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On Constructing Delaunay Triangulations
for Sets Constrained by Line Segments

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Abstract. In this paper, we propose a simple algorithm for constructing a Delaunay triangulation for a finite set of points in the plane constrained by a finite collection of line segments. This algorithm constructs first a Delaunay triangulation for the set, and then generates from it a sequence of triangulations as each line segment is incorporated into the previously obtained triangulation. An expected time analysis shows the algorithm to be linear if the number of line segments is kept constant.

Key words: algorithm; computational complexity; computational geometry; constrained Delaunay triangulation; expected time analysis; multiply-connected polygonal regions; Voronoi diagram

AMS(MOS) subject classifications. 68U05

1. Introduction

Let S be a finite set of points in the plane. By a *triangulation for S* we mean a finite collection of triangles in the plane having pair-wise disjoint interiors, each of which intersects S exactly at its vertices, and the union of which is the convex hull of S. Given a triangulation T for S, we say that T is a *Delaunay triangulation for S* if for each triangle in T there does not exist a point of S inside the circumcircle of the triangle. Delaunay triangulations have been thoroughly studied in [8,11,12].

We say that a region in the plane is a *multiply-connected polygonal region* if it is a bounded and closed set whose interior is nonempty and connected, and whose boundary is the union of a finite number of closed line segments. Here and in the sequel, given a multiply-connected polygonal region R, we denote by $INT(R)$, $BND(R)$, and $VTX(R)$, respectively, the interior of R, the boundary of R, and the set of vertices of the boundary of R. In addition, given points P and Q in the plane, $P \neq Q$, we denote by $[P,Q]$, (P,Q), $[P,Q)$, and $(P,Q]$, respectively, the closed, open, and half-open line segments that P and Q determine.

De Floriani, Falcidieno, and Pienovi [5] have studied the concept of a Delaunay triangulation for a finite set of points in the plane constrained by the boundary of a multiply-connected polygonal region. Let R be a multiply-connected polygonal region, and let S be a finite subset of R that contains $VTX(R)$. De Floriani, *et al.* define a *triangulation for S constrained by $BND(R)$* as a finite collection of triangles having pair-wise disjoint interiors, each of which intersects S exactly at its vertices, and the union of which is R. Given a triangulation T for S constrained by $BND(R)$, De Floriani, *et al.* call T a *Delaunay triangulation for S constrained by $BND(R)$* if for each triangle in T there does not exist a point of S inside the circumcircle of the triangle such that the line segments that connect the point and the vertices of the triangle are contained in R.

Let T be a triangulation for a set S constrained by the boundary of a multiply-connected polygonal region R. Given a triangle t in T, we denote by $ADJ(t)$ the set of points in S that are vertices of triangles in T adjacent to t, and say that t satisfies the *circle criterion in T* if there does not exist a point in $ADJ(t)$ inside the circumcircle of t. De Floriani, *et al.* state that using arguments similar to those developed in [6,8], it can be shown that T is a Delaunay triangulation for S constrained by $BND(R)$ if each triangle in T satisfies the circle criterion in T.

In this paper, we study a more general type of triangulation. Let S be a finite set of points in the plane, and let E be a finite collection, possibly empty, of line segments that intersect only at points in S and whose endpoints belong to S. We say that a triangulation

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T for S is a *triangulation for S constrained by E* if for each e in E and each t in T, e does not intersect $INT(t)$. Given T, a triangulation for S constrained by E, we say that T is a *Delaunay triangulation for S constrained by E* if for each t in T there does not exist a point P of S inside the circumcircle of t such that no e in E intersects the interior of the convex hull of $t \cup \{P\}$.

Delaunay triangulations constrained by line segments have been studied by Lee [7] and Chew [3,4] for line segments that do not cross. Lee and Chew have also shown how to construct them in $O(N (\log N)^2)$ and $O(N \log N)$ worst-case complexity, respectively, where N is the number of points in S. In this paper, we present an algorithm, based on a different approach, that computes this type of triangulation in expected linear time for fixed E. It consists of two steps. In the first step, a Delaunay triangulation for S is constructed by applying to S one of several existing algorithms [1,2,9,10]. In the second step, a sequence of triangulations for S, the last of which is the desired triangulation, is generated from the initial triangulation as each line segment in E is incorporated into the previously obtained triangulation.

Given a simple polygon R, and points P_0 and P_1 in $VTX(R)$ such that $[P_0, P_1]$ is a linear component of $BND(R)$, we say that R is *star-shaped relative to* $[P_0, P_1]$ if for each point P in $R \setminus [P_0, P_1]$, there exists a point Q in (P_0, P_1) such that (P, Q) is contained in $INT(R)$. We say that R is *edge star-shaped* if it is star-shaped relative to at least one of the linear components of $BND(R)$. Figure 1 illustrates an edge star-shaped simple polygon.

Given a region R in the plane, we say that R is a *generalized multiply-connected polygonal region* if it is a bounded and closed set whose interior is nonempty, and whose boundary is the union of a finite number of closed line segments. Given a generalized multiply-connected polygonal region R, for some unique positive integer k, let R_i, $i = 1, \ldots, k$, be the unique multiply-connected polygonal regions that are the closures of the connected components of the interior of R. Let S be a finite subset of R that contains $VTX(R_i)$ for each i, $i = 1, \ldots, k$, and for each i, $i = 1, \ldots, k$, let S_i be the intersection of S and R_i. Given for each i, $i = 1, \ldots, k$, a triangulation T_i for S_i constrained by $BND(R_i)$, we call the union of the triangulations T_i, $i = 1, \ldots, k$, a *triangulation for S constrained by $BND(R)$*, where $BND(R)$ denotes the boundary of R. We call it a *Delaunay triangulation for S constrained by $BND(R)$*, if for each i, $i = 1, \ldots, k$, T_i is a Delaunay triangulation for S_i constrained by $BND(R_i)$.

A crucial aspect of the algorithm we propose involves edge star-shaped simple polygons and generalized multiply-connected polygonal regions: the incorporation of each line segment of E into the previously obtained triangulation for S. To do this, the algorithm first identifies those triangles in the triangulation whose interiors intersect the line segment. A
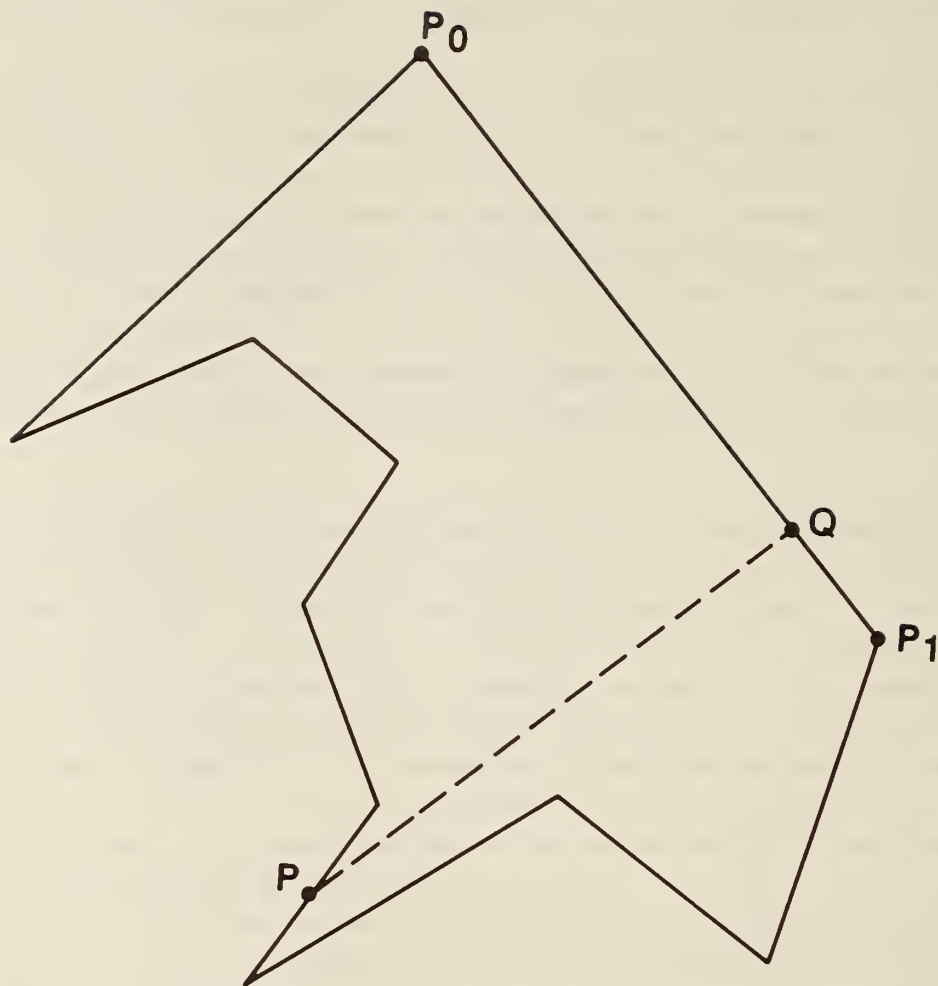
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Figure 1: An edge star-shaped simple polygon. For each point P in the complement of $[P_0, P_1]$ with respect to the polygon, there exists a point Q in (P_0, P_1) such that (P, Q) is contained in the interior of the polygon.

finite collection of edge star-shaped simple polygons with pair-wise disjoint interiors is then produced as the union of the identified triangles is intersected with each closed half-plane that the line segment determines. Next, for each polygon in this collection, a monotonically increasing (ordered by inclusion) finite sequence of generalized multiply-connected polygonal regions that converge to the polygon is produced; a Delaunay triangulation for the intersection of S with each of these regions constrained by the boundary of the region is computed; and in the limit a Delaunay triangulation for the intersection of S with the polygon constrained by the boundary of the polygon is obtained. Finally, a new triangulation for S results as the triangulations obtained in the limit as above are appended to the previously obtained triangulation for S, and the triangles identified as having interiors that intersect the line segment in E are eliminated from this triangulation.

Three algorithms, which we call SEGMNT, EDGSTR, and UPDTRI, are outlined and justified in this paper. Algorithm SEGMNT, which makes use of algorithm EDGSTR, constructs a Delaunay triangulation for a finite set of points in the plane constrained by a finite collection of line segments. Algorithm EDGSTR, which makes use of algorithm UPDTRI, computes a Delaunay triangulation for a specialized subset of the boundary of an edge star-shaped simple polygon constrained by the boundary of the polygon. Finally, algorithm UPDTRI constructs a Delaunay triangulation for a set constrained by the boundary of a generalized multiply-connected polygonal region from a specialized triangulation for the set constrained by the same boundary. In Sections 2, 3, and 4, we outline and justify algorithms UPDTRI, EDGSTR, and SEGMNT, respectively. In Section 5, we show SEGMNT to be of expected linear complexity if the cardinality of the collection of line segments is kept constant. Finally, in Section 6, we offer a summary of the results presented in this paper.

2. The UPDTRI algorithm

Consider a generalized multiply-connected polygonal region R, and for some positive integer k, let R_i, $i = 1, \ldots, k$, be the multiply-connected polygonal regions that are the closures of the connected components of the interior of R. Let S be a finite subset of R that contains $VTX(R_i)$ for each i, $i = 1, \ldots, k$. For each i, $i = 1, \ldots, k$, let S_i be the intersection of S and R_i, and let T_i be a triangulation for S_i constrained by $BND(R_i)$ such that for an integer l, $1 \leq l \leq k$, a triangle t^* in T_l, and a vertex P^* of t^*, the following conditions hold:

1. If for some triangle t' in T_l, $t' \neq t^*$, P^* is inside the circumcircle of t', and the line segments that connect P^* and the vertices of t' are contained in R_l, then one of these line segments intersects $INT(t^*)$.

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2. For each triangle t' in T_l, $t' \neq t^*$, there does not exist a point of S_l different from P^* inside the circumcircle of t' such that the line segments that connect the point and the vertices of t' are contained in R_l.

3. For each i, $1 \leq i \leq k$, $i \neq l$, T_i is a Delaunay triangulation for S_i constrained by $BND(R_i)$.

With T defined as the union of the triangulations T_i, $i = 1, \ldots, k$, we present and justify algorithm UPDTRI, which produces from T, t^*, and P^*, a Delaunay triangulation T^* for S constrained by $BND(R)$. Here and in the sequel, given a triangle t with vertices P_1, P_2, P_3, in the order in which they appear in $BND(t)$ in a counterclockwise direction around $INT(t)$, we denote t by $\triangle P_1 P_2 P_3$ and say that $\triangle P_1 P_2 P_3$ identifies t. Based on this convention, we let P' and P'' be points in S_l for which $\triangle P^* P' P''$ identifies t^*. The outline of UPDTRI follows.

```
procedure UPDTRI(T, T*, P*, P', P'')
    begin
    T* := T;  P_adj := P';  P_cur := P'';
    flag := 1;
    while (flag = 1) do
        begin
        if (there does not exist P̂ such that △P̂ P_cur P_adj
        identifies a triangle in T* or (there exists P̂ such that
        △P̂ P_cur P_adj identifies a triangle in T* and
        P̂ is not inside the circumcircle of △P* P_adj P_cur)) then
            if (P_adj ≠ P') then
                begin
                P̂ := point such that △P* P̂ P_adj identifies
                a triangle in T*;
                P_cur := P_adj;  P_adj := P̂
                end
            else flag := 0
        else
            begin
            P̂ := point such that △P̂ P_cur P_adj identifies a triangle in T*;
            T* := (T* \ {△P* P_adj P_cur, △P̂ P_cur P_adj})∪
            {△P* P_adj P̂, △P* P̂ P_cur};
```

$$P_{adj} := \hat{P}$$
$$\text{end}$$
$$\text{end}$$
$$\text{end}$$

Justification of UPDTRI. That the execution of UPDTRI produces a triangulation T^* from T follows from the outline of the algorithm. Thus, assume that UPDTRI has been executed. For each i, $i = 1, \ldots, k$, let T_i^* be the set of triangles in T^* whose vertices are all in S_i. Since R is a generalized multiply-connected polygonal region, it follows from the outline of UPDTRI that for each i, $i = 1, \ldots, k$, $i \neq l$, T_i^* equals T_i, so that by the third condition stated above, each triangle in T_i^* satisfies the circle criterion in T_i^*. Also, from the outline of UPDTRI, by the first and second condition stated above, each triangle in $T_l^* \cap T_l$ satisfies the circle criterion in T_l^*. Thus, if $T_l^* \neq T_l$, it suffices to show that each triangle in $T_l^* \setminus T_l$ satisfies the circle criterion in T_l^*. We assume this is not the case. Therefore, since P^* is a vertex of every triangle in $T_l^* \setminus T_l$, points Q', Q'', and Q''' exist in S_l such that $\triangle P^*Q'Q''$ and $\triangle P^*Q''Q'''$ identify triangles in $T_l^* \setminus T_l$ with Q''' inside the circumcircle of $\triangle P^*Q'Q''$. Clearly, it follows that a triangle can be identified with $\triangle Q'Q''Q'''$ and P^* is not inside the circumcircle of this triangle.

Let $\hat{Q}'$ and $\hat{Q}'''$ be points in S_l such that $\triangle \hat{Q}'Q''\hat{Q}'''$ identifies the first triangle in T_l eliminated by the algorithm with Q'' as a vertex.

If $\hat{Q}'$ equals Q' and $\hat{Q}'''$ equals Q''' then P^* is not inside the circumcircle of $\triangle \hat{Q}'Q''\hat{Q}'''$, a contradiction to the definition of $\triangle \hat{Q}'Q''\hat{Q}'''$.

If $\hat{Q}'$ does not equal Q' then from the outline of UPDTRI and the definitions of Q' and Q'', there exists a point Q^* in S_l separated from Q'' by the line through P^* and Q' such that $\triangle Q^*Q'Q''$ identifies the first triangle in T_l eliminated by the algorithm with Q' as a vertex. Figure 2 illustrates an example of this situation.

Clearly, $[Q', Q''']$ and $[Q'', Q''']$ are contained in R_l, and we show that so is $[Q^*, Q''']$. Let C_1 and C_2 be the quadrilaterals with vertices P^*, Q', Q'', Q''', and P^*, Q^*, Q', Q'', respectively. Since the line through P' and P'' separates P^* from Q', Q'', Q''', and Q^*, and C_1 and C_2 are convex, it follows that $C_1 \cup C_2$ is convex so that $[Q^*, Q''']$ is contained in $C_1 \cup C_2$. Thus, $[Q^*, Q''']$ is contained in R_l, and by the second condition stated above Q''' is not inside the circumcircle of $\triangle Q^*Q'Q''$. In addition, since triangles can be identified with $\triangle Q^*Q'Q''$, $\triangle Q'Q''Q'''$, and $\triangle P^*Q'Q''$, Q^* and Q''' lie in the half-plane that contains P^* and that is determined by the line through Q' and Q''. Therefore, as shown by the example in Figure 3, the circumcircle of $\triangle Q'Q''Q'''$ contains the part of the circumcircle of $\triangle Q^*Q'Q''$ that is contained in this half-plane. Thus, by the definition of $\triangle Q^*Q'Q''$, P^* is inside the
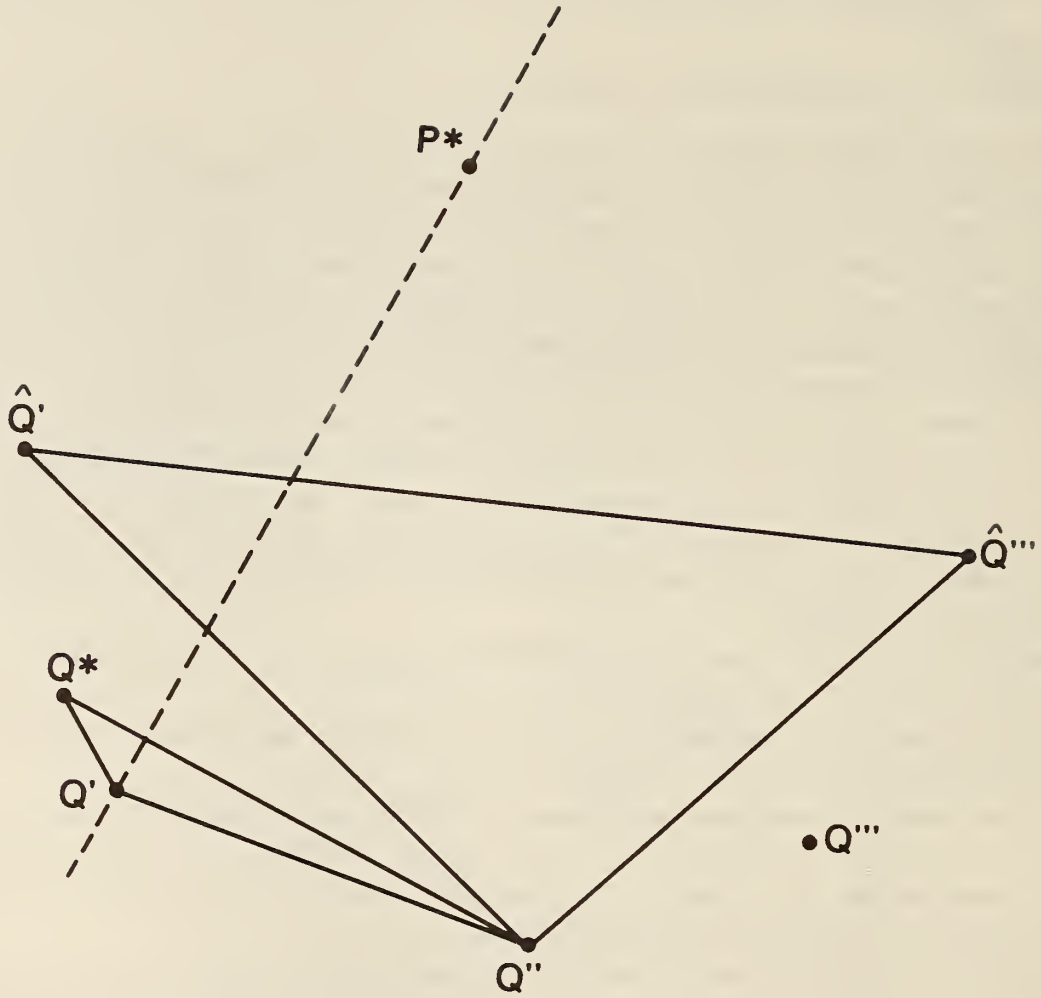
Figure 2: $\triangle \hat{Q}'Q''\hat{Q}'''$ and $\triangle Q^*Q'Q''$ when $\hat{Q}'$ does not equal Q'. The line through P^* and Q' separates Q^* and Q''.
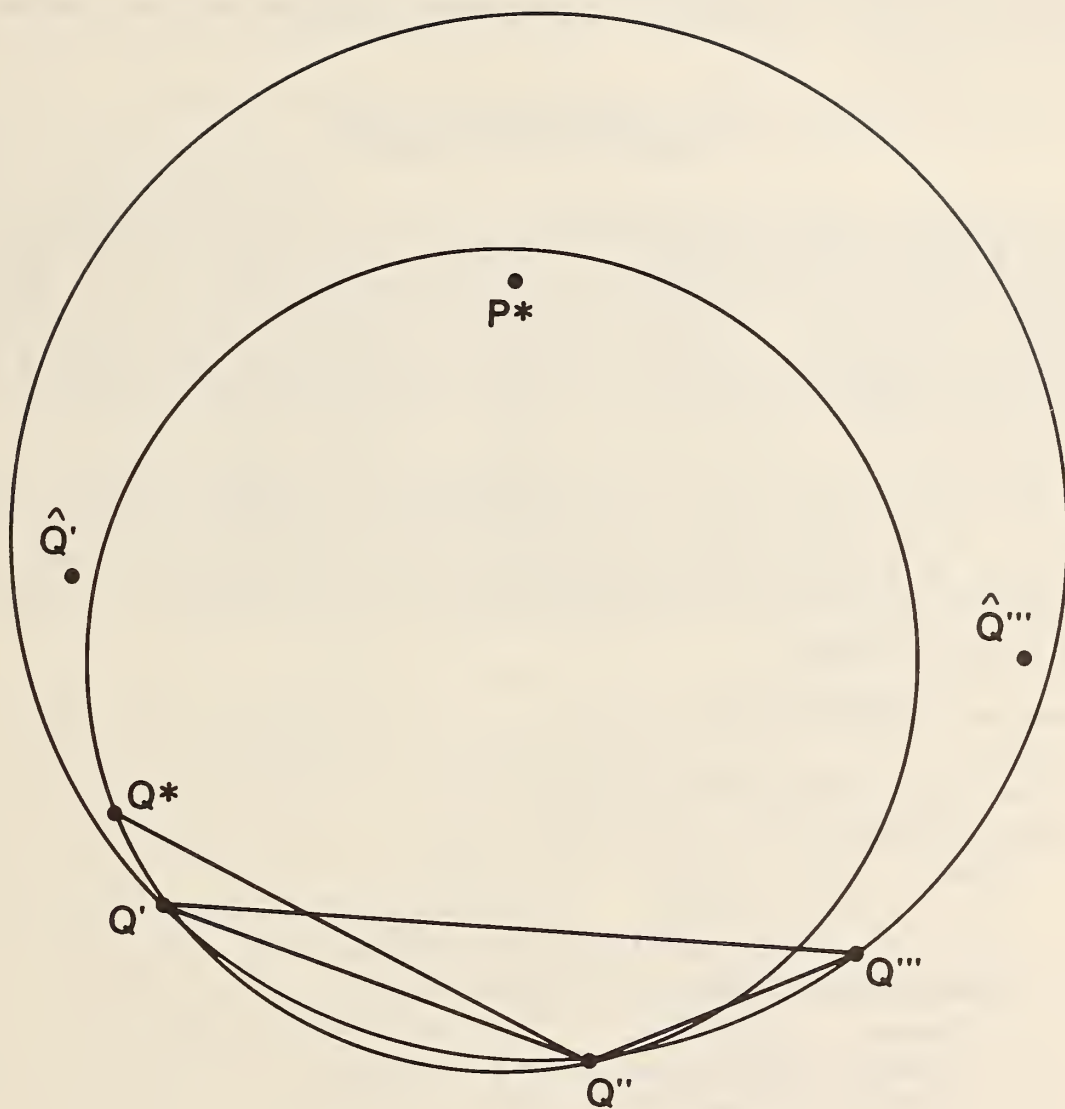
Figure 3: The circumcircles of $\Delta Q'Q''Q'''$ and $\Delta Q^*Q'Q''$.

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circumcircle of $\triangle Q'Q''Q'''$, a contradiction.

Similarly, if $\hat{Q}'''$ does not equal Q''', a contradiction is obtained, and the justification of UPDTRI is complete.

3. The EDGSTR algorithm

Let R be an edge star-shaped simple polygon, and let e be a linear component of $BND(R)$ such that R is star-shaped relative to e. Let S be a finite subset of $BND(R)$ such that S contains $VTX(R)$ and does not intersect the relative interior of e. Let r be the number of points in S, and let $P_1, \ldots, P_r$ be the points in S in the order in which they appear in $BND(R)$ in a counterclockwise direction around $INT(R)$ with $[P_1, P_r]$ equal to e. Define a one-to-one function F from $\{1, \ldots, r\}$ onto S by setting $F(i)$ equal to P_i for each i, $i = 1, \ldots, r$.

In what follows, we present and justify algorithm EDGSTR which computes a Delaunay triangulation T for S constrained by $BND(R)$. Here, given points Q_1, Q_2, Q_3 in the plane, $Q_2 \neq Q_3$ and $Q_2 \neq Q_1$, we let $m(Q_2\vec{Q}_3, Q_2\vec{Q}_1)$ represent the size in radians of the angle produced by a counterclockwise rotation around Q_2 from ray $Q_2\vec{Q}_3$ to ray $Q_2\vec{Q}_1$. The outline of EDGSTR follows.

```
        procedure EDGSTR(T, F, r)
            begin
1.          T := ∅; G(1) := F(1); G(2) := F(2); J := 2;
2.          for I := 3 until r do
                begin
3.              J := J + 1; G(J) := F(I);
4.              Q₁ := G(J − 2); Q₂ := G(J − 1); Q₃ := G(J);
5.              while (m(Q₂Q₃, Q₂Q₁) < π and J ≠ 2) do
                    begin
6.                  T := T ∪ {ΔQ₁Q₂Q₃};
7.                  UPDTRI(T, T, Q₃, Q₁, Q₂);
8.                  J := J − 1; G(J) := Q₃;
9.                  if (J ≠ 2) then
                        begin
10.                     Q₁ := G(J − 2); Q₂ := G(J − 1)
                        end
                    end
```

end

    end

Justification of EDGSTR. We show that the execution of EDGSTR produces a collection of triangles T that is a Delaunay triangulation for S constrained by $BND(R)$. First, we must prove the following observation:

Given integers j, k, l, $1 \leq j < k < l \leq r$, if R contains $[P_j, P_k]$ and $[P_k, P_l]$, $m(P_k\vec{P_l}, P_k\vec{P_j})$ is less than π, and $[P_j, P_k]$ and $[P_k, P_l]$ intersect S exactly at P_j and P_k, and P_k and P_l, respectively, then $\triangle P_j P_k P_l$ identifies a triangle that is contained in R and that intersects S exactly at P_j, P_k, and P_l. In addition, given a finite collection of triangles T with pair-wise disjoint interiors, each of which is contained in R and intersects S exactly at its vertices, if (P_j, P_k) and (P_k, P_l) do not intersect the interior of each triangle in T, and $INT(\triangle P_j P_k P_l)$ does not intersect the interior of each triangle in T with P_k as a vertex, then $INT(\triangle P_j P_k P_l)$ does not intersect the interior of any triangle in T.

Figure 4 illustrates an example of R, P_j, P_k, P_l, P_1, P_r, as described above.

In what follows, we prove the first part of the observation. The second part will follow from it.

That $\triangle P_j P_k P_l$ identifies a triangle follows from the fact that $m(P_k\vec{P_l}, P_k\vec{P_j})$ is less than π. It follows from the ordering of P_i, $i = 1, \ldots, r$, in $BND(R)$, that in order to prove the rest of the first part of the observation, it suffices to show that for each integer i, $1 \leq i \leq j$ or $l \leq i \leq r$, P_i does not belong to $C \equiv \triangle P_j P_k P_l \setminus ([P_j, P_k] \cup [P_k, P_l])$. We assume the contrary and let H be the set of integers i at which $m(P_l\vec{P_i}, P_l\vec{P_k})$ attains its minimum value, with $1 \leq i \leq j$ or $l \leq i \leq r$, and P_i in C. Clearly, each i in H must be different from j and l, so that either $1 \leq i < j$ or $l < i \leq r$ for each i in H.

If $l < i \leq r$ for some i in H, we show that for each point Q in (P_1, P_r), (P_l, Q) is not entirely contained in $INT(R)$, a contradiction to the fact that R is star-shaped relative to $[P_1, P_r]$. Thus, let i^* be an integer in H such that $l < i^* \leq r$. It follows that the ray $P_l\vec{P_{i^*}}$ intersects $[P_j, P_k)$. Thus, there exists an integer n, $j \leq n < k$, and a point P^* in $BND(R)$, such that $P_l\vec{P_{i^*}}$ intersects $[P_n, P_{n+1})$ exactly at P^*, and $[P^*, P_{i^*}]$ is contained in R. Let R' be the simple polygon whose vertices are contained in $\{P^*\} \cup \{P_i, n < i \leq i^*\}$ and whose boundary contains P^*, P_i, $i = n + 1, \ldots, i^*$, in this order in a counterclockwise direction around its interior. Figure 5 illustrates an example of R', P_j, P_k, P_l, P_{i^*}, P_n, P_{n+1}, P^*, P_1, P_r. Let Q be a point in (P_1, P_r) and assume that (P_l, Q) is contained in $INT(R)$. It follows from the definition of R' that $[P_l, Q]$ must intersect $INT(R')$. However, from the definitions of i^*, P^*, and the ordering of P_i, $i = 1, \ldots, r$, in BND(R), Q is not contained in $INT(R')$, so that $[P_l, Q]$ must intersect $BND(R')$ at a point other than P_l. Every point in $BND(R')$
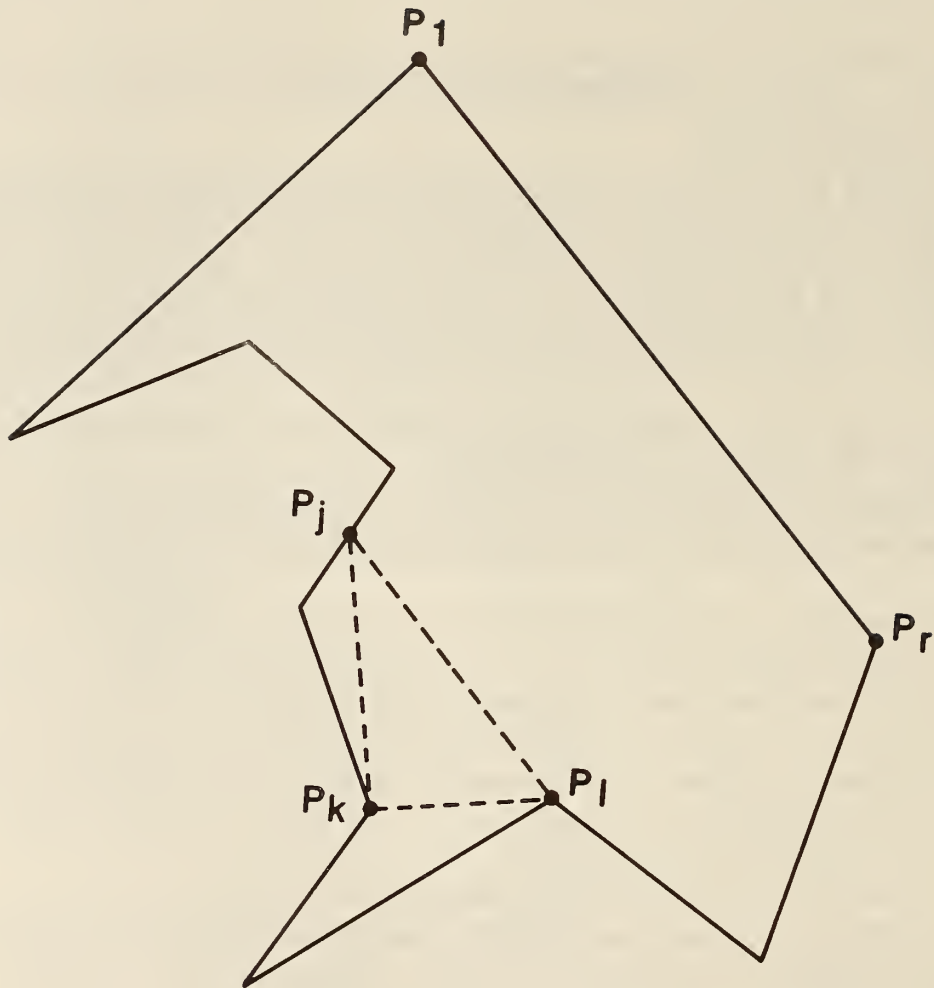
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Figure 4: The polygon R and the points P_j, P_k, P_l, P_1, P_r. $\triangle P_j P_k P_l \subseteq R$ and $\triangle P_j P_k P_l \cap S = \{P_j, P_k, P_l\}$.
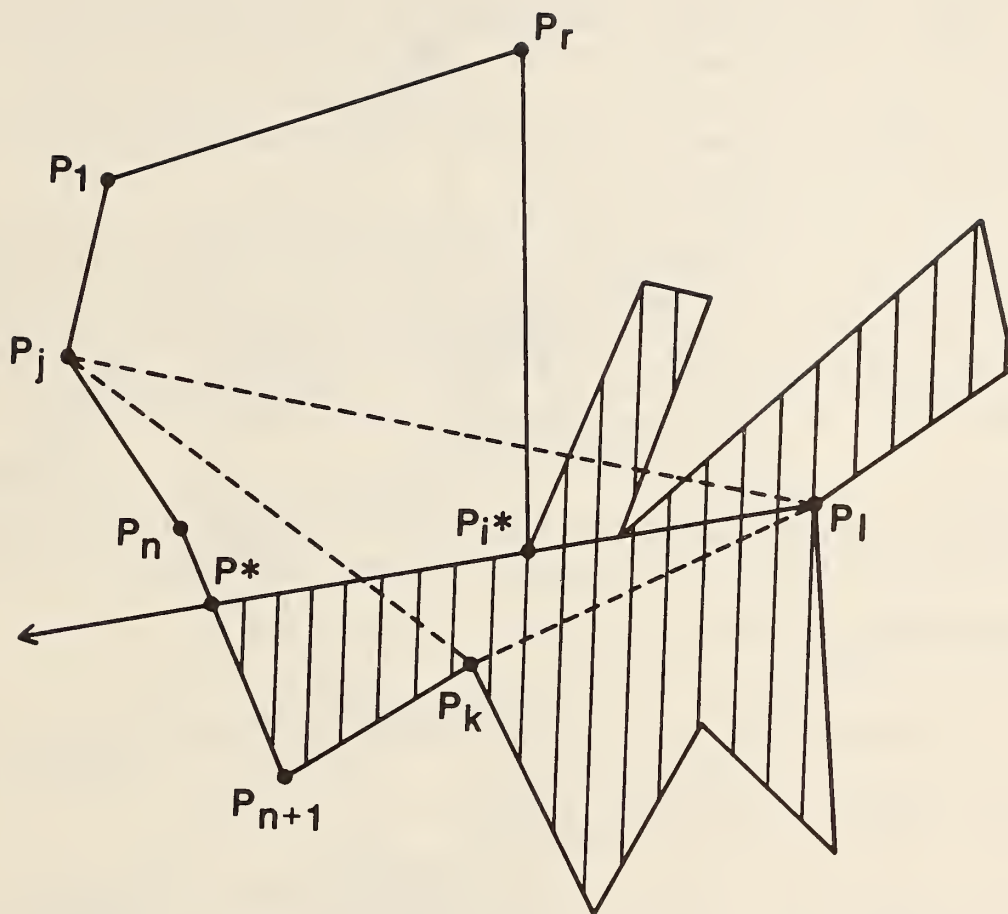
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Figure 5: The polygon R' (shaded region) and the points P_j, P_k, P_l, P_{i^*}, P_n, P_{n+1}, P^*, P_1, P_r. The ray $P_l\vec{P}_{i^*}$ intersects $[P_n, P_{n+1})$ at P^*.

that is not in (P^*, P_{i^*}) belongs to $BND(R)$ and differs from Q. Thus, $[P_l, Q]$ must intersect (P^*, P_{i^*}). But P^*, P_{i^*}, and P_l are colinear with P_{i^*} in (P^*, P_l), so that (P_l, Q) must contain P_{i^*}. Since Q is an arbitrary point in (P_1, P_r), we have the desired contradiction.

Similarly, if $1 \le i < j$ for each i in H, it can be shown that for each point Q in (P_1, P_r), (P_j, Q) is not entirely contained in $INT(R)$. This is also a contradiction, and it completes the proof of the first part of the observation.

Let I, J, G, and T be as defined in the outline of EDGSTR. Let K be a variable which is set to zero at the start of the execution of EDGSTR and which is incremented by one at the start of each execution of line 6 of EDGSTR. It follows from the outlines of EDGSTR and UPDTRI, the above observation, the definition of R, and an induction process on K, that at the start of each execution of line 6, the criteria listed below are met. Figure 6 illustrates an example of R, F, G, T, I, J that satisfies these criteria. The criteria follow:

1. $\{G(j), \; j = 1, \ldots, J\} \subseteq \{F(i), \; i = 1, \ldots, I\}$, $G(1)$ and $G(J)$ equal $F(1)$ and $F(I)$, respectively, and given integers k, l, k', l', $1 \le k$, $l \le I$, $1 \le k'$, $l' \le J$, such that $F(k)$ and $F(l)$ equal $G(k')$ and $G(l')$, respectively, then $k < l$ if and only if $k' < l'$.

2. If $J > 3$ then $m(G(j)\vec{G}(j+1), G(j)\vec{G}(j-1)) \ge \pi$ for each j,
   $j = 2, \ldots, J - 2$.

3. The triangles in T have pair-wise disjoint interiors.

4. Each triangle in T is contained in R and intersects
   $\{F(i), \; i = 1, \ldots, I\}$ exactly at its vertices.

5. Given integers k, l, $1 \le k < l \le I$ such that $[F(k), F(l)]$ is an edge of a triangle in T, if l equals $k + 1$ or if for some integer j, $1 \le j \le J - 1$, $F(k)$ and $F(l)$ equal $G(j)$ and $G(j + 1)$, respectively, then $[F(k), F(l)]$ is an edge of exactly one triangle in T. Otherwise, it is an edge of two triangles in T.

6. Given an integer i, $1 \le i \le I - 1$, $[F(i), F(i+1)]$ is not an edge of a triangle in T if and only if for some integer j, $1 \le j \le J - 1$, $F(i)$ and $F(i + 1)$ equal $G(j)$ and $G(j + 1)$, respectively.

7. Given an integer j, $1 \le j \le J - 1$, $[G(j), G(j + 1)]$ is not an edge of a triangle in T if and only if for some integer i, $1 \le i \le I - 1$, $G(j)$ and $G(j + 1)$ equal $F(i)$ and $F(i + 1)$, respectively.
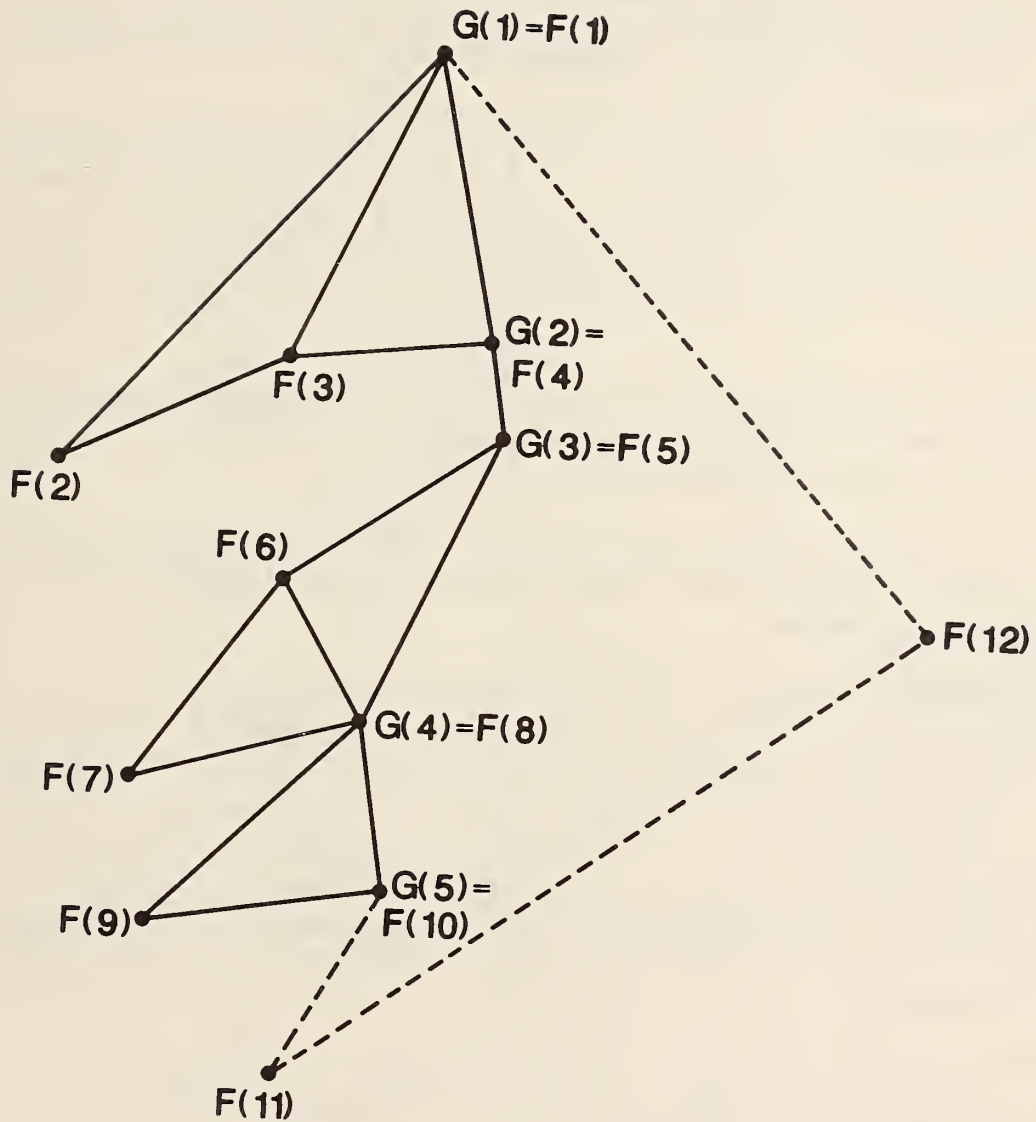
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Figure 6: A polygon R, points $F(i)$, $i = 1, \ldots, r$, for r equal to 12, and the corresponding triangulation T and points $G(j)$, $j = 1, \ldots, J$, at the start of the execution of line 6 of EDGSTR for I and J equal to 10 and 5, respectively.

8. Given t in T, there exist integers j, k, l, i_1, i_2, i_3, $1 \le j \le J-1$, $1 \le k \le i_1 < i_2 < i_3 \le l \le I$, such that $G(j)$ and $G(j+1)$ equal $F(k)$ and $F(l)$, respectively, and $\triangle F(i_1)F(i_2)F(i_3)$ identifies t.

9. Given integers j, k, l, $1 \le j \le J-1$, $1 \le k < l \le I$, such that $G(j)$ and $G(j+1)$ equal $F(k)$ and $F(l)$, respectively, if $k+1 < l$ then (i) the union of the triangles in T of the form $\triangle F(i_1)F(i_2)F(i_3)$, $k \le i_1 < i_2 < i_3 \le l$, is a simple polygon that is star-shaped relative to $[G(j), G(j+1)]$, and (ii) the closed half-plane determined by a clockwise rotation of π radians of the ray $G(j)\vec{G}(j+1)$ around $G(j)$ contains $F(i)$ for each i, $k \le i \le l$.

10. Given an integer k, $1 \le k < I$, such that $G(J-1)$ equals $F(k)$, the closed half-plane determined by a clockwise rotation of π radians of the ray $G(J-2)\vec{G}(J-1)$ around $G(J-2)$ contains $F(i)$ for each i, $k \le i < I$.

11. T is a Delaunay triangulation for a subset of $\{F(i), \ i = 1, \ldots, I\}$ constrained by the boundary of the generalized multiply-connected polygonal region that the union of the triangles in T defines.

Let j_r be the initial value of J the first time line 6 is executed for I equal to r. From the first criterion above and lines 3, 4 and 8 of EDGSTR, at the start of the execution of line 6 for I equal to r and J equal to j_r, we must have that $G(1)$, $G(J-1)$, and $G(J)$ equal P_1, P_{r-1}, and P_r, respectively. This, together with the second criterion (for the same I and J) and the ordering of P_i, $i = 1, \ldots, r$, in $BND(R)$, implies that $m(G(j)\vec{G}(J), G(j)\vec{G}(j-1))$ is less than π for each j, $j = 2, \ldots, J-1$. Figure 7 illustrates this situation for the same R and F of Figure 6. Thus, from the remaining criteria (for the same I and J), it can be seen from lines 6-10 that at the end of the execution of EDGSTR for I equal to r, the following conditions are met:

1. The triangles in T have pair-wise disjoint interiors.

2. Each triangle in T is contained in R.

3. For each i, $i = 1, \ldots, r-1$, there exists a triangle in T with $[P_i, P_{i+1}]$ as an edge.

4. There exists a triangle in T with $[P_1, P_r]$ as an edge.

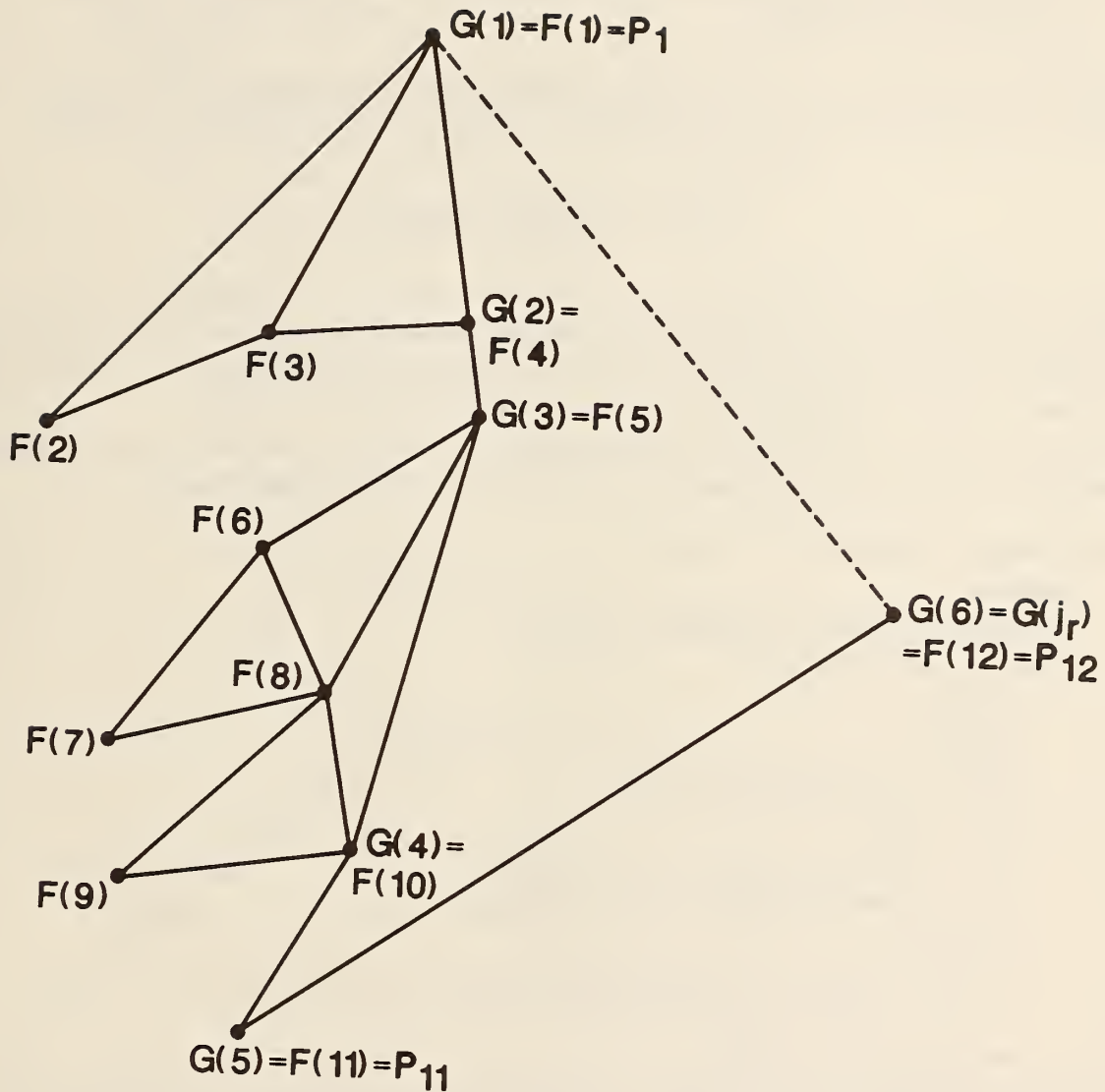5. Given an edge of a triangle in T, if it is not contained in $BND(R)$ then it is an edge of two triangles in T.

Figure 7: A polygon R, points $F(i)$, $i = 1, \ldots, r$, for r equal to 12, and the corresponding triangulation T and points $G(j)$, $j = 1, \ldots, J$, at the start of the execution of line 6 of EDGSTR for I and J equal to r and j_r, respectively.

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6. T is a Delaunay triangulation for a subset of S constrained by the boundary of the union of the triangles in T.

Since these conditions imply that the execution of EDGSTR produces a Delaunay triangulation T for S constrained by $BND(R)$, the justification of EDGSTR is complete.

4. The SEGMNT algorithm

Let S be a finite set of points in the plane. Given a positive integer M, let e_i, $i = 1, \ldots, M$, be line segments in the plane that intersect only at points in S and whose endpoints belong to S. Define a one-to-one function D from $\{1, \ldots, M\}$ onto $E \equiv \{e_i, \ i = 1, \ldots, M\}$ by setting $D(i)$ equal to e_i for each i, $i = 1, \ldots, M$.

In this section, we present and justify algorithm SEGMNT which computes a Delaunay triangulation T^* for S constrained by E. The outline of SEGMNT follows.

    procedure SEGMNT(T^*, S, D, M)
        begin
1.   obtain a Delaunay triangulation T^* for S;
2.   **for** $I := 1$ **until** M **do**
          begin
3. $T_I^* := \emptyset$;
4. $Q_0, \ Q_1 :=$ endpoints of $D(I)$;
5. $Q_2 := Q_0$;
6. $H_1 :=$ closed half-plane to 'left' of $Q_0\vec{Q}_1$;
7. $H_2 :=$ closed half-plane to 'left' of $Q_1\vec{Q}_2$;
8. $flag1 := 1$;
9.   **if** (each triangle in T^* with Q_0 as a vertex is contained in H_1)
          then $flag1 := 0$ **else** $J := 0$;
10.   **while** $(flag1 = 1)$ **do**
          begin
11. $J := J + 1$;
12. $Q'', \ Q''' :=$ points in S for which $\triangle Q_{J-1}Q''Q'''$
          identifies a triangle in T^* with $Q'' \ \epsilon \ H_J$ and $Q''' \ \notin H_J$;
13.   **if** $(J = 1$ and $Q'' \ \epsilon \ INT(H_J))$ **then**
          $T_I^* := T_I^* \cup \{\triangle Q_{J-1}Q''Q'''\}$;
14. $F_J(1) := Q_{J-1}$; $F_J(2) := Q''$;

```
15.                L := 2; Q' := Q'''; flag2 := 1;
16.                while (flag2 = 1) do
                        begin
17.                     Q''' := point in S for which ΔQ'Q''Q'''
                        identifies a triangle in T*;
18.                     if (J = 1 and (Q'' ∈ INT(H_J) or Q''' ∈ INT(H_J)))
                            then T_I^* := T_I^* ∪ {ΔQ'Q''Q'''};
19.                     if (Q''' ∉ H_J) then Q' := Q''' else
                            begin
20.                         L := L + 1; F_J(L) := Q''';
21.                         if (Q''' ≠ Q_J) then Q'' := Q'''
                            else if (J = 1 and T_I^* = ∅) then flag1, flag2 := 0
                            else
                                begin
22.                             r_J := L;
23.                             EDGSTR(T_J, F_J, r_J);
24.                             if (J = 1) then flag2 := 0
                                else
                                    begin
25.                                 T* := (T* \ T_I^*) ∪ T_1 ∪ T_2;
26.                                 flag1, flag2 := 0
                                    end
                                end
                            end
                        end
                    end
                end
            end
```

Justification of SEGMNT. We show that the execution of SEGMNT produces a collection of triangles T^* that is a Delaunay triangulation for S constrained by E. Here, given a subset E' of E, a triangulation T for S constrained by E', and triangles t and t' in T, we say that t' is *adjacent to t in T relative to E'* if the intersection of t and t' is a line segment that is not contained in any of the line segments in E'. In addition, we denote by $ADJ(t, E')$ the set of points in S that are vertices of triangles in T adjacent to t in T relative to E', and say that t satisfies the *circle criterion in T relative to E'* if there does not exist a point of

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$ADJ(t, E')$ inside the circumcircle of t. As in [5], we note that using arguments similar to those developed in [6,8], it can be shown that if each triangle in T satisfies the circle criterion in T relative to E' then T is a Delaunay triangulation for S constrained by E'.

Let I, J, T^*, T_I^*, H_J, T_J, F_J, r_J be as defined in the outline of SEGMNT. We show that at the end of each execution of line 25 of SEGMNT, T^* is a Delaunay triangulation for S constrained by $\{e_i : 1 \le i \le I\}$. Clearly, for I equal to M this is the desired conclusion. From line 1, at the start of the execution of line 3 for I equal to 1, T^* is a Delaunay triangulation for S constrained by $\{e_i : 1 \le i \le I - 1\}$. Thus, we assume by induction on I that given k, $1 \le k \le M$, at the start of the execution of line 3 for I equal to k, T^* is a Delaunay triangulation for S constrained by $\{e_i : 1 \le i \le I - 1\}$, and show that at the end of the execution of line 25 for I equal to k, T^* is a Delaunay triangulation for S constrained by $\{e_i : 1 \le i \le I\}$.

For each J, $J = 1, 2$, at the end of each execution of line 22, let R_J be defined as the intersection of H_J with the union of the triangles in T_I^*. From the definitions of T_I^*, H_J, F_J, and r_J, for each J, $J = 1, 2$, if R_J is not empty for I equal to k, we may assume without any loss of generality that R_J is a simple polygon that is star-shaped relative to $[F_J(1), F_J(r_J)] = D(I)$, $\{F_J(L), L = 1, \ldots, r_J\}$ equals $S \cap R_J$ and contains $VTX(R_J)$, the points $F_J(L)$, $L = 1, \ldots, r_J$, appear in $BND(R_J)$ in a counterclockwise direction around $INT(R_J)$, and $D(I)$ intersects S exactly at its vertices. Thus, for I equal to k and for each J, $J = 1, 2$, if R_J is not empty EDGSTR can be executed in line 23 for F_J and r_J to produce a Delaunay triangulation T_J for $S \cap R_J$ constrained by $BND(R_J)$, and at the end of the execution of line 25 for the same I, T^* is a triangulation for S. In addition, since $D(I)$ does not intersect the interior of each triangle in T^*, it follows from the induction hypothesis and the definitions of E, T_1, and T_2, that T^* is a triangulation for S constrained by $\{e_i : 1 \le i \le I\}$. Finally, let t be a triangle in T^* at the end of the execution of line 25 for I equal to k, and assume R_J for the same I is not empty for each J, $J = 1, 2$. If t belongs to $T^* \setminus (T_1 \cup T_2)$ then from the induction hypothesis it follows that t satisfies the circle criterion in T^* relative to $\{e_i : 1 \le i \le I\}$. Else, if t belongs to T_1 then all triangles adjacent to t in T^* relative to $\{e_i : 1 \le i \le I\}$ must belong to $T^* \setminus T_2$, and since T_1 is a Delaunay triangulation for $S \cap R_1$ constrained by $BND(R_1)$, we must have that t satisfies the circle criterion in T^* relative to $\{e_i : 1 \le i \le I\}$. Similarly, if t belongs to T_2 the same conclusion is obtained. Thus, by induction on I, we have shown that at the end of each execution of line 25, T^* is a Delaunay triangulation for S constrained by $\{e_i : 1 \le i \le I\}$, and the justification of SEGMNT is complete.

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5. Proof of complexity

In this section, we assume that S, E, M, e_i, $i = 1, \ldots, M$, are as defined in the previous section, and that the points in S have been chosen independently from a uniform distribution on a square in the plane. Letting N represent the cardinality of S, we then prove that for fixed M the expected time required by SEGMNT to produce a Delaunay triangulation for S constrained by E is $O(N)$.

First, we introduce additional terminology. Let B denote the square from which S has been chosen, and let d denote the Euclidean length of each of its sides. Let K be defined as the *floor* of $N^{1/2}$, i. e. the largest integer less than or equal to $N^{1/2}$, and assume that B has been divided into K^2 equal-sized square *cells*. Let $L(N)$ be defined as the floor of $\log N$ where log denotes the natural logarithm, and let B' denote the square obtained by surrounding B with $L(N) + 1$ layers of cells of the type into which B has been divided. Let S' denote the set of points that are centroids of cells in the $L(N) + 1$ new layers. Given e in E, let $|e|$ denote the Euclidean length of e. Given a finite set X of points in the plane and points P and P' in X, $P \neq P'$, call P' a *Delaunay neighbor of P relative to X* if in some Delaunay triangulation for X there exists a triangle with P and P' as vertices. Given G, a finite nonempty set in the plane, and G_1, G_2, nonempty subsets of G, let $V(G, G_1, G_2)$ denote the set of points in G_2 that are Delaunay neighbors of points in G_1 relative to G. Given e in E, let $V(e)$ represent the union of the sets $V(S \cup \{P\}, \{P\}, S)$ for P in e, let $V'(e)$ represent the union of the sets $V(S \cup S' \cup \{P\}, \{P\}, S)$ for P in e, and let $\overline{S(e)}$ represent the union of the sets $V(S \cup S' \cup \{P\}, S', S)$ for P in e. Finally, let $\overline{S}$ represent the set $V(S \cup S', S', S)$.

Next, based on the terminology just defined, we list and prove some observations, the last of which establishes our assertion about the complexity of SEGMNT. Here, I, J, F_J, and r_J are as defined in the outline of SEGMNT.

1. For each e in E, $\overline{S(e)} \subseteq \overline{S}$.

2. For each e in E, $V(e) \subseteq V'(e) \cup \overline{S(e)} \subseteq V'(e) \cup \overline{S}$.

3. For each e in E, $S \cap e \subseteq V(e)$.

4. Given points Q_1 and Q_2 in S, if Q_1 is a Delaunay neighbor of Q_2 relative to S then Q_1 and Q_2 belong to $V(S \cup \{P\}, \{P\}, S)$ for each point P in (Q_1, Q_2).

5. Given an integer i, $1 \leq i \leq M$, for I equal to i the points $F_J(L)$, $L = 1, \ldots, r_J$, $J = 1, 2$, are contained in $\cup_{l=1}^{i} V(e_l) \subseteq (\cup_{l=1}^{i} V'(e_l)) \cup \overline{S}$.

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6. For each e in E, the expected number of points in $V'(e)$ is bounded above by $(|e|/d) \cdot O(N^{1/2})$.

7. The expected number of points in $\overline{S}$ is $O(N^{1/2})$.

8. For fixed M, the expected time required by SEGMNT to produce a Delaunay triangulation for S constrained by E is bounded above by

$$O(N) + \sum_{i=1}^{M} (((\sum_{l=1}^{i} |e_l|/d) + 1)^2 \cdot O(N)) = (1 + O(M^3)) \cdot O(N) = O(N).$$

Proof of first observation. Let e be a line segment in E, and let Q be a point in $\overline{S(e)}$. By the definition of $\overline{S(e)}$, Q is in S and for some point P in e there exists a point Q' in S' such that Q is a Delaunay neighbor of Q' relative to $S \cup S' \cup \{P\}$. Thus, we must have that either (i) Q and Q' are adjacent points on the convex hull of $S \cup S' \cup \{P\}$ or (ii) the line segment $[Q, Q']$ is an edge of two triangles in some Delaunay triangulation T' for $S \cup S' \cup \{P\}$. If (i) occurs then Q and Q' are also adjacent points on the convex hull of $S \cup S'$ so that Q is a Delaunay neighbor of Q' relative to $S \cup S'$ and, therefore, Q is a point in $\overline{S}$. Else, if (ii) occurs then for some point Q'' in $S \cup S'$ there exists a triangle t in T' such that Q, Q', and Q'' are vertices of t. By the definition of T' there does not exist a point of $S \cup S'$ inside the circumcircle of t. Thus, t is a triangle in some Delaunay triangulation for $S \cup S'$ and, therefore, Q belongs to $\overline{S}$.

Proof of second observation. Through slight modifications of a proof in [2], it can be shown that if X and Y are finite nonempty sets in the plane and X' is a nonempty subset of X then

$$V(X, X', X) \subseteq V(X \cup Y, X', X) \cup V(X \cup Y, Y, X).$$

Thus, given e in E and P in e, we must have that

$$V(S \cup \{P\}, \{P\}, S) = V(S \cup \{P\}, \{P\}, S \cup \{P\})$$

$$\subseteq V(S \cup S' \cup \{P\}, \{P\}, S \cup \{P\}) \cup V(S \cup S' \cup \{P\}, S', S \cup \{P\})$$

$$\subseteq V(S \cup S' \cup \{P\}, \{P\}, S) \cup V(S \cup S' \cup \{P\}, S', S) \cup \{P\},$$

and since P does not belong to $V(S \cup \{P\}, \{P\}, S)$ it follows that

$$V(S \cup \{P\}, \{P\}, S) \subseteq V(S \cup S' \cup \{P\}, \{P\}, S) \cup V(S \cup S' \cup \{P\}, S', S),$$

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which together with the first observation implies the desired result.

Proof of third observation. Let e be a line segment in E, and let P' be a point in $S \cap e$. The assertion follows, since for P in e sufficiently close to P', P' is in $V(S \cup \{P\}, \{P\}, S)$.

Proof of fourth observation. Let Q_1 and Q_2 be points in S, and assume that Q_1 is a Delaunay neighbor of Q_2 relative to S. Let P be a point in (Q_1, Q_2), let T' be a Delaunay triangulation for $S \cup \{P\}$, and let $T'(Q_1)$ be the collection of triangles in T' with Q_1 as a vertex. If Q_1 does not belong to $V(S \cup \{P\}, \{P\}, S)$ then P is neither inside nor on the circumcircle of each triangle in $T'(Q_1)$. Thus, (i) the points that are vertices of triangles in $T'(Q_1)$ belong to S, and (ii) Q_2 is neither inside nor on the circumcircle of each triangle in $T'(Q_1)$. But this is a contradiction, since from (i) and the definition of T' a Delaunay triangulation for S exists in which the collection of triangles with Q_1 as a vertex equals $T'(Q_1)$ so that from (ii) Q_1 can not be a Delaunay neighbor of Q_2 relative to S. Thus, Q_1 belongs to $V(S \cup \{P\}, \{P\}, S)$, and similarly, so does Q_2.

Proof of fifth observation. Let T^* be as defined in the outline of SEGMNT. Given an integer i, $1 \le i \le M$, let T^{*i} equal T^* at the start of the execution of line 3 of SEGMNT for I equal to i, and let S^i equal the set $\{F_J(L),\ L = 1, \ldots, r_J,\ J = 1, 2\}$ for I equal to i. In what follows, we show by induction on i that $S^i \subseteq \cup_{l=1}^{i} V(e_l)$ for each i, $1 \le i \le M$. The rest of the observation will follow from the second observation above.

Given i, $1 \le i \le M$, let Q be a point in S^i. It follows that either (1) Q is a point in $S \cap e_i$, or (2) for some point Q' in S and some triangle t in T^{*i}, Q and Q' are distinct vertices of t, and (Q, Q') intersects the relative interior of e_i. If (1) occurs then by the third observation above Q is in $V(e_i)$. Else, if (2) occurs, we assume inductively that if P_1 and P_2 are distinct vertices of a triangle in T^{*i} then either (a) $i \ge 1$ and P_1 is a Delaunay neighbor of P_2 relative to S, or (b) $i > 1$, and P_1 and P_2 belong to $\cup_{l=1}^{i-1} V(e_l)$. It then follows from (a), (b), and the fourth observation above that Q is in $\cup_{l=1}^{i} V(e_l)$, and from the definition of T^{*i}, for $i < M$ the induction assumption holds when i is replaced by $i + 1$.

Proof of sixth observation. As in [1], given a point P in the plane, a finite subset X of the plane, and a collection O of octants around P as shown in Figure 8, we say that P is *closed in X relative to O* if at least one point in X different from P is contained in each of the octants in O.

Let e be a line segment in E, let $K(e)$ represent the floor of $(|e|/d) \cdot N^{1/2}$, and without regard to any specific order, let Q_0 and $Q_{K(e)}$ denote the endpoints of e. For each k, $k = 0, \ldots, K(e)$, let P_k denote the point

$$Q_0 + (k/K(e)) \cdot (Q_{K(e)} - Q_0),$$

and let O_k denote the collection of octants around P_k for which the interior of each octant
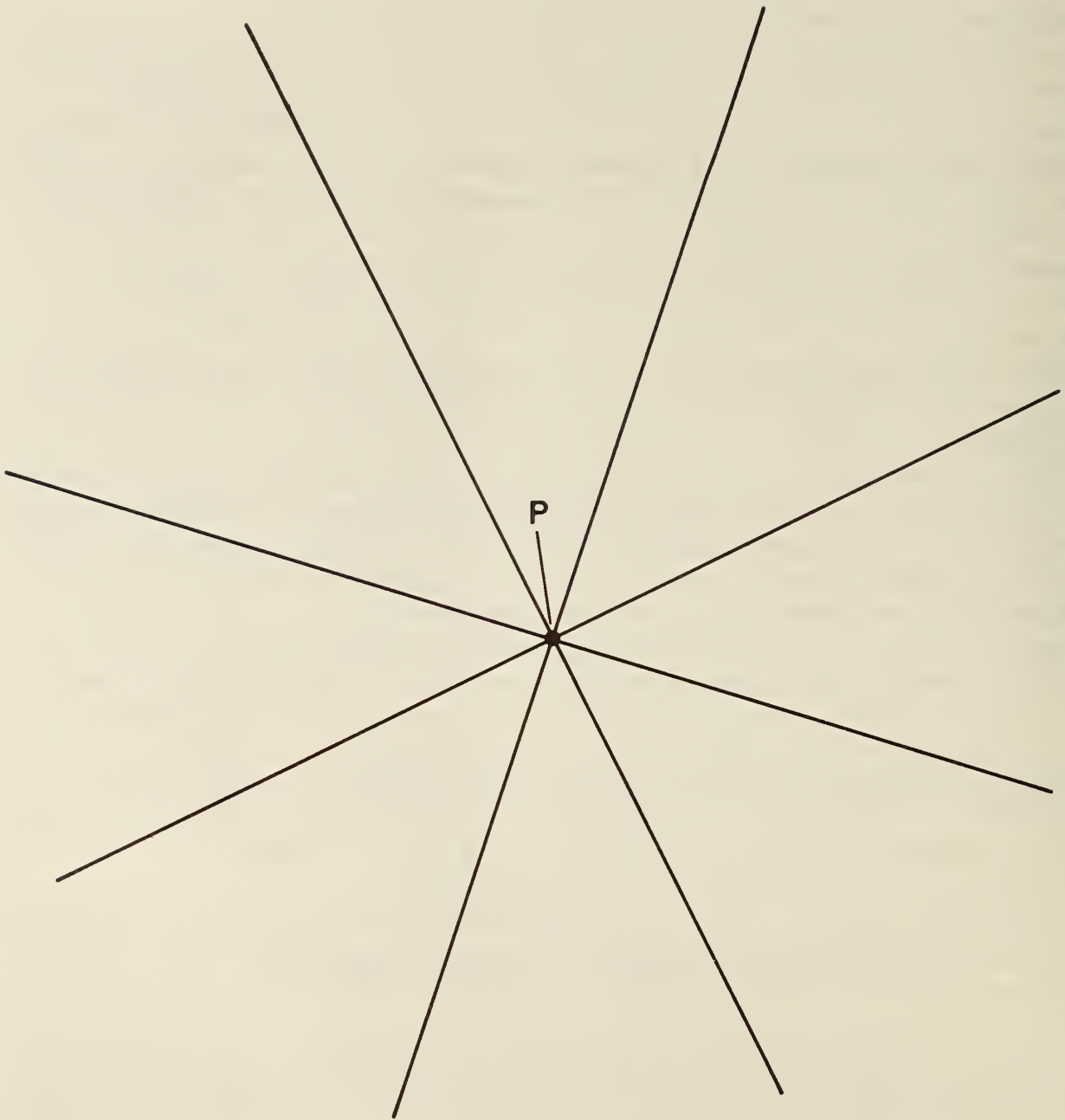
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Figure 8: A point P in the plane and a collection of octants around it.

does not intersect e. Figure 9 illustrates an example of P_k, O_k, $k = 1, \ldots, K(e)$.

Let p_1 be the probability that for each k, $k = 0, \ldots, K(e)$, P_k is closed in $S \cup S'$ relative to O_k, and let t_1 be the expected number of points in $V'(e)$ when for each k, $k = 0, \ldots, K(e)$, P_k is closed in $S \cup S'$ relative to O_k. Let p_2 and t_2 be similarly defined, respectively, for the event in which for some k, $0 \leq k \leq K(e)$, P_k is not closed in $S \cup S'$ relative to O_k.

Given k, $0 \leq k \leq K(e)$, P_k is not in any of the outermost $L(N)$ layers of cells of B'. Thus, it follows from the selection of S and S', and results in [1], that if P_k is closed in $S \cup S'$ relative to O_k then the expected number of points in $V(S \cup S' \cup \{P_k\}, \{P_k\}, S \cup S')$ is $O(1)$. In addition, we note that for each k, $k = 1, \ldots, K(e)$, the length of $[P_{k-1}, P_k]$ equals $|e|/K(e) \approx d/N^{1/2}$ (the length of each side of a cell), and e equals $\cup_{k=1}^{K(e)}[P_{k-1}, P_k]$. Hence, it follows from the geometry of the octants around the points P_k, $k = 0, \ldots, K(e)$, as illustrated in Figure 9, that if for each k, $k = 0, \ldots, K(e)$, P_k is closed in $S \cup S'$ relative to O_k, then the expected number of points in $\cup_{P \in e} V(S \cup S' \cup \{P\}, \{P\}, S \cup S')$ is $K(e) \cdot O(1)$. Therefore,

$$t_1 = K(e) \cdot O(1) = (|e|/d) \cdot O(N^{1/2}).$$

On the other hand, if for some k, $0 \leq k \leq K(e)$, P_k is not closed in $S \cup S'$ relative to O_k, then using an argument from [1], the probability of this occurring is bounded above by $8 \cdot (K(e) + 1) \cdot e^{-O(L(N)^2)}$. Hence,

$$p_2 \leq 8 \cdot (K(e) + 1) \cdot e^{-O(L(N)^2)}.$$

Finally, t_2 is clearly bounded above by the number of points in S. Therefore,

$$t_2 \leq N.$$

Thus, if t is the expected number of points in $V'(e)$, then

$$
\begin{aligned}
t &= p_1 \cdot t_1 + p_2 \cdot t_2 \\
&= 1 \cdot (|e|/d) \cdot O(N^{1/2}) + 8 \cdot (K(e) + 1) \cdot e^{-O(L(N)^2)} \cdot N \\
&= (|e|/d) \cdot O(N^{1/2}) + 8 \cdot (K(e) + 1) \cdot O(1) \\
&= (|e|/d) \cdot O(N^{1/2}) + (|e|/d) \cdot O(N^{1/2}) \\
&= (|e|/d) \cdot O(N^{1/2}).
\end{aligned}
$$

Proof of seventh observation. Let S'' represent the set of points in S' contained in the first layer of cells in B' that surrounds B. From the definition of S', points in S'' are the only points in S' that can be Delaunay neighbors of points in S relative to $S \cup S'$. Since points
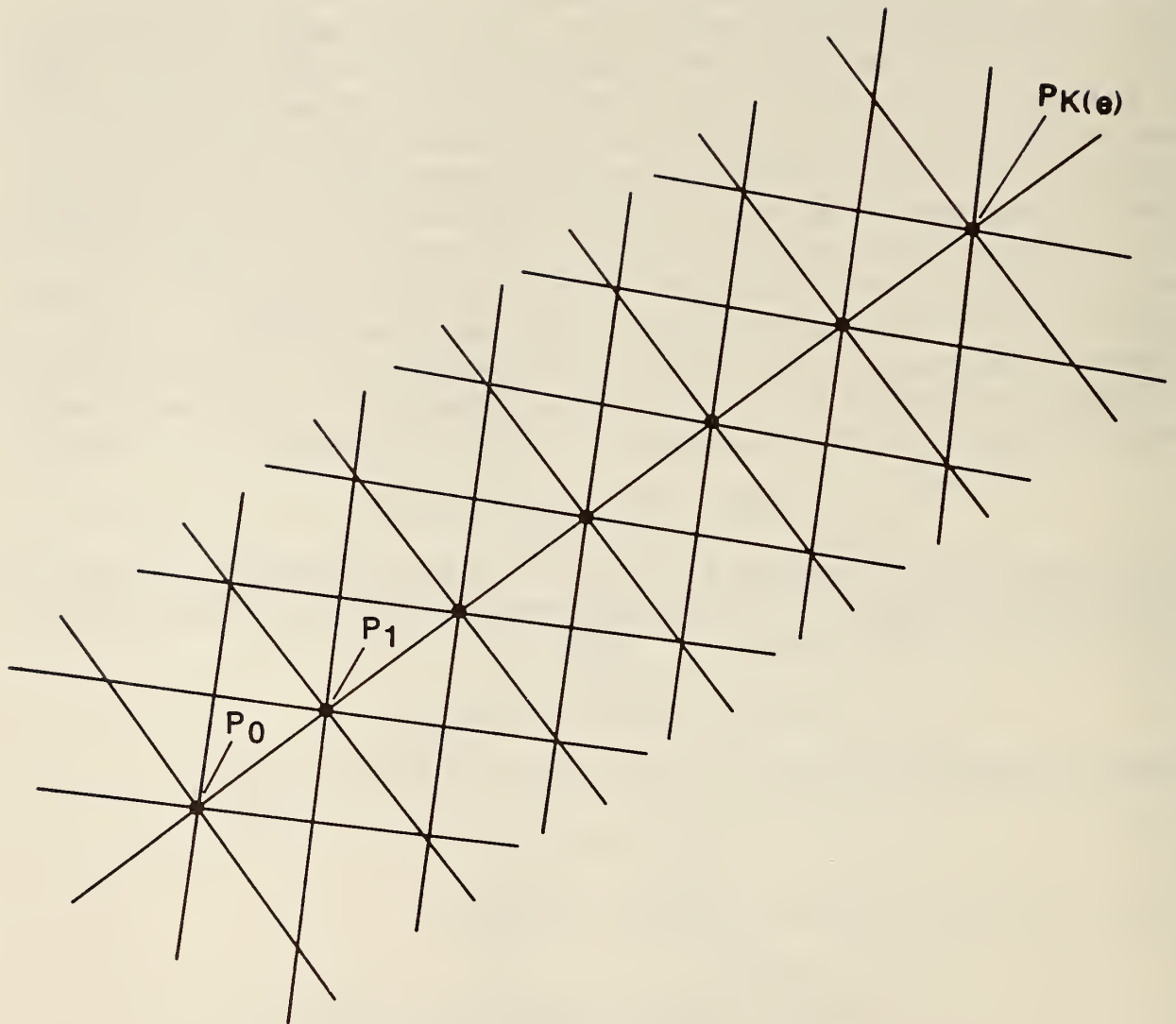
Figure 9: The points and collections of octants P_k, O_k, $k = 0, \ldots, K(e)$.

in S'' are not contained in any of the outermost $L(N)$ layers of cells of B', it follows from results in [1] that the expected number of points in $S \cup S'$ that are Delaunay neighbors of a given point in S'' relative to $S \cup S'$ is $O(1)$. Therefore, since the number of points in S'' is $O(N^{1/2})$, we must have that the expected number of points in

$$\overline{S} = V(S \cup S', S', S) = V(S \cup S', S'', S) \subseteq V(S \cup S', S'', S \cup S')$$

is $O(N^{1/2}) \cdot O(1) = O(N^{1/2})$.

Proof of eighth observation. Let T_J be as defined in the outline of SEGMNT. Given i, $1 \leq i \leq M$, it follows from the fifth observation above that for I equal to i the points $F_J(L)$, $L = 1, \ldots, r_J$, $J = 1, 2$, are contained in $\cup_{l=1}^{i} V(e_l) \subseteq (\cup_{l=1}^{i} V'(e_l)) \cup \overline{S}$. In addition, from the outline of EDGSTR, we obtain that for I equal to i the worst-case time required by the execution of EDGSTR to produce T_J, $J = 1, 2$, in line 23 of SEGMNT is $O(r_J^2)$. Therefore, from the sixth and seventh observations above, we must have that for I equal to i the expected time required by the execution of EDGSTR to produce T_J, $J = 1, 2$, is bounded above by

$$((\sum_{l=1}^{i}((|e_l|/d) \cdot O(N^{1/2}))) + O(N^{1/2}))^2 \;\; = \;\; (((\sum_{l=1}^{i}|e_l|/d) + 1) \cdot O(N^{1/2}))^2$$

$$= \;\; ((\sum_{l=1}^{i}|e_l|/d) + 1)^2 \cdot O(N).$$

The result now follows by assuming that the Delaunay triangulation for S obtained in line 1 of SEGMNT is constructed by applying to S an expected $O(N)$ algorithm [1,2,9,10].

6. Summary

We have presented a simple algorithm for computing a Delaunay triangulation for a finite set S of points in the plane constrained by a finite collection E of line segments. The algorithm is based on the simple idea of constructing first a Delaunay triangulation for S, and then 'repairing' this triangulation as each line segment in E is incorporated into the previously obtained triangulation. As it turns out, the 'repairing' of the triangulation is an easy process, since it involves computing the Delaunay triangulation of subsets of edge star-shaped simple polygons constrained by the boundaries of the polygons. Given that N is the number of points in S, and M is the number of line segments in E, we have also shown that if the points in S are chosen independently from a uniform distribution on a square, then the

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expected time required by the algorithm to produce the desired triangulation is bounded above by $O(M^3 \cdot N)$. Actually, depending on the lengths of the line segments and their locations with respect to the boundary of the square, we expect this time to be $O(M \cdot N)$ or better for N large relative to M, since in this case most of the triangles that are eliminated during the 'repairing' of the triangulation belong to the Delaunay triangulation for S. At any rate, for fixed M, the expected time required by the algorithm to produce a Delaunay triangulation for S constrained by E is $O(N)$. Finally, we report that an implementation of the algorithm has just been completed at the National Bureau of Standards.

Acknowledgments

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10. SUPPLEMENTARY NOTES

☐ Document describes a computer program; SF-185, FIPS Software Summary, is attached.

11. ABSTRACT (A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here)

In this paper, we propose a simple algorithm for constructing a Delaunay triangulation for a finite set of points in the plane constrained by a finite collection of line segments. This algorithm constructs first a Delaunay triangulation for the set, and then generates from it a sequence of triangulations as each line segment is incorporated into the previously obtained triangulation. An expected time analysis shows the algorithm to be linear if the number of line segments is kept constant.

12. KEY WORDS (Six to twelve entries; alphabetical order; capitalize only proper names; and separate key words by semicolons)

algorithm; computational complexity; computational geometry; constrained Delaunay triangulation; expected time analysis; multiply-connected polygonal regions; Voronoi diagram

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