = \frac{1}{2}\left[s_{++}^T(\vec{k},\omega) + s_{+-}^T(\vec{k},\omega) + s_{-+}^T(\vec{k},\omega) + s_{--}^T(\vec{k},\omega)\right]$$

$$s_{xy}^T(\vec{k},\omega) = \frac{i}{2}\left[s_{++}^T(\vec{k},\omega) - s_{+-}^T(\vec{k},\omega) + s_{-+}^T(\vec{k},\omega) - s_{--}^T(\vec{k},\omega)\right]$$

$$s^T_{yx}(\vec{k},\omega) = -\frac{i}{2}\left[s^T_{++}(\vec{k},\omega) + s^T_{+-}(\vec{k},\omega) - s^T_{-+}(\vec{k},\omega) - s^T_{--}(\vec{k},\omega) \right]$$

$$s^T_{yy}(\vec{k},\omega) = \frac{1}{2}\left[s^T_{++}(\vec{k},\omega) - s^T_{+-}(\vec{k},\omega) - s^T_{-+}(\vec{k},\omega) + s^T_{--}(\vec{k},\omega) \right]$$

(VIII-18)

$$s^T_{xz}(\vec{k},\omega) = \frac{1}{2}\left[s^T_{+z}(\vec{k},\omega) + s^T_{-z}(\vec{k},\omega) \right]$$

$$s^T_{yz}(\vec{k},\omega) = -\frac{i}{2}\left[s^T_{+z}(\vec{k},\omega) - s^T_{-z}(\vec{k},\omega) \right]$$

$$s^T_{zx}(\vec{k},\omega) = \left[s^T_{z+}(\vec{k},\omega) + s^T_{z-}(\vec{k},\omega) \right]$$

$$s^T_{zy}(\vec{k},\omega) = i\left[s^T_{z+}(\vec{k},\omega) - s^T_{z-}(\vec{k},\omega) \right] .$$

For convenience, we define the Fourier transform of the applied current density $\Delta\vec{j}^A(1)$ by

$$\Delta\vec{j}^A(1) = \int \frac{d^3\vec{k}\,d\omega}{(2\pi)^4}\,\vec{I}(\vec{k},\omega)\,e^{i\vec{k}\cdot\vec{r}_1 - i\omega t_1} . \qquad \text{(VIII-19)}$$

From equations (II-12), (VIII-9), (VIII-19), and the wave equation (II-29), we obtain the Fourier transform of the wave equation

$$\begin{bmatrix} \vec{k}^2 - k_x^2 - \frac{\omega^2}{c^2} - i\frac{4\pi\omega}{c^2} s^T_{xx}(\vec{k},\omega) & -k_x k_y - i\frac{4\pi\omega}{c^2} s^T_{xy}(\vec{k},\omega) & -k_x k_z - i\frac{4\pi\omega}{c^2} s^T_{xz}(\vec{k},\omega) \\ -k_y k_x - i\frac{4\pi\omega}{c^2} s^T_{yx}(\vec{k},\omega) & \vec{k}^2 - k_y^2 - \frac{\omega^2}{c^2} - i\frac{4\pi\omega}{c^2} s^T_{yy}(\vec{k},\omega) & -k_y k_z - i\frac{4\pi\omega}{c^2} s_{yz}(\vec{k},\omega) \\ -k_z k_x - i\frac{4\pi\omega}{c^2} s^T_{zx}(\vec{k},\omega) & -k_z k_y - i\frac{4\pi\omega}{c^2} s^T_{zy}(\vec{k},\omega) & \vec{k}^2 - k_z^2 - \frac{\omega^2}{c^2} - i\frac{4\pi\omega}{c^2} s^T_{zz}(\vec{k},\omega) \end{bmatrix} \begin{bmatrix} e^T_x(\vec{k},\omega) \\ e^T_y(\vec{k},\omega) \\ e^T_z(\vec{k},\omega) \end{bmatrix}$$

(VIII-20)

$$= i\frac{4\pi\omega}{c^2} \begin{bmatrix} I_x(\vec{k},\omega) \\ I_y(\vec{k},\omega) \\ I_z(\vec{k},\omega) \end{bmatrix}$$

It is convenient to use the following definitions:

$$k_+ = k_x + i k_y$$

$$k_- = k_x - i k_y$$

$$I_+ = I_x + i I_y$$

$$I_- = I_x - i I_y \qquad \text{(VIII-21)}$$

$$e^T_+ = e^T_x + i e^T_y$$

$$e^T_- = e^T_x - i e^T_y \; .$$

Then the wave equation (VIII-20) may be written in the form

$$\begin{bmatrix} \frac{k_+k_-}{2}+k_z^2-i\frac{4\pi\omega}{c^2}s^T_{++}(\vec{k},\omega)-\frac{\omega^2}{c^2} & -\frac{k_+^2}{2}-i\frac{4\pi\omega}{c^2}s^T_{+-}(\vec{k},\omega) & -k_+k_z-i\frac{4\pi\omega}{c^2}s^T_{+z}(\vec{k},\omega) \\ -\frac{k_-^2}{2}-i\frac{4\pi\omega}{c^2}s^T_{-+}(\vec{k},\omega) & \frac{k_-k_+}{2}+k_z^2-i\frac{4\pi\omega}{c^2}s^T_{--}(\vec{k},\omega)-\frac{\omega^2}{c^2} & -k_-k_z-i\frac{4\pi\omega}{c^2}s^T_{-z}(\vec{k},\omega) \\ -\frac{k_zk_-}{2}-i\frac{4\pi\omega}{c^2}s^T_{z+}(\vec{k},\omega) & -\frac{k_zk_+}{2}-i\frac{4\pi\omega}{c^2}s^T_{z-}(\vec{k},\omega) & k_+k_--i\frac{4\pi\omega}{c^2}s^T_{zz}(\vec{k},\omega)-\frac{\omega^2}{c^2} \end{bmatrix} \begin{bmatrix} e^T_+ \\ e^T_- \\ e^T_z \end{bmatrix} \tag{VIII-22}$$

$$= i\frac{4\pi\omega}{c^2}\begin{bmatrix} I_+ \\ I_- \\ I_z \end{bmatrix} .$$

From equations (VIII-20, 22) it is apparent that for plane wave propagation not in the direction of the applied magnetic field ($k_x \neq 0$ or $k_y \neq 0$ or both), the "plasma oscillations" of the gas are generally coupled to the "transverse" electromagnetic fields.

Some Calculations of the "Conductance Tensor"

The quantities $s^T_{\ell m}(\vec{k}, \omega)$ (VIII-9) are difficult to calculate in general, but there are special cases of interest which are relatively easy. One of these is for plane wave propagation in the direction of the applied magnetic field (z-axis). In this case, one need evaluate $s^T_{\ell m}(\vec{k}, \omega)$ for non-zero values of k_z only. With the aid of the identities (6-2) one obtains

$$\begin{aligned} s^T_{+-}(k_z, \omega) &= s^T_{-+}(k_z, \omega) \\ = s^T_{+z}(k_z, \omega) &= s^T_{z+}(k_z, \omega) \qquad \text{(VIII-23)} \\ = s^T_{-z}(k_z, \omega) &= s^T_{z-}(k_z, \omega) \equiv 0 , \end{aligned}$$

while

$$-i\omega s^T_{++}(k_z, \omega) = \frac{e^2 n_e}{m}$$

$$+ \frac{e^2 \omega_B^2}{4\pi^2 \hbar} \int_{-\infty}^{\infty} d\lambda \sum_{n=0}^{\infty} (n+1) \left[\frac{\Gamma_1(n, k_z + \lambda) W_1(n+1, \lambda) + \Gamma_2(n, k_z + \lambda) W_2(n+1, \lambda)}{\omega + i\varepsilon + \omega_B - \frac{\hbar}{2m}(k_z^2 + 2k_z \lambda)} \right.$$

$$\left. - \frac{W_1(n, k_z + \lambda) \Gamma_1(n+1, \lambda) + W_2(n, k_z + \lambda) \Gamma_2(n+1, \lambda)}{\omega - i\varepsilon + \omega_B - \frac{\hbar}{2m}(k_z^2 + 2k_z \lambda)} \right]$$

$$+ \frac{\mu^2 e^2 \omega_B}{8\pi^2 m} k_z^2 \int_{-\infty}^{\infty} d\lambda \sum_{n=0}^{\infty} \left[\frac{\Gamma_2(n, k_z + \lambda) W_1(n, \lambda)}{\omega + i\varepsilon + \mu\omega_B - \frac{\hbar}{2m}(k_z^2 + 2k_z \lambda)} \right.$$

$$\left. - \frac{W_2(n, k_z + \lambda) \Gamma_1(n, \lambda)}{\omega - i\varepsilon + \mu\omega_B - \frac{\hbar}{2m}(k_z^2 + 2k_z \lambda)} \right]_{\varepsilon \to 0^+}$$

$$-i\omega s^{T}_{--}(k_z,\omega) = \frac{e^2 n_e}{m} \tag{VIII-24}$$

$$+ \frac{e^2\omega_B^2}{4\pi^2\hbar}\int_{-\infty}^{\infty} d\lambda \sum_{n=0}^{\infty} n \left[\frac{\Gamma_1(n,k_z+\lambda)\,W_1(n-1,\lambda) + \Gamma_2(n,k_z+\lambda)\,W_2(n-1,\lambda)}{\omega + i\varepsilon - \omega_B - \frac{\hbar}{2m}(k_z^2 + 2k_z\lambda)} \right.$$

$$\left. - \frac{W_1(n,k_z+\lambda)\,\Gamma_1(n-1,\lambda) + W_2(n,k_z+\lambda)\,\Gamma_2(n-1,\lambda)}{\omega - i\varepsilon - \omega_B - \frac{\hbar}{2m}(k_z^2 + 2k_z\lambda)} \right]$$

$$+ \frac{\mu^2 e^2\omega_B^2}{8\pi^2 m} k_z^2 \int_{-\infty}^{\infty} d\lambda \sum_{n=0}^{\infty} \left[\frac{\Gamma_1(n,k_z+\lambda)\,W_2(n,\lambda)}{\omega + i\varepsilon - \mu\omega_B - \frac{\hbar}{2m}(k_z^2 + 2k_z\lambda)} \right.$$

$$\left. - \frac{W_1(n,k_z+\lambda)\,\Gamma_2(n,\lambda)}{\omega - i\varepsilon - \mu\omega_B - \frac{\hbar}{2m}(k_z^2 + 2k_z\lambda)} \right]_{\varepsilon \to 0^+}$$

$$-i\omega s^{T}_{zz}(k_z,\omega) = \frac{e^2 n_e}{m}$$

$$+ \frac{e^2\omega_B}{16\pi^2 m}\int_{-\infty}^{\infty} d\lambda\,(k_z + 2\lambda)^2 \sum_{n=0}^{\infty} \left[\frac{\Gamma_1(n,k_z+\lambda)\,W_1(n,\lambda) + \Gamma_2(n,k_z+\lambda)\,W_2(n,\lambda)}{\omega + i\varepsilon - \frac{\hbar}{2m}(k_z^2 + 2k_z\lambda)} \right.$$

$$\left. - \frac{W_1(n,k_z+\lambda)\,\Gamma_1(n,\lambda) + W_2(n,k_z+\lambda)\,\Gamma_2(n,\lambda)}{\omega - i\varepsilon - \frac{\hbar}{2m}(k_z^2 + 2k_z\lambda)} \right]$$

In this case ($k_x = k_y \equiv 0$) the "symmetry" of the "conductance tensor" $s^{T}_{\ell m}(\vec{k},\omega)$ is such that it has rotational invariance for rotations about the z-axis (direction of the applied magnetic field). From equations (VIII-18, 23) one obtains the results

$$s^T_{xz}(k_z, \omega) = s^T_{zx}(k_z, \omega) = s^T_{yz}(k_z, \omega) = s^T_{zy}(k_z, \omega) \equiv 0,$$

$$s^T_{xx}(k_z, \omega) = s^T_{yy}(k_z, \omega) = \frac{1}{2}\left[s^T_{++}(k_z, \omega) + s^T_{--}(k_z, \omega)\right] \quad \text{(VIII-25)}$$

$$s^T_{xy}(k_z, \omega) = -s^T_{yx}(k_z, \omega) = \frac{i}{2}\left[s^T_{++}(k_z, \omega) - s^T_{--}(k_z, \omega)\right].$$

As an aid to the evaluation of equations (VIII-24), the well known identity

$$\lim_{\varepsilon \to 0^+} \frac{1}{x \pm i\varepsilon} = P\left(\frac{1}{x}\right) \mp i\pi\,\delta(x) \qquad \text{(VIII-26)}$$

may be used (see reference (1), equation (3. 31), reference (46), and Mathematical Appendix VII, Part 1).

For a non-degenerate gas ($e^{\beta\zeta} << 1$) one obtains the following expressions with the aid of the expressions (VII-12), and equations (VIII-24):

$$-i\omega s^T_{++}(k_z, \omega) \cong \frac{e^2 n_e}{m}\Bigg\{1$$

$$+ \left(\frac{1}{2\pi m\beta}\right)^{\frac{1}{2}} \frac{\beta\hbar\omega_B}{2} \operatorname{csch} \frac{\beta\hbar\omega_B}{2} \int_{-\infty}^{\infty} d\lambda \left[\frac{e^{-\frac{\beta\hbar\omega_B}{2} - \frac{\beta\hbar^2\lambda^2}{2m}}}{\omega + i\varepsilon + \omega_B - \frac{\hbar}{2m}(k_z^2 + 2k_z\lambda)}\right.$$

$$\left. - \frac{e^{\frac{\beta\hbar\omega_B}{2} - \frac{\beta\hbar^2}{2m}(k_z + \lambda)^2}}{\omega - i\varepsilon + \omega_B - \frac{\hbar}{2m}(k_z^2 + 2k_z\lambda)}\right]$$

$$+\left(\frac{1}{2\pi m\beta}\right)^{\frac{1}{2}} \frac{\mu^2\beta\hbar^2 k_z^2}{4m}\,\mathrm{sech}\,\frac{\mu\beta\hbar\omega_B}{2}\int_{-\infty}^{\infty} d\lambda\left[\frac{e^{-\frac{\mu\beta\hbar\omega_B}{2}-\frac{\beta\hbar^2\lambda^2}{2m}}}{\omega+i\varepsilon+\mu\omega_B-\frac{\hbar}{(2m)}(k_z^2+2k_z\lambda)}\right.$$

$$\left.-\frac{e^{\frac{\mu\beta\hbar\omega_B}{2}-\frac{\beta\hbar^2}{2m}(k_z+\lambda)^2}}{\omega-i\varepsilon+\mu\omega_B-\frac{\hbar}{2m}(k_z^2+2k_z\lambda)}\right]\Bigg\}_{\varepsilon\to 0^+}$$

$$-i\omega\, s^T_{--}(k_z,\omega) \cong \frac{e^2 n_e}{m}\Bigg\{1$$

$$+\left(\frac{1}{2\pi m\beta}\right)^{\frac{1}{2}} \frac{\beta\hbar\omega_B}{2}\,\mathrm{csch}\,\frac{\beta\hbar\omega_B}{2}\int_{-\infty}^{\infty} d\lambda\left[\frac{e^{\frac{\beta\hbar\omega_B}{2}-\frac{\beta\hbar^2\lambda^2}{2m}}}{\omega+i\varepsilon-\omega_B-\frac{\hbar}{2m}(k_z^2+2k_z\lambda)}\right. \qquad \text{(VIII-27)}$$

$$\left.-\frac{e^{-\frac{\beta\hbar\omega_B}{2}-\frac{\beta\hbar^2}{2m}(k_z+\lambda)^2}}{\omega-i\varepsilon-\omega_B-\frac{\hbar}{2m}(k_z^2+2k_z\lambda)}\right]$$

$$+\left(\frac{1}{2\pi m\beta}\right)^{\frac{1}{2}} \frac{\mu^2\beta\hbar^2 k_z^2}{4m}\,\mathrm{sech}\,\frac{\mu\beta\hbar\omega_B}{2}\int_{-\infty}^{\infty} d\lambda\left[\frac{e^{\frac{\mu\beta\hbar\omega_B}{2}-\frac{\beta\hbar^2\lambda^2}{2m}}}{\omega+i\varepsilon-\mu\omega_B-\frac{\hbar}{2m}(k_z^2+2k_z\lambda)}\right.$$

$$\left.-\frac{e^{-\frac{\mu\beta\hbar\omega_B}{2}-\frac{\beta\hbar^2}{2m}(k_z+\lambda)^2}}{\omega-i\varepsilon-\mu\omega_B-\frac{\hbar}{2m}(k_z^2+2k_z\lambda)}\right]\Bigg\}_{\varepsilon\to 0^+}$$

$$-i\omega\, s^T_{zz}(k_z,\omega) \cong \frac{e^2 n_e}{m}\Bigg\{1$$

$$+ \left(\frac{1}{2\pi m\beta}\right)^{\frac{1}{2}} \frac{\beta\hbar^2}{4m} \int_{-\infty}^{\infty} d\lambda (k_z + 2\lambda)^2 \left[\frac{e^{-\frac{\beta\hbar^2\lambda^2}{2m}}}{\omega + i\varepsilon - \frac{\hbar}{2m}(k_z^2 + 2k_z\lambda)} \right.$$

$$\left. - \frac{e^{-\frac{\beta h^2}{2m}(k_z+\lambda)^2}}{\omega - i\varepsilon - \frac{\hbar}{2m}(k_z^2 + 2k_z\lambda)} \right]\Bigg\}_{\varepsilon \to 0^+} .$$

In the case of the fully degenerate gas ($\beta \to \infty$) the calculations are more difficult, since equations (VIII-24) must be evaluated piecemeal for each value of "cutoff" number N_c. For $\vec{k} = 0$, however, the procedure is straightforward and the following results are obtained with the aid of the equations (VII-12, 22, 23) ("cutoff" number N_c unrestricted):

$$- i\omega s^T_{++}(0, \omega) = \frac{e^2 n_e}{m} \left[\frac{\omega}{\omega + \omega_B} - i\pi\omega_B\, \delta(\omega + \omega_B) \right]$$

$$- i\omega s^T_{--}(0, \omega) = \frac{e^2 n_e}{m} \left[\frac{\omega}{\omega - \omega_B} - i\pi\omega_B\, \delta(\omega - \omega_B) \right] \qquad \text{(VIII-28)}$$

$$- i\omega s^T_{zz}(0, \omega) = \frac{e^2 n_e}{m} .$$

Another special case of interest in the evaluation of the quantities $s^T_{\ell m}(\vec{k}, \omega)$ is for "weak spatial dispersion." By series expansion $s^T_{\ell m}(\vec{k}, \omega)$ may be evaluated up to a given order in $\vec{k}$. If one is concerned only with plane waves of small propagation constant ($|\vec{k}| \cong 0$), or if one is concerned with superpositions (groups) of such plane waves, then the series expansion will be a good approximation.

The identities (5-50), (6-2), and (6-19) from the Mathematical Appendices can be used for calculations of $s^{T}_{\ell m}(\vec{k}, \omega)$ up to the second order in $\vec{k}$.

Complex Frequencies

For fields which vary harmonically in time with exponential damping (complex frequencies), one has

$$\delta \vec{E}^{T}(\vec{r}, t) = \delta \vec{\mathcal{E}}^{T}(\vec{r})\, e^{-i\Omega t} \quad \text{for } t > 0$$

where (VIII-29)

$$\Omega = \omega' - i\omega'', \quad \omega'' > 0 \quad (\omega' \text{ and } \omega'' \text{ both real}).$$

We are dealing with the linearized expressions for ρ, $\vec{j}$, and the wave equation here, therefore the use of complex quantities for the electric field is quite proper. The real parts of $\delta \vec{E}^{T}$, ρ, $\vec{j}$, and the wave equation so calculated are the physically observable parts. Since the current density generally depends upon the entire temporal history of the fields, it is convenient to make (or postulate) the physically reasonable ansatz

$$\delta \vec{E}^{T}(\vec{r}, t) = \delta \vec{\mathcal{E}}^{T}(\vec{r})\, e^{-i\Omega_1^* t} \quad \text{for } t < 0$$

where (VIII-30)

$$\Omega_1^* = \omega_1' + i\omega_1'', \quad \omega_1'' > 0 \quad (\omega_1' \text{ and } \omega_1'' \text{ both real}).$$

This choice is made so the current density will have the same form of time dependence as the electric field. There may be other choices giving the same result, but nevertheless this choice is one suitable for examination. The complex frequency Ω_1^* ("growing fields" for $t < 0$), the fields, and the applied current density $\Delta\vec{j}^{A}(\vec{r}, t)$ are chosen so as to mutually satisfy the wave equation (II-29) for times $t < 0$, and equations (VIII-29, 30) guarantee the continuity of $\delta\vec{E}^{T}$ at $t = 0$. The frequency Ω is so chosen as to satisfy the wave equation (II-29) for times $t > 0$ with $\Delta\vec{j}^{A} = 0$.

The Fourier transform of the function $\delta\vec{\mathcal{E}}(\vec{r})$ in equations (VIII-29, 30) may be defined by

$$\delta\vec{\mathcal{E}}^{T}(\vec{r}) \equiv \int \frac{d^3k}{(2\pi)^3} \vec{\mathcal{E}}^{T}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}} \tag{VIII-31}$$

Then the Fourier transform $\vec{e}^{T}(\vec{k}, \omega)$ of $\delta\vec{E}^{T}(\vec{r}, t)$ is given from equations (II-12), and (VIII-29, 30) by

$$\vec{e}^{T}(\vec{k}, \omega) = \vec{\mathcal{E}}^{T}(\vec{k}) \left\{\frac{1}{i(\omega - \Omega_1^*)} - \frac{1}{i(\omega - \Omega)}\right\} . \tag{VIII-32}$$

Thus, one obtains in this case the result

$$-i\omega j_{\ell}(\vec{k}, \omega) = -i\omega\, s^{T}_{\ell m}(\vec{k}, \omega)\, \mathcal{E}^{T}_{m}(\vec{k}) \left[\frac{1}{i(\omega - \Omega_1^*)} - \frac{1}{i(\omega - \Omega)}\right], \tag{VIII-33}$$

or

$$\frac{\partial}{\partial t_1} \delta j_{\ell}(1) = \iint d^3\vec{r}_2 \frac{d^3\vec{k}\, d\omega}{(2\pi)^4} \left\{-i\omega \right.$$

$$\times \left[\frac{1}{i(\omega - \Omega_1^*)} - \frac{1}{i(\omega - \Omega)}\right] s^{T}_{\ell m}(\vec{k}, \omega)\Big\}\, \delta\mathcal{E}^{T}_{m}(\vec{r}_2)\, e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2) - i\omega t_1} \tag{VIII-34}$$

One obtains the same result directly from the current density equation (VIII-7) and from equations (VIII-29, 30).

The integrations over the frequency ω in equation (VIII-34) may be performed by use of the Cauchy Integral Theorem (see Mathematical Appendix VII, Part 2). The results are given by

$$\frac{\partial}{\partial t_1}\delta j_\ell(1) = \begin{cases} \iint d^3\vec{r}_2 \frac{d^3\vec{k}}{(2\pi)^3}(-i\Omega_1^*) s^T_{\ell m}(k, \Omega_1^*)_- \delta \mathcal{E}^T_m(\vec{r}_2) e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)-i\Omega_1^* t_1}, & t_1 < 0 \\ \iint d^3\vec{r}_2 \frac{d^3\vec{k}}{(2\pi)^3}(-i\Omega)\, s^T_{\ell m}(\vec{k},\Omega)_+ \delta \mathcal{E}^T_m(\vec{r}_2) e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)-i\Omega t_1}, & t_1 > 0, \end{cases} \tag{VIII-35}$$

where

$$\begin{aligned} -i\Omega_1^* s^T_{\ell m}(\vec{k}, \Omega_1^*)_- &\equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\{\frac{-i\omega\, s^T_{\ell m}(\vec{k}, \omega)}{i(\omega - \Omega_1^*)}\right\} e^{-i(\omega-\Omega_1^*)t_1}, \quad t_1 < 0 \\ -i\Omega\, s^T_{\ell m}(\vec{k}, \Omega)_+ &\equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\{\frac{-i\omega\, s^T_{\ell m}(\vec{k}, \omega)}{-i(\omega - \Omega)}\right\} e^{-i(\omega-\Omega)t_1}, \quad t_1 > 0 . \end{aligned} \tag{VIII-36}$$

For plane wave fields with frequency $\Omega (t > 0)$ one has

$$\delta \vec{E}^T(\vec{r}, t) = \vec{\mathcal{E}}^T e^{i\vec{k}\cdot\vec{r} - i\Omega t}, \tag{VIII-37}$$

$$\delta j_\ell(\vec{r}, t) = s^T_{\ell m}(\vec{k}, \Omega)_+ \delta E^T_m(\vec{r}, t), \tag{VIII-38}$$

and the wave equation is given by the expression

$$\left[\delta_{\ell m}\vec{k}^2 - k_\ell k_m - i\frac{4\pi\Omega}{c^2} s^T_{\ell m}(\vec{k},\Omega)_+ - \delta_{\ell m}\frac{\Omega^2}{c^2}\right]\mathcal{E}^T_m = 0. \qquad \text{(VIII-39)}$$

The relations formed from the determinant of the coefficients of $\mathcal{E}^T_m$ in (VIII-39),

$$\det\left|\delta_{\ell m}\vec{k}^2 - k_\ell k_m - i\frac{4\pi\Omega}{c^2} s^T_{\ell m}(\vec{k},\Omega)_+ - \delta_{\ell m}\frac{\Omega^2}{c^2}\right| = 0\,, \qquad \text{(VIII-40)}$$

give rise to the dispersion relations for propagation.

CHAPTER IX

"DIELECTRIC SCREENING" BY THE ELECTRON GAS

The effective ("screened") potential of a static test charge placed in an electron gas has been considered elsewhere using the Green's function technique for an isotropic unmagnetized gas (see reference (3)) and for the non-degenerate magnetized gas (see reference (16)). The essential features of the technique are briefly discussed here.

From the equations (I-9) and (I-12) the total potential is given by

$$U^T(1) = U^A(1) + \int d^3\vec{r}_2 \, \frac{\rho\left(\vec{r}_2, t_1 - \frac{r_{12}}{c}\right)}{r_{12}}. \qquad \text{(IX-1)}$$

If time retardation is neglected in equation (IX-1), the first variation of $U^T(1)$ with respect to $U^A(3)$ is given by

$$\frac{\delta U^T(1)}{\delta U^A(3)_o} = \delta^4(1, 3) + \int d(2) v(1, 2) \frac{\delta \rho(2)}{\delta U^A(3)_o}$$

where (IX-2)

$$v(1, 2) \equiv \frac{1}{r_{12}} \delta(t_1 - t_2) .$$

By the "chain rule" for differentiation

$$\frac{\delta \rho(2)}{\delta U^{A}(3)_{o}} = \int d(4) \frac{\delta \rho(2)}{\delta U^{T}(4)_{o}} \frac{\delta U^{T}(4)}{\delta U^{A}(3)_{o}} , \qquad \text{(IX-3)}$$

where the terms arising from variations with respect to $\vec{A}^{T}$ are omitted because they are presumably negligible in the "static" limit. Thus, to this order of approximation

$$\frac{\delta U^{T}(1)}{\delta U^{A}(3)_{o}} \equiv K(1, 3)$$

$$= \delta^{4}(1, 3) + \iint d(2) d(4) v(1, 2) \frac{\delta \rho(2)}{\delta U^{T}(4)_{o}} K(4, 3) . \qquad \text{(IX-4)}$$

From the equations (V-50)

$$\frac{\delta \rho(1)}{\delta U^{T}(2)_{o}} = -\frac{ie^{2}}{\hbar} \mathrm{Tr}\, G_{1}(1, 2)_{o} G_{1}(2, 1)_{o},$$

and from the form (VI-54) for $G_{1}(1, 2)_{o} \cong G_{o}(1, 2)$ (IX-5)

$$\frac{\delta \rho(1)}{\delta U^{T}(2)_{o}} = -\frac{ie^{2}}{\hbar} e^{-\varepsilon |t_{1}-t_{2}|} \overline{G}_{o}(1, 2) \overline{G}_{o}(2, 1), \varepsilon \rightarrow 0^{+} .$$

Now,

$$\frac{\delta \rho(1)}{\delta U^{T}(2)_{o}}$$

depends upon the differences $(\vec{r}_{1} - \vec{r}_{2}; t_{1} - t_{2})$ of all co-ordinates, and its Fourier transform is given by

$$\overline{\overline{f}}(\vec{k}, \omega) = \int d(2) \frac{\delta \rho(1)}{\delta U^T(2)_o} e^{-i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)+i\omega(t_1-t_2)} \qquad \text{(IX-6)}$$

when we have used (II-13) and (VIII-11). The function v(1, 2) can also be expressed by Fourier expansion as

$$v(1, 2) = \int \frac{d^3\vec{k}\, d\omega}{(2\pi)^4} \frac{4\pi}{\vec{k}^2} e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)-i\omega(t_1-t_2)}. \qquad \text{(IX-7)}$$

For K(1, 3), we write

$$K(1, 3) = \int \frac{d^3\vec{k}\, d\omega}{(2\pi)^4} \mathcal{K}(\vec{k}, \omega)\, e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_3)-i\omega(t_1-t_3)} \qquad \text{(IX-8)}$$

Now, we may solve the integral equation (IX-4) by Fourier expansion to obtain the result

$$\mathcal{K}(\vec{k}, \omega) = 1 + \frac{4\pi}{\vec{k}^2} \overline{\overline{f}}(\vec{k}, \omega) \mathcal{K}(\vec{k}, \omega),$$

which when solved for $K(\vec{k}, \omega)$ yields (IX-9)

$$\mathcal{K}(\vec{k}, \omega) = \frac{\vec{k}^2}{\vec{k}^2 - 4\pi \overline{\overline{f}}(\vec{k}, \omega)} .$$

Thus

$$K(1,2) = \int \frac{d^3\vec{k}\,d\omega}{(2\pi)^4} \left(\frac{\vec{k}^2}{\vec{k}^2 - 4\pi \overline{\overline{f}}(\vec{k},\omega)} \right) e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)-i\omega(t_1-t_2)} . \quad \text{(IX-10)}$$

Then, for the first variation of the effective potential, one has

$$\delta U^T(1) = \int d(2) \frac{\delta U^T(1)}{\delta U^A(2)}_o \delta U^A(2)$$

$$= \int d(2)\, K(1,2)\, \delta U^A(2) \quad \text{(IX-11)}$$

if one ignores the contribution from $\delta \vec{A}^A$ (valid in the "static" limit). We take for the potential $\delta U^A(2)$ the Coulomb potential

$$\delta U^A(2) = \begin{cases} 0, & t_2 < t_0 \\ \dfrac{q}{r_{20}}, & t_2 > t_0 \end{cases} \quad \text{(IX-12)}$$

where $r_{20} = |\vec{r}_2 - \vec{r}_0|$. The potential $\delta U^A(2)$ may be expressed in "closed" form with the aid of the integral representation for the step function (see reference (1), equation (3. 30)):

$$\eta_+(1,2) = i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_1-t_2)}}{(\omega + i\varepsilon)} = \begin{cases} 1, & t_1 > t_2 \\ 0, & t_1 < t_2 \end{cases}$$

$$\eta_-(1,2) = i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_1-t_2)}}{(\omega - i\varepsilon)} = \begin{cases} 0, & t_1 > t_2 \\ 1, & t_1 < t_2 \end{cases} , \quad \text{(IX-13)}$$

where ε is an "infinitesimal" ($\varepsilon > 0$). Thus

$$\delta U^A(2) = i\frac{q}{r_{20}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t_2-t_0)}}{(\omega + i\varepsilon)}$$

$$= i\,4\pi q \int_{-\infty}^{\infty} \frac{d^3\vec{k}\,d\omega}{(2\pi)^4} \frac{e^{i\vec{k}\cdot(\vec{r}_2-\vec{r}_0)-i\omega(t_2-t_0)}}{\vec{k}^2(\omega+i\varepsilon)}. \tag{IX-14}$$

By substituting the expression (IX-14) into equation (IX-11), one obtains

$$\delta U^T(1) = i\,4\pi q \int \frac{d^3\vec{k}\,d\omega}{(2\pi)^4} \frac{\mathcal{K}(\vec{k},\omega)\, e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_0)-i\omega(t_1-t_0)}}{\vec{k}^2(\omega+i\varepsilon)}$$

$$= i\,4\pi q \int \frac{d^3\vec{k}\,d\omega}{(2\pi)^4} \frac{e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_0)-i\omega(t_1-t_0)}}{(\omega+i\varepsilon)[\vec{k}^2 - 4\pi \overline{\overline{f}}(\vec{k},\omega)]}. \tag{IX-15}$$

Since we are concerned only with the static potential, we consider $\delta U^T(1)$ in the limit $(t_1 - t_2) \to \infty$ and make the mild assumption that all singularities in the lower half ω-plane arising from the expression

$$\mathcal{K}(\vec{k},\omega)\, e^{-i\omega(t_1-t_0)}$$

vanish ("damp out") in this limit. One can equate the integral

$$\lim_{(t_1-t_0)\to\infty} \left(-\frac{1}{2\pi i}\right) \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega(t_1-t_0)}}{[\vec{k}^2 - 4\pi \overline{\overline{f}}(\vec{k},\omega)](\omega+i\varepsilon)}, \quad \varepsilon \to 0^+$$

to a contour integral with the contour closed by a semicircle of "infinite" radius in the lower half ω-plane (see Mathematical Appendix VII). Thus

$$\lim_{(t_1-t_0)\to\infty} \delta U^T(1) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{4\pi q\, e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_0)}}{[\vec{k}^2 - 4\pi \overline{\overline{f}}(\vec{k}, 0)]} . \qquad \text{(IX-16)}$$

The solution of equation (IX-16) from (IX-6) for the magnetized electron gas was given by Bonch-Bruevich and Mironov (see reference (16)) for the non-degenerate gas. What they obtained is essentially a modification of the Debye-Hückel "screened" potential for a "Coulombic" charge, with the equipotential surfaces "warped" about an axis of symmetry coinciding with the direction of the applied magnetic field.

Equation (IX-16) may also be obtained from the "source" equation (II-30) in a more rigorous fashion. Equation (II-30) is related to the theory of plasma oscillations. Consider first the relations

$$\frac{\delta j_\ell(1)}{\delta A^T_m(2)_o} .$$

From equations (VIII-1) through (VIII-6) one can see by inspection (the correlations all depend upon the differences of the co-ordinates) that these relations are symmetrical with respect to the interchange of co-ordinates and Cartesian indices:

$$\frac{\delta j_\ell(1)}{\delta A^T_m(2)_o} = \frac{\delta j_m(2)}{\delta A^T_\ell(1)_o} . \qquad \text{(IX-17)}$$

The symmetry condition (IX-17) is reflected in the Fourier transforms by

$$\bar{\bar{q}}_{\ell m}(\vec{k}, \omega) = \bar{\bar{q}}_{m\ell}(-\vec{k}, -\omega). \qquad \text{(IX-18)}$$

Now, one must remember that $\bar{\bar{q}}_{\ell m}$ is dependent upon the applied magnetic field. If one designates

$$\vec{\omega}_B \equiv \vec{i}_z \, \omega_B \, , \qquad \text{(IX-19)}$$

then the tensor $\bar{\bar{q}}_{\ell m}$ may be expressed most generally in the following form (for Cartesian co-ordinates only):

$$\bar{\bar{q}}_{\ell m}(\vec{k}, \omega) = \delta_{\ell m} \, q_1(\vec{k}^2, \omega^2) + \left[k_\ell k_m - \vec{k}^2 \delta_{\ell m} \right] q_2(\vec{k}^2, \omega^2)$$

$$+ \, i \omega q_3(\vec{k}^2, \omega^2) \sum_{n=1}^{3} \varepsilon_{\ell m n} (\vec{\omega}_B)_n . \qquad \text{(IX-20)}$$

The quantity $\varepsilon_{\ell mn}$ is given by

$$\varepsilon_{\ell mn} = \begin{cases} 1 & \text{if } \ell, \ m, \text{ and } n \text{ are a "cyclic" permutation of positive integers} \\ -1 & \text{if } \ell, \ m, \ n \text{ are an "anticyclic" permutation of positive integers} \\ 0 & \text{otherwise .} \end{cases} \qquad \text{(IX-21)}$$

Equation (IX-20) satisfies both the condition (IX-18) and the condition for rotational invariance in form for rotations about the direction of the magnetic field. The combinations $\delta_{\ell m}$, $k_\ell k_m$, and

$$\sum_{n=1}^{3} \varepsilon_{\ell mn} (\vec{\omega}_B)_n$$

exhaust the possibilities satisfying all the required conditions for the tensor $\bar{q}_{\ell m}$.

The Fourier transform of the expression

$$\frac{\delta \rho(1)}{\delta A_m^T(2)_o}$$

is given from (II-13) and (VIII-11) by

$$\frac{\delta \rho(1)}{\delta A_m^T(2)_o} = \int \frac{d^3\vec{k}\, d\omega}{(2\pi)^4} \bar{\bar{g}}_m(\vec{k}, \omega)\, e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2) - i\omega(t_1 - t_2)} . \qquad \text{(IX-22)}$$

The Fourier transform of the "source" equation (II-30) is given from (II-25) and from the definition

$$\Delta\rho^A(1) = \int \frac{d^3\vec{k}\, d\omega}{(2\pi)^4} \bar{\rho}^A(\vec{k}, \omega)\, e^{i\vec{k}\cdot\vec{r}_1 - i\omega t_1} \qquad \text{(IX-23)}$$

as

$$\omega k_m e_m^T(\vec{k}, \omega) = -4\pi c\, \bar{\bar{g}}_m(\vec{k}, \omega)\, e_m^T(\vec{k}, \omega) - i\, 4\pi\omega\, \bar{\rho}^A(\vec{k}, \omega). \qquad \text{(IX-24)}$$

With the aid of the relations (VIII-12), formula (IX-20), and the definitions

$$e_{\parallel}^{T}(\vec{k},\omega) \equiv \frac{\vec{k}\cdot\vec{e}^{\,T}(\vec{k},\omega)}{|\vec{k}|} = \frac{k_m\, e_m^{T}(\vec{k},\omega)}{|\vec{k}|}$$

$$\vec{e}_{\perp}^{\,T}(\vec{k},\omega) \equiv \left[\vec{e}^{\,T}(\vec{k},\omega) - \frac{\vec{k}\, e_{\parallel}(\vec{k},\omega)}{|\vec{k}|}\right], \qquad \text{(IX-25)}$$

expressions (IX-24) may be reduced to the form

$$\omega|\vec{k}|\, e_{\parallel}^{T}(\vec{k},\omega) = -4\pi c\, \overline{\overline{g}}_m(\vec{k},\omega)\left[e_{\perp m}^{T}(\vec{k},\omega) + \frac{k_m\, e_{\parallel}^{T}(\vec{k},\omega)}{|\vec{k}|}\right] - i\,4\pi\omega\,\overline{\rho}^{A}(\vec{k},\omega)$$

$$\text{(IX-26)}$$

$$= -4\pi c\,\overline{\overline{g}}_m(\vec{k},\omega)\, e_{\perp m}^{T} + \frac{4\pi\omega\,\overline{f}(\vec{k},\omega)}{|\vec{k}|}\, e_{\parallel}^{T} - i\,4\pi\omega\,\overline{\rho}^{A}(\vec{k},\omega)$$

$$i|\vec{k}|\, e_{\parallel}^{T}(\vec{k},\omega) = i\vec{k}\cdot\vec{e}^{\,T}(\vec{k},\omega) = \frac{4\pi\vec{k}^{2}\left[\overline{\rho}^{A}(\vec{k},\omega) - i\frac{c}{\omega}\overline{\overline{g}}_m(\vec{k},\omega)\, e_{\perp m}^{T}(\vec{k},\omega)\right]}{\left[\vec{k}^{2} - 4\pi\overline{\overline{f}}(\vec{k},\omega)\right]}$$

$$= \frac{4\pi\vec{k}^{2}\left[\overline{\rho}^{A}(\vec{k},\omega) - i\frac{c}{\omega^{2}}\, k_{\ell}\,\overline{\overline{q}}_{\ell m}(\vec{k},\omega)\, e_{\perp m}^{T}(\vec{k},\omega)\right]}{\left[\vec{k}^{2} - 4\pi\,\overline{f}(\vec{k},\omega)\right]}$$

$$\text{(IX-27)}$$

$$= \frac{4\pi\vec{k}^{2}\left[\overline{\rho}^{A}(\vec{k},\omega) + \frac{c}{\omega}\, q_3(\vec{k}^{2},\omega^{2})\,\vec{\omega}_B\cdot(\vec{k}\times\vec{e}_{\perp}^{\,T}(\vec{k},\omega))\right]}{\left[\vec{k}^{2} - 4\pi\overline{f}(\vec{k},\omega)\right]}$$

$$= \frac{4\pi\vec{k}^{2}\left[\overline{\rho}^{A}(\vec{k},\omega) + q_3(\vec{k}^{2},\omega^{2})\,\vec{\omega}_B\cdot\vec{b}^{\,T}(\vec{k},\omega)\right]}{\left[\vec{k}^{2} - 4\pi\overline{f}(\vec{k},\omega)\right]}$$

In terms of the potentials one obtains from equations (IX-24, 27) and from the Lorentz condition (I-3) the results:

$$i\vec{k}\cdot\vec{e}^{\,T}(\vec{k},\omega) = i\vec{k}\cdot\left[-i\vec{k}\,u^T(\vec{k},\omega) + \frac{i\omega}{c}\vec{a}^{\,T}(\vec{k},\omega)\right]$$

$$= \left(\vec{k}^2 - \frac{\omega^2}{c^2}\right) u^T(\vec{k},\omega)$$

$$= \frac{4\pi\vec{k}^2\left[\bar{\rho}^A(\vec{k},\omega) + i\,q_3(\vec{k}^2,\omega^2)\,\vec{\omega}_B\cdot(\vec{k}\times\vec{a}^{\,T}(\vec{k},\omega))\right]}{\left[\vec{k}^2 - 4\pi\overline{\overline{f}}(\vec{k},\omega)\right]}$$

and

$$i\vec{k}\cdot\vec{e}^{\,T}(\vec{k},\omega) = \left(\vec{k}^2 - \frac{\omega^2}{c^2}\right) u^T(\vec{k},\omega) \qquad \text{(IX-28)}$$

$$= -i\frac{4\pi c}{\omega}\overline{\overline{g}}_m(\vec{k},\omega)\left[-ik_m\,u^T(\vec{k},\omega) + \frac{i\omega}{c}a^T_m(\vec{k},\omega)\right] + 4\pi\bar{\rho}^A(\vec{k},\omega)$$

$$= 4\pi\overline{\overline{f}}(\vec{k},\omega)\,u^T(\vec{k},\omega) + \frac{4\pi}{\omega}k_\ell\overline{\overline{q}}_{\ell m}(\vec{k},\omega)\,a^T_m(\vec{k},\omega) + 4\pi\bar{\rho}^A(\vec{k},\omega)$$

or

$$\left(\vec{k}^2 - \frac{\omega^2}{c^2} - 4\pi\overline{\overline{f}}(\vec{k},\omega)\right)u^T(\vec{k},\omega) = 4\pi\left[\bar{\rho}^A(\vec{k},\omega) + \frac{k_\ell}{\omega}\overline{\overline{q}}_{\ell m}(\vec{k},\omega)\,a^T_m(\vec{k},\omega)\right]$$

$$u^T(\vec{k},\omega) = \frac{4\pi\left[\bar{\rho}^A(\vec{k},\omega) + \frac{k_\ell}{\omega}\overline{\overline{q}}_{\ell m}(\vec{k},\omega)\,a^T_m(\vec{k},\omega)\right]}{\left[\vec{k}^2 - \frac{\omega^2}{c^2} - 4\pi\overline{\overline{f}}(\vec{k},\omega)\right]}.$$

The charge density of the test charge is

$$\Delta \rho^{A}(1) = \begin{cases} 0, & t_1 < 0 \\ q, & t_1 > 0 \end{cases} \tag{IX-29}$$

where q is the charge. The Fourier transform of $\Delta \rho^{A}(1)$ is given by

$$\bar{\rho}^{A}(\vec{k}, \omega) = \frac{-q}{i(\omega + i\varepsilon)}, \quad \varepsilon \to 0^{+} \tag{IX-30}$$

By combining (IX-30) with (IX-27) or (IX-28), one obtains more accurate expressions for equations (IX-15, 16). Equations (IX-27, 28) are the field relations for "plasma oscillations," and it is evident from these that transverse oscillations may be coupled to the longitudinal oscillations.

CHAPTER X

SPIN MAGNETIZATION OF THE ELECTRON GAS

In Chapter III an operator $\vec{\hat{M}}_s$ is identified as the "spin magnetization" operator for the electron gas. The "magnetic polarization" or "magnetization" given by the expectation value $\vec{M}_s$ of $\vec{\hat{M}}_s$ is expressed by equation (V-22) in the form

$$\vec{M}_s(1) = \frac{i\mu\hbar e}{2mc} \operatorname{Tr} \vec{\sigma}\, G_1(1, 1_+) . \tag{X-1}$$

In the present chapter we derive the equation of motion for $\vec{M}_s$, that is, we obtain an expression for

$$\frac{\partial \vec{M}_s}{\partial t} .$$

Following this discussion, we calculate the value of $\vec{M}_s(1)$ for the initial equilibrium state of the system.

From (X-1), one obtains the equation

$$\begin{aligned}\frac{\partial \vec{M}_s(1)}{\partial t_1} &= \frac{\mu e}{2mc} \vec{\sigma}_{\alpha\beta} \left\{ i\hbar \frac{\partial}{\partial t_1} G_{1\beta\alpha}(1, 2) + i\hbar \frac{\partial}{\partial t_2} G_{1\beta\alpha}(1, 2) \right\}_{2 \to 1_+} \\ &= \frac{\mu e}{2mc} \operatorname{Tr} \vec{\sigma} \left\{ i\hbar \frac{\partial}{\partial t_1} G_1(1, 2) + i\hbar \frac{\partial}{\partial t_2} G_1(1, 2) \right\}_{2 \to 1_+}\end{aligned} \tag{X-2}$$

By substitution of equations (V-31, 33) into (X-2), we obtain the expression

$$\frac{\partial \vec{M}_s(1)}{\partial t_1} = \frac{\mu e}{2mc} \mathrm{Tr}\, \vec{\sigma} \left\{ \frac{1}{2m} \left[\vec{\pi}^T(1)^2 - \vec{\pi}^{T*}(2)^2 \right] G_1(1,2) \right.$$

$$\left. + \frac{\mu \hbar e}{2mc} \vec{B}^T(1) \cdot \left[\vec{\sigma}\, G_1(1,2) - G_1(1,2)\, \vec{\sigma} \right] \right\}_{2 \to 1_+} \tag{X-3}$$

$$+ \frac{i \mu e^3}{2mc} \mathrm{Tr}\, \vec{\sigma} \int \frac{d^3 \vec{r}_3}{r_{13}} \left[G_1(\vec{r}_1, t_1; \vec{r}_3, t_1+) + G_1(\vec{r}_1, t_1+; \vec{r}_3, t_1) \right] G_1(\vec{r}_3, t_1; \vec{r}_1, t_1+) .$$

From the identities (j, k, ℓ represent Cartesian co-ordinates)

$$\sigma_j \sigma_k = -\sigma_k \sigma_j = i \sigma_\ell \quad (j, k, \ell \text{ in cyclic order})$$

and (X-4)

$$\sigma_j^2 = I$$

for the Pauli spin matrices, we obtain the relations

$$\vec{\sigma}_{\alpha\beta} \left[\vec{B}^T(1) \cdot \vec{\sigma}_{\beta\lambda} \right] = \delta_{\alpha\lambda} \vec{B}^T(1) - i \left[\vec{\sigma}_{\alpha\lambda} \times \vec{B}^T(1) \right]$$

and (X-5)

$$\vec{\sigma}_{\alpha\beta} \left[\vec{B}^T(1) \cdot \vec{\sigma}_{\lambda\alpha} \right] = \delta_{\lambda\beta} \vec{B}^T(1) + i \left[\vec{\sigma}_{\lambda\beta} \times \vec{B}^T(1) \right] .$$

Thus, because of relations (X-5), equation (X-3) may be rewritten in the following form:

$$\frac{\partial \vec{M}_s(1)}{\partial t_1}$$

$$= -\frac{\mu e}{2mc}\left\{\frac{i\mu\hbar e}{2mc}\,\mathrm{Tr}\,\vec{\sigma}\,G_1(1,1_+)\right\} \times \vec{B}^T(1)$$

$$+ \frac{\mu e}{2mc}\,\mathrm{Tr}\,\frac{\vec{\sigma}}{2m}\left\{\left[\vec{\pi}^T(1)^2 - \vec{\pi}^{T*}(2)^2\right] G_1(1,2)\right\}_{2\to 1_+}$$

$$+ \frac{i\mu e^3}{2mc}\,\mathrm{Tr}\,\vec{\sigma}\int\frac{d(3)}{r_{13}}\left[\delta(t_3-t_1+) + \delta(t_3-t_1-)\right] G_1(1,3)\,G_1(3,1_{++}) \tag{X-6}$$

$$= -\frac{\mu e}{2mc}\,\vec{M}_s(1)\times\vec{B}^T(1) + \frac{\mu e}{2mc}\,\mathrm{Tr}\,\frac{\vec{\sigma}}{2m}\left\{\left[\vec{\pi}^T(1) + \vec{\pi}^{T*}(2)\right]\cdot\left[\vec{\pi}^T(1) - \vec{\pi}^{T*}(2)\right] G_1(1,2)\right\}_{2\to 1_+}$$

$$+ \frac{i\mu e^3}{2mc}\,\mathrm{Tr}\,\vec{\sigma}\int\frac{d(3)}{r_{13}}\left[\delta(t_3-t_1+) + \delta(t_3-t_1-)\right] G_1(1,3)\,G_1(3,1_{++}) \; .$$

Equation (X-6) takes a familiar form in the first term on the right hand side of the equation. The second term on the right arises from the motions of the electrons; this expression is not so readily reduced as was the first term on the right, although its significance can be shown quite easily. Consider the definition

$$\rho_e(1)\,\vec{V}_e \equiv \vec{j}_e(1)_v$$

where

$$\rho_e(1) = -e\,\langle \psi_a^\dagger(1)\,\psi_a(1)\rangle = +\,i\,e\,\mathrm{Tr}\,G_1(1,1_+) \tag{X-7}$$

and

$$\vec{j}_e(1)_v \equiv \frac{ie}{2m} \mathrm{Tr} \left\{ \left[\vec{\pi}^T(1) + \vec{\pi}^{T*}(2) \right] G_1(1,2) \right\}_{2 \to 1_+}$$

The quantity ρ_e is the electron charge density while $\vec{j}_e(1)_v$ is that part of the electron current density due to the motion of the electrons alone. Thus, the "average" electron velocity $\vec{V}_e$ is given by

$$\vec{V}_e(1) = \frac{\vec{j}_e(1)_v}{\rho_e(1)} = \frac{1}{2m} \frac{\mathrm{Tr} \left\{ \left[\vec{\pi}^T(1) + \vec{\pi}^{T*}(2) \right] G_1(1,2) \right\}_{2 \to 1_+}}{\mathrm{Tr}\, G_1(1, 1_+)} . \tag{X-8}$$

From (X-1), one obtains the result

$$\begin{aligned} \nabla_1 \vec{M}_s(1) &= -\frac{\mu e}{2mc} \mathrm{Tr} \left\{ \left[\frac{h}{i} \nabla_1 + \frac{h}{i} \nabla_2 \right] G_1(1,2) \vec{\sigma} \right\}_{2 \to 1_+} \\ &= -\frac{\mu e}{2mc} \mathrm{Tr} \left\{ \left[\vec{\pi}^T(1) - \vec{\pi}^{T*}(2) \right] G_1(1,2) \vec{\sigma} \right\}_{2 \to 1_+} \end{aligned} \tag{X-9}$$

For convenience, we make the definition

$$\vec{\sum}_s(1) \equiv \frac{\mu e}{2mc} \mathrm{Tr} \frac{\vec{\sigma}}{2m} \left\{ \left[\vec{\pi}^T(1) + \vec{\pi}^{T*}(2) \right] \cdot \left[\vec{\pi}^T(1) - \vec{\pi}^{T*}(2) \right] G_1(1,2) \right\}_{2 \to 1_+} \tag{X-10}$$

If we compare $\vec{\Sigma}_s(1)$ to the combination $-[\vec{V}_e(1) \cdot \nabla_1] \vec{M}_s(1)$ using the equations (X-8, 9), we find that the two terms are similar in appearance to each other except for the fact that $\vec{\Sigma}_s(1)$ is a trace over products, while $-[\vec{V}_e(1) \cdot \nabla_1] \vec{M}_s(1)$ is a product of traces.

The value of $\vec{M}_s(1)_o$, in the "self-consistent field approximation" is given from (X-1) by

$$\vec{M}_s(1)_o = \frac{i\mu\hbar e}{2mc} \mathrm{Tr}\, \vec{\sigma}\, G_o(1, 1_+). \tag{X-11}$$

Since G_o is diagonal (see (VI-54)), the terms $\mathrm{Tr}\, \sigma_x G_o(1, 1_+)$ and $\mathrm{Tr}\, \sigma_y G_o(1, 1_+)$ both vanish, and hence

$$\vec{M}_s(1)_o = \vec{i}_z M_{so},$$

where (X-12)

$$M_{so} = \frac{i\mu\hbar e}{2mc} \mathrm{Tr}\, \sigma_z G_o(1, 1_+) .$$

From (VI-54), $G_{o\alpha\beta}(1, 1_+) = G_{<\alpha,\beta}(1, 1)$ and therefore

$$M_{so} = \frac{i\mu\hbar e}{2mc}\left[G_{<11}(1, 1) - G_{<22}(1, 1)\right]$$

$$= \frac{\mu\hbar e}{2mc}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left[W_2(n, k) - W_1(n, k)\right] v_{n,\ell,k}(\vec{r}_1) v^*_{n,\ell,k}(\vec{r}_1)$$

(X-13)

$$= \frac{\mu e\omega_B}{8\pi^2 c}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} dk \left[W_2(n, k) - W_1(n, k)\right] .$$

For the non-degenerate gas, $e^{\beta\zeta} << 1$, and

$$M_{so} \cong \frac{\mu e\omega_B}{8\pi^2 c} e^{\beta\zeta}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} dk \left\{e^{-\beta E_{2,n,k}} - e^{-\beta E_{1,n,k}}\right\}$$

(X-14)

$$= \frac{\mu e \omega_B}{4\pi^2 c}\, e^{\beta\zeta - \frac{\beta\hbar\omega_B}{2}} \sinh \frac{\mu\beta\hbar\omega_B}{2} \left[\sum_{n=0}^{\infty} e^{-n\beta\hbar\omega_B}\right] \int_{-\infty}^{\infty} dk\, e^{-\frac{\beta\hbar^2 k^2}{2m}},$$

so with the aid of the equations (VII-11, 12, 13), one obtains

$$M_{so} \cong \frac{\mu e\hbar n_e}{2mc} \tanh \frac{\mu\beta\hbar\omega_B}{2} \tag{X-15}$$

For the case of the fully degenerate gas in the ground state ($\beta \to \infty$), a treatment similar to that for (VII-22, 23) results in

$$M_{so} = \frac{\mu\hbar e}{2mc} \frac{\omega_B (2m)^{\frac{3}{2}}}{(2\pi\hbar)^2} \begin{cases} \left[\sum_{n=0}^{N_c} \sqrt{\zeta-(n-g)\hbar\omega_B} - \sum_{n=0}^{N_c-2} \sqrt{\zeta-(n+1+g)\hbar\omega_B}\right] \\ \quad \text{for } (N_c-g)\hbar\omega_B \leq \zeta < (N_c+g)\hbar\omega_B, \\ \left[\sum_{n=0}^{N_c} \sqrt{\zeta-(n-g)\hbar\omega_B} - \sum_{n=0}^{N_c-1} \sqrt{\zeta-(n+1+g)\hbar\omega_B}\right] \\ \quad \text{for } (N_c+g)\hbar\omega_B \leq \zeta < (N_c+1-g)\hbar\omega_B \end{cases} \tag{X-16}$$

or

$$M_{so} = \frac{\mu \hbar e n_e}{2mc} \begin{cases} \dfrac{\sum_{n=0}^{N_c} \sqrt{\zeta - (n-g)\hbar\omega_B} - \sum_{n=0}^{N_c-2} \sqrt{\zeta - (n+1+g)\hbar\omega_B}}{\sum_{n=0}^{N_c} \sqrt{\zeta - (n-g)\hbar\omega_B} + \sum_{n=0}^{N_c-2} \sqrt{\zeta - (n+1+g)\hbar\omega_B}} \\ \quad \text{for } (N_c - g)\hbar\omega_B \leq \zeta < (N_c+g)\hbar\omega_B, \\ \dfrac{\sum_{n=0}^{N_c} \sqrt{\zeta - (n-g)\hbar\omega_B} - \sum_{n=0}^{N_c-1} \sqrt{\zeta - (n+1+g)\hbar\omega_B}}{\sum_{n=0}^{N_c} \sqrt{\zeta - (n-g)\hbar\omega_B} + \sum_{n=0}^{N_c-1} \sqrt{\zeta - (n+1+g)\hbar\omega_B}} \\ \quad \text{for } (N_c + g)\hbar\omega_B \leq \zeta < (N_c+1-g)\hbar\omega_B \end{cases} \qquad \text{(X-17)}$$

CHAPTER XI

DIAGRAMMATIC TECHNIQUES

In recent years the use of diagrams in connection with perturbation theories and in the solution of integral equations by iterative expansions has become rather popular. To some extent the description of many body problems in terms of Green's functions circumvents the necessity of such an approach; however, the exchange and correlation contributions to the Green's functions themselves may be described in terms of diagrams.

Two examples of the diagrammatic technique are considered here, not for the purpose of obtaining solutions, but rather to illustrate the topological structure of the diagrams.

The first example will be the integral equation (IX-4):

$$K(1, 2) = \delta^4(1, 2) + \iint d(3)\, d(4)\, v(1, 3) \frac{\delta\rho(3)}{\delta U^T(4)_0} K(4, 2). \tag{XI-1}$$

By iteration K(1, 2) may be expressed in terms of the infinite series

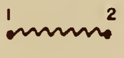
$$K(1, 2) = \delta^4(1, 2) + \int d(3)\, v(1, 3) \frac{\delta\rho(3)}{\delta U^T(2)_0}$$

$$+ \int d(3)\, d(4)\, v(1, 3) \frac{\delta\rho(3)}{\delta U^T(4)_0} v(4, 5) \frac{\delta\rho(5)}{\delta U^T(2)_0} \tag{XI-2}$$

$$+ \cdots$$

Each of the integrals in (XI-2) may be expressed in terms of equivalent diagrams. We associate with each of the quantities v(1, 2) and

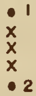
$\dfrac{\delta\rho(1)}{\delta U^T(2)_o}$ an "elementary" diagram. The interaction term v(1, 2) may be represented by a "wavy" line:

$$v(1,\ 2) \Longleftrightarrow$$  (XI-3)

and the quantity $\dfrac{\delta\rho(1)}{\delta U^T(2)_o}$, which we designate as a "propagator" (or "generator") may be represented by a "bubble":

$$\frac{\delta\rho(1)}{\delta U^T(2)_o} \Longleftrightarrow$$  (XI-4)

For convenience, we also define the diagrams

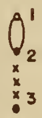
$$K(1,\ 2) \Longleftrightarrow$$ 

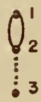
and (XI-5)

$$\delta^4(1,\ 2) \Longleftrightarrow$$ 

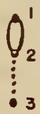
A "chain" diagram of "elementary" diagrams joined at the "end" points (or "vertices") represents a product of the associated quantities, along with an integration over the space-time co-ordinates of the joined "vertices." For example

 $$\Longleftrightarrow \int d(2)\ v(1,\ 2)\ \frac{\delta\rho(2)}{\delta U^T(3)_o}\ ,$$

 $$\Longleftrightarrow \int d(2)\ \frac{\delta\rho(1)}{\delta U^T(2)_o}\ K(2,\ 3)\ ,$$ (XI-6)

 $$\Longleftrightarrow \int d(2)\ \frac{\delta\rho(1)}{\delta U^T(2)_o}\ \delta^4(2,\ 3)\ .$$

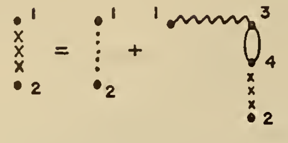
Observe that in the last diagram of (XI-6) the integration over the delta function may be performed, and thus

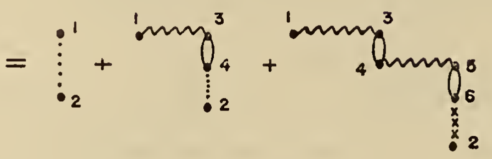
 $$\Longleftrightarrow \frac{\delta\rho(1)}{\delta U^T(3)_o} \Longleftrightarrow$$ 

(XI-7)

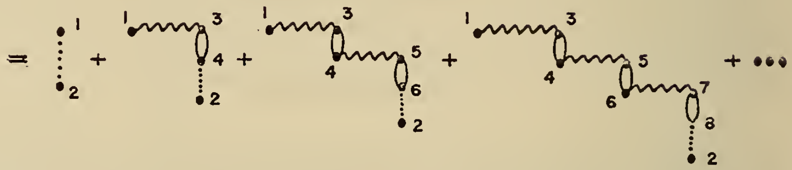
 $\equiv$ 

Therefore, we may represent equation (XI-1) by the series of diagrams obtained by iteration

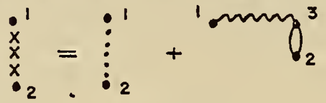
K(1, 2) <==> 



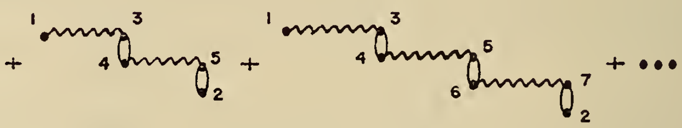
(XI-8)



From (XI-7), we see that the diagrams in (XI-8) reduce to the form

K(1, 2) <==> 

(XI-9)



The series of diagrams in (XI-9) represents the series (XI-2) exactly. The solution (IX-10) of equation (XI-1) is precisely the solution which should be obtained for the diagram equation (XI-9) if it could be evaluated. The interesting thing here in the structure of the diagrams (XI-9) is that they bear a curious resemblance to the "ring diagrams" for the Grand Partition Function. As a matter of fact, there is indeed a direct relationship between the two sets of diagrams; both correspond to the "self-consistent field approximation" (sometimes known as the "random-phase approximation") see reference (42)).

The second example of the diagrammatic technique will be the integral equation (VI-8) for the Green's function:

$$G_1(1, 2) = G_o(1, 2) + \int d(3)\, d(4)\, V(3, 4)\, G_o(1, 3)\, G_1(3, 4)_o\, G_1(4, 2)_o \,, \qquad \text{(XI-10)}$$

where

$$V(1, 2) \equiv \frac{ie^2}{h}\, \delta(t_1 + \varepsilon - t_2)\, \frac{1}{r_{12}}, \quad \varepsilon \rightarrow 0^+$$

Here we designate $G_o(1, 2)$ as the "propagator" (or "generator"), and the representative diagram will be

$G_o(1, 2) \Longleftrightarrow$  (XI-11)

The interaction term V(1, 2) will be represented by a "wavy" line in this case also:

$V(1, 2) \Longleftrightarrow$  (XI-12)

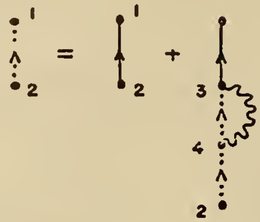
The Green's function $G_1(1, 2)_o$ will be represented by the notation

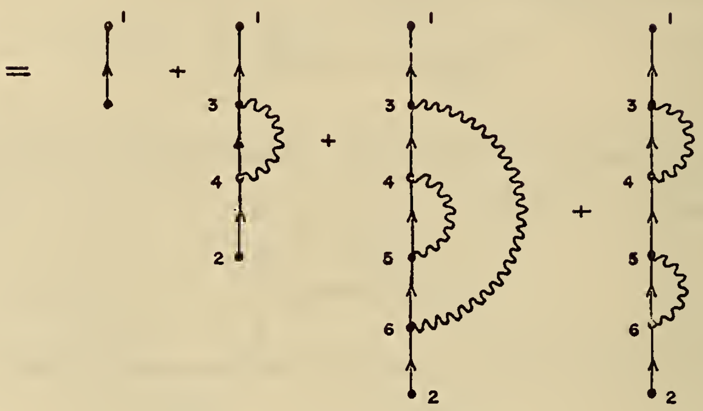
$G_1(1, 2)_o \Longleftrightarrow$  (XI-13)

As before, when the "elementary" diagrams are joined at some "vertex point," this represents a product of the related quantities and an integration over the co-ordinates of the vertex point. There is always an "incoming" and an "outgoing" propagator joined with an interaction at each vertex point in the expansion of (XI-10);

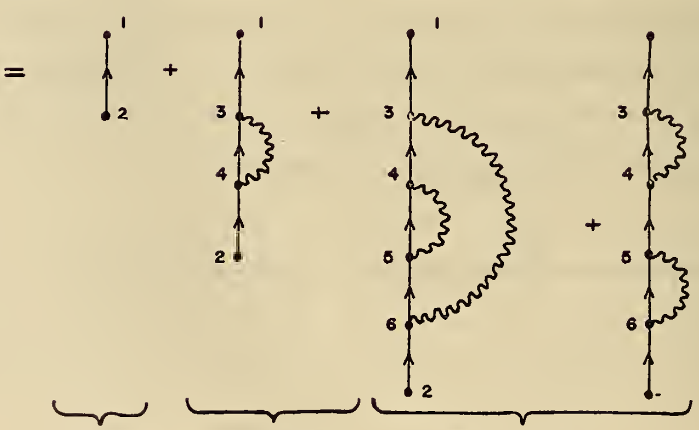
$\Longleftrightarrow \int d(2)\, G_o(1, 2)\, G_o(2, 3)\, V(2, 4)\,.$ (XI-14)

With the "elementary" diagrams (XI-11, 12, 13), the equation (XI-10) is represented by the iterated series

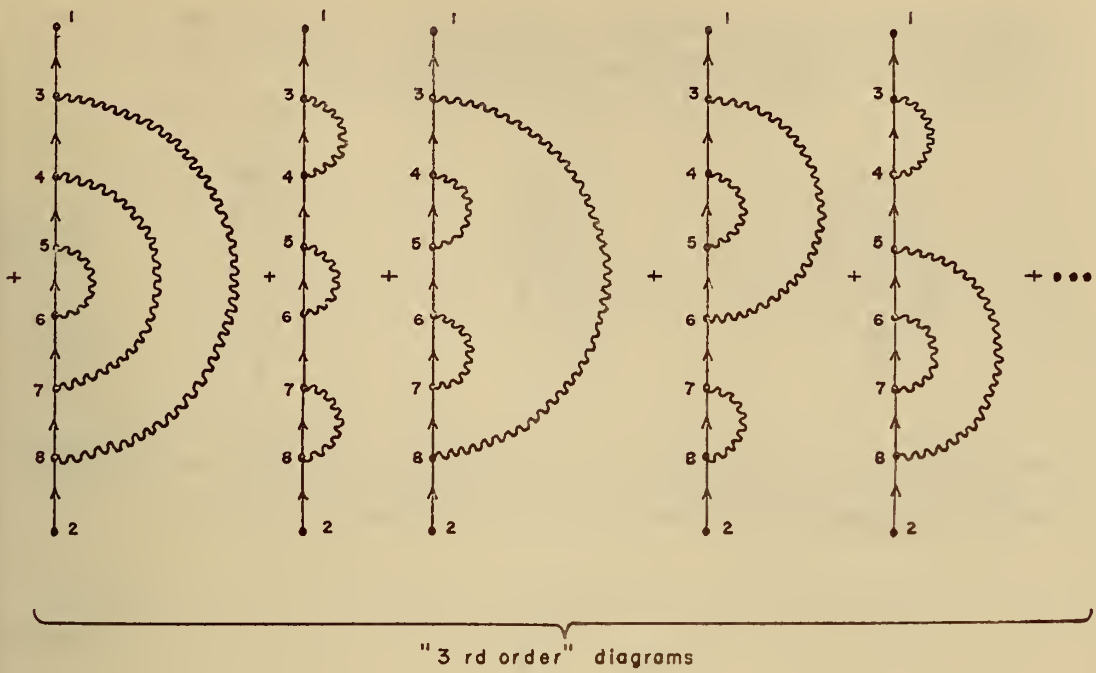
$G_1(1, 2)_o \Longleftrightarrow$ 



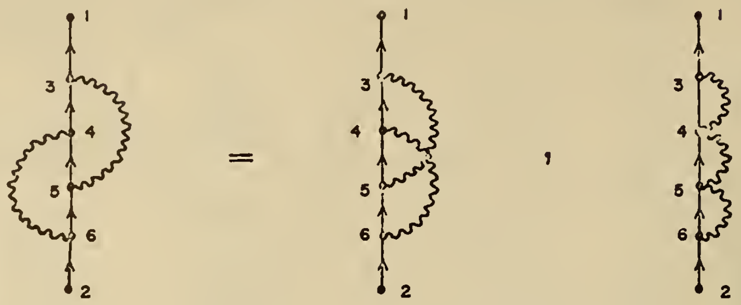
(XI-15)



"Zeroth order" diagrams — "1 st order" diagrams — "2 nd order" diagrams



One of the nice features of this diagrammatic expansion of G_1 (XI-15) is that a particular form of diagram appears once and only once. A classification of these diagrams is possible on the basis of the number of "propagators" in any one diagram; we observe that there is always an odd number of "propagators" in any one diagram, thus we designate an nth order diagram as one with $(2n+1)$ "propagators." The rule for constructing all nth order diagrams is a simple one; all "joined" vertices must contain two propagators and an interaction (XI-14), and all interactions must join to two different vertices without any two interactions "crossing through" each other or joining at the same vertex. That is, there are no diagrams of the types



(XI-16)

Moreover, since all diagrams must comprise of continuously "linked" chains of propagators; there can be no diagrams of the type

(XI-17)

We emphasize that the directions indicated for the propagators by the arrows is important since the propagators are matrices; the propagators are arranged from right to left in the order of the directed diagrams. For example

$\Longleftrightarrow G_o(1, 2)\, G_o(2, 3)\, G_o(3, 4)$ (XI-18)

Diagrams of the types

(XI-19)

where the arrows point in the opposite senses relative to a vertex are meaningless.

We consider two approximate "linear" equations for G_1 to illustrate the classes of diagrams belonging to each:

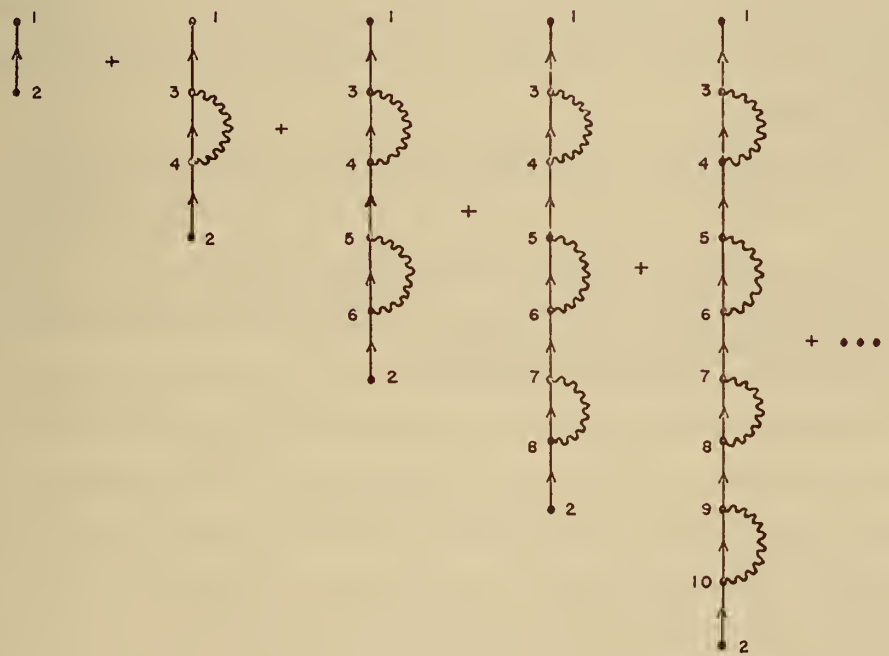
$$\left[G_1(1,2)_o\right]_A \equiv G_o(1,2) + \int d(3)\, d(4)\, V(3,4)\, G_o(1,3)\, G_o(3,4) \left[G_1(4,2)_o\right]_A ,$$

and (XI-20)

$$\left[G_1(1,2)_o\right]_B \equiv G_o(1,2) + \int d(3)\, d(4)\, V(3,4)\, G_o(1,3) \left[G_1(3,4)_o\right]_B G_o(4,2) .$$

Now $\left[G_1(1,2)_o\right]_A$ is represented by the series of diagrams

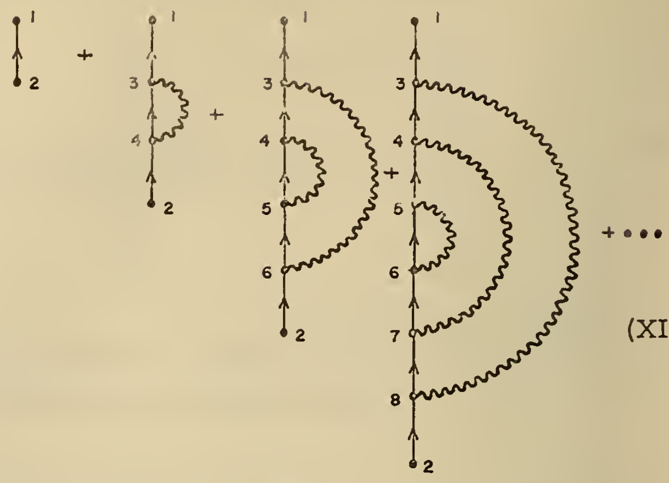
$\left[G_1(1,2)_o\right]_A \Longleftrightarrow$



(XI-21)

while $\left[G_1(1, 2)_o \right]_B$ is represented by the following series of diagrams:

$$\left[G_1(1, 2)_o \right]_B \Longleftrightarrow$$



(XI-22)

The "zeroth" and "first" order diagrams are the same in both approximations, but the higher order diagrams differ. Each approximation exhibits a definite symmetry or topology, and we have the "peculiar" result that the "zeroth" and "first" order diagrams go equally well with both cases.

The well defined symmetries of the two approximations (XI-20, 21, 22) are occasioned directly by the fact that these equations are "linear." If the solutions of (XI-20) were known, they could be used as generating functions along with G_o to obtain solutions corresponding to the "non-linear" diagrams.

CHAPTER XII

SUMMARY AND SUGGESTED LINES OF FURTHER STUDY

The functional series expansions developed in Chapter II for charge and current density from fundamental principles of electromagnetic theory apply generally. The series may be extended to non-linear orders in the field by the method outlined. This result amounts to a formal theory of the electromagnetic properties of physical systems in terms of charge transport and of charge density fluctuations.

It was shown how the use of a Darwin Hamiltonian in "second quantized" form leads to expressions for the charge and current densities of the electron gas, and to expressions for temperature dependent Green's functions, all of which depend upon the applied electromagnetic fields. It was further shown how these expressions are "renormalized" into functionals of the total electromagnetic fields, relating the charge and current densities both explicitly and implicitly to the fields.

In order to facilitate calculations, the functional series for ρ and $\vec{j}$ obtained in Chapter II were used to express the charge and current densities of the electron gas explicitly to an order linear in the perturbing electric field. Calculations were obtained in the "self-consistent field" approximation in terms of a "conductance tensor". The nine components of the "conductance tensor" may be expressed as Fourier transforms generally covering all ranges of temperatures in the non-relativistic gas, although actual calculations are quite difficult except in a few special cases, i.e., for plane wave propagation in the direction of the applied magnetic field and for the "weak spatial dispersion" limit, at zero temperature and at the classical high temperatures. The results at classical high temperatures correspond to quantities obtained through Maxwell-Boltzmann statistics. The results obtained for zero temperatures (degenerate gas) very definitely show quantum effects. It is interesting that these latter results permit calculation of the current density and other quantities quite readily at very high applied magnetic fields, but the procedure becomes more difficult at low values of magnetic field. Of course, all of the more difficult calculations could be programmed for computation and numerical tabulation by a high speed computing machine.

Although a considerable amount of effort is required to bring about the formulation of the expressions for the charge and current densities and to give them in a form showing explicit dependence upon the perturbing fields, the results given here should facilitate examination of the exchange effects and other higher order quantum contributions. In this case, the solution of the appropriate non-linear integral equations is required. Moreover, the treatment is readily extended to include effects of non-linear orders in the field.

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MATHEMATICAL APPENDICES

MATHEMATICAL APPENDIX I

A brief discussion is given here regarding the definition of a functional and of a functional series expansion. To simplify matters, we consider functions of only one variable.

Suppose that one has an arbitrary function U dependent upon the variable x:

$$U = U(x) \; ; \tag{1-1}$$

and let us further suppose that we have another function Y such that Y depends upon the variable x explicitly, and also upon the function U explicitly:

$$Y = Y[x, U(x)] \, . \tag{1-2}$$

The function Y is designated by the terminology

$$Y = \underline{\text{functional}} \text{ of } U \, . \tag{1-3}$$

Let us assume that Y is known for some particular function $U = U_o(x)$:

$$Y_o = Y[x, U_o(x)] \text{ known} \, . \tag{1-4}$$

Then, let us designate any other function U and the associated Y by the notation

$$U(x) = U_o(x) + \delta U(x) \, ,$$

and (1-5)

$$\Delta Y[x, \delta U] = Y[x, U(x)] - Y[x, U_o(x)] \, .$$

Assume that it is possible to express $\Delta Y[x, \delta U]$ in terms of an infinite series of the type

$$\Delta Y[x, \delta U] = \int dx_1\, A_1[x, U_o(x); x_1]\, \delta U(x_1)$$

$$+ \frac{1}{2!}\iint dx_1\, dx_2\, A_2[x, U_o(x); x_1, x_2]\, \delta U(x_1)\, \delta U(x_2) \tag{1-6}$$

$$+ \frac{1}{3!}\iiint dx_1\, dx_2\, dx_3\, A_3[x, U_o(x); x_1, x_2, x_3]\, \delta U(x_1)\, \delta U(x_2)\, \delta U(x_3)$$

$$+ \cdot \cdot \cdot ,$$

where the integrations range over the region for which the function Y is defined, and where A_1, A_2, A_3, etc., are coefficients dependent only upon the variables x, x_1, x_2, . . . , and upon U_o. If one takes for Y the function U, then obviously in this case

$$A_1[x, U_o(x); x_1] = \delta(x - x_1)$$

$$A_2 = A_3 = A_4 = . . . = 0 . \tag{1-7}$$

We define a linear variational operation upon any functional $Y[x, U(x)]$ of U(x) such that

$$\frac{\delta Y[x, U(x)]}{\delta U(x_j)} = \text{the variational derivative of } Y[x, U(x)] \text{ with respect to } U(x_j) ;$$

and (1-8)

$$\frac{\delta Y[x, U_o(x)]}{\delta U(x_j)} = 0 ,$$

$$\frac{\delta U_o(x)}{\delta U(x_j)} = 0 ,$$

$$\frac{\delta U(x)}{\delta U(x_j)} = \delta(x - x_j) .$$

Higher order "variational derivatives" of $Y[x, U(x)]$ may be defined with nomenclature similar to that of ordinary derivatives. In general

$$\frac{\delta^n Y[x, U(x)]}{\delta U(x_1)\, \delta U(x_2) \ldots \delta U(x_n)} = \text{the } \underline{\text{nth}} \text{ order variational derivative of } Y[x, U(x)] ,$$

with (1-9)

$$\frac{\delta^n Y[x, U_o(x)]}{\delta U(x_1)\, \delta U(x_2) \ldots \delta U(x_n)} = 0 ,$$

$$\frac{\delta^n U_o(x)}{\delta U(x_1)\, \delta U(x_2) \ldots \delta U(x_n)} = 0 ,$$

$$\frac{\delta^n U(x)}{\delta U(x_1)\, \delta U(x_2) \ldots \delta U(x_n)} = \begin{cases} \delta(x - x_1), & n = 1 \\ 0 & , n > 1 . \end{cases}$$

Thus (1-10)

$$\frac{\delta^n Y[x, U(x)]}{\delta U(x_1)\, \delta U(x_2) \ldots \delta U(x_n)} = \frac{\delta^n \Delta Y[x, \delta U(x)]}{\delta U(x_1)\, \delta U(x_2) \ldots \delta U(x_n)} ,$$

and from (1-6, 8, 10) one obtains (1-11)

$$\frac{\delta Y[x, U(x)]}{\delta U(x_j)} = \int dx_1\, A_1[x, U_o(x); x]\, \delta(x_1 - x_j)$$

$$+ \frac{1}{2!} \iint dx_1\, dx_2\, A_2[x, U_o(x); x_1, x_2] \times$$

$$\{\delta(x_1 - x_j)\, \delta U(x_2) + \delta(x_2 - x_j)\, \delta U(x_1)\}$$

$$+ \ldots .$$

It is easily seen that (1-12)

$$\lim_{(\delta U = 0)} \frac{\delta Y[x, U(x)]}{\delta U(x_j)} \equiv \frac{\delta Y(x)}{\delta U(x_j)_o} = A_1[x, U_o(x); x_j] .$$

Similarly,

$$\frac{\delta^2 Y[x, U(x)]}{\delta U(x_k)\, \delta U(x_j)} = \frac{\delta}{\delta U(x_k)} \left[\frac{\delta Y[x, U(x)]}{\delta U(x_j)} \right] \tag{1-13}$$

and from (1-11) we obtain the result

$$\lim_{(\delta U = 0)} \frac{\delta^2 Y[x, U(x)]}{\delta U(x_k)\, \delta U(x_j)} \equiv \frac{\delta^2 Y(x)}{\delta U(x_k)\, \delta U(x_j)_o} = A_2[x, U_o(x); x_j, x_k] \; . \tag{1-14}$$

(In general, we assume $A_n[x, U_o(x); x_1, x_2, \ldots, x_n]$ to be "symmetric" in the co-ordinates $x_1, x_2, \ldots, x_n$. That is, A_n is independent of the order in which $x_1, x_2, \ldots, x_n$ occur.)

Generally

$$\lim_{(\delta U = 0)} \frac{\delta^n Y[x, U(x)]}{\delta U(x_1)\, \delta U(x_2) \ldots \delta U(x_n)} \equiv \frac{\delta^n Y(x)}{\delta U(x_1)\, \delta U(x_2) \ldots \delta U(x_n)_o}$$

(1-15)

$$= A_n[x, U_o(x); x_1, x_2, \ldots, x_n] \; .$$

It should be clear by now that the operations which have been defined are analogous to ordinary differentiation operations with the designation of the functions U as the "variables", and U_o as "constants." Thus, the series for Y may be expressed in the form

$$Y[x, U(x)] = Y[x, U_o(x)] + \Delta Y[x, \delta U(x)] \; ,$$

with (1-16)

$$\Delta Y[x, \delta U(x)] = \delta Y[x, \delta U] + \delta^2 Y[x, \delta U] + \delta^3 Y[x, \delta U] + \ldots ,$$

and

$$\delta Y[x, \delta U] = \int dx_1 \frac{\delta Y(x)}{\delta U(x_1)_o} \delta U(x_1) \; ,$$

$$\delta^2 Y[x, \delta U] = \frac{1}{2!} \iint dx_1\, dx_2 \frac{\delta^2 Y(x)}{\delta U(x_2)\, \delta U(x_1)_o} \delta U(x_1)\, \delta U(x^2) ,$$

etc.

The symbol $\delta^n Y[x, \delta U]$ is called the nth variation of Y with respect to U, and the quantities (1-15) are called functional derivatives, or variational derivatives.

All the usual rules of ordinary differentiation apply here, as for instance, the "chain rule" of differentiation for changes of variables:

$$\frac{\delta g[x_1, u(x_1)]}{\delta v(x_2)} = \int dx_3 \frac{\delta g[x_1, u(x_1)]}{\delta u(x_3)} \frac{\delta u(x_3)}{\delta v(x_2)} , \tag{1-17}$$

The results of this discussion may be generalized to the case of functionals of more than one variable, and the same philosophy applies throughout.

More exact and complete discussions concerning functionals can be found in a number of sources; see for instance, the reference 28.

MATHEMATICAL APPENDIX II

The equation of continuity (I-7) and the gauge transformation (I-5, 6) lead to connecting relationships between the functional derivatives of ρ and $\vec{j}$ with respect to the field potentials $\vec{A}^T$ and U^T. The first order variational derivatives of the equation of continuity are given by

$$\frac{\delta}{\delta U^T(2)_o} \frac{\partial \rho(1)}{\partial t_1} + \frac{\delta}{\delta U^T(2)_o} \nabla_1 \cdot \vec{j}(1) = 0 .$$

or

$$\frac{\partial}{\partial t_1} \frac{\delta \rho(1)}{\delta U^T(2)_o} + \nabla_1 \cdot \frac{\delta \vec{j}(1)}{\delta U^T(2)_o} = 0 ,$$

and (2-1)

$$\frac{\delta}{\delta A_q^T(2)_o} \frac{\partial \rho(1)}{\partial t_1} + \frac{\delta}{\delta A_q^T(2)_o} \nabla_1 \cdot \vec{j}(1) = 0$$

or

$$\frac{\partial}{\partial t_1} \frac{\delta \rho(1)}{\delta A_q^T(2)_o} + \nabla_1 \cdot \frac{\delta \vec{j}(1)}{\delta A_q^T(2)_o} = 0 .$$

One may obtain similar results for higher order functional derivates. The same results (2-1) may be obtained in another way. As a functional series, the equation of continuity becomes

$$0 = \frac{\partial \rho_o(1)}{\partial t_1} + \nabla_1 \cdot \vec{j}_o(1)$$

$$+ \int d(2) \left\{ \left[\frac{\partial}{\partial t_1} \frac{\delta \rho(1)}{\delta U^T(2)_o} + \nabla_1 \cdot \frac{\delta \vec{j}(1)}{\delta U^T(2)_o} \right] \delta U^T(2) \right. \tag{2-2}$$

$$\left. + \left[\frac{\partial}{\partial t_1} \frac{\delta \rho(1)}{\delta A_q^T(2)_o} + \nabla_1 \cdot \frac{\delta \vec{j}(1)}{\delta A_q^T(2)_o} \right] \delta A_q^T(2) . \right\}$$

$$+ \ . \ . \ . \ .$$

Now, ρ_o and $\vec{j}_o$ represent the charge and current densities for the system in some initially "unperturbed" state ($\delta U^T = \delta A_q^T = 0$), and thus

$$\frac{\partial \rho_o(1)}{\partial t_1} + \nabla_1 \cdot \vec{j}_o(1) = 0 \tag{2-3}$$

The functions δU^T and δA_q^T are here considered "independent" and arbitrary, thus each of their coefficients in (2-2) must vanish separately, again resulting in the equation (2-1).

Further connecting relations are found from the requirement of gauge invariance. The densities ρ and $\vec{j}$ are invariant under the change of gauge

$$\delta U^T \rightarrow \delta U^T + \frac{1}{c} \frac{\partial}{\partial t} \delta \Lambda^T$$

(2-4)

$$\delta \vec{A}^T \rightarrow \delta \vec{A}^T - \nabla \delta \Lambda^T ,$$

where $\delta \Lambda^T$ is an arbitrary function satisfying the equation

$$- \nabla^2 \delta \Lambda^T + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \delta \Lambda^T = 0 . \tag{2-5}$$

The functional series for the charge density ρ and the current density $\vec{j}$ are given by

$$\rho(1) = \rho_o(1) + \int d(2) \left[\frac{\delta \rho(1)}{\delta U^T(2)_o} \delta U^T(2) + \frac{\delta \rho(1)}{\delta A_q^T(2)_o} \delta A_q^T(2) \right] + \ldots ,$$

and (2-6)

$$\vec{j}(1) = \vec{j}_o(1) + \int d(2) \left[\frac{\delta \vec{j}(1)}{\delta U^T(2)_o} \delta U^T(2) + \frac{\delta \vec{j}(1)}{\delta A_q^T(2)_o} \delta A_q^T(2) \right] + \ldots .$$

If we apply the gauge transformation (2-5) to the equations (2-7), we obtain

$$\rho(1) = \rho_o(1) + \int d(2) \left[\frac{\delta \rho(1)}{\delta U^T(2)_o} \delta U^T(2) + \frac{\delta \rho(1)}{\delta A_q^T(2)_o} \delta A_q^T(2) \right] + \ldots$$

$$+ \int d(2) \left[\frac{\delta \rho(1)}{\delta U^T(2)_o} \frac{1}{c} \frac{\partial}{\partial t_2} \delta \Lambda^T(2) - \frac{\delta \rho(1)}{\delta A_q^T(2)_o} \nabla_{2q} \delta \Lambda^T(2) \right] + \ldots ,$$

and (2-7)

$$\vec{j}(1) = \vec{j}_o(1) + \int d(2) \left[\frac{\delta \vec{j}(1)}{\delta U^T(2)_o} \delta U^T(2) \quad \frac{\delta \vec{j}(1)}{\delta A_q^T(2)_o} \delta A_q^T(2) \right] + \ldots$$

$$+ \int d(2) \left[\frac{\delta \vec{j}(1)}{\delta U^T(2)_o} \frac{1}{c} \frac{\partial}{\partial t_2} \delta \Lambda^T(2) - \frac{\delta \vec{j}(1)}{\delta A_q^T(2)_o} \nabla_{2q} \delta \Lambda^T(2) \right] + \ldots .$$

Now, if we assume that the functions $\frac{\delta\rho(1)}{\delta U^T(2)_o}$, $\frac{\delta\rho(1)}{\delta A_q^T(2)_o}$, $\frac{\delta\vec{j}(1)}{\delta U^T(2)_o}$, $\frac{\delta\vec{j}(1)}{\delta A_q^T(2)_o}$, etc., all vanish identically at the boundaries $|\vec{r}_2| = \infty$ and $t_2 = \infty$ for finite $|\vec{r}_1|$ and t_1, then by integration by parts we obtain the integrals

$$0 = \int d(2) \left[-\frac{1}{c}\frac{\partial}{\partial t_2}\frac{\delta\rho(1)}{\delta U^T(2)_o} + \nabla_{2q}\frac{\delta\rho(1)}{\delta A_q^T(2)_o} \right] \delta\Lambda^T(2) + \ldots$$

$$0 = \int d(2) \left[-\frac{1}{c}\frac{\partial}{\partial t_2}\frac{\delta\vec{j}(1)}{\delta U^T(2)_o} + \nabla_{2q}\frac{\delta\vec{j}(1)}{\delta A_q^T(2)_o} \right] \delta\Lambda^T(2) + \ldots \qquad (2\text{-}8)$$

from a comparison of the two sets of equations (2-6) and (2-7). Since the function $\delta\Lambda^T$ is "independent", the coefficients of each of the combinations of terms $\delta\Lambda^T(2)$, $\delta\Lambda^T(2) \times \delta\Lambda^T(3)$, etc., in the integrals must vanish individually. Thus

$$-\frac{1}{c}\frac{\partial}{\partial t_2}\frac{\delta\rho(1)}{\delta U^T(2)_o} + \nabla_{2q}\frac{\delta\rho(1)}{\delta A_q^T(2)_o} = 0$$

$$-\frac{1}{c}\frac{\partial}{\partial t_2}\frac{\delta\vec{j}(1)}{\delta U^T(2)_o} + \nabla_{2q}\frac{\delta\vec{j}(1)}{\delta A_q^T(2)_o} = 0 , \qquad (2\text{-}9)$$

and the higher order equations are similarly formed.

The equations (2-1) and (2-9) are precisely the connecting relationships desired (equations II-11).

MATHEMATICAL APPENDIX III

The Hamiltonian given in the equations (III-2) includes the non time retarded form of the "magnetic" interactions. It may not be readily apparent or immediately obvious how this form of Hamiltonian is obtained and we consider here its derivation.

We begin with the ordinary Hamiltonian operator for two electrons. The "spin" is an unnecessary complication in the derivation, however, and we ignore all spin contributions until we have obtained the "correct" prescription for constructing the Hamiltonian.

From the equations (39.14) of reference 35 one has for the non-relativistic Hamiltonian of two electrons the expression

$$H_{nr} = \frac{1}{2m}\left[\vec{P}(\vec{r}_1, t)\right]^2 + \frac{1}{2m}\left[\vec{P}(\vec{r}_2, t)\right]^2 - e\,U^A(\vec{r}_1, t) - e\,U^A(\vec{r}_2, t) + \frac{e^2}{r_{12}},$$

where (3-1)

$$\vec{P}(\vec{r}_1, t) \equiv \frac{h}{i}\nabla_1 + \frac{e}{c}\,\vec{A}^A(\vec{r}_1, t)\,.$$

The interaction of the electrons with the applied fields are included along with the "electrostatic" interaction between the electrons. Now the part of the time retarded "magnetic" interactions (Darwin terms; see references (31) and (35)) due to the motions of the electrons alone is

$$H_m = -\frac{e^2}{2m^2c^2}\frac{1}{r_{12}}\left\{\vec{P}(\vec{r}_1, t)\cdot\vec{P}(\vec{r}_2, t) + \frac{\vec{r}_{12}}{r_{12}^3}\cdot\left[\vec{r}_{12}\cdot\vec{P}(\vec{r}_1, t)\right]\vec{P}(\vec{r}_2, t)\right\}$$

(3-2)

The operator H_m (3-2) is Hermitian as is also H_{nr} (3-1); i.e., both H_m and H_{nr} satisfy the condition

$$\int d^3\vec{r}_1 d^3\vec{r}_2 \, \phi^*(H\psi) = \int d^3\vec{r}_1 d^3\vec{r}_2 \, \psi \, (H\phi)^* \, , \qquad (3\text{-}3)$$

where ϕ and ψ are arbitrary functions of the time and the two sets $(\vec{r}_1)$ and $(\vec{r}_2)$ of spatial co-ordinates [see reference (43)]. Moreover, a combination Hamiltonian

$$H_t = H_{nr} + H_m \qquad (3\text{-}4)$$

results in an "equation of continuity" of the form

$$\frac{\partial}{\partial t} R(\vec{r}_1, \vec{r}_2, t) + \nabla_1 \cdot \vec{J}_1(\vec{r}_1, \vec{r}_2, t) + \nabla_2 \cdot \vec{J}_2(\vec{r}_1, \vec{r}_2, t) = 0 \quad ,$$

where (3-5)

$$R(\vec{r}_1, \vec{r}_2, t) \equiv \psi^*(\vec{r}_1, \vec{r}_2, t)\, \psi(\vec{r}_1, \vec{r}_2, t) \quad ,$$

and ψ is some eigenfunction of H. From the Schrödinger equation

$$i h \frac{\partial}{\partial t} \psi = H\psi$$

and (3-6)

$$-i h \frac{\partial}{\partial t} \psi = (H\psi)^* \quad ,$$

we find $\vec{J}_1$ and $\vec{J}_2$ to be given by

$$
\begin{aligned}
\vec{J}_1(\vec{r}_1,\vec{r}_2,t) \equiv\ & \psi^* \frac{\vec{P}(\vec{r}_1,t)}{2m}\psi + \psi \frac{\vec{P}^*(\vec{r}_1,t)}{2m}\psi^* \\
& - \left(\frac{e}{2mc}\right)^2 \frac{1}{r_{12}} \Big\{ \psi^* \vec{P}(\vec{r}_2,t)\psi + \psi \vec{P}^*(\vec{r}_2,t)\psi^* \\
& + \psi^* \frac{\vec{r}_{12}}{r_{12}^2}[\vec{r}_{12}\cdot\vec{P}(\vec{r}_2,t)\psi] + \psi \frac{\vec{r}_{12}}{r_{12}^2}[\vec{r}_{12}\cdot\vec{P}^*(\vec{r}_2,t)\psi^*]\Big\} \\
\vec{J}_2(\vec{r}_1,\vec{r}_2,t) \equiv\ & \psi^* \frac{\vec{P}(\vec{r}_2,t)}{2m}\psi + \psi \frac{\vec{P}^*(\vec{r}_2,t)}{2m}\psi^* \\
& - \left(\frac{e}{2mc}\right)^2 \frac{1}{r_{21}} \Big\{ \psi^* \vec{P}(\vec{r}_1,t)\psi + \psi \vec{P}^*(\vec{r}_1,t)\psi^* \\
& + \psi^* \frac{\vec{r}_{21}}{r_{21}^2}[\vec{r}_{21}\cdot\vec{P}(\vec{r}_1,t)\psi] + \psi \frac{\vec{r}_{21}}{r_{21}^2}[\vec{r}_{21}\cdot\vec{P}^*(\vec{r}_1,t)\psi^*]\Big\} .
\end{aligned}
\tag{3-7}
$$

The non-time retarded equivalent of H_m (3-2) must also be a Hermitian operator (3-3), and it must correspond to the classical limit. These requirements are sufficient to give an expression for the "correct" Hamiltonian in the non-time retarded case. If we define operators $\vec{A}_{12}$ and $\vec{A}_{21}$ such that

$$\vec{A}_{12}(\vec{r}_1, \vec{r}_2, t) \equiv -\frac{e}{2mc}\left[\frac{1}{r_{12}}\vec{P}(\vec{r}_2, t) + \vec{P}(\vec{r}_2, t)\frac{1}{r_{12}}\right]$$

$$\vec{A}_{21}(\vec{r}_1, \vec{r}_2, t) \equiv -\frac{e}{2mc}\left[\frac{1}{r_{21}}\vec{P}(\vec{r}_1, t) + \vec{P}(\vec{r}_1, t)\frac{1}{r_{12}}\right], \tag{3-8}$$

then we may write the non-time retarded Hamiltonian H_{mn} for the "magnetic" interactions in the form

$$H_{mn} \equiv \frac{\vec{P}(\vec{r}_1, t)}{2m} \cdot \frac{e}{c}\vec{A}_{12} + \frac{e}{c}\vec{A}_{12} \cdot \frac{\vec{P}(\vec{r}_1, t)}{2m}$$

$$= \frac{\vec{P}(\vec{r}_2, t)}{2m} \cdot \frac{e}{c}\vec{A}_{21} + \frac{e}{c}\vec{A}_{21} \cdot \frac{\vec{P}(\vec{r}_2, t)}{2m} \tag{3-9}$$

$$= -\left(\frac{e}{2mc}\right)^2 \left\{\vec{P}(\vec{r}_1, t) \cdot \frac{1}{r_{12}}\vec{P}(\vec{r}_2, t) + \vec{P}(\vec{r}_1, t) \cdot \vec{P}(\vec{r}_2, t)\frac{1}{r_{12}}\right.$$

$$\left. + \frac{1}{r_{12}}\vec{P}(\vec{r}_2, t) \cdot \vec{P}(\vec{r}_1, t) + \vec{P}(\vec{r}_2, t)\frac{1}{r_{12}} \cdot \vec{P}(\vec{r}_1, t)\right\} .$$

The operator H_{mn} satisfies the Hermiticity condition (3-3) and it corresponds to the classical limit. This latter condition we can see by expressing the wave function in the form (see reference 39, section 15)

$$\psi = a e^{i/\hbar \mathcal{S}} , \tag{3-10}$$

where a and $\mathcal{S}$ are both real functions of $\vec{r}_1$, $\vec{r}_2$, and t. With the expression (3-10), and with the total Hamiltonian defined as

$$H = H_{nr} + H_{mn} \tag{3-11}$$

the Schrödinger equation (3-6) goes over into a form in which the real and imaginary quantities become manifestly distinct from each other.

The real and imaginary parts of the Schrodinger equation can therefore be written in the form of two separate equations:

$$-a\left\{\frac{\partial S}{\partial t}+\frac{1}{2m}\left[\nabla_1 S+\frac{e}{c}\vec{A}^A(\vec{r}_1,t)\right]^2+\frac{1}{2m}\left[\nabla_2 S+\frac{e}{c}\vec{A}^A(\vec{r}_2,t)\right]^2-eU^A(\vec{r}_1,t)\right.$$

$$-eU^A(\vec{r}_2,t)+\frac{e^2}{r_{12}}-\left(\frac{e}{mc}\right)^2\frac{1}{r_{12}}\left[\nabla_1 S+\frac{e}{c}\vec{A}^A(\vec{r}_1,t)\right]\cdot\left[\nabla_2 S+\frac{e}{c}\vec{A}^A(\vec{r}_2,t)\right]$$

$$\left.-\frac{\hbar^2}{2m}\frac{\nabla_1^2 a}{a}-\frac{\hbar^2}{2m}\frac{\nabla_2^2 a}{a}+\left(\frac{e\hbar}{mc}\right)^2\frac{1}{r_{12}}\frac{\nabla_1\cdot\nabla_2 a}{a}+\frac{1}{2}\left(\frac{e\hbar}{mc}\right)^2\left[\nabla_1 a\cdot\nabla_2\frac{1}{r_{21}}+\nabla_2 a\cdot\nabla_1\frac{1}{r_{12}}\right]\right\} \tag{3-12}$$

$$= 0 \quad ,$$

and

$$i\hbar\left\{\frac{\partial a}{\partial t}+\frac{a}{2m}\nabla_1^2 S+\frac{1}{m}\nabla_1 a\cdot\nabla_1 S+\frac{ea}{2mc}\nabla_1\cdot\vec{A}^A(\vec{r}_1,t)+\frac{e}{mc}\vec{A}^A(\vec{r}_1,t)\cdot\nabla_1 a\right.$$

$$+\frac{a}{2m}\nabla_2^2 S+\frac{1}{m}\nabla_2 a\cdot\nabla_2 S+\frac{ea}{2mc}\nabla_2\cdot\vec{A}^A(\vec{r}_2,t)+\frac{e}{mc}\vec{A}^A(\vec{r}_2,t)\cdot\nabla_2 a$$

$$-\left(\frac{e}{mc}\right)^2\frac{1}{r_{12}}\left[\nabla_1 S+\frac{e}{c}\vec{A}^A(\vec{r}_1,t)\right]\cdot\left[\nabla_2 a+\frac{a}{2}\nabla_2\frac{1}{r_{12}}\right] \tag{3-13}$$

$$\left.-\left(\frac{e}{mc}\right)^2\left[\nabla_2 S+\frac{e}{c}\vec{A}^A(\vec{r}_2,t)\right]\cdot\left[\nabla_1 a+\frac{a}{2}\nabla_1\frac{1}{r_{12}}\right]-a\left(\frac{e}{mc}\right)^2\frac{1}{r_{12}}\nabla_1\cdot\nabla_2 S\right\} = 0 .$$

The equation (3-12) is precisely the classical Hamilton-Jacobi equation in the limit $\hbar \to 0$ where the identifications

$$-\frac{\partial S}{\partial t} = E$$

$$\nabla_1 S = \vec{p}_1 \qquad (3\text{-}14)$$

$$\nabla_2 S = \vec{p}_2$$

are made (see reference (32)). In the non-time retarded limit, the Lagrangian is given by

$$L = \frac{m\vec{V}_1^{\,2}}{2} + \frac{m\vec{V}_2^{\,2}}{2} + eU^A(\vec{r}_1, t) + eU^A(\vec{r}_2, t) - \frac{e^2}{r_{12}} + \left(\frac{e}{c}\right)^2 \frac{\vec{V}_1 \cdot \vec{V}_2}{r_{12}}$$

$$- \frac{e}{c}\vec{A}^A(\vec{r}_1, t)\cdot\vec{V}_1 - \frac{e}{c}\vec{A}^A(\vec{r}_2, t)\cdot\vec{V}_2 \ . \qquad (3\text{-}15)$$

This Lagrangian corresponds to the Hamilton-Jacobi equation (3-12) if we assume that

$$E = \vec{V}_1 \cdot \frac{\partial L}{\partial \vec{V}_1} + \vec{V}_2 \cdot \frac{\partial L}{\partial \vec{V}_2} - L$$

$$\vec{p}_1 \equiv \frac{\partial L}{\partial \vec{V}_1} \cong m\vec{V}_1 - \frac{e}{c}\vec{A}^A(\vec{r}_1, t) \qquad (3\text{-}16)$$

$$\vec{p}_2 \equiv \frac{\partial L}{\partial \vec{V}_2} \cong m\vec{V}_2 - \frac{e}{c}\vec{A}^A(\vec{r}_2, t) \ .$$

(See the procedure outlined in reference (32).)

The "probability current densities" $\vec{J}_1$ and $\vec{J}_2$ as derived from the Hamiltonian (3-11) are given by

$$\vec{J}_1(\vec{r}_1, \vec{r}_2, t) = \psi^* \frac{\vec{P}(\vec{r}_1, t)}{2m} \psi + \psi \frac{\vec{P}^*(\vec{r}_1, t)}{2m} \psi^*$$

$$- \frac{1}{2r_{12}} \left(\frac{e}{mc}\right)^2 \left[\psi^* \vec{P}(\vec{r}_2, t) \psi + \psi \vec{P}^*(\vec{r}_2, t) \psi^*\right]$$

(3-17)

$$\vec{J}_2(\vec{r}_1, \vec{r}_2, t) = \psi^* \frac{\vec{P}(\vec{r}_2, t)}{2m} \psi + \psi \frac{\vec{P}^*(\vec{r}_2, t)}{2m} \psi^*$$

$$- \frac{1}{2r_{12}} \left(\frac{e}{mc}\right)^2 \left[\psi^* \vec{P}(\vec{r}_1, t) \psi + \psi \vec{P}^*(\vec{r}_1, t) \psi^*\right].$$

The results which have been obtained for the two electron system can be generalized to the case of an N electron system $(N \geqslant 2)$ as follows:

$$H = \sum_{j=1}^{N} \left[\frac{1}{2m} \vec{P}(\vec{r}_j, t)^2 - e\, U^A(\vec{r}_j, t)\right] + \frac{1}{2} \sum_{j \neq \ell} \frac{e^2}{r_{j\ell}}$$

$$+ \frac{1}{2} \frac{e}{2mc} \sum_{j \neq \ell} \left[\vec{P}(\vec{r}_j, t) \cdot \vec{A}_{j\ell}(\vec{r}_j, \vec{r}_\ell, t) + \vec{A}_{j\ell}(\vec{r}_j, \vec{r}_\ell, t) \cdot \vec{P}(\vec{r}_j, t)\right]$$

where (3-18)

$$\vec{A}_{j\ell}(\vec{r}_j, \vec{r}_\ell, t) \equiv - \frac{e}{2mc} \left[\frac{1}{r_{j\ell}} \vec{P}(\vec{r}_\ell, t) + \vec{P}(\vec{r}_\ell, t) \frac{1}{r_{j\ell}}\right].$$

Since relativistic and time retardation corrections of order $\left(\frac{v}{c}\right)^2$ are missing from (3-18), one must always remember this equation to be incomplete.

From equations (3-8, 9) we see that the "prescription" for the "magnetic" interactions part of the Hamiltonian is such that the operator $\vec{A}_{12}$ for two electrons is incorporated into the Hamiltonian in a manner similar to that for the vector potential $\vec{A}^A$ for a single particle. Since the operator $\vec{A}_{12}$ takes the form of a vector potential, we must add to it all contributions from "magnetic dipoles," i. e., electron "spins." This is done in the manner indicated by Slater (see reference (36)):

$$\vec{A}_{j\ell}(\vec{r}_j, \vec{r}_\ell, t) = -\frac{e}{2mc}\left[\frac{1}{r_{j\ell}}\vec{P}(\vec{r}_\ell, t) + \vec{P}(\vec{r}_\ell, t)\frac{1}{r_{j\ell}} + \mu h \vec{\sigma}_\ell \times \frac{\vec{r}_{j\ell}}{r_{j\ell}^3}\right] \tag{3-19}$$

where $\vec{\sigma}_\ell$ operates on the ℓth particle only. The interaction between spin and magnetic field is given by

$$H_s = \frac{\mu h e}{2mc}\sum_{\ell=1}^{N}\vec{\sigma}_\ell \cdot \left[\nabla_\ell \times \vec{A}^A(\vec{r}_\ell, t) + \frac{1}{2}\sum_{j\neq\ell}\nabla_\ell \times \vec{A}_{\ell j}(\vec{r}_\ell, \vec{r}_j, t)\right]. \tag{3-20}$$

In a straightforward manner, the Hamiltonian (3-18, 19, 20) may be converted into a "second quantized" operator (see reference (29), Chapter 6) resulting in equation (III-2). We observe that the "second quantized" Hamiltonian (III-2) is "symmetrical" in all its components.

The fact that an equation of continuity of the form (3-5) does exist for the Hamiltonian (3-18) (as a generalization of (3-11)) indicates that one can define a current density for the system in the "second quantization" formalism.

MATHEMATICAL APPENDIX IV

The derivative of a "step discontinuity" can be represented by means of the Dirac delta function. Suppose we have a function f(x) such that

$$f(x) = \begin{cases} g(x), & x < x_o \\ h(x), & x > x_o \end{cases} , \tag{4-1}$$

where $g(x_o) \neq h(x_o)$ and both g and h are continuous functions in the neighborhood of x_o. We can define the "derivative" $\left.\frac{\partial f}{\partial x}\right]_{x=x_o}$ from the expression

$$\begin{aligned} f(x_{o+}) - f(x_{o-}) &= \lim_{\varepsilon \to 0^+} \int_{x_o - \varepsilon}^{x_o + \varepsilon} \frac{\partial f}{\partial x} dx \\ &= h(x_o) - g(x_o) \end{aligned} \tag{4-2}$$

where $\varepsilon > 0$. The equation (4-2) shows that one can uniquely define $\left.\frac{\partial f}{\partial x}\right]_{x=x_o}$ as

$$\left.\frac{\partial f}{\partial x}\right]_{x=x_o} = \delta(x - x_o) \left[f(x_{o+}) - f(x_{o-})\right] = \delta(x - x_o) \left[h(x_o) - g(x_o)\right] . \tag{4-3}$$

MATHEMATICAL APPENDIX V

Solutions have been obtained elsewhere (see reference (39) and (44)) for the Schrödinger equation of an electron in a constant magnetic field $\vec{B}_o^T = \vec{i}_z B_o$:

$$i\hbar \frac{\partial \Phi}{\partial t} = H\Phi$$

where (5-1)

$$H = \frac{1}{2m} \vec{\pi}(\vec{r}, t)^2 + \frac{\mu\hbar\omega_B}{2} \sigma_z$$

$$\Phi = U e^{-\frac{i}{\hbar} Et}$$

$$U = \begin{bmatrix} f(\vec{r}) \\ g(\vec{r}) \end{bmatrix} .$$

We outline the solution of equation (5-1) here and obtain a number of useful identities. The vector potential $\vec{A}_o^T$ is taken to be

$$\vec{A}_o^T = \frac{1}{2} \vec{B}_o^T \times \vec{r} . \qquad (5\text{-}2)$$

In the gauge (5-2) the operator $\vec{\pi}$ is given by

$$\pi_x = \frac{\hbar}{i} \frac{\partial}{\partial x} - \frac{m\omega_B}{2} y$$

$$\pi_y = \frac{\hbar}{i} \frac{\partial}{\partial y} + \frac{m\omega_B}{2} x \qquad (5\text{-}3)$$

$$\pi_z = \frac{\hbar}{i}\frac{\partial}{\partial z} ,$$

or by

$$\pi_+ = \frac{2\hbar}{i}\frac{\partial}{\partial x_-} + i\frac{m\omega_B}{2} x_+$$

$$\pi_- = \frac{2\hbar}{i}\frac{\partial}{\partial x_+} - i\frac{m\omega_B}{2} x_- \qquad (5\text{-}4)$$

$$\pi_z = \frac{\hbar}{i}\frac{\partial}{\partial z}$$

From (5-4) we obtain the relations

$$\pi_+\pi_- = \left(\frac{2\hbar}{i}\right)^2 \frac{\partial^2}{\partial x_-\partial x_+} + m\hbar\omega_B\left(x_+\frac{\partial}{\partial x_+} - x_-\frac{\partial}{\partial x_-}\right) + \left(\frac{m\omega_B}{2}\right)^2 x_+x_- - m\hbar\omega_B$$

$$(5\text{-}5)$$

$$\pi_-\pi_+ = \left(\frac{2\hbar}{i}\right)^2 \frac{\partial^2}{\partial x_+\partial x_-} + m\hbar\omega_B\left(x_+\frac{\partial}{\partial x_+} - x_-\frac{\partial}{\partial x_-}\right) + \left(\frac{m\omega_B}{2}\right)^2 x_-x_+ + m\hbar\omega_B .$$

Now

$$\pi_x^2 + \pi_y^2 = \frac{1}{2}\left[\pi_+\pi_- + \pi_-\pi_+\right] ,$$

$$(5\text{-}6)$$

$$= \left(\frac{2\hbar}{i}\right)^2 \frac{\partial^2}{\partial x_+\partial x_-} + m\hbar\omega_B\left(x_+\frac{\partial}{\partial x_+} - x_-\frac{\partial}{\partial x_-}\right) + \left(\frac{m\omega_B}{2}\right)^2 x_+x_- .$$

Thus

$$\pi_+\pi_- = \pi_x^2 + \pi_y^2 - m\hbar\omega_B$$

$$(5\text{-}7)$$

$$\pi_-\pi_+ = \pi_x^2 + \pi_y^2 + m\hbar\omega_B ,$$

or

$$\pi_x^2 + \pi_y^2 = \pi_+ \pi_- + m\hbar\omega_B = \pi_- \pi_+ - m\hbar\omega_B \, . \tag{5-8}$$

The Hamiltonian may be written in the forms

$$\begin{aligned} H &= \frac{1}{2m}(\pi_x^2 + \pi_y^2 + \pi_z^2) + \frac{\mu\hbar\omega_B}{2}\sigma_z \\ &= \frac{1}{2m}(\pi_+ \pi_- + m\hbar\omega_B + \pi_z^2) + \frac{\mu\hbar\omega_B}{2}\sigma_z \\ &= \frac{1}{2m}(\pi_- \pi_+ - m\hbar\omega_B + \pi_z^2) + \frac{\mu\hbar\omega_B}{2}\sigma_z \, . \end{aligned} \tag{5-9}$$

From (5-9) we obtain the identity

$$\pi_- \pi_+ - \pi_+ \pi_- = 2m\hbar\omega_B \, . \tag{5-10}$$

We shall find that π_+ has the properties of a creation operator, while π_- corresponds to an annihilation operator.

From (5-1) we obtain the equations

$$\begin{aligned} E_1 f(\vec{r}) &= \left[\frac{1}{2m}\vec{\pi}^2 + \frac{\mu\hbar\omega_B}{2}\right] f(\vec{r}) \\ E_2 g(\vec{r}) &= \left[\frac{1}{2m}\vec{\pi}^2 - \frac{\mu\hbar\omega_B}{2}\right] g(\vec{r}) \end{aligned} \tag{5-11}$$

The equations (5-11) infer that the eigenfunctions of (5-1) are $U_1 = \binom{1}{0} v(\vec{r})$ with energy eigenvalue of spin index $\alpha = 1$, and

$$U_2 = \binom{0}{1} v(\vec{r}) \tag{5-12}$$

with energy eigenvalue of spin index $\alpha = 2$, and where

$$\frac{\vec{\pi}^2}{2m} v(\vec{r}) = \left[E_{(\alpha=1)} - \frac{\mu\hbar\omega_B}{2}\right] v(\vec{r}) = \left[E_{(\alpha=2)} + \frac{\mu\hbar\omega_B}{2}\right] v(\vec{r}) . \quad (5\text{-}13)$$

The solution $v(\vec{r})$ is separable for the z-direction, in which case we assume a plane wave solution

$$v(\vec{r}) \propto e^{ikz} \quad (5\text{-}14)$$

Then, from (5-9, 12, 13, and 14) we obtain

$$\frac{\pi_+ \pi_-}{2m} v(\vec{r}) = \left[E_{(\alpha=1)} - \frac{\mu\hbar\omega_B}{2} - \frac{\hbar\omega_B}{2} - \frac{\hbar^2 k^2}{2m}\right] v(\vec{r})$$

$$= \left[E_{(\alpha=2)} + \frac{\mu\hbar\omega_B}{2} - \frac{\hbar\omega_B}{2} - \frac{\hbar^2 k^2}{2m}\right] v(\vec{r})$$

and (5-15)

$$\frac{\pi_- \pi_+}{2m} v(\vec{r}) = \left[E_{(\alpha=1)} - \frac{\mu\hbar\omega_B}{2} + \frac{\hbar\omega_B}{2} - \frac{\hbar^2 k^2}{2m}\right] v(\vec{r})$$

$$= \left[E_{(\alpha=2)} + \frac{\mu\hbar\omega_B}{2} + \frac{\hbar\omega_B}{2} - \frac{\hbar^2 k^2}{2m}\right] v(\vec{r}) .$$

By inspection of the equations (5-10, 15) we find the eigenvalue of the operator $\pi_+ \pi_-$ to be less than the eigenvalue of the operator $\pi_- \pi_+$ by $2m\hbar\omega_B$, thus we infer that

$$\frac{\pi_+}{\sqrt{2m\hbar\omega_B}}$$

is a creation operator and

$$\frac{\pi_-}{\sqrt{2m\hbar\omega_B}}$$

is an annihilation operator. By analogy with the harmonic oscillator problem, we seek some function $v_o(\vec{r})$ such that

$$\pi_- v_o(\vec{r}) = 0 . \tag{5-16}$$

The function

$$v_o(\vec{r}) = \sqrt{\frac{m\omega_B}{2\pi\hbar}}\; e^{-\frac{m\omega_B}{4\hbar} x_+ x_-}\; e^{ikz} \tag{5-17}$$

satisfies the condition (5-16) and it satisfies the boundary conditions

$$\begin{aligned} v_o(\vec{r}) \to 0, \text{ for } |x| \to \infty \\ \text{or for } |y| \to \infty , \end{aligned} \tag{5-18}$$

and finally v_o satisfies the normalization condition

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, |v_o(\vec{r})|^2 = 1 . \tag{5-19}$$

The normalization condition (5-19) is easily proved by a change of co-ordinate system to polar cylindrical co-ordinates:

$$\begin{aligned} x &= \rho \cos \theta \\ y &= \rho \sin \theta \\ z &= z . \end{aligned} \tag{5-20}$$

Thus

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, |v_o(\vec{r})|^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \frac{m\omega_B}{2\pi\hbar} \, e^{-\frac{m\omega_B}{2\hbar}(x^2+y^2)}$$

$$= \frac{m\omega_B}{2\pi h} \int_0^{2\pi} d\theta \int_0^{\infty} \rho \, d\rho \, e^{-\frac{m\omega_B}{2\hbar}\rho^2} \tag{5-21}$$

$$= 1 \; .$$

Finally, the function (5-17) satisfies the equations (5-15) with

$$E_{(a=1)} \equiv E_{1,0,k} = (1+g)\hbar\omega_B + \frac{\hbar^2 k^2}{2m}$$

$$E_{(a=2)} \equiv E_{2,0,k} = -g\hbar\omega_B + \frac{\hbar^2 k^2}{2m} \tag{5-22}$$

since $\mu \equiv 1 + 2g$. The function $v_o(\vec{r})$ corresponds to the ground states of the electron (two spin states). Now, the function

$$v_o(\vec{r})_F = F(x_-)\, v_o(\vec{r}) \, , \tag{5-23}$$

where $F(x_-)$ is an arbitrary function of x_-, satisfies the condition (5-16) equally well. The angular momentum operator (z-component)

$$L_z = \frac{\hbar}{i}\left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)$$

$$= \hbar \left(x_+ \frac{\partial}{\partial x_+} - x_- \frac{\partial}{\partial x_-}\right) \tag{5-24}$$

$$= \frac{\hbar}{i} \frac{\partial}{\partial \theta}$$

commutes with the Hamiltonian H, and we can remove the arbitrariness of the function F by allowing U to be an eigenfunction of L_z as well as of H. Thus, if we take

$$L_z v_o(\vec{r})_F = -\ell\hbar\, v_o(\vec{r})_F$$

where ℓ is any positive integer (5-25)

$$\ell = 0, 1, 2, 3, \cdots, \infty,$$

then we obtain the solution

$$F(x_-) = x_-^{\ell} = (x - iy)^{\ell} = \rho^{\ell} e^{-i\ell\theta} \qquad (5\text{-}26)$$

which satisfies boundary conditions. Thus, the function $v_o(\vec{r})$ is generalized to an eigenfunction of L_z with angular momentum quantum number ℓ and energy quantum number k:

$$v_o(\vec{r})_F \equiv v_{o,\ell,k}(\vec{r}) = \frac{1}{\sqrt{\pi \ell!}} \left(\frac{m\omega_B}{2\hbar}\right)^{\frac{\ell+1}{2}} (x_-)^{\ell}\, e^{-\frac{m\omega_B}{4\hbar} x_+ x_-}\, e^{ikz}$$

$$= \frac{1}{\sqrt{\pi \ell!}} \left(\frac{m\omega_B}{2\hbar}\right)^{\frac{\ell+1}{2}} (x - iy)^{\ell}\, e^{-\frac{m\omega_B}{4\hbar}(x^2 + y^2)}$$

$$= \frac{1}{\sqrt{\pi \ell!}} \left(\frac{m\omega_B}{2\hbar}\right)^{\frac{\ell+1}{2}} \rho^{\ell}\, e^{-i\ell\theta}\, e^{-\frac{m\omega_B}{4\hbar}\rho^2}\, e^{ikz},$$

where $v_{o,\ell,k}$ is normalized

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, |v_{o,\ell,k}(\vec{r})|^2 = \int_{0}^{2\pi} d\theta \int_{0}^{\infty} \rho \cdot d\rho \, |v_{o,\ell,k}(\vec{r})|^2 = 1 .$$

It is easy to see from the polar co-ordinate form that $v_{o,\ell,k}$ is orthonormal in ℓ:

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; v^*_{o,\ell_1,k_1}(\vec{r}) \, v_{o,\ell_2,k_2}(\vec{r})$$

$$= \int_{0}^{2\pi} d\theta \int_{0}^{\infty} \rho \, d\rho \; v^*_{o,\ell_1,k_1}(\vec{r}) \, v_{o,\ell_2,k_2}(\vec{r}) = \delta_{\ell_1,\ell_2} \, e^{i(k_2 - k_1)z} \tag{5-28}$$

The functions $v_{o,\ell,k}(\vec{r})$ exhibit a many-fold degeneracy in the energy eigenvalues, i.e., for a given value of k the eigenvalues (5-22) are the same for all values of ℓ.

As noted before, the problem is analogous to that of the simple-harmonic oscillator (see reference (39), section 21) and therefore we introduce the energy quantum numbers n with the functions $v_{n,\ell,k}(\vec{r})$ such that

$$\frac{\pi_- \pi_+}{2m\hbar\omega_B} v_{n,\ell,k}(\vec{r}) = (n+1) \, v_{n,\ell,k}(\vec{r})$$

$$\frac{\pi_+ \pi_-}{2m\hbar\omega_B} v_{n,\ell,k}(\vec{r}) = n \, v_{n,\ell,k}(\vec{r}) \quad ,$$

$$\frac{\pi_+}{\sqrt{2m\hbar\omega_B}} v_{n,\ell,k}(\vec{r}) = \sqrt{n+1} \, v_{n+1,\ell,k}(\vec{r}) \tag{5-29}$$

$$\frac{\pi_-}{\sqrt{2m\hbar\omega_B}} v_{n,\ell,k}(\vec{r}) = \sqrt{n} \, v_{n-1,\ell,k}(\vec{r})$$

where the numbers n are positive integers, n = 0, 1, 2, ⋯, ∞. Using the equations (5-29), we find that

$$\left(\frac{1}{2m\hbar\omega_B}\right)^{\frac{1}{2}} \pi_+ v_{o,\ell,k}(\vec{r}) = \sqrt{1}\, v_{1,\ell,k}(\vec{r})$$

$$\left(\frac{1}{2m\hbar\omega_B}\right)^{\frac{1}{2}} \pi_+^2 v_{o,\ell,k}(\vec{r}) = \sqrt{1\cdot 2}\, v_{2,\ell,k}(\vec{r}) \qquad (5\text{-}30)$$

$$\vdots$$

$$\left(\frac{1}{2m\hbar\omega_B}\right)^{\frac{n}{2}} \pi_+^n v_{o,\ell,k}(r) = \sqrt{n!}\, v_{n,\ell,k}(\vec{r}) ,$$

and thus

$$v_{n,\ell,k}(\vec{r}) = \frac{1}{\sqrt{n!}}\left(\frac{1}{2m\hbar\omega_B}\right)^{\frac{n}{2}} \pi_+^n v_{o,\ell,k}(\vec{r}) . \qquad (5\text{-}31)$$

Now, one can easily prove the identity

$$\pi_+ e^{\frac{m\omega_B}{4\hbar}x_+x_-} = e^{\frac{m\omega_B}{4\hbar}x_+x_-} \frac{2\hbar}{i}\frac{\partial}{\partial x_-} , \qquad (5\text{-}32)$$

and repeated use of the identity (5-32) results in the "more general" identity

$$\pi_+^n e^{\frac{m\omega_B}{4\hbar}x_+x_-} = e^{\frac{m\omega_B}{4\hbar}x_+x_-} \left(\frac{2\hbar}{i}\frac{\partial}{\partial x_-}\right)^n . \qquad (5\text{-}33)$$

Thus, if we apply (5-33) to (5-31) we obtain

$$v_{n,\ell,k}(\vec{r}) = \frac{(-i)^n (2\hbar)^n}{\sqrt{n!}}\left(\frac{1}{2m\hbar\omega_B}\right)^{\frac{n}{2}} e^{\frac{m\omega_B}{4\hbar}x_+x_-}\left(\frac{\partial}{\partial x_-}\right)^n e^{-\frac{m\omega_B}{4\hbar}x_+x_-} v_{o,\ell,k}(\vec{r}),$$

(5-34)

and from (5-27) then one has the result

$$v_{n,\ell,k}(\vec{r}) = \frac{(-i)^n}{\sqrt{\pi n!\ell!}} \left(\frac{m\omega_B}{2\hbar}\right)^{\frac{1+\ell-n}{2}} e^{ikz}\, e^{\frac{m\omega_B}{4\hbar}x_+x_-} \left(\frac{\partial}{\partial x_-}\right)^n (x_-)^\ell\, e^{-\frac{m\omega_B}{2\hbar}x_+x_-}. \tag{5-35}$$

Now, from the easily proven commutation relations

$$\begin{aligned} (x_+)^\ell \left(\frac{\partial}{\partial x_-}\right)^n - \left(\frac{\partial}{\partial x_-}\right)^n (x_+)^\ell &= 0 \\ (x_-)^\ell \left(\frac{\partial}{\partial x_+}\right)^n - \left(\frac{\partial}{\partial x_+}\right)^n (x_-)^\ell &= 0\,, \end{aligned} \tag{5-36}$$

we may write (5-35) in the form

$$v_{n,\ell,k}(\vec{r}) = \frac{(-i)^n}{\sqrt{\pi n!\ell!}} \left(\frac{m\omega_B}{2\hbar}\right)^{\frac{1+\ell-n}{2}} e^{ikz}\, e^{\frac{m\omega_B}{4\hbar}x_+x_-} (x_+)^{n-\ell} \left(\frac{1}{x_+}\frac{\partial}{\partial x_-}\right)^n (x_+x_-)^\ell\, e^{-\frac{m\omega_B}{2\hbar}x_+x_-} \tag{5-37}$$

From the definitions (5-20) we obtain the identities

$$\begin{aligned} x_+ &= x + iy = \rho e^{i\theta} \\ x_- &= x - iy = \rho e^{-i\theta} \\ x_+x_- &= x_-x_+ = \rho^2 \\ 2\frac{\partial}{\partial x_+} &= e^{-i\theta}\left[\frac{\partial}{\partial \rho} - i\frac{1}{\rho}\frac{\partial}{\partial \theta}\right] \end{aligned} \tag{5-38}$$

$$2\frac{\partial}{\partial x_-} = e^{i\theta}\left[\frac{\partial}{\partial \rho} + i\frac{1}{\rho}\frac{\partial}{\partial \theta}\right]$$

$$\frac{1}{x_+}\frac{\partial}{\partial x_-} = \frac{1}{2\rho}\left[\frac{\partial}{\partial \rho} + i\frac{1}{\rho}\frac{\partial}{\partial \theta}\right]$$

$$\frac{1}{x_-}\frac{\partial}{\partial x_+} = \frac{1}{2\rho}\left[\frac{\partial}{\partial \rho} - i\frac{1}{\rho}\frac{\partial}{\partial \theta}\right] .$$

Combining (5-37) and (5-38) results in

$$v_{n,\ell,k}(\vec{r})$$

$$= \frac{(-i)^n}{\sqrt{\pi n!\ell!}}\left(\frac{m\omega_B}{2\hbar}\right)^{\frac{1+\ell-n}{2}} e^{ikz}\, e^{\frac{m\omega_B}{4\hbar}\rho^2}\, \rho^{n-\ell}\, e^{i(n-\ell)\theta}\left[\frac{1}{2\rho}\left(\frac{\partial}{\partial\rho}+i\frac{1}{\rho}\frac{\partial}{\partial\theta}\right)\right]^n \rho^{2\ell}\, e^{-\frac{m\omega_B}{2\hbar}\rho^2}$$

(5-39)

Since the operator

$$\left[\frac{1}{2\rho}\left(\frac{\partial}{\partial\rho}+i\frac{1}{\rho}\frac{\partial}{\partial\theta}\right)\right]^n$$

in (5-39) operates upon the function

$$\rho^{2\ell}\, e^{-\frac{m\omega_B}{2\hbar}\rho^2} ,$$

which is independent of θ, we may make the replacement

$$\left[\frac{1}{2\rho}\left(\frac{\partial}{\partial\rho}+i\frac{1}{\rho}\frac{\partial}{\partial\theta}\right)\right]^n \rho^{2\ell}\, e^{-\frac{m\omega_B}{2\hbar}\rho^2} = \left[\frac{1}{2\rho}\frac{\partial}{\partial\rho}\right]^n \rho^{2\ell}\, e^{-\frac{m\omega_B}{2\hbar}\rho^2}$$

(5-40)

and since

$$\frac{\partial}{\partial \rho^2} = \frac{1}{2\rho}\frac{\partial}{\partial \rho} \,, \tag{5-41}$$

we obtain the result

$$v_{n,\ell,k}(\vec{r}) = \frac{(i)^n}{\sqrt{\pi n!\ell!}}\left(\frac{m\omega_B}{2\hbar}\right)^{\frac{1}{2}} e^{ikz}\, e^{i(n-\ell)\theta}\, \xi^{\frac{n-\ell}{2}}\, e^{-\frac{\xi}{2}}\, \mathscr{L}_{n,\ell}(\xi)$$

where (5-42)

$$\xi \equiv \left(\frac{m\omega_B}{2\hbar}\right)\rho^2$$

$$\mathscr{L}_{n,\ell}(\xi) \equiv (-1)^n e^{\xi}\left(\frac{\partial}{\partial \xi}\right)^n e^{-\xi}\,\xi^{\ell}$$

(see reference (44)). The orthonormality of the functions $v_{n,\ell,k}(\vec{r})$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; v^*_{n_1,\ell_1,k_1}(\vec{r})\, v_{n_2,\ell_2,k_2}(\vec{r}) = \delta_{n_1,n_2}\,\delta_{\ell_1,\ell_2}\, e^{i(k_2-k_1)z}$$

(5-43)

$$= \int_0^{2\pi} d\theta \int_0^{\infty} \rho\, d\rho\; v^*_{n_1,\ell_1,k_1}(\vec{r})\, v_{n_2,\ell_2,k_2}(\vec{r})$$

can be verified with reasonable simplicity if one uses the polar co-ordinate form of $v_{n,\ell,k}(\vec{r})$ (5-42). The expressions (5-35, 42) do in fact satisfy the creation and annihilation relations (5-29) as one can easily verify by mathematical induction. For convenience, one may write

$$v_{n,\ell,k}(\vec{r}) = \begin{cases} e^{ikz}\, w_{n,\ell}(x,y) & \text{in Cartesian co-ordinates,} \\ e^{ikz}\, u_{n,\ell}(\rho,\theta) & \text{in polar co-ordinates,} \end{cases} \tag{5-44}$$

where

$w_{n,\ell}(x, y)$

$$\equiv \frac{(-i)^n}{2^n\sqrt{\pi n!\,\ell!}}\left(\frac{m\omega_B}{2\hbar}\right)^{\frac{1+\ell-n}{2}} e^{\frac{m\omega_B}{4\hbar}(x^2+y^2)}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^n (x-iy)^\ell\, e^{-\frac{m\omega_B}{2\hbar}(x^2+y^2)} \tag{5-45}$$

and

$u_{n,\ell}(\rho, \theta)$

$$\equiv \frac{(i)^n}{\sqrt{\pi n!\,\ell!}}\left(\frac{m\omega_B}{2\hbar}\right)^{\frac{1}{2}} e^{i(n-\ell)\theta}\,\xi^{\frac{n-\ell}{2}}\, e^{-\frac{\xi}{2}}\mathscr{L}_{n,\ell}(\xi) \ . \tag{5-46}$$

Clearly, the functions $w_{n,\ell}$ and $u_{n,\ell}$ have the following orthonormality conditions:

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ w^*_{n_1,\ell_1}(x,y)\, w_{n_2,\ell_2}(x,y) = \delta_{n_1,n_2}\,\delta_{\ell_1,\ell_2}$$

and

$$\int_0^{2\pi} d\theta \int_0^{\infty} \rho\, d\rho\ u^*_{n_1,\ell_1}(\rho,\theta)\, u_{n_2,\ell_2}(\rho,\theta) = \delta_{n_1,n_2}\,\delta_{\ell_1,\ell_2} \ . \tag{5-47}$$

The energy eigenvalues for (5-13) are given in general by

$$E_{(\alpha=1)} \equiv E_{1,n,k} = (n+1+g)\hbar\omega_B + \frac{\hbar^2k^2}{2m}$$

$$E_{(\alpha=2)} \equiv E_{2,n,k} = (n-g)\hbar\omega_B + \frac{\hbar^2k^2}{2m} \ . \tag{5-48}$$

The designation of the energy eigenvalue is

$$E_{(\alpha)} \rightarrow E_{\alpha, n, k} \tag{5-49}$$

where the index α is the "spin" index, n is the "harmonic oscillator" quantum number, and k refers to the wave number in the z-direction.

From equations (5-4, 29, 35), one can easily prove the following identities:

$$\pi_+ = \frac{2\hbar}{i}\frac{\partial}{\partial x_-} + i\frac{m\omega_B}{2}x_+$$

$$\pi_- = \frac{2\hbar}{i}\frac{\partial}{\partial x_+} - i\frac{m\omega_B}{2}x_-$$

$$\pi_z = \frac{\hbar}{i}\frac{\partial}{\partial z}$$

$$\pi_+^* = -\frac{2\hbar}{i}\frac{\partial}{\partial x_+} - i\frac{m\omega_B}{2}x_-$$

$$\pi_-^* = -\frac{2\hbar}{i}\frac{\partial}{\partial x_-} + i\frac{m\omega_B}{2}x_+$$

$$\pi_z^* = -\frac{\hbar}{i}\frac{\partial}{\partial z}$$

$$\pi_+^* = -\pi_- - im\omega_B x_-$$

$$\pi_-^* = -\pi_+ + im\omega_B x_+$$

$$\pi_z^* = -\pi_z$$

$$\pi_+ v_{n,\ell,k} = \left(\frac{2\hbar}{i}\frac{\partial}{\partial x_-} + i\frac{m\omega_B}{2}x_+\right) v_{n,\ell,k} = \sqrt{2m\hbar\omega_B}\sqrt{n+1}\, v_{n+1,\ell,k}$$

$$\pi_- v_{n,\ell,k} = \left(\frac{2\hbar}{i}\frac{\partial}{\partial x_+} - i\frac{m\omega_B}{2}x_-\right) v_{n,\ell,k} = \sqrt{2m\hbar\omega_B}\sqrt{n}\, v_{n-1,\ell,k}$$

$$\pi_z v_{n,\ell,k} = \frac{\hbar}{i}\frac{\partial}{\partial z} v_{n,\ell,k} = \hbar k\, v_{n,\ell,k}$$

$$\pi_+^* v_{n,\ell,k}^* = \left(-\frac{2\hbar}{i}\frac{\partial}{\partial x_+} - i\frac{m\omega_B}{2}x_-\right) v_{n,\ell,k}^* = \sqrt{2m\hbar\omega_B}\sqrt{n+1}\, v_{n+1,\ell,k}^*$$

$$\pi_-^* v_{n,\ell,k}^* = \left(-\frac{2\hbar}{i}\frac{\partial}{\partial x_-} + i\frac{m\omega_B}{2}x_+\right) v_{n,\ell,k}^* = \sqrt{2m\hbar\omega_B}\sqrt{n}\, v_{n-1,\ell,k}^*$$

$$\pi_z^* v_{n,\ell,k}^* = -\frac{\hbar}{i}\frac{\partial}{\partial z} v_{n,\ell,k}^* = \hbar k\, v_{n,\ell,k}^* \qquad (5\text{-}50)$$

$$\pi_+^* v_{n,\ell,k} = \left(-\frac{2\hbar}{i}\frac{\partial}{\partial x_+} - i\frac{m\omega_B}{2}x_-\right) v_{n,\ell,k} = -i\sqrt{2m\hbar\omega_B}\sqrt{\ell+1}\, v_{n,\ell+1,k}$$

$$\pi_-^* v_{n,\ell,k} = \left(-\frac{2\hbar}{i}\frac{\partial}{\partial x_-} + i\frac{m\omega_B}{2}x_+\right) v_{n,\ell,k} = i\sqrt{2m\hbar\omega_B}\sqrt{\ell}\, v_{n,\ell-1,k}$$

$$\pi_z^* v_{n,\ell,k} = -\frac{\hbar}{i}\frac{\partial}{\partial z} v_{n,\ell,k} = -\hbar k\, v_{n,\ell,k}$$

$$\pi_+ v_{n,\ell,k}^* = \left(\frac{2\hbar}{i}\frac{\partial}{\partial x_-} + i\frac{m\omega_B}{2}x_+\right) v_{n,\ell,k}^* = i\sqrt{2m\hbar\omega_B}\sqrt{\ell+1}\, v_{n,\ell+1,k}^*$$

$$\pi_- v_{n,\ell,k}^* = \left(\frac{2\hbar}{i}\frac{\partial}{\partial x_+} - i\frac{m\omega_B}{2}x_-\right) v_{n,\ell,k}^* = -i\sqrt{2m\hbar\omega_B}\sqrt{\ell}\, v_{n,\ell-1,k}^*$$

$$\pi_z v_{n,\ell,k}^* = \frac{\hbar}{i}\frac{\partial}{\partial z} v_{n,\ell,k}^* = -\hbar k\, v_{n,\ell,k}^*$$

$$\frac{2\hbar}{i}\frac{\partial}{\partial x_+} v_{n,\ell,k} = \frac{\sqrt{2m\hbar\omega_B}}{2}\left[\sqrt{n}\, v_{n-1,\ell,k} + i\sqrt{\ell+1}\, v_{n,\ell+1,k}\right]$$

$$\frac{2\hbar}{i}\frac{\partial}{\partial x_-} v_{n,\ell,k} = \frac{\sqrt{2m\hbar\omega_B}}{2}\left[\sqrt{n+1}\, v_{n+1,\ell,k} - i\sqrt{\ell}\, v_{n,\ell-1,k}\right]$$

$$\frac{im\omega_B}{2} x_+ v_{n,\ell,k} = \frac{\sqrt{2m\hbar\omega_B}}{2}\left[\sqrt{n+1}\, v_{n+1,\ell,k} + i\sqrt{\ell}\, v_{n,\ell-1,k}\right]$$

$$\frac{im\omega_B}{2} x_- v_{n,\ell,k} = \frac{\sqrt{2m\hbar\omega_B}}{2}\left[-\sqrt{n}\, v_{n-1,\ell,k} + i\sqrt{\ell+1}\, v_{n,\ell+1,k}\right]$$

$$-\frac{2\hbar}{i}\frac{\partial}{\partial x_+} v^*_{n,\ell,k} = \frac{\sqrt{2m\hbar\omega_B}}{2}\left[\sqrt{n+1}\, v^*_{n+1,\ell,k} + i\sqrt{\ell}\, v^*_{n,\ell-1,k}\right]$$

$$-\frac{2\hbar}{i}\frac{\partial}{\partial x_-} v^*_{n,\ell,k} = \frac{\sqrt{2m\hbar\omega_B}}{2}\left[\sqrt{n}\, v^*_{n-1,\ell,k} - i\sqrt{\ell+1}\, v^*_{n,\ell+1,k}\right]$$

$$-i\frac{m\omega_B}{2} x_+ v^*_{n,\ell,k} = \frac{\sqrt{2m\hbar\omega_B}}{2}\left[-\sqrt{n}\, v^*_{n-1,\ell,k} - i\sqrt{\ell+1}\, v^*_{n,\ell+1,k}\right]$$

$$-i\frac{m\omega_B}{2} x_- v^*_{n,\ell,k} = \frac{\sqrt{2m\hbar\omega_B}}{2}\left[\sqrt{n+1}\, v^*_{n+1,\ell,k} - i\sqrt{\ell}\, v^*_{n,\ell-1,k}\right]$$

Thus we see that the operator

$$\frac{i\pi^*_+}{\sqrt{2m\hbar\omega_B}}$$

behaves as a creation operator for the function $v_{n,\ell,k}$ terms of the angular momentum states, while the operator

$$\frac{-i\pi^*_-}{\sqrt{2m\hbar\omega_B}}$$

serves as an annihilation operator.

MATHEMATICAL APPENDIX VI

We give here a proof of the following identities:

$$\sum_{\ell=0}^{\infty} v_{n,\ell,k}(\vec{r}_1)v^*_{n,\ell,k}(\vec{r}_2) \equiv \frac{m\omega_B}{2\pi\hbar n!} e^{ik(z_1 - z_2)}$$

$$\times\ e^{\frac{im\omega_B}{2\hbar}(x_1y_2 - x_2y_1)}\, e^{-\frac{v_{12}}{2}} L_n(v_{12}),$$

$$\sum_{\ell=0}^{\infty} v_{n+1,\ell,k}(\vec{r}_1)v^*_{n,\ell,k}(\vec{r}_2) \equiv \frac{i(x_{2+} - x_{1+})}{\pi(n+1)!\,\sqrt{n+1}}\left(\frac{m\omega_B}{2\hbar}\right)^{\frac{3}{2}}$$

$$\times\ e^{ik(z_1 - z_2)}\, e^{\frac{im\omega_B}{2\hbar}(x_1y_2 - x_2y_1)}\, e^{-\frac{v_{12}}{2}} \frac{d}{dv_{12}} L_{n+1}(v_{12}$$

$$\sum_{\ell=0}^{\infty} v_{n,\ell,k}(\vec{r}_1)v^*_{n+1,\ell,k}(\vec{r}_2) \equiv \frac{i(x_{1-} - x_{2-})}{\pi(n+1)!\,\sqrt{n+1}}\left(\frac{m\omega_B}{2\hbar}\right)^{\frac{3}{2}} \qquad (6\text{-}1)$$

$$\times\ e^{ik(z_1 - z_2)}\, e^{\frac{im\omega_B}{2\hbar}(x_1y_2 - x_2y_1)}\, e^{-\frac{v_{12}}{2}} \frac{d}{dv_{12}} L_{n+1}(v_{12}),$$

$$\sum_{\ell=0}^{\infty} v_{n+2,\ell,k}(\vec{r}_1)v^*_{n,\ell,k}(\vec{r}_2) \equiv \frac{-(x_{2+} - x_{1+})^2}{\pi(n+2)!\,\sqrt{(n+1)(n+2)}}\left(\frac{m\omega_B}{2\hbar}\right)^{2}$$

$$\times\ e^{ik(z_1 - z_2)}\, e^{\frac{im\omega_B}{2\hbar}(x_1y_2 - x_2y_1)}\, e^{-\frac{v_{12}}{2}} \frac{d^2}{dv_{12}^2} L_{n+2}(v_{12}),$$

$$\sum_{\ell=0}^{\infty} v_{n,\ell,k}(\vec{r}_1) v^*_{n+2,\ell,k}(\vec{r}_2) \equiv \frac{-(x_{1-}-x_{2-})^2}{\pi(n+2)!\sqrt{(n+1)(n+2)}} \left(\frac{m\omega_B}{2\hbar}\right)^2$$

$$\times\ e^{ik(z_1 - z_2)}\ e^{\frac{im\omega_B}{2\hbar}(x_1 y_2 - x_2 y_1)}\ e^{-\frac{v_{12}}{2}} \frac{d^2}{dv_{12}^2} L_{n+2}(v_{12}) ,$$

and

$$\sum_{\ell=0}^{\infty} v_{n,\ell,k}(\vec{r}_1) v_{n,\ell,k}(\vec{r}_1) = \frac{m\omega_B}{2\pi\hbar} ,$$

$$\sum_{\ell=0}^{\infty} v_{n+1,\ell,k}(\vec{r}_1) v^*_{n,\ell,k}(\vec{r}_1) = \sum_{\ell=0}^{\infty} v_{n,\ell,k}(\vec{r}_1) v^*_{n+1,\ell,k}(\vec{r}_1) = 0 ,$$

$$\sum_{\ell=0}^{\infty} v_{n+2,\ell,k}(\vec{r}_1) v^*_{n,\ell,k}(\vec{r}_1) = \sum_{\ell=0}^{\infty} v_{n,\ell,k}(\vec{r}_1) v^*_{n+2,\ell,k}(\vec{r}_1) = 0 , \qquad (6-2)$$

where the $v_{n,\ell,k}(\vec{r})$ are the functions developed in Appendix V (5-35, 42),

$$v_{12} = \frac{m\omega_B}{2\hbar^2}\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]$$

$$= \frac{m\omega_B}{2\hbar}(x_{1+} - x_{2+})(x_{1-} - x_{2-}) , \qquad (6-3)$$

and $L_n(v)$ is the Laguerre polynomial

$$L_n(v) = e^v \frac{d^n}{dv^n} (e^{-v} v^n)$$
$$= n! \sum_{k=0}^{n} \frac{(-1)^k n! v^k}{(n-k)! (k!)^2} , \qquad (6\text{-}4)$$

which satisfies the differential equation

$$v \frac{d^2 L_n(v)}{dv^2} + (1 - v) \frac{dL_n(v)}{dv} + n L_n(v) = 0 ,$$

and the recurrence relation

$$\frac{d}{dv} L_n(v) - L_n(v) = \frac{1}{(n+1)} \frac{d}{dv} L_{n+1}(v) . \qquad (6\text{-}5)$$

(See reference (45).)

From equation (5-35) we obtain the relation

$$v_{n,\ell,k}(\vec{r}_1) v^*_{n,\ell,k}(\vec{r}_2) = \frac{1}{\pi n! \ell !} \left(\frac{m\omega_B}{2\hbar}\right)^{1+\ell-n} e^{ik(z_1 - z_2)} \times$$
$$e^{\frac{m\omega_B}{4\hbar}(x_{1+}x_{1-} + x_{2-}x_{2+})} \left(\frac{\partial^2}{\partial x_{1-} \partial x_{2+}}\right)^n (x_{1-}x_{2+})^\ell \, e^{-\frac{m\omega_B}{4\hbar}(x_{1+}x_{1-} + x_{2-}x_{2+})} \qquad (6\text{-}6)$$

Thus

$$\sum_{\ell=0}^{\infty} v_{n,\ell,k}(\vec{r}_1) v^{*}_{n,\ell,k}(\vec{r}_2) = \frac{1}{\pi n!}\left(\frac{m\omega_B}{2\hbar}\right) e^{ik(z_1 - z_2)}$$

$$\times\ e^{\frac{m\omega_B}{4\hbar}(x_{1+}x_{1-} + x_{2-}x_{2+})}\left(\frac{2\hbar}{m\omega_B}\frac{\partial^2}{\partial x_{1-}\partial x_{2+}}\right)^n \qquad (6\text{-}7)$$

$$\times\ e^{\frac{m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})},$$

since

$$\sum_{\ell=0}^{\infty} \frac{x^{\ell}}{\ell!} = e^{x},$$

and

$$\sum_{\ell=0}^{\infty} \frac{1}{\ell!}\left(\frac{m\omega_B}{2\hbar} x_{1-}x_{2+}\right)^{\ell} = e^{\frac{m\omega_B}{2\hbar} x_{1-}x_{2+}} .$$

Now, the following identity will be proven by the method of mathematical induction:

$$e^{\frac{-m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})}\left(\frac{2\hbar}{m\omega_B}\frac{\partial^2}{\partial x_{1-}\partial x_{2+}}\right)^n$$

$$\times\ e^{\frac{m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})} \qquad (6\text{-}8)$$

$$\equiv L_n\left[\frac{-m\omega_B}{2\hbar}(x_{1+} - x_{2+})(x_{1-} - x_{2-})\right].$$

The expression (6-8) certainly holds true for $n = 0$ since $L_o(v) = 1$ from (6-4). Now, if we assume (6-8) for arbitrary positive integer n, then

$$e^{\frac{-m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})}\left(\frac{2\hbar}{m\omega_B}\frac{\partial^2}{\partial x_{1-}\partial x_{2+}}\right)^{n+1}$$

$$\times\; e^{\frac{m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})}$$

$$= e^{\frac{-m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})}\left(\frac{2\hbar}{m\omega_B}\frac{\partial^2}{\partial x_{1-}\partial x_{2+}}\right)$$

$$\times\; e^{\frac{m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})}\; L_n(v_{12}) \tag{6-9}$$

$$= \left\{(1 - v_{12})\, L_n(v_{12}) + (x_{1-} - x_{2-})\frac{\partial L_n(v_{12})}{\partial x_{1-}} + (x_{2+} - x_{1+})\frac{\partial L_n(v_{12})}{\partial x_{2+}}\right.$$

$$\left. + \frac{2\hbar}{m\omega_B}\frac{\partial^2 L_n(v_{12})}{\partial x_{1-}\partial x_{2+}}\right\}$$

$$= \left\{(1 - v_{12})\, L_n(v_{12}) + \left[\frac{2\hbar}{m\omega_B}\frac{\partial^2 v_{12}}{\partial x_{1-}\partial x_{2+}} + (x_{1-} - x_{2-})\frac{\partial v_{12}}{\partial x_{1-}}\right.\right.$$

$$\left. + (x_{2+} - x_{1+})\frac{\partial v_{12}}{\partial x_{2+}}\right]\frac{dL_n(v_{12})}{dv_{12}}$$

$$\left. + \frac{2\hbar}{m\omega_B}\frac{\partial v_{12}}{\partial x_{1-}}\frac{\partial v_{12}}{\partial x_{2+}}\frac{d^2 L_n(v_{12})}{dv_{12}^2}\right\}$$

where

$$v_{12} = \frac{m\omega_B}{2\hbar}(x_{1-} - x_{2-})(x_{1+} - x_{2+}) .$$

Now, from (6-3)

$$\frac{\partial v_{12}}{\partial x_{1-}} = \frac{m\omega_B}{2\hbar}(x_{1+} - x_{2+}) ,$$

$$\frac{\partial v_{12}}{\partial x_{2+}} = \frac{m\omega_B}{2\hbar}(x_{2-} - x_{1-}) , \qquad (6\text{-}10)$$

$$\frac{\partial^2 v_{12}}{\partial x_{1-}\partial x_{2+}} = \frac{m\omega_B}{2\hbar}$$

and thus the expression (6-9) becomes

$$e^{\frac{-m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})}\left(\frac{2\hbar}{m\omega_B}\frac{\partial^2}{\partial x_{1-}\partial x_{2+}}\right)^{n+1}$$

$$\times\, e^{\frac{m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})} \qquad (6\text{-}11)$$

$$= \left\{(1 - v_{12})L_n(v_{12}) + (2v_{12} - 1)\frac{dL_n(v_{12})}{dv_{12}} - v_{12}\frac{d^2L_n(v_{12})}{dv_{12}^2}\right\} .$$

Using the series expression (6-4) for $L_n(v)$, one obtains

$$(1 - v)L_n(v) + (2v - 1)\frac{dL_n(v)}{dv} - v\frac{d^2L_n(v)}{dv^2}$$

$$= \sum_{k=0}^{n}\frac{(-1)^k(n!)^2\left[(1 + 2k)v^k - v^{k+1} + k^2v^{k-1}\right]}{(n-k)!\,(k!)^2} \qquad (6\text{-}12)$$

$$= \sum_{k=0}\frac{(-1)^k(n+1)!^2\,v^k}{(n+1-k)!\,(k!)^2} = L_{n+1}(v) ,$$

and thus equation (6-11) reduces to the form

$$e^{\frac{-m\omega_B}{2\hbar}(x_{1-}x_{2+}-x_{1+}x_{1-}-x_{2-}x_{2+})}\left(\frac{2h}{m\omega_B}\frac{\partial^2}{\partial x_{1-}\partial x_{2+}}\right)^{n+1} \tag{6-13}$$

$$\times\ e^{\frac{m\omega_B}{2\hbar}(x_{1-}x_{2+}-x_{1+}x_{1-}-x_{2-}x_{2+})} = L_{n+1}(v_{12})\ .$$

Thus, since the result (6-13) is equivalent to (6-8), one may conclude that the expression (6-8) has been proven valid for arbitrary positive integer n. Therefore, equation (6-7) may be given as

$$\sum_{\ell=0}^{\infty} v_{n,\ell,k}(\vec{r}_1)v^{*}_{n,\ell,k}(\vec{r}_2) = \frac{m\omega_B}{2\pi\hbar n!}\ e^{ik(z_1-z_2)}$$

$$\times\ e^{\frac{m\omega_B}{4\hbar}(2x_{1-}x_{2+}-x_{1+}x_{1-}-x_{2-}x_{2+})}\ L_n(v_{12})$$

$$= \frac{m\omega_B}{2\pi\hbar n!}\ e^{ik(z_1-z_2)}\ e^{\frac{m\omega_B}{4\hbar}(x_{2+}x_{1-}-x_{1+}x_{2-})}$$

$$\times\ e^{\frac{-v_{12}}{2}}\ L_n(v_{12}) \tag{6-14}$$

$$= \frac{m\omega_B}{2\pi\hbar n!}\ e^{ik(z_1-z_2)}\ e^{\frac{im\omega_B}{2\hbar}(x_1y_2-x_2y_1)}$$

$$\times\ e^{\frac{-v_{12}}{2}}\ L_n(v_{12})$$

Equation (6-14) is the first of the identities (6-11).

From (5-35) one has the relation

$$\sum_{\ell=0}^{\infty} v_{n+1,\ell,k}(\vec{r}_1) v^*_{n,\ell,k}(\vec{r}_2) = \frac{-i}{\sqrt{n+1}} \left(\frac{m\omega_B}{2\pi\hbar n!}\right) e^{ik(z_1 - z_2)} \tag{6-15}$$

$$\times\ e^{\frac{m\omega_B}{4\hbar}(x_{1+}x_{1-} + x_{2-}x_{2+})} \left(\sqrt{\frac{2\hbar}{m\omega_B}} \frac{\partial}{\partial x_{1-}}\right)\left(\frac{2\hbar}{m\omega_B} \frac{\partial^2}{\partial x_{1-}\partial x_{2+}}\right)^n$$

$$\times\ e^{\frac{m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})}$$

by a procedure similar to that for (6-7).

By comparison with (6-8), we see that (6-15) may be written in the form

$$\sum_{\ell=0}^{\infty} v_{n+1,\ell,k}(\vec{r}_1) v^*_{n,\ell,k}(\vec{r}_2) = \frac{-i}{\sqrt{n+1}} \left(\frac{m\omega_B}{2\pi\hbar n!}\right) e^{ik(z_1 - z_2)}$$

$$\times\ e^{\frac{m\omega_B}{4\hbar}(x_{1+}x_{1-} - x_{2-}x_{2+})} \left(\frac{2\hbar}{m\omega_B} \frac{\partial}{\partial x_{1-}}\right)$$

$$\times\ e^{\frac{m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})} L_n(v_{12}) \tag{6-16}$$

$$= \frac{-i(x_{2+} - x_{1+})}{\pi n! \sqrt{n+1}} \left(\frac{m\omega_B}{2\hbar}\right)^{\frac{3}{2}} e^{ik(z_1 - z_2)}$$

$$\times\ e^{\frac{m\omega_B}{4\hbar}(2x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})} \left[L_n(v_{12}) + \frac{2\hbar}{m\omega_B} \frac{1}{(x_{2+} - x_{1+})} \frac{\partial v_{12}}{\partial x_{1-}} \frac{dL_n(v_{12})}{dv_{12}}\right]$$

$$= \frac{i(x_{2+} - x_{1+})}{\pi (n+1)! \sqrt{n+1}} \left(\frac{m\omega_B}{2\hbar}\right)^{\frac{3}{2}} e^{ik(z_1 - z_2)}$$

$$\times\ e^{\frac{im\omega_B}{2\hbar}(x_1 y_2 - x_2 y_1)} e^{-\frac{v_{12}}{2}} \frac{dL_{n+1}(v_{12})}{dv_{12}} ,$$

from (6-5) and (6-10). Equation (6-16) is precisely the second of the identities (6-1). The third identity in (6-1) is found from the complex conjugate of (6-16), with the subscripts 1 and 2 interchanged.

The fourth of the expressions (6-1) is given from (5-35) as

$$\sum_{\ell=0}^{\infty} v_{n+2,\ell,k}(\vec{r}_1) v^*_{n,\ell,k}(\vec{r}_2) = -\frac{1}{\pi n! \sqrt{(n+1)(n+2)}} \left(\frac{m\omega_B}{2\hbar}\right)$$

$$\times\ e^{ik(z_1 - z_2)} e^{\frac{m\omega_B}{4\hbar}(x_{1+}x_{1-} - x_{2-}x_{2+})} \left(\sqrt{\frac{2\hbar}{m\omega_B}} \frac{\partial}{\partial x_{1-}}\right)^2 \left(\frac{2\hbar}{m\omega_B} \frac{\partial^2}{\partial x_{1-} \partial x_{2+}}\right)^n$$

$$\times\ e^{\frac{m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})} \qquad (6\text{-}17)$$

$$= -\frac{1}{\pi n! \sqrt{(n+1)(n+2)}} \left(\frac{m\omega_B}{2\hbar}\right) e^{ik(z_1 - z_2)} e^{\frac{m\omega_B}{4\hbar}(x_{1+}x_{1-} + x_{2-}x_{2+})}$$

$$\times \left(\sqrt{\frac{2\hbar}{m\omega_B}} \frac{\partial}{\partial x_{1-}}\right)^2 e^{\frac{m\omega_B}{2\hbar}(x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})} L_n(v_{12})$$

$$= \frac{1}{\pi n! \sqrt{(n+1)(n+2)}} \left(\frac{m\omega_B}{2\hbar}\right) e^{ik(z_1 - z_2)} e^{\frac{m\omega_B}{4\hbar}(2x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})}$$

$$\times \left\{ \frac{2\hbar}{m\omega_B} \left(\frac{\partial v_{12}}{\partial x_{1-}}\right)^2 \frac{d^2 L_n(v_{12})}{dv_{12}^2} + \left[\frac{2\hbar}{m\omega_B}\left(\frac{\partial^2 v_{12}}{\partial x_{1-}^2}\right) + 2(x_{2+} - x_{1+})\frac{\partial v_{12}}{\partial x_{1-}}\right] \frac{dL_n(v_{12})}{dv_{12}} \right.$$

$$\left. + \frac{m\omega_B}{2\hbar}(x_{2+} - x_{1+})^2 L_n(v_{12}) \right\}$$

$$= \frac{-(x_{2+} - x_{1+})^2}{\pi n! \sqrt{(n+1)(n+2)}} \left(\frac{m\omega_B}{2\hbar}\right)^2 e^{ik(z_1 - z_2)} e^{\frac{m\omega_B}{4\hbar}(2x_{1-}x_{2+} - x_{1+}x_{1-} - x_{2-}x_{2+})}$$

$$\times \left\{ \frac{d}{dv_{12}} \left[\frac{dL_n(v_{12})}{dv_{12}} - L_n(v_{12})\right] - \left[\frac{dL_n(v_{12})}{dv_{12}} - L_n(v_{12})\right] \right\}$$

$$= \frac{-(x_{2+} - x_{1+})^2}{\pi (n+2)! \sqrt{(n+1)(n+2)}} \left(\frac{m\omega_B}{2\hbar}\right)^2 e^{ik(z_1 - z_2)} e^{i\frac{m\omega_B}{2\hbar}(x_1 y_2 - x_2 y_1)} \frac{d^2 L_{n+2}(v_{12})}{dv_{12}^2}$$

$$\times e^{-\frac{v_{12}}{2}}$$

by the same type of argument as was used for (6-7). Equation (6-17) is the form of the fourth identity (6-1). The fifth identity in (6-1) is obtained from the complex conjugate of (6-17) with the subscripts 1 and 2 interchanged.

The identities (6-2) are easily obtained from the expressions (6-1). The second of the equations (6-4) leads to the following equalities:

$$\lim_{v \to 0} L_n(v) = n!$$

$$\lim_{v \to 0} \frac{d}{dv} L_n(v) = -n!\, n \tag{6-18}$$

$$\lim_{v \to 0} \frac{d^2}{dv^2} L_n(v) = n! \frac{n(n-1)}{2}.$$

Thus, in the case $\vec{r}_2 = \vec{r}_1 = \vec{r}$, the equations (6-1) reduce to the relations (6-2).

The identities

$$\sum_{\ell=0}^{\infty} \sqrt{\ell}\, v^*_{n,\ell-1,k}(\vec{r})\, v_{n,\ell,k}(\vec{r}) = \left(\frac{m\omega_B}{2\hbar}\right)^{\frac{3}{2}} \frac{x_-}{\pi}$$

$$\sum_{\ell=0}^{\infty} \sqrt{\ell}\, v^*_{n,\ell,k}(\vec{r})\, v_{n,\ell-1,k}(\vec{r}) = \left(\frac{m\omega_B}{2\hbar}\right)^{\frac{3}{2}} \frac{x_+}{\pi}$$

$$\sum_{\ell=0}^{\infty} \sqrt{\ell+1}\, v^*_{n,\ell+1,k}(\vec{r})\, v_{n,\ell,k}(\vec{r}) = \left(\frac{m\omega_B}{2\hbar}\right)^{\frac{3}{2}} \frac{x_+}{\pi}$$

$$\sum_{\ell=0}^{\infty} \sqrt{\ell+1}\, v^*_{n,\ell,k}(\vec{r})\, v_{n,\ell+1,k}(\vec{r}) = \left(\frac{m\omega_B}{2\hbar}\right)^{\frac{3}{2}} \frac{x_-}{\pi} \qquad (6\text{-}19)$$

$$\sum_{\ell=0}^{\infty} \sqrt{\ell(\ell-1)}\, v^*_{n,\ell-2,k}(\vec{r})\, v_{n,\ell,k}(\vec{r}) = \left(\frac{m\omega_B}{2\hbar}\right)^{2} \frac{x_-^2}{\pi}$$

$$\sum_{\ell=0}^{\infty} \sqrt{\ell(\ell-1)}\, v^*_{n,\ell,k}(\vec{r})\, v_{n,\ell-2,k}(\vec{r}) = \left(\frac{m\omega_B}{2\hbar}\right)^{2} \frac{x_+^2}{\pi}$$

$$\sum_{\ell=0}^{\infty} \sqrt{(\ell+1)(\ell+2)}\, v^*_{n,\ell+2,k}(\vec{r})\, v_{n,\ell,k}(\vec{r}) = \left(\frac{m\omega_B}{2\hbar}\right)^{2} \frac{x_+^2}{\pi}$$

$$\sum_{\ell=0}^{\infty} \sqrt{(\ell+1)(\ell+2)}\, v^*_{n,\ell,k}(\vec{r})\, v_{n,\ell+2,k}(\vec{r}) = \left(\frac{m\omega_B}{2\hbar}\right)^{2} \frac{x_-^2}{\pi}$$

are obtained from (5-50) and (6-2), by taking simple combinations of terms.

MATHEMATICAL APPENDIX VII

Part 1:

We give here a proof of the identity

$$\lim_{\varepsilon \to 0^+} \frac{1}{x \mp i\varepsilon} \equiv P\left(\frac{1}{x}\right) \pm i\pi \, \delta(x), \; \varepsilon > 0 \; , \tag{7-1}$$

where ε is an "infinitesimal." Given some function f(x) which is continuous in the neighborhood of x = 0, we consider the following integral:

$$\int_{-\infty}^{\infty} dx \, \frac{1}{x \mp i\varepsilon} \, f(x) = \int_{-\infty}^{\infty} \frac{x \, f(x)}{(x^2 + \varepsilon^2)} \, dx \pm i\varepsilon \int_{-\infty}^{\infty} \frac{f(x) \, dx}{(x^2 + \varepsilon^2)} \; . \tag{7-2}$$

Now, by definition

$$P\int_{-\infty}^{\infty} \frac{f(x)}{x} \, dx \equiv \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \frac{x \, f(x) \, dx}{(x^2 + \varepsilon^2)} = \lim_{\varepsilon \to 0^+} \left[\int_{-\infty}^{-\varepsilon} \frac{f(x)}{x} \, dx + \int_{+\varepsilon}^{\infty} \frac{f(x)}{x} dx \right]$$

$$\equiv \int_{-\infty}^{\infty} dx \, f(x) \, P\left(\frac{1}{x}\right) \; . \tag{7-3}$$

With the change of variable

$$x = \varepsilon z \; . \tag{7-4}$$

we obtain the result

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \frac{\varepsilon \, f(x)}{x^2 + \varepsilon^2} \, dx = \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \frac{f(\varepsilon z) \, dz}{1 + z^2}$$

$$= f(0) \int_{-\infty}^{\infty} \frac{dz}{1 + z^2} \tag{7-5}$$

$$= \pi \, f(0) \; .$$

Therefore

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} dx \, \frac{1}{x \mp i\varepsilon} \, f(x) = P \int_{-\infty}^{\infty} dx \, \frac{f(x)}{x} \pm i\pi \, f(0) \, , \qquad (7\text{-}6)$$

and we see that

$$\frac{1}{x \mp i\varepsilon} \to P\left(\frac{1}{x}\right) \pm i\pi \, \delta(x), \; \varepsilon \to 0^+ \, . \qquad (7\text{-}7)$$

Part 2:

The Cauchy Integral Theorem may be utilized in many integrations of importance in physical theory. We are particularly concerned here with integrals containing "complex exponentials" in the integrand.

From complex variable theory, we have (see reference (41))

$$f(z) = \frac{1}{2\pi i} \oint \frac{d\omega \, f(\omega)}{\omega - z} = - \frac{1}{2\pi i} \oint \frac{d\omega \, f(\omega)}{\omega - z} \, . \qquad (7\text{-}8)$$

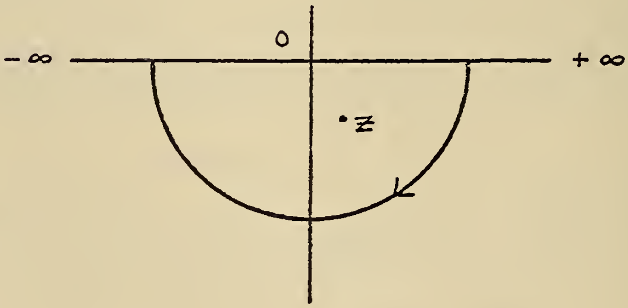
Now integrals of the form

$$\int_{-\infty}^{\infty} \frac{d\omega \, g(\omega) \, e^{-i\omega t}}{(\omega - z)} \, , \qquad (7\text{-}9)$$

where $g(\omega)$ is analytic in the complex ω plane, can be evaluated via (7-8). For $t > 0$, we choose an integration contour in the lower half ω plane of the form of a semicircle:

$$\oint \frac{d\omega \, g(\omega) \, e^{-i\omega t}}{(\omega - z)} = \lim_{R \to \infty} \int_{-R}^{R} \frac{d\omega \, g(\omega) \, e^{-i\omega t}}{(\omega - z)} \qquad (7\text{-}10)$$

$$+ \lim_{R \to \infty} \int_{0}^{\pi} \frac{-i \, R \, e^{-i\theta} \, d\theta \, g(R \, e^{-i\theta}) \, e^{-iRe^{-i\theta} t}}{(R \, e^{-i\theta} - z)} \, , \quad t > 0 \, ,$$



The exponential in the integrand (7-10) causes a vanishing contribution to the integral about the "infinite" semicircle, thus

$$\int_{-\infty}^{\infty} \frac{d\omega\, g(\omega)\, e^{-i\omega t}}{(\omega - z)} = \oint \frac{d\omega\, g(\omega)\, e^{-i\omega t}}{(\omega - z)}, \quad t>0$$

$$= \begin{cases} -i\, 2\pi\ g(z)\, e^{-izt}, & z \text{ in the lower half plane} \\ 0 & , z \text{ in the upper half plane} \end{cases} \qquad (7\text{-}11)$$

from (7-9). Similarly, for $t<0$ an integration in the upper half plane results in

$$\int_{-\infty}^{\infty} \frac{d\omega\, g(\omega)\, e^{-i\omega t}}{(\omega - z)} = + \oint \frac{d\omega\, g(\omega)\, e^{-i\omega t}}{(\omega - z)}, \quad t<0$$

$$= \begin{cases} i\, 2\pi\ g(z)\, e^{-izt}, & z \text{ in the upper half plane} \\ 0 & , z \text{ in the lower half plane.} \end{cases} \qquad (7\text{-}12)$$

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