Precise theory of levels of hydrogen and deuterium: The Coulomb vacuum polarization correction

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The Coulomb vacuum polarization contribution to the energy levels of hydrogen and deuterium atoms is calculated for states with $n \le 200$. The Uehling potential is the dominant contribution of the vacuum polarization to the energy levels. Numerical values for this contribution are given to nine figure precision in this paper for states with $l \le 2$. This correction is negligible for higher values of l. These results are available on the NIST Physics Laboratory Web site at physics.nist.gov/hdel.

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I. INTRODUCTION

Energy levels in hydrogen and deuterium are determined primarily by the eigenvalues of the Dirac equation. However, to obtain accurate values for the levels, it is necessary to take into account many additional corrections, including those due to quantum electrodynamic (QED) effects. In hydrogen and deuterium, the largest QED effect is the self-energy. The next largest is the vacuum polarization, which is due to the creation of a virtual electron-positron pair in the exchange of photons between the electron and the nucleus.

II. VACUUM POLARIZATION

The second-order vacuum-polarization level shift is written as

$$E^{(2)}_{\rm VP} = \frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3} H(Z\alpha)\, m_{\rm e}c^2, \tag{1}$$

where Z is the charge of the nucleus, which in the case of hydrogen and deuterium, is 1, α is the fine structure constant, n is the principal quantum number of the state, $m_{\rm e}$ is the mass of the electron, c is the speed of light, and $H(Z\alpha)$ is a dimensionless function of order 1. In order to facilitate the calculation, that function is divided into two parts, corresponding to the Uehling potential [1, 2], denoted here by $H^{(1)}(Z\alpha)$ and a higher-order remainder $H^{(\rm R)}(Z\alpha) = H^{(3)}(Z\alpha) + H^{(5)}(Z\alpha) + \cdots$, where the superscript denotes the order in powers of the external field. The individual terms may be expanded in a power series in $Z\alpha$ as

$$\begin{aligned} H^{(1)}(Z\alpha) &= V_{40} + V_{50}\,(Z\alpha) + V_{61}\,(Z\alpha)^2 \ln(Z\alpha)^{-2} \\ &\quad + G^{(1)}_{\rm VP}(Z\alpha)\,(Z\alpha)^2 \qquad (2)\\ H^{(\rm R)}(Z\alpha) &= G^{(\rm R)}_{\rm VP}(Z\alpha)\,(Z\alpha)^2\ , \qquad (3)\end{aligned}$$

with

$$\begin{aligned} V_{40} &= -\frac{4}{15}\,\delta_{l0} \\ V_{50} &= \frac{5}{48}\pi\,\delta_{l0} \qquad (4)\\ V_{61} &= -\frac{2}{15}\,\delta_{l0}, \end{aligned}$$

where

$$\delta_{l0} = \begin{cases} 1 & \text{if } l = 0 \\ 0 & \text{if } l \ge 1 \end{cases}, \tag{5}$$

and l is the orbital angular momentum of the state. The remaining part $G^{(1)}_{\rm VP}(Z\alpha)$ that arises from the Uehling potential is readily calculated numerically, as described in Sec. III.

The higher-order remainder $G^{(\rm R)}_{\rm VP}(Z\alpha)$ has been considered by Wichmann and Kroll, and the leading terms in powers of $Z\alpha$ are [3–5]

$$\begin{aligned} G^{(\rm R)}_{\rm VP}(Z\alpha) &= \left(\frac{19}{45} - \frac{\pi^2}{27}\right)\delta_{l0} \\ &\quad + \left(\frac{1}{16} - \frac{31\pi^2}{2880}\right)\pi(Z\alpha)\delta_{l0} + \cdots\ . \end{aligned} \tag{6}$$

Higher-order terms omitted from Eq. (6) are negligible in hydrogen and deuterium.

III. CALCULATION

The numerical evaluation of the Uehling potential correction has been described before [6, 7] and is reviewed in this section.. The Uehling potential in a Coulomb field, in units where $m_{\rm e}$, c, and $\hbar$ are replaced by 1, is [1, 2]

$$\begin{aligned} U(x) &= -\frac{\alpha}{\pi}Z\alpha\int_1^\infty {\rm d}t\,\left(t^2-1\right)^{1/2} \\ &\quad \times \left(\frac{2}{3t^2} + \frac{1}{3t^4}\right)\frac{\exp(-2tx)}{x}\ , \end{aligned} \tag{7}$$

and the energy-level shift is

$$E_{\rm U} = \int {\rm d}\boldsymbol{x}\,|\phi_n(\boldsymbol{x})|^2\, U(x) \tag{8}$$

where $\phi_n(\boldsymbol{x})$ is the Dirac wave function for hydrogen. This can be written as

$$E_{\rm U} = -\frac{Z\alpha^2}{\pi}\int_1^\infty {\rm d}t\,\left(t^2-1\right)^{1/2}\left(\frac{2}{3t^2} + \frac{1}{3t^4}\right)L_n(2t), \tag{9}$$

where

$$L_n(u) = \int \mathrm{d}\boldsymbol{x}\, |\phi_n(\boldsymbol{x})|^2 \, \frac{\exp(-ux)}{x}$$
$$= \int_0^\infty \mathrm{d}x\, x \left[f_1^2(x) + f_2^2(x)\right] \exp(-ux) \,, \quad (10)$$

where f_1 and f_2 are the radial wave functions as defined in Ref. [6]

The transform $L_n(u)$ is an elementary function for any state, and is given explicitly for the states with $n = 1, 2$ in Ref. [6]. However, to evaluate $L_n(u)$ for a wide range of values of n, it is convenient to numerically integrate the expression in Eq. (10). The integrand in Eq. (10) is of the form

$$x^{1-\epsilon} P(x) \exp(-ax), \quad (11)$$

where $\epsilon << 1$ and $P(x)$ is a polynomial with degree of order $2n$. An integral of this form is evaluated exactly by applying Gauss-Laguerre quadrature of the general form that includes the fractional power in the weight function, provided a sufficient number of abscissas and weights are employed [8]. In principle, this number is of order n in order to obtain the exact result, but in fact an accurate evaluation results for a very small number of integration points, of order 10 or less, even when $n = 200$, because the dominant contribution comes from the first few powers of x. The radial wave functions f_1 and f_2 in Eq. (10) are readily evaluated with the aid of Ref. [9], which gives recursion relations for the polynomial terms in the wave functions.

The integral over t in Eq. (9) is evaluated by making a change of variable to $x = 1 - (1 - t^{-2})^{1/2}$ to obtain

$$E_{\mathrm{U}} = -\frac{Z\alpha^2}{3\pi} \int_0^1 \mathrm{d}x\, (1-x)^2 \frac{2 + 2x - x^2}{2x - x^2}$$
$$\times L_n\left[\frac{2}{(2x - x^2)^{1/2}}\right], \quad (12)$$

and finally evaluating the integral with Gauss-Legendre quadrature applied to the variable s, where $x = s^4$. The values obtained here result from a 70 point evaluation, although this is well above the number of integration points needed to obtain the quoted uncertainty.

The accuracy of the two-dimensional integration was monitored by comparing the results of 8×60 point integration and 10×70 point integration for each evaluation. In all cases, the difference is well below the quoted precision.

The results for $G_{\mathrm{VP}}^{(1)}(Z\alpha)$ are obtained by computing

$$G_{\mathrm{VP}}^{(1)}(Z\alpha) = (Z\alpha)^{-2}\big[H^{(1)}(Z\alpha) - V_{40} - V_{50}\,(Z\alpha) - V_{61}\,(Z\alpha)^2 \ln(Z\alpha)^{-2}\big] \quad (13)$$

where $H^{(1)}(Z\alpha)$ is given by

$$H^{(1)}(Z\alpha) = \left[\frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3} m_{\mathrm{e}} c^2\right]^{-1} E_{\mathrm{U}}, \quad (14)$$

IV. RESULTS

The results of the numerical evaluation of $G_{\mathrm{VP}}^{(1)}(Z\alpha)$ for $Z = 1$ and $\alpha = 1/137.036$ are given in Table I. All figures shown in the table are significant, and the numbers have been rounded to the nearest last figure.

The Uehling potential correction for states with angular momentum $l \geq 3$ is zero to the number of figures shown in the table.

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TABLE I: Values of the function $G^{(1)}_{\rm VP}(\alpha)$.

n	$S_{1/2}$	$P_{1/2}$	$P_{3/2}$	$D_{3/2}$	$D_{5/2}$
1	−0.618 723 915				
2	−0.808 872 415	−0.064 005 731	−0.014 132 445		
3	−0.814 530 258	−0.075 858 689	−0.016 749 740	−0.000 000 046	−0.000 000 015
4	−0.806 578 623	−0.080 007 050	−0.017 665 786	−0.000 000 065	−0.000 000 021
5	−0.798 362 423	−0.081 927 062	−0.018 089 778	−0.000 000 075	−0.000 000 024
6	−0.791 450 317	−0.082 969 986	−0.018 320 090	−0.000 000 080	−0.000 000 026
7	−0.785 810 647	−0.083 598 809	−0.018 458 958	−0.000 000 083	−0.000 000 027
8	−0.781 196 818	−0.084 006 922	−0.018 549 086	−0.000 000 086	−0.000 000 027
9	−0.777 380 712	−0.084 286 710	−0.018 610 877	−0.000 000 087	−0.000 000 028
10	−0.774 184 470	−0.084 486 833	−0.018 655 075	−0.000 000 088	−0.000 000 028
11	−0.771 474 587	−0.084 634 896	−0.018 687 775	−0.000 000 089	−0.000 000 028
12	−0.769 151 196	−0.084 747 505	−0.018 712 646	−0.000 000 089	−0.000 000 029
13	−0.767 138 988	−0.084 835 137	−0.018 732 001	−0.000 000 090	−0.000 000 029
14	−0.765 380 486	−0.084 904 668	−0.018 747 359	−0.000 000 090	−0.000 000 029
15	−0.763 831 250	−0.084 960 760	−0.018 759 748	−0.000 000 091	−0.000 000 029
16	−0.762 456 474	−0.085 006 665	−0.018 769 887	−0.000 000 091	−0.000 000 029
17	−0.761 228 557	−0.085 044 708	−0.018 778 291	−0.000 000 091	−0.000 000 029
18	−0.760 125 367	−0.085 076 588	−0.018 785 333	−0.000 000 091	−0.000 000 029
19	−0.759 128 960	−0.085 103 566	−0.018 791 292	−0.000 000 091	−0.000 000 029
20	−0.758 224 651	−0.085 126 599	−0.018 796 380	−0.000 000 092	−0.000 000 029
21	−0.757 400 309	−0.085 146 420	−0.018 800 758	−0.000 000 092	−0.000 000 029
22	−0.756 645 832	−0.085 163 599	−0.018 804 553	−0.000 000 092	−0.000 000 029
23	−0.755 952 738	−0.085 178 586	−0.018 807 864	−0.000 000 092	−0.000 000 029
24	−0.755 313 862	−0.085 191 738	−0.018 810 770	−0.000 000 092	−0.000 000 029
25	−0.754 723 106	−0.085 203 343	−0.018 813 333	−0.000 000 092	−0.000 000 029
26	−0.754 175 248	−0.085 213 635	−0.018 815 607	−0.000 000 092	−0.000 000 029
27	−0.753 665 794	−0.085 222 804	−0.018 817 633	−0.000 000 092	−0.000 000 029
28	−0.753 190 855	−0.085 231 008	−0.018 819 445	−0.000 000 092	−0.000 000 030
29	−0.752 747 048	−0.085 238 378	−0.018 821 074	−0.000 000 092	−0.000 000 030
30	−0.752 331 416	−0.085 245 023	−0.018 822 542	−0.000 000 092	−0.000 000 030
31	−0.751 941 364	−0.085 251 035	−0.018 823 870	−0.000 000 092	−0.000 000 030
32	−0.751 574 608	−0.085 256 492	−0.018 825 076	−0.000 000 092	−0.000 000 030
33	−0.751 229 124	−0.085 261 460	−0.018 826 174	−0.000 000 092	−0.000 000 030
34	−0.750 903 117	−0.085 265 997	−0.018 827 176	−0.000 000 092	−0.000 000 030
35	−0.750 594 988	−0.085 270 149	−0.018 828 094	−0.000 000 092	−0.000 000 030
36	−0.750 303 309	−0.085 273 961	−0.018 828 936	−0.000 000 092	−0.000 000 030
37	−0.750 026 798	−0.085 277 467	−0.018 829 711	−0.000 000 092	−0.000 000 030
38	−0.749 764 303	−0.085 280 700	−0.018 830 425	−0.000 000 092	−0.000 000 030
39	−0.749 514 788	−0.085 283 688	−0.018 831 085	−0.000 000 092	−0.000 000 030
40	−0.749 277 313	−0.085 286 454	−0.018 831 697	−0.000 000 092	−0.000 000 030
41	−0.749 051 029	−0.085 289 020	−0.018 832 264	−0.000 000 092	−0.000 000 030
42	−0.748 835 163	−0.085 291 405	−0.018 832 791	−0.000 000 092	−0.000 000 030
43	−0.748 629 014	−0.085 293 626	−0.018 833 281	−0.000 000 092	−0.000 000 030
44	−0.748 431 940	−0.085 295 696	−0.018 833 739	−0.000 000 092	−0.000 000 030
45	−0.748 243 355	−0.085 297 630	−0.018 834 167	−0.000 000 092	−0.000 000 030
46	−0.748 062 724	−0.085 299 440	−0.018 834 566	−0.000 000 092	−0.000 000 030
47	−0.747 889 553	−0.085 301 135	−0.018 834 941	−0.000 000 092	−0.000 000 030
48	−0.747 723 391	−0.085 302 725	−0.018 835 292	−0.000 000 092	−0.000 000 030
49	−0.747 563 820	−0.085 304 218	−0.018 835 623	−0.000 000 093	−0.000 000 030
50	−0.747 410 457	−0.085 305 623	−0.018 835 933	−0.000 000 093	−0.000 000 030
51	−0.747 262 946	−0.085 306 946	−0.018 836 225	−0.000 000 093	−0.000 000 030
52	−0.747 120 958	−0.085 308 193	−0.018 836 501	−0.000 000 093	−0.000 000 030
53	−0.746 984 191	−0.085 309 371	−0.018 836 761	−0.000 000 093	−0.000 000 030
54	−0.746 852 360	−0.085 310 483	−0.018 837 007	−0.000 000 093	−0.000 000 030
55	−0.746 725 204	−0.085 311 536	−0.018 837 240	−0.000 000 093	−0.000 000 030
56	−0.746 602 479	−0.085 312 532	−0.018 837 460	−0.000 000 093	−0.000 000 030
57	−0.746 483 956	−0.085 313 476	−0.018 837 669	−0.000 000 093	−0.000 000 030
58	−0.746 369 425	−0.085 314 372	−0.018 837 867	−0.000 000 093	−0.000 000 030
59	−0.746 258 686	−0.085 315 223	−0.018 838 055	−0.000 000 093	−0.000 000 030
60	−0.746 151 555	−0.085 316 032	−0.018 838 234	−0.000 000 093	−0.000 000 030
61	−0.746 047 858	−0.085 316 801	−0.018 838 404	−0.000 000 093	−0.000 000 030

TABLE I: *continued*

n	$S_{1/2}$	$P_{1/2}$	$P_{3/2}$	$D_{3/2}$	$D_{5/2}$
62	−0.745 947 433	−0.085 317 533	−0.018 838 566	−0.000 000 093	−0.000 000 030
63	−0.745 850 127	−0.085 318 230	−0.018 838 720	−0.000 000 093	−0.000 000 030
64	−0.745 755 797	−0.085 318 895	−0.018 838 867	−0.000 000 093	−0.000 000 030
65	−0.745 664 309	−0.085 319 530	−0.018 839 007	−0.000 000 093	−0.000 000 030
66	−0.745 575 537	−0.085 320 136	−0.018 839 141	−0.000 000 093	−0.000 000 030
67	−0.745 489 361	−0.085 320 715	−0.018 839 269	−0.000 000 093	−0.000 000 030
68	−0.745 405 669	−0.085 321 268	−0.018 839 392	−0.000 000 093	−0.000 000 030
69	−0.745 324 355	−0.085 321 798	−0.018 839 509	−0.000 000 093	−0.000 000 030
70	−0.745 245 320	−0.085 322 305	−0.018 839 621	−0.000 000 093	−0.000 000 030
71	−0.745 168 469	−0.085 322 791	−0.018 839 728	−0.000 000 093	−0.000 000 030
72	−0.745 093 712	−0.085 323 257	−0.018 839 831	−0.000 000 093	−0.000 000 030
73	−0.745 020 966	−0.085 323 703	−0.018 839 930	−0.000 000 093	−0.000 000 030
74	−0.744 950 150	−0.085 324 132	−0.018 840 025	−0.000 000 093	−0.000 000 030
75	−0.744 881 189	−0.085 324 544	−0.018 840 116	−0.000 000 093	−0.000 000 030
76	−0.744 814 010	−0.085 324 939	−0.018 840 203	−0.000 000 093	−0.000 000 030
77	−0.744 748 545	−0.085 325 319	−0.018 840 287	−0.000 000 093	−0.000 000 030
78	−0.744 684 730	−0.085 325 685	−0.018 840 368	−0.000 000 093	−0.000 000 030
79	−0.744 622 503	−0.085 326 037	−0.018 840 446	−0.000 000 093	−0.000 000 030
80	−0.744 561 806	−0.085 326 375	−0.018 840 521	−0.000 000 093	−0.000 000 030
81	−0.744 502 582	−0.085 326 701	−0.018 840 593	−0.000 000 093	−0.000 000 030
82	−0.744 444 779	−0.085 327 016	−0.018 840 663	−0.000 000 093	−0.000 000 030
83	−0.744 388 347	−0.085 327 319	−0.018 840 730	−0.000 000 093	−0.000 000 030
84	−0.744 333 236	−0.085 327 611	−0.018 840 794	−0.000 000 093	−0.000 000 030
85	−0.744 279 402	−0.085 327 893	−0.018 840 857	−0.000 000 093	−0.000 000 030
86	−0.744 226 800	−0.085 328 165	−0.018 840 917	−0.000 000 093	−0.000 000 030
87	−0.744 175 389	−0.085 328 428	−0.018 840 975	−0.000 000 093	−0.000 000 030
88	−0.744 125 129	−0.085 328 682	−0.018 841 031	−0.000 000 093	−0.000 000 030
89	−0.744 075 981	−0.085 328 927	−0.018 841 086	−0.000 000 093	−0.000 000 030
90	−0.744 027 908	−0.085 329 165	−0.018 841 138	−0.000 000 093	−0.000 000 030
91	−0.743 980 877	−0.085 329 394	−0.018 841 189	−0.000 000 093	−0.000 000 030
92	−0.743 934 854	−0.085 329 616	−0.018 841 238	−0.000 000 093	−0.000 000 030
93	−0.743 889 805	−0.085 329 831	−0.018 841 285	−0.000 000 093	−0.000 000 030
94	−0.743 845 702	−0.085 330 039	−0.018 841 331	−0.000 000 093	−0.000 000 030
95	−0.743 802 514	−0.085 330 240	−0.018 841 376	−0.000 000 093	−0.000 000 030
96	−0.743 760 213	−0.085 330 436	−0.018 841 419	−0.000 000 093	−0.000 000 030
97	−0.743 718 773	−0.085 330 625	−0.018 841 461	−0.000 000 093	−0.000 000 030
98	−0.743 678 166	−0.085 330 808	−0.018 841 502	−0.000 000 093	−0.000 000 030
99	−0.743 638 369	−0.085 330 986	−0.018 841 541	−0.000 000 093	−0.000 000 030
100	−0.743 599 358	−0.085 331 159	−0.018 841 579	−0.000 000 093	−0.000 000 030
101	−0.743 561 108	−0.085 331 326	−0.018 841 616	−0.000 000 093	−0.000 000 030
102	−0.743 523 599	−0.085 331 489	−0.018 841 652	−0.000 000 093	−0.000 000 030
103	−0.743 486 808	−0.085 331 647	−0.018 841 687	−0.000 000 093	−0.000 000 030
104	−0.743 450 716	−0.085 331 800	−0.018 841 721	−0.000 000 093	−0.000 000 030
105	−0.743 415 303	−0.085 331 949	−0.018 841 754	−0.000 000 093	−0.000 000 030
106	−0.743 380 549	−0.085 332 094	−0.018 841 786	−0.000 000 093	−0.000 000 030
107	−0.743 346 437	−0.085 332 235	−0.018 841 817	−0.000 000 093	−0.000 000 030
108	−0.743 312 949	−0.085 332 371	−0.018 841 848	−0.000 000 093	−0.000 000 030
109	−0.743 280 068	−0.085 332 505	−0.018 841 877	−0.000 000 093	−0.000 000 030
110	−0.743 247 778	−0.085 332 634	−0.018 841 906	−0.000 000 093	−0.000 000 030
111	−0.743 216 062	−0.085 332 760	−0.018 841 933	−0.000 000 093	−0.000 000 030
112	−0.743 184 906	−0.085 332 883	−0.018 841 961	−0.000 000 093	−0.000 000 030
113	−0.743 154 294	−0.085 333 002	−0.018 841 987	−0.000 000 093	−0.000 000 030
114	−0.743 124 214	−0.085 333 118	−0.018 842 013	−0.000 000 093	−0.000 000 030
115	−0.743 094 651	−0.085 333 231	−0.018 842 038	−0.000 000 093	−0.000 000 030
116	−0.743 065 591	−0.085 333 342	−0.018 842 062	−0.000 000 093	−0.000 000 030
117	−0.743 037 023	−0.085 333 449	−0.018 842 086	−0.000 000 093	−0.000 000 030
118	−0.743 008 933	−0.085 333 554	−0.018 842 109	−0.000 000 093	−0.000 000 030
119	−0.742 981 310	−0.085 333 656	−0.018 842 132	−0.000 000 093	−0.000 000 030
120	−0.742 954 143	−0.085 333 755	−0.018 842 154	−0.000 000 093	−0.000 000 030
121	−0.742 927 419	−0.085 333 852	−0.018 842 175	−0.000 000 093	−0.000 000 030
122	−0.742 901 129	−0.085 333 947	−0.018 842 196	−0.000 000 093	−0.000 000 030

TABLE I: *continued*

n	$S_{1/2}$	$P_{1/2}$	$P_{3/2}$	$D_{3/2}$	$D_{5/2}$
123	−0.742 875 262	−0.085 334 040	−0.018 842 217	−0.000 000 093	−0.000 000 030
124	−0.742 849 807	−0.085 334 130	−0.018 842 237	−0.000 000 093	−0.000 000 030
125	−0.742 824 756	−0.085 334 218	−0.018 842 256	−0.000 000 093	−0.000 000 030
126	−0.742 800 098	−0.085 334 304	−0.018 842 275	−0.000 000 093	−0.000 000 030
127	−0.742 775 824	−0.085 334 388	−0.018 842 294	−0.000 000 093	−0.000 000 030
128	−0.742 751 925	−0.085 334 470	−0.018 842 312	−0.000 000 093	−0.000 000 030
129	−0.742 728 394	−0.085 334 550	−0.018 842 330	−0.000 000 093	−0.000 000 030
130	−0.742 705 220	−0.085 334 628	−0.018 842 347	−0.000 000 093	−0.000 000 030
131	−0.742 682 397	−0.085 334 704	−0.018 842 364	−0.000 000 093	−0.000 000 030
132	−0.742 659 916	−0.085 334 779	−0.018 842 380	−0.000 000 093	−0.000 000 030
133	−0.742 637 770	−0.085 334 852	−0.018 842 397	−0.000 000 093	−0.000 000 030
134	−0.742 615 951	−0.085 334 923	−0.018 842 412	−0.000 000 093	−0.000 000 030
135	−0.742 594 452	−0.085 334 993	−0.018 842 428	−0.000 000 093	−0.000 000 030
136	−0.742 573 266	−0.085 335 061	−0.018 842 443	−0.000 000 093	−0.000 000 030
137	−0.742 552 386	−0.085 335 128	−0.018 842 458	−0.000 000 093	−0.000 000 030
138	−0.742 531 806	−0.085 335 193	−0.018 842 472	−0.000 000 093	−0.000 000 030
139	−0.742 511 520	−0.085 335 257	−0.018 842 486	−0.000 000 093	−0.000 000 030
140	−0.742 491 520	−0.085 335 320	−0.018 842 500	−0.000 000 093	−0.000 000 030
141	−0.742 471 802	−0.085 335 381	−0.018 842 514	−0.000 000 093	−0.000 000 030
142	−0.742 452 358	−0.085 335 441	−0.018 842 527	−0.000 000 093	−0.000 000 030
143	−0.742 433 185	−0.085 335 499	−0.018 842 540	−0.000 000 093	−0.000 000 030
144	−0.742 414 274	−0.085 335 557	−0.018 842 553	−0.000 000 093	−0.000 000 030
145	−0.742 395 623	−0.085 335 613	−0.018 842 565	−0.000 000 093	−0.000 000 030
146	−0.742 377 224	−0.085 335 668	−0.018 842 577	−0.000 000 093	−0.000 000 030
147	−0.742 359 074	−0.085 335 722	−0.018 842 589	−0.000 000 093	−0.000 000 030
148	−0.742 341 167	−0.085 335 775	−0.018 842 601	−0.000 000 093	−0.000 000 030
149	−0.742 323 498	−0.085 335 827	−0.018 842 612	−0.000 000 093	−0.000 000 030
150	−0.742 306 062	−0.085 335 878	−0.018 842 624	−0.000 000 093	−0.000 000 030
151	−0.742 288 855	−0.085 335 927	−0.018 842 635	−0.000 000 093	−0.000 000 030
152	−0.742 271 873	−0.085 335 976	−0.018 842 646	−0.000 000 093	−0.000 000 030
153	−0.742 255 111	−0.085 336 024	−0.018 842 656	−0.000 000 093	−0.000 000 030
154	−0.742 238 564	−0.085 336 071	−0.018 842 667	−0.000 000 093	−0.000 000 030
155	−0.742 222 230	−0.085 336 117	−0.018 842 677	−0.000 000 093	−0.000 000 030
156	−0.742 206 103	−0.085 336 162	−0.018 842 687	−0.000 000 093	−0.000 000 030
157	−0.742 190 179	−0.085 336 206	−0.018 842 696	−0.000 000 093	−0.000 000 030
158	−0.742 174 456	−0.085 336 250	−0.018 842 706	−0.000 000 093	−0.000 000 030
159	−0.742 158 929	−0.085 336 292	−0.018 842 716	−0.000 000 093	−0.000 000 030
160	−0.742 143 594	−0.085 336 334	−0.018 842 725	−0.000 000 093	−0.000 000 030
161	−0.742 128 448	−0.085 336 375	−0.018 842 734	−0.000 000 093	−0.000 000 030
162	−0.742 113 488	−0.085 336 415	−0.018 842 743	−0.000 000 093	−0.000 000 030
163	−0.742 098 709	−0.085 336 455	−0.018 842 752	−0.000 000 093	−0.000 000 030
164	−0.742 084 110	−0.085 336 494	−0.018 842 760	−0.000 000 093	−0.000 000 030
165	−0.742 069 686	−0.085 336 532	−0.018 842 769	−0.000 000 093	−0.000 000 030
166	−0.742 055 434	−0.085 336 569	−0.018 842 777	−0.000 000 093	−0.000 000 030
167	−0.742 041 352	−0.085 336 606	−0.018 842 785	−0.000 000 093	−0.000 000 030
168	−0.742 027 436	−0.085 336 642	−0.018 842 793	−0.000 000 093	−0.000 000 030
169	−0.742 013 683	−0.085 336 678	−0.018 842 801	−0.000 000 093	−0.000 000 030
170	−0.742 000 091	−0.085 336 712	−0.018 842 809	−0.000 000 093	−0.000 000 030
171	−0.741 986 657	−0.085 336 747	−0.018 842 816	−0.000 000 093	−0.000 000 030
172	−0.741 973 377	−0.085 336 780	−0.018 842 824	−0.000 000 093	−0.000 000 030
173	−0.741 960 250	−0.085 336 813	−0.018 842 831	−0.000 000 093	−0.000 000 030
174	−0.741 947 273	−0.085 336 846	−0.018 842 838	−0.000 000 093	−0.000 000 030
175	−0.741 934 443	−0.085 336 878	−0.018 842 845	−0.000 000 093	−0.000 000 030
176	−0.741 921 758	−0.085 336 909	−0.018 842 852	−0.000 000 093	−0.000 000 030
177	−0.741 909 215	−0.085 336 940	−0.018 842 859	−0.000 000 093	−0.000 000 030
178	−0.741 896 812	−0.085 336 970	−0.018 842 866	−0.000 000 093	−0.000 000 030
179	−0.741 884 546	−0.085 337 000	−0.018 842 872	−0.000 000 093	−0.000 000 030
180	−0.741 872 416	−0.085 337 029	−0.018 842 879	−0.000 000 093	−0.000 000 030
181	−0.741 860 418	−0.085 337 058	−0.018 842 885	−0.000 000 093	−0.000 000 030
182	−0.741 848 552	−0.085 337 086	−0.018 842 891	−0.000 000 093	−0.000 000 030
183	−0.741 836 814	−0.085 337 114	−0.018 842 898	−0.000 000 093	−0.000 000 030

TABLE I: *continued*

n	$S_{1/2}$	$P_{1/2}$	$P_{3/2}$	$D_{3/2}$	$D_{5/2}$
184	−0.741 825 203	−0.085 337 142	−0.018 842 904	−0.000 000 093	−0.000 000 030
185	−0.741 813 717	−0.085 337 169	−0.018 842 910	−0.000 000 093	−0.000 000 030
186	−0.741 802 353	−0.085 337 195	−0.018 842 916	−0.000 000 093	−0.000 000 030
187	−0.741 791 110	−0.085 337 221	−0.018 842 921	−0.000 000 093	−0.000 000 030
188	−0.741 779 986	−0.085 337 247	−0.018 842 927	−0.000 000 093	−0.000 000 030
189	−0.741 768 979	−0.085 337 272	−0.018 842 933	−0.000 000 093	−0.000 000 030
190	−0.741 758 087	−0.085 337 297	−0.018 842 938	−0.000 000 093	−0.000 000 030
191	−0.741 747 308	−0.085 337 322	−0.018 842 944	−0.000 000 093	−0.000 000 030
192	−0.741 736 640	−0.085 337 346	−0.018 842 949	−0.000 000 093	−0.000 000 030
193	−0.741 726 082	−0.085 337 369	−0.018 842 954	−0.000 000 093	−0.000 000 030
194	−0.741 715 633	−0.085 337 393	−0.018 842 959	−0.000 000 093	−0.000 000 030
195	−0.741 705 290	−0.085 337 416	−0.018 842 965	−0.000 000 093	−0.000 000 030
196	−0.741 695 051	−0.085 337 439	−0.018 842 970	−0.000 000 093	−0.000 000 030
197	−0.741 684 916	−0.085 337 461	−0.018 842 974	−0.000 000 093	−0.000 000 030
198	−0.741 674 883	−0.085 337 483	−0.018 842 979	−0.000 000 093	−0.000 000 030
199	−0.741 664 949	−0.085 337 505	−0.018 842 984	−0.000 000 093	−0.000 000 030
200	−0.741 655 115	−0.085 337 526	−0.018 842 989	−0.000 000 093	−0.000 000 030