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PHASE CHARACTERISTICS AND TIME RESPONSES OF UNKNOWN LINEAR SYSTEMS DETERMINED FROM MEASURED CW AMPLITUDE DATA

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Abstract

An alternative but simpler technique for calculating the complete time and frequency characteristics of an unknown linear system from the measured amplitude response to cw excitations is described. The associated system transfer function so determined may or may not be at minimum phase. A comparison of the time responses shows the worst case. Results also indicate that the susceptibility of the minimum-phase system to damage by pulsed excitation is the greatest during the initial period of excitation.

Key words: all-pass function; Hilbert transform; Laplace transform; minimum phase; non-minimum phase; system transfer function; time response.

1. INTRODUCTION

The time response of a linear system to a given excitation (whether it be cw or pulse) can be uniquely determined, if the transfer function of this system is known, by use of the inverse Laplace transform. However, for complicated systems involved in practical applications, the system transfer function may be unknown. The time response of such a system to pulsed excitations can be obtained only by sophisticated time-domain measurements or by derivations using measurements of the amplitude <u>and</u> phase responses to cw excitations. These measurement results can then be used to assess (a) whether or not the system may survive an assumed pulsed excitation, and (b) if not, what hardening is required for the system to withstand the threat. Such time-domain or frequency-domain phase measurements typically require expensive equipment and special considerations on radiation hazards and regulatory compliance (if performed outdoors). On the other hand, taking cw amplitude responses of the same complicated system due to low levels of escitation is relatively easy and less costly. If the measured cw amplitude could be used to predict the system's phase and thus its transfer function, the time response to a general excitation could then be determined.

In this report, we present an alternative but simpler technique to determine the time and frequency characteristics of an unknown, linear system, based only on a given set of cw amplitude responses. The associated system transfer function so determined may or may not be at minimum phase. The time responses corresponding to the minimum phase and non-minimum phases thus obtained can then be used to assess the susceptibility of the system to damage by pulsed excitations. The theoretical background of relationship between the amplitude and phase of a minimum-phase transfer function is outlined in section 2. The conventional numerical approach practiced in the past for calculating the phase information and the time response of the linear minimum-phase systems from the measured cw amplitude, and the accuracy involved in this process are reviewed and discussed in section 3.

Our new approach for obtaining a set of transfer functions (both minimum and non-minimum phases) and the corresponding impulse responses directly from the given cw amplitudes is presented in section 4. This technique does not require knowledge of the necessary phase information. Of course, this phase information is automatically obtained after the system's transfer function is determined. It can also be determined analytically, if it is required, even before the minimum-phase transfer function is obtained. Physical meanings of the minimum-phase results are discussed. A few examples with various orders of transfer functions and existing measured data to demonstrate this idea are given. Energy contents associated with the system are discussed in section 5. Some concluding remarks and extensions for future work are presented in the final section.

2. THEORETICAL BACKGROUND

A well designed, stable, linear system with time-invariant and lumped-constant elements exists only when its transfer function $H(s)$ has no poles in the right half of the complex frequency s-plane. In other words, $H(s)$ is analytic in $Re(s) \geq 0$, where Re stands for the "real part of" [1]. We address only the stable system in this report, because otherwise the system is not useful in application. In general, $H(s)$ is a rational function of s or a ratio of two polynomials with the degree of the numerator polynomial lower than the degree of the denominator polynomial. When this transfer function is evaluated at the real radian frequency $s = j\omega$, $H(j\omega)$ is a complex function of ω. It then consists of a real part $R(\omega)$ and an imaginary part $X(\omega)$, or a magnitude $|H(\omega)|$ and a phase $\theta(\omega)$. That is,

$$H(j\omega) = R(\omega) + j\,X(\omega) = |H(j\omega)|\,e^{-j\theta(\omega)}, \tag{1}$$

where the convention of assigning a minus sign to the phase is adopted. The magnitude function $|H(j\omega)|$ may also be expressed in terms of the attenuation function $\alpha(\omega)$:

$$|H(j\omega)| = e^{-\alpha(\omega)}. \tag{2}$$

When $H(s)$ is analytic, the real and imaginary parts of $H(j\omega)$ are related by the Hilbert transform pair [2],

$$X(\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{R(y)}{\omega^2 - y^2}\,dy,$$

and (3)

$$R(\omega) = \frac{2}{\pi}\int_0^\infty \frac{yX(y)}{\omega^2 - y^2}\,dy.$$

In other words, if one part is given or specified, the other part can be uniquely determined by performing one of the integrals shown in (3). The complex function $H(j\omega)$ is then completely obtained. In practice, however, (3) is not useful because we cannot just measure $R(\omega)$ or $X(\omega)$.

If, in addition, H(s) has no zeros in Re(s) ≥ 0, the transfer function is said to be at minimum phase, herein denoted by $H_m(s)$. Under this condition, the attenuation and phase functions are related by another pair of Hilbert transforms [2, 3]:

$$\theta(\omega) = \frac{\omega}{\pi} \int_{-\infty}^{\infty} \frac{\alpha(y)}{y^2 - \omega^2}\, dy \;=\; - \frac{\omega}{\pi} \int_{-\infty}^{\infty} \frac{\ell n|H_m(jy)|}{y^2 - \omega^2}\, dy, \tag{4a}$$

and

$$\alpha(\omega) = \alpha(0) - \frac{\omega^2}{\pi} \int_{-\infty}^{\infty} \frac{\theta(y)}{y(y^2 - \omega^2)}\, dy. \tag{4b}$$

Equation (4b) shows that the attenuation function can be determined completely from a given phase function only when $\alpha(0)$ is also known. But, for our application, only (4a) is useful because we assume that the magnitude (or attenuation) function is given. Once $\theta(\omega)$ is determined, the entire complex $H_m(j\omega)$ is obtained from (1), because $|H_m(j\omega)|$ or $\alpha(\omega)$ is already known. The impulse response of the system is then traditionally calculated by the inverse Fourier transform,

$$h_m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_m(j\omega)\, e^{j\omega t} dt. \tag{5}$$

The system's time response to a general excitation can subsequently be computed by the convolution integral $h_m(t)*e(t)$, where e(t) is an arbitrary excitation, cw or pulse, applied to the system. The success of determining the time response in this case is based on the assumption that the system's transfer function is at minimum phase.

3. CONVENTIONAL APPROACH

Since the type of integral in (4a) is considered extremely difficult to calculate, the approach has been to apply another transformation known as the Wiener-Lee transform [2] to $\alpha(\omega)$ or $|H(j\omega)|$, and then to use numerical procedures [4] to obtain the necessary $\theta(\omega)$, $H(j\omega)$, and h(t).

When the Wiener-Lee transform [2],

$$\omega = - \tan(\delta/2), \tag{6}$$

is applied, we are able to transform the entire ω range, $(-\infty, \infty)$ into $(-\pi, \pi)$ for δ. Because of this transformation, the original attenuation and phase functions, $\alpha(\omega)$ and $\theta(\omega)$, will become respectively $\alpha'(\delta)$ and $\theta'(\delta)$. Furthermore, since the attenuation function $\alpha(\omega) = - \ell n|H(j\omega)| = - \frac{1}{2} \ell n [R^2(\omega) + X^2(\omega)]$ is an even function, and the phase function $\theta(\omega) = \tan^{-1}[X(\omega)/R(\omega)]$ is an odd function, of ω in $(-\infty, \infty)$, their respective transforms $\alpha'(\delta)$ and $\theta'(\delta)$ are also even and odd functions of δ in $(-\pi, \pi)$. As such, they may be expanded into Fourier series,

$$\alpha'(\delta) = a_0 + a_1 \cos\delta + \ldots + a_n \cos n\delta + \ldots,$$

and (7)

$$\theta'(\delta) = b_1 \sin\delta + \ldots + b_n \sin n\delta + \ldots ,$$

where the expansion coefficients are determined by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \alpha'(\delta)\, d\delta, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \alpha'(\delta) \cos n\delta \, d\delta, \tag{8a}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \theta'(\delta) \sin n\delta \, d\delta. \tag{8b}$$

When the system under consideration is such that h(t) is causal [h(t) = 0 when $t \le 0$], as is usually so in practice, it is known that a_n and b_n are simply related by [2]

$$b_n = - a_n. \tag{9}$$

Therefore, the determination of a_n from (8a) automatically yields b_n, which in turn gives $\theta'(\delta)$, $\theta(\omega)$, and thus $H(j\omega)$. The impulse response is then obtained from (5).

The justification for using the Wiener-Lee transform and the derivation as outlined above seems reasonable and straightforward. The important question we should ask is then: if the integral in (4a) is difficult to compute before the Wiener-Lee transform is applied because the given $|H(j\omega)|$ or $\alpha(\omega)$ are very complicated, is it much simpler to compute the Fourier series expansion coefficients a_n as required in (8a) after the Wiener-Lee transform is used? If so, what kind of penalty is paid in terms of the final accuracy? After all, the entire procedure requires (i) conversion of the given $\alpha(\omega)$ data into $\alpha'(\delta)$, (ii) numerical computation of a_n from $\alpha'(\delta)$, (iii) construction of a set of $\theta'(\delta)$ by including a finite number of terms in the Fourier sine series of (7) with $b_n = - a_n$, (iv) numerical determination of $H(j\omega)$ based on the given attenuation data and the newly retrieved phase data, and (v) numerical computation of $h_m(t)$ by performing the discrete inverse Fourier transform. Each of these five steps involves approximations and thus introduces inaccuracies.

4. AN ALTERNATIVE BUT SIMPLER APPROACH

Here, we review modern passive network theory together with a re-examination of the Hilbert transform (4a) to assess whether or not a simple and analytical method may be devised to handle the problem at hand without going through unnecessary numerical procedures. Since H(s) for the system under study is a rational function of s, the square of the magnitude of $H(j\omega)$, $|H(j\omega)|^2$, is a ratio of two polynomials of even order in ω. As such, the numerator and denominator polynomials in $|H(j\omega)|^2$ may contain one or more, or any combination, of the following elementary factors:

$$\text{(i) } \omega^2 + a^2 \quad \text{and} \quad \text{(ii) } \omega^4 + b\omega^2 + c^2, \tag{10}$$

where a, b, and c are real numbers. In addition, since $|H(j\omega)|^2 \geq 0$ for all ω, we require, for the factor (ii) in (10),

$$b^2 \leq 4c^2. \tag{11}$$

The number of the above elementary factors contained in $|H(j\omega)|^2$ depends on the complexity of the transfer function being studied. The first-order transfer function has just one factor in form (i) in the denominator of $|H(j\omega)|^2$, and its numerator is a constant. For a second-order transfer function, the denominator of $|H(j\omega)|^2$ contains a factor in form (ii), or two factors in form (i), and the numerator has either a factor in form (i) or a constant. For a third-order transfer function, the denominator can be expressed as a product of form (i) and form (ii) [or as a product of three factors in form (i)], and the numerator as a factor of form (ii), form (i), or a constant. Thus, for an nth-order transfer function, the denominator of $|H(j\omega)|^2$ has a special form of polynomial such as $\omega^{2n} + a_{n-1}\omega^{2n-2} + a_{n-2}\omega^{2n-4} + \ldots + a_0$, and the numerator is an even polynomial of the order of (2n - 2) or lower.

When the magnitude function $|H(j\omega)|$ can be described by a mathematical expression, H(s) can be deduced directly and exactly from it in a straightforward manner [5], without having to find the phase function first. For example, consider the simplest, first-order case described by

$$|H(j\omega|^2 = \frac{1}{\omega^2 + a^2}, \qquad a > 0. \tag{12}$$

We can use the relationship, $|H(j\omega)|^2 = H(s)H(-s)|_{s = j\omega}$, to obtain

$$H(s)H(-s) = \frac{1}{-s^2 + a^2} = \frac{1}{(-s + a)\ (s + a)}, \tag{13}$$

yielding

$$H_m(s) = 1/(s + a), \tag{14}$$

which has a simple pole (s = -a) on the left-hand real axis. Since it has no zeros, the transfer function in this particular form is automatically at minimum phase. The associated impulse function is then determined directly by the inverse Laplace transform as $h_m(t) = \exp(-at)$, $t > 0$ without having to find the phase function first. It can readily be verified that (14) indeed gives $|H(j\omega)|^2$ in (12). Note that the other factor (-s + a) in (13) is not allowable in H(s) because the resulting system will then be unstable.

The expression in (12) may represent the output of a low-pass filter with its maximum at $\omega = 0$ and with its half-power (or 3 dB) point at $\omega = a$. This same parameter "a" is also the decay rate of the time response.

Clearly, the phase function associated with the system's transfer function in (14) is,

$$\theta(\omega) = \tan^{-1}(\omega/a). \tag{15}$$

If, however, the phase information is desired before the system transfer function in (14) is obtained, it can also be found analytically by the following procedure. Since

$$\alpha(\omega) = -\ell n\,|H(j\omega)| = -\frac{1}{2}\,\ell n\,|H(j\omega)|^2 = \frac{1}{2}\,\ell n(\omega^2 + a^2), \tag{16}$$

the required phase function for the minimum-phase case, in accordance with (4a), is

$$\theta(\omega) = \frac{\omega}{\pi}\int_{-\infty}^{\infty} \frac{\ell n(y^2 + a^2)\,dy}{2(y^2 - \omega^2)} = \frac{\omega}{\pi}\int_{0}^{\infty} \frac{\ell n(y^2 + a^2)dy}{y^2 - \omega^2}$$

$$= \tan^{-1}(\omega/a), \tag{17}$$

which agrees with (15). The last step in (17) is achieved by using the definite integral [6]:

$$\int_0^{\infty} \frac{\ell n(p^2 + q^2x^2)\,dx}{g^2x^2 - h^2} = \frac{\pi}{gh}\tan^{-1}\left(\frac{qh}{pg}\right), \quad p,\ q,\ g,\ h > 0. \tag{18}$$

Another method for getting $h_m(t)$ is through the inverse Fourier integral given in (5), since we already know the complex $H_m(j\omega) = |H_m(j\omega)| \exp[-j\theta(\omega)]$.

That is,

$$h_m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_m(j\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H_m(j\omega)|\, e^{j[\omega t - \theta(\omega)]}\, d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{\cos[\omega t - \theta(\omega)]d\omega}{(\omega^2 + a^2)^{1/2}}$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{a\cos(\omega t)\,d\omega}{\omega^2 + a^2} + \frac{1}{\pi}\int_0^{\infty} \frac{\omega\sin(\omega t)\,d\omega}{\omega^2 + a^2}$$

$$= \frac{1}{2}\,e^{-at} + \frac{1}{2}\,e^{-at} = e^{-at}, \tag{19}$$

which is the same as obtained above by a much simpler procedure (inverse Laplace transform). The last line in (19) results from the application of the following two integrals [6]:

$$\int_0^\infty \frac{\cos(\alpha x)\, dx}{x^2 + \beta^2} = \frac{\pi}{2\beta} e^{-\alpha\beta} \quad \text{and} \quad \int_0^\infty \frac{x \sin(\alpha x)\, dx}{x^2 + \beta^2} = \frac{\pi}{2} e^{-\alpha\beta},$$

$$\alpha > 0 \quad \text{and} \quad \mathrm{Re}(\beta) > 0. \tag{20}$$

This method of inverse Fourier integral is the one used in the numerical approach discussed in the last section.

We wish to note another important point about the transfer function. Even though the particular form in (14) is taken for analysis, it does not necessarily imply that the system under study is minimum phase. In fact, the original unknown system may possibly be represented by many transfer functions with non-minimum phases, because a product of $H_m(s)$ and an all-pass function yields a transfer function with non-minimum phase but still with the same $|H(j\omega)|^2$. That is, $H_n(s) = H_m(s)\, H_a(s)$, where $H_n(s)$ is the non-minimum-phase transfer function, and $H_a(s)$ is the all-pass function defined as

$$H_a(s) = \Pi\,(s - \alpha_i)/\,\Pi(s + \alpha_i), \tag{21}$$

with each zero in the right-half s-plane the mirror image of the corresponding pole in the left-half s-plane [5]. In (21), the symbol Π denotes successive product, and the parameter α_i is either positive real or complex. When α_i is complex, (21) must also have another complex conjugate factor. The fact that $|H_a(j\omega)|^2 = 1$ explains why $|H_n(j\omega)|^2 = |H_m(j\omega)|^2$.

The corresponding impulse response is then

$$h_n(t) = h_m(t) * h_a(t), \tag{22}$$

where $h_a(t)$ is the inverse Laplace transform of $H_a(s)$, and * represents convolution integral. For example, consider the case with a = 1 and $H_a(s) = (s - 2)/(s + 2)$. We then have $h_m(t) = e^{-t}$, $t > 0$, and

$$\begin{aligned} h_n(t) &= h_m(t)*h_a(t) = h_m(t)*[\delta(t) - 4e^{-2t}] \\ &= h_m(t) - r(t), \qquad t > 0, \end{aligned} \tag{23}$$

where

$$r(t) = 4e^{-t}*e^{-2t} = 4(e^{-t} - e^{-2t}). \tag{24}$$

Since $r(t) \geq 0$ for all $t \geq 0$, we conclude that $h_n(t) \leq h_m(t)$. This implies that, for this particular example, if the minimum-phase system can survive a threat from a given pulsed excitation, the non-minimum-phase system can also successfully withstand the same threat.

Before we proceed with an example of a second-order transfer function, we note that the impulse response $h_n(t)$ given in (23) can also be obtained by first performing a partial fraction expansion of $H_n(s)$,

$$H_n(s) = H_m(s)H_a(s) = \frac{s - 2}{(s + 1)(s + 2)} = \frac{-3}{s + 1} + \frac{4}{s + 2}, \qquad (25)$$

and then doing an inverse Laplace transform [7] to get

$$h_n(t) = -3e^{-t} + 4e^{-2t} = h_m(t) - r(t). \qquad (26)$$

We will now present another example using a second-order transfer function whose squared magnitude is given by

$$|H(j\omega)|^2 = \frac{\omega^2 + 1}{\omega^4 + \omega^2 + 16}, \qquad (27)$$

a plot of which is shown in figure 1 with its resonant frequency at $\omega^2 = 3$ and with a half-power bandwidth of $\Delta\omega^2 = 12.689$. From (27), we have

$$H(s)H(-s) = \frac{-s^2 + 1}{s^4 - s^2 + 16} = \frac{(s + 1)(-s + 1)}{(s^2 + 3s + 4)(s^2 - 3s + 4)}, \qquad (28)$$

thus yielding the minimum-phase transfer function

$$H_m(s) = \frac{s + 1}{s^2 + 3s + 4}, \qquad (29)$$

and the simplest possible non-minimum-phase transfer function

$$H_n(s) = \frac{-s + 1}{s^2 + 3s + 4}. \qquad (30)$$

As a check, both (29) and (30) give the same $|H(j\omega)|^2$ in (27). Of course, the other possible non-minimum-phase transfer functions having the same $|H(j\omega)|^2$ in (27) are the product of $H_m(s)$ in (29) [or $H_n(s)$ in (30)] and the all-pass function $H_a(s)$ in (21).

The phases associated with the transfer functions given in (29) and (30) are respectively

$$\theta_m(\omega) = \tan^{-1}[3\omega/(4 - \omega^2)] - \tan^{-1}(\omega), \qquad (31)$$

and

$$\theta_n(\omega) = \tan^{-1}[3\omega/(4 - \omega^2)] + \tan^{-1}(\omega). \tag{32}$$

The impulse responses of (29) and (30) can be determined, respectively, to be [7]

$$h_m(t) = e^{-1.5t}(\cos 1.32288t - 0.37796 \sin 1.32288t), \tag{33}$$

and

$$h_n(t) = e^{-1.5t}(- \cos 1.32288t + 1.88982 \sin 1.32288t)$$

$$= h_m(t) - r(t), \tag{34}$$

where

$$r(t) = 2e^{-1.5t}(\cos 1.32288t - 1.13389 \sin 1.32288t) \geq 0$$

in $0 \leq t \leq 0.54634$. This means that $h_n(t) \leq h_m(t)$ shortly after an excitation is applied to the system. The minimum-phase response will not underestimate the system behavior.

Thus, we have shown once again that (i) the minimum-phase transfer function for an unknown system and the resultant impulse response can be determined directly from the given $|H(j\omega)|^2$ without having to obtain the associated phase function, and (ii) the minimum-phase time response $h_m(t)$ so determined is the most conservative estimate as far as the initial threat to the system by an excitation is concerned.

The phase function corresponding to the minimum phase can also be determined from (4a) before its transfer function in (29) is obtained. We begin with the denominator of (27),

$$\omega^4 + \omega^2 + 16 = (\omega^2 + a_1^2)(\omega^2 + a_2^2), \tag{35}$$

where a_1^2 and a_2^2 are a complex conjugate pair. Even though it is straightforward to determine a_1^2 and a_2^2 [in this case, $a_1^2 = (1 + j\sqrt{63})/2$, and $a_2^2 = (1 - j\sqrt{63})/2$], we only need to know, as shown later, $a_1 a_2$ and $a_1 + a_2$. We know from (35) that $a_1^2 + a_2^2 = 1$ and $a_1^2 a_2^2 = 16$. We obtain $a_1 a_2 = 4$, and $(a_1 + a_2)^2 = a_1^2 + a_2^2 + 2a_1 a_2 = 1 + 8 = 9$, or $a_1 + a_2 = 3$. Then we have

$$\ell n|H(j\omega)| = - \frac{1}{2} \ell n(\omega^2 + a_1^2) - \frac{1}{2} \ell n(\omega^2 + a_2^2) + \frac{1}{2} \ell n(\omega^2 + 1). \tag{36}$$

Applying (4a) we obtain

$$\theta(\omega) = \theta_1(\omega) + \theta_2(\omega) - \theta_3(\omega), \tag{37}$$

where, with the aid of (18),

$$\theta_1(\omega) = \frac{\omega}{\pi}\int_0^\infty \frac{\ln(y^2 + a_1^2)}{y^2 - \omega^2}\,dy = \tan^{-1}(\omega/a_1), \tag{38}$$

$$\theta_2(\omega) = \frac{\omega}{\pi}\int_0^\infty \frac{\ln(y^2 + a_2^2)}{y^2 - \omega^2}\,dy = \tan^{-1}(\omega/a_2), \tag{39}$$

and

$$\theta_3(\omega) = \frac{\omega}{\pi}\int_0^\infty \frac{\ln(y^2 + 1)}{y^2 - \omega^2}\,dy = \tan^{-1}(\omega). \tag{40}$$

However,

$$\begin{aligned}\theta_1(\omega) + \theta_2(\omega) &= \tan^{-1}(\omega/a_1) + \tan^{-1}(\omega/a_2)\\ &= \tan^{-1}[(a_1 + a_2)\omega/(a_1 a_2 - \omega^2)]\\ &= \tan^{-1}[3\omega/(4 - \omega^2)].\end{aligned} \tag{41}$$

Thus, we have

$$\theta(\omega) = \tan^{-1}[3\omega/(4 - \omega^2)] - \tan^{-1}(\omega), \tag{42}$$

which checks with (31).

The impulse response given in (33) can also be obtained by the inverse Fourier integral method, as we demonstrated in the first example for the first-order transfer function, with a much more involved procedure.

We make two comments about the examples presented above. First, in obtaining (38) and (39), we have extended the application of (18) to the case with a complex number for p. Evidently, (18) is still valid as long as a complex conjugate pair appears together. This is certainly satisfied in our study since the given $|H(j\omega)|^2$ is a rational function of ω^2.

Second, the variable ω has been used loosely as frequency. In (27) we have identified $\omega^2 = 3$ (or $\omega = \sqrt{3}$) as the resonant frequency for the second-order case. Strictly speaking, ω should be treated as the normalized radian frequency. It can be translated into any frequency of interest by a simple frequency transformation [5],

$$\omega' = A\omega, \tag{43}$$

where A is a normalization constant.

If, for example, the actual resonant frequency occurs at $\sqrt{3}$ MHz, we then use $A = 2\pi(10)^6$ to translate $\omega = \sqrt{3}$ to $\omega' = \sqrt{3} \times 2\pi(10)^6$. Substituting $\omega = \omega'/A$ into (27) gives a new squared magnitude function,

$$|G(j\omega')|^2 = \frac{A^2(\omega'^2 + A^2)}{\omega'^4 + A^2\omega'^2 + 16A^4} , \tag{44}$$

yielding the $|G(j\omega')|^2$ vs ω'^2 curve identical to that shown in figure 1. The corresponding minimum-phase transfer function, impulse response, and phase function then become respectively:

$$G_m(s') = \frac{A(s' + A)}{s'^2 + 3As' + 4A^2} , \tag{45}$$

$$g_m(t) = A\ e^{-1.5At}(\cos 1.32288At - 0.37796 \sin 1.32288At), \tag{46}$$

and

$$\theta(\omega') = \tan^{-1}[3A\omega'/(4A^2 - \omega'^2)] - \tan^{-1}(\omega'/A). \tag{47}$$

The same principle and procedures apply to higher-order cases. The results on extracting $H_m(s)$, $h_m(t)$, and $\theta(\omega)$ from a given $|H(j\omega)|^2$ and the procedures involved, as demonstrated in the two examples, are exact, and no approximations have been exercised. Approximations are needed only when an analytical expression for $|H(j\omega)|^2$ is unknown. In practice, when only the measured data of $|H(j\omega)|$ are available, all we have to do is to obtain an experimental curve for $|H(j\omega)|^2$ and then to approximate it with a realizable, mathematical expression. By realizability, we mean (i) $|H(j\omega)|^2 = N(\omega^2)/D(\omega^2) \geq 0$ for all ω, (ii) the numerator and denominator polynomials, $N(\omega^2)$ and $D(\omega^2)$ are in even orders of ω with some restrictions on their real coefficients, and (iii) the degree of $N(\omega^2)$ is lower than the degree of $D(\omega^2)$ by at least 2. This can be achieved by the available approximation theory and curve-fitting techniques. In fact, this is the only step involving approximations. The remaining procedures will be exact.

The last example to demonstrate the applicability of the approach presented in this section is now given, and is based on the measured data shown in figure 2 (solid curve). The data represent normalized, vertically polarized electric fields reflected from a helicopter when it is irradiated by an impulse. By examining the solid curve in figure 2 more carefully, we notice at least three resonant frequencies at 16.5 MHz, 25.6 MHz, and 54.3 MHz. Obviously, this system is too complicated to be represented by a simple second-order transfer function. However, for demonstration purpose, we only consider the most important resonant frequency at 25.6 MHz. This means that we treat the dashed curve in figure 2 as "the given" one, and then try to approximate it by a second-order transfer function with minimum phase in the form of

$$H_m(s) = B/(s^2 + as + b), \tag{48}$$

where B, a, and b are three unknown constants to be determined.

From (48), we have

$$|H_m(j\omega)|^2 = \frac{B^2}{\omega^4 - (2b - a^2)\,\omega^2 + b^2}\,, \tag{49}$$

which has the resonant frequency at $\omega_0^2 = b - a^2/2$.

Expressing (49) in terms of ω_0 gives

$$|H_m(j\omega)|^2 = \frac{B^2}{\omega^4 - 2\omega_0^2\,\omega^2 + b^2}\,. \tag{50}$$

Two essential conditions are then imposed on (50):

$$\omega_0^2 = b - a^2/2 = (2\pi \times 25.6 \times 10^6)^2\,, \tag{51}$$

and

$$|H_m(j\omega_0)|^2 = 53^2 \quad \text{(or 34.48 dB as given in fig. 2)} \tag{52}$$

A third condition may be imposed at either (i) 20 MHz or (ii) 30 MHz. Indeed, any other condition based on a physical reasoning may be used. If we choose (i) with $\omega_1 = 2\pi \times 20 \times (10)^6 = 1.2566 \times 10^8$, and use

$$|H_m(j\omega_1)|^2 = 13^2 \quad \text{(or 22.28 dB as indicated in fig. 2)}, \tag{53}$$

we have the solutions from (51), (52), and (53):

$$B = 13.5185 \times 10^{16}\,, \quad a = 0.1584 \times 10^8, \quad b = 2.5998 \times 10^{16}. \tag{54}$$

The $|H_m(j\omega)|^2$ in (50) with the values of B, a, and b given in (54) is plotted in figure 2 as the curve marked with dots. Not surprisingly, the frequency region from 10 MHz to 25.6 MHz is very accurately matched to the dashed curve, while the frequency region greater than 25.6 MHz is not.

If we choose (ii) for matching at 30 MHz (which is a 3-dB point), we have $\omega_2 = 1.8850 \times 10^8$. The condition

$$|H_m(j\omega_2)|^2 = 53^2/2 \quad \text{(or 31.48 dB)}, \tag{55}$$

with (51) and (52), yields

$$B = 51.1874 \times 10^{16}, \quad a = 0.5906 \times 10^8, \quad b = 2.7616 \times 10^{16}. \tag{56}$$

The $|H_m(j\omega)|^2$ in (50) with this new set of B, a, and b values is presented in figure 2 as the curve marked with crosses. Here we see a better approximation for frequencies higher than the resonant frequency, but a poorer approximation for lower frequencies.

Even though these two approximations may not be wholly satisfactory, they are still considered effective because the complicated experimental curve is approximated by a simple second-order transfer function. A higher-order (4th or 6th) transfer function, or even a different kind of a second-order transfer function such as $H_m(s) = B(s + c)/(s^2 + as + b)$ with an additional control parameter c will certainly give a better approximation. The use of higher-order transfer functions and additional control parameters should be topics for future work. The objective of this report is primarily to demonstrate the feasibility of solving this kind of problem with a simpler approach.

The impulse responses for the two approximations under discussion are:

(i) with the matching point at 20 MHz,

$$h_{m1}(t) = 8.4347 \times 10^8 e^{-\alpha_1 t} \sin \beta_1 t, \quad t > 0, \tag{57}$$
$$\text{with } \alpha_1 = 0.0792 \times 10^8, \text{ and } \beta_1 = 1.6104 \times 10^8;$$

and

(ii) with the matching point at 30 MHz,

$$h_{m2}(t) = 31.0755 \times 10^8 e^{-\alpha_2 t} \sin \beta_2 t, \quad t > 0, \tag{58}$$
$$\text{with } \alpha_2 = 0.2953 \times 10^8, \text{ and } \beta_2 = 1.6354 \times 10^8 .$$

The normalized time responses are shown in figure 3 with curves marked respectively with dots and crosses, along with the normalized measured time response (the unmarked curve). On the first look, the predicted minimum-phase time responses clearly do not agree well with the measured curve. This implies that the actual system (helicopter plus its attached instruments and equipment) may have a non-minimum-phase transfer function with one or more zeros in the right half s-plane. This fact is also indicated by the initial negative response appearing in the experimental curve. On the other hand, the general shape of a damped sinusoid with a period between $3.84 \times (10)^{-8}$ and $3.90 \times (10)^{-8}$ seconds agrees well with the shape of the experimental curve. In addition, the rate of decay for $h_{m2}(t)$ with $\alpha_2 = 0.2953 \times 10^8$ also approaches the experimental curve. This indicates that matching at 30 MHz not only improves $|H_m(j\omega)|^2$ at higher frequencies, but also yields a better approximation to the measured time response. This is desirable because the receiving antenna used in the actual measurement also works more accurately in the upper frequency range.

The corresponding phases are:

$$\theta_1(\omega) = \tan^{-1}[0.1584 \times 10^8 \, \omega/(2.5998 \times 10^{16} - \omega^2)], \tag{59}$$

and

$$\theta_2(\omega) = \tan^{-1}[0.5906 \times 10^8\ \omega/(2.7616 \times 10^{16} - \omega^2)]. \tag{60}$$

5. CONSIDERATION OF ENERGY CONTENTS

To assess the ability of a system to withstand damage from an excitation, it is often useful to compute the energy associated with an impulse response. Indeed, if h(t) represents a voltage waveform across a 1 Ω resistor, the quantity

$$E = \int_0^\infty h^2(t)\ dt \tag{61}$$

equals the total energy delivered to the resistor by the excitation [2]. Equation (61) also represents the area under the curve $h^2(t)$.

The energy E may also be computed, in view of Parseval's theorem [2], by

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2\ d\omega. \tag{62}$$

Since the minimum-phase impulse response $h_m(t)$ and the associated non-minimum-phase impulse response $h_n(t) = h_m(t)*h_a(t)$ have the same $|H(j\omega)|^2$, their respective total energies [in $0 \le t \le \infty$] are equal even though $h_n(t) \le h_m(t)$ during the initial period near $t = 0^+$ as demonstrated in section 4.

For the first example considered in section 4, we have

$$|H_m(j\omega)|^2 = 1/(\omega^2 + a^2), \quad \text{and} \quad h_m(t) = e^{-at}.$$

The energy is then, from (62),

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 + a^2} = \frac{1}{\pi} \int_0^{\infty} \frac{d\omega}{\omega^2 + a^2} = 1/(2a); \tag{63}$$

or, from (61),

$$E = \int_0^\infty h_m^2(t)\ dt = \int_0^\infty e^{-2at}\ dt = 1/(2a). \tag{64}$$

Referring to the example with a non-minimum-phase impulse response presented in (23), we have $h_m(t) = e^{-t}$ (with a = 1), $h_a(t) = \delta(t) - 4e^{-2t}$, and $h_n(t) = h_m(t) * h_a(t) = -3e^{-t} + 4e^{-2t}$. The energy is then

$$E = \int_0^\infty h_n^2(t)\ dt = \int_0^\infty (9e^{-2t} - 24e^{-3t} + 16e^{-4t})\ dt$$

$$= 1/2 = 1/(2a) \text{ with } a = 1, \qquad (65)$$

which is indeed the same as in (63) and (64).

For the second example as given in (27), we have

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(\omega^2 + 1)d\omega}{\omega^4 + \omega^2 + 16} = \frac{1}{\pi}\int_0^{\infty} \frac{a_1 d\omega}{2(\omega^2 + \beta_1)} + \frac{1}{\pi}\int_0^{\infty} \frac{a_1^* d\omega}{2(\omega^2 + \beta_1^*)}$$

$$= \frac{1}{2}\{a_1/\sqrt{\beta_1} + a_1^*/\sqrt{\beta_1^*}\} = 5/24, \qquad (66)$$

where

$$a_1 = \frac{1 + j/\sqrt{63}}{2}, \qquad \text{and} \qquad \beta_1 = \frac{1 + j\sqrt{63}}{2}. \qquad (67)$$

The same result can be obtained by computing

$$E = \int_0^{\infty} h_m^2(t)\, dt \qquad \text{or} \qquad E = \int_0^{\infty} h_n^2(t)\, dt, \qquad (68)$$

where $h_m(t)$ and $h_n(t)$ are given respectively in (33) and (34).

For our third example involving experimental data, the energies are: (i) when the frequency point is matched at 20 MHz,

$$E_1 = \int_0^{\infty} h_{m1}^2(t)\, dt = 2.229 \times (10)^{10},$$

and (ii) when the frequency point is matched at 30 MHz,

$$E_2 = \int_0^{\infty} h_{m2}^2(t)\, dt = 7.978 \times (10)^{10},$$

where $h_{m1}(t)$ and $h_{m2}(t)$ are given respectively in (57) and (58).

6. CONCLUSIONS

We have shown an alternative and simpler method to determine the complete transfer function (including the associated phase information) and the impulse response for an unknown linear system from a given cw amplitude response, without having to retrieve the phase function first. The solutions (both minimum and non-minimum phases) are exact only if an analytical expression for the cw amplitude response is known. If not, approximations can be used to derive a realizable squared magnitude describing the cw system response. The remaining procedures for obtaining the complete system characteristics are exact. Three examples, using known analytical expressions and measured data for the cw magnitude, have been examined and shown to validate the proposed technique. We also have shown that the minimum-phase time response represents the most conservative estimate as far as the response to an initial threat to the system by an excitation is concerned. Examination of the basic properties of transfer functions with higher orders and with different forms for each order, and study of accuracies involved in the approximation (when the measured data are available) to cover more complicated cases should be topics for future research.

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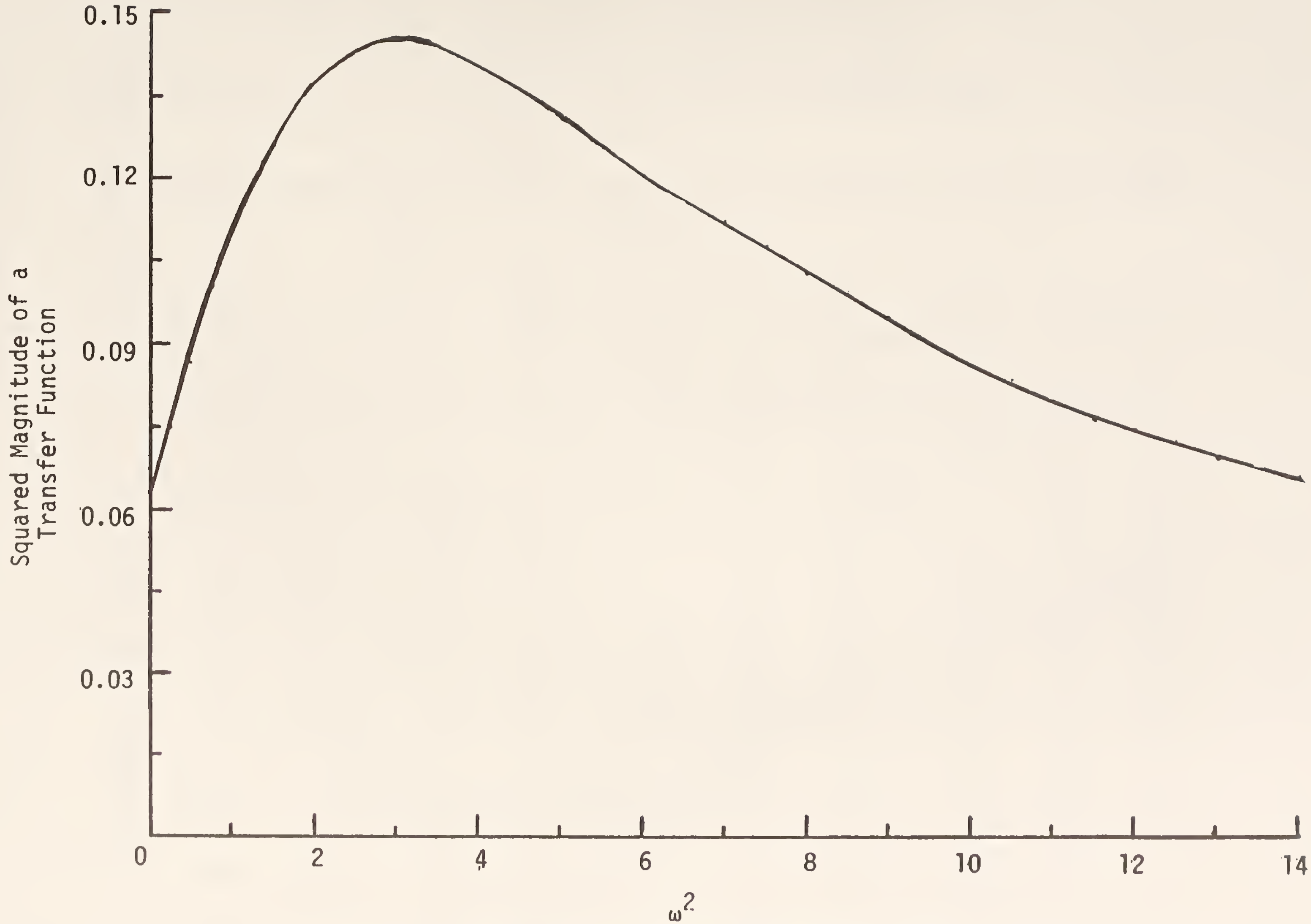

Figure 1. A given second-order $|H(j\omega)|^2$ vs. ω^2.

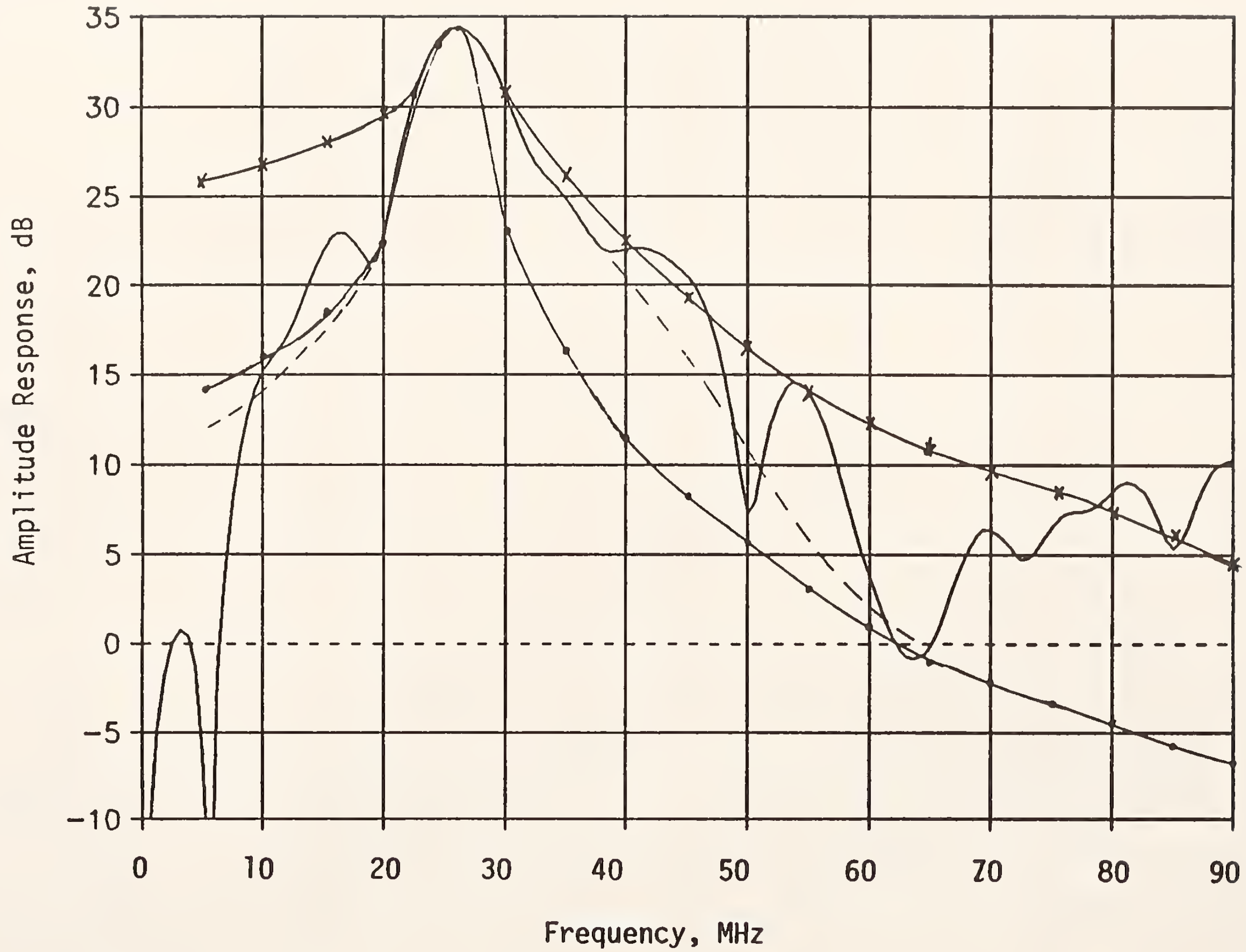

Figure 2. Vertically polarized electric field response of an unknown system:
——— measured results,
– – – modified response from the measured data,
•–•–• approximation by matching at 20 MHz,
×–×–× approximation by matching at 30 MHz.

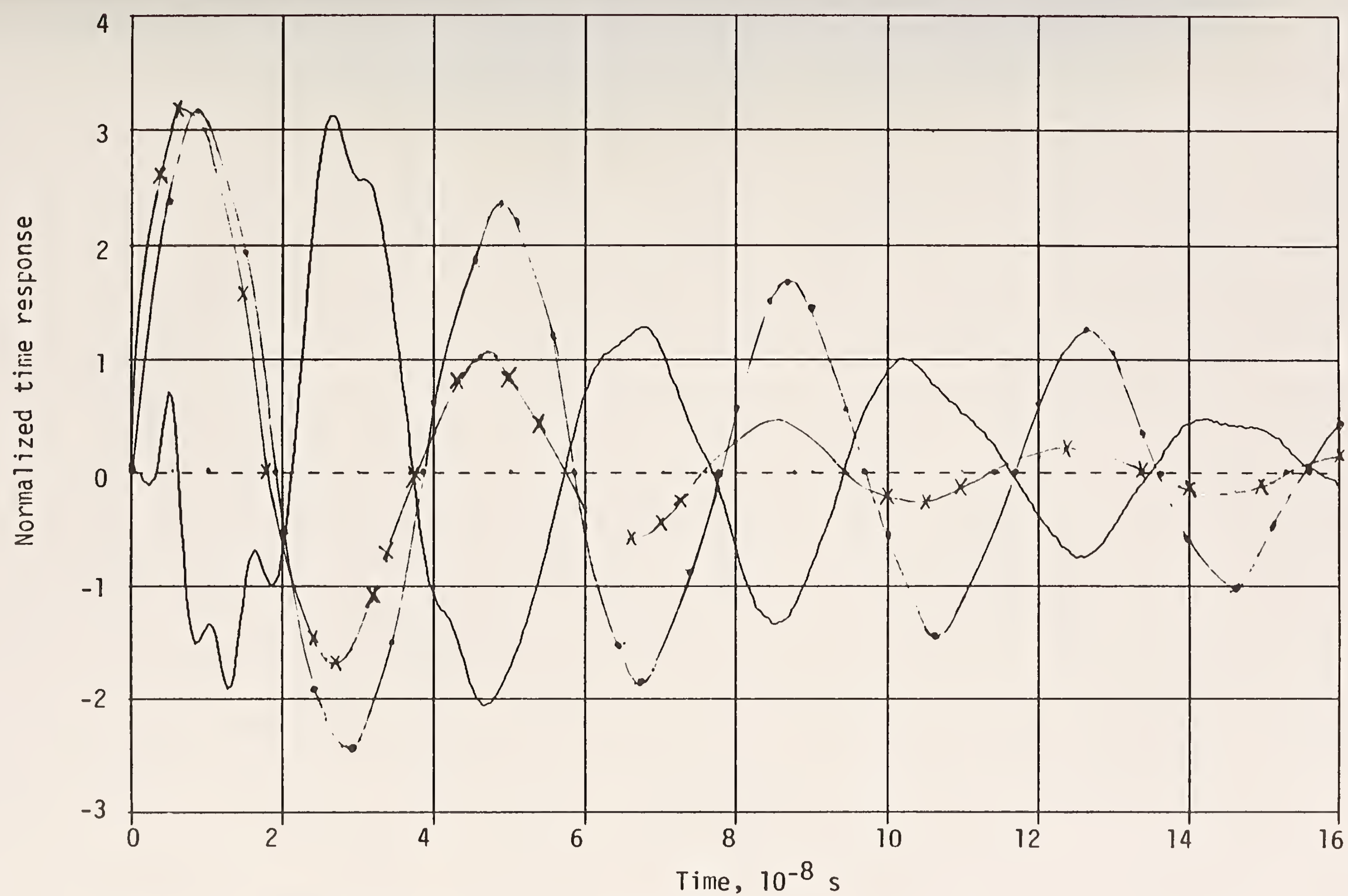

Figure 3. Impulse responses of an unknown system:

measured results,

theoretically predicted results with minimum-phase assumption and with matching point at 20 MHz,

theoretically predicted results with minimum-phase assumption and with matching at 30 MHz.

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An alternative but simpler technique for calculating the complete time and frequency characteristics of an unknown linear system from the measured amplitude response to cw excitations is described. The associated system transfer function so determined may or may not be at minimum phase. A comparison of the time responses shows the worst case. Results also indicate that the susceptibility of the minimum-phase system to damage by pulsed excitation is the greatest during the initial period of excitation.

12. KEY WORDS (6 TO 12 ENTRIES; ALPHABETICAL ORDER; CAPITALIZE ONLY PROPER NAMES; AND SEPARATE KEY WORDS BY SEMICOLONS)

all-pass function; Hilbert transform; Laplace transform; minimum phase; non-minimum phase; system transfer function; time response.

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