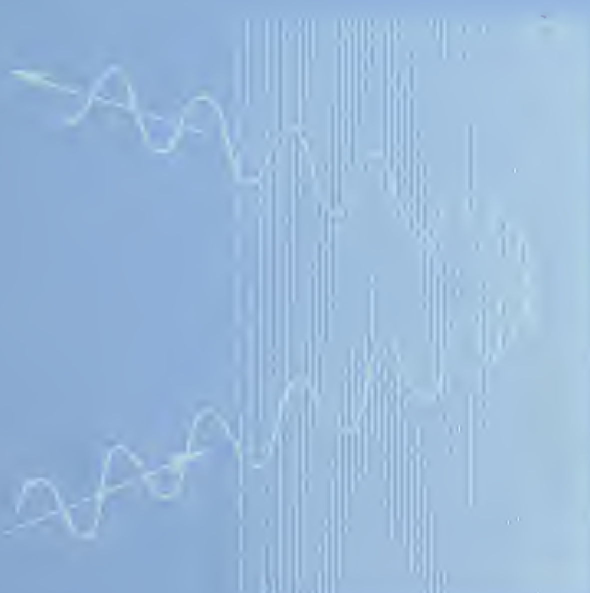
NBS TECHNICAL NOTE 1171

U.S. DEPARTMENT OF COMMERCE/National Bureau of Standards

An Electromagnetic Formulation for Treating Optical Reflections from Graded-Material Surfaces



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An Electromagnetic Formulation for Treating Optical Reflections from Graded-Material Surfaces

NBS technical note

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CONTENTS

AN ELECTROMAGNETIC FORMULATION FOR TREATING OPTICAL REFLECTIONS FROM GRADED-MATERIAL SURFACES

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ABSTRACT

The reflection of a monochromatic plane wave falling obliquely upon the surface of an arbitrary, flat, depth-dependent material is investigated theoretically. The complex reflection coefficient for either principal (s or p) polarization of the field is shown to satisfy a non-linear differential equation of the Ricatti type. An alternate formulation based on the wave immittance (i.e., impedance or admittance) functions is also presented. The immittance functions are shown to satisfy Ricatti differential equations of their own. The reflection coefficient formulation and the wave immittance formulations are related via a bilinear algebraic transformation. Singularities appearing in the reflection coefficient formulation may be suppressed in the immittance formulation, and vice-versa. The advantage of either formulation is that the reflection coefficients for an arbitrary, depth-dependent medium can be obtained directly without having to solve Maxwell's equations for the internal field configurations.

Key Words: electromagnetic waves; graded materials; inhomogeneous media; jellium; optical reflections; reflection coefficient; Ricatti equation; surface reflections; wave immittance.

1. INTRODUCTION

A general formulation is presented describing the field reflection coefficients for an obliquely-incident monochromatic plane wave reflected from the surface of a medium having arbitary depth-dependent properties. The reflection coefficient for each principal linear polarization of the incident field is shown to satisfy a nonlinear, ordinary differential equation of the Ricatti type. The computation of a reflection coefficient by this means avoids the sometimes impossibly difficult solution of Maxwell's equations for the vector fields. The present method is significantly advantageous for dealing with inhomogeneous media because the interior fields are not simply configured except for a few special cases. Fortunately the interior field configuration is rarely of interest and the present formulation provides direct access to the plane wave reflection coefficients.

The methodology developed in this paper is rooted in a theoretical analysis that Foersterling [1][1] provided in 1913 and 1914 to explain the Lippmann color photographic process [2]. In those articles, and in a later one [3] treating radio wave propagation within the Heaviside layer of the ionosphere, Foersterling showed that the component of the local propagation vector along the normal to the plane of stratification obeys a Ricatti differential equation. Geffcken in 1941 [4] went on to show that for normal incidence a characteristic function could be defined which also satisfied a Ricatti differential equation and from which, outside the inhomogeneous medium, the usual plane wave reflection coefficient could be recovered. Kofink [5] later extended Geffcken's work to include oblique incidence. The

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[1] Numbers in brackets indicate literature references listed at the end of this report.

utility of these various formulations was, however, handicapped by the non-linearity of the differential equations and the absence of today's high speed computers. A number of analytical papers consequently followed [6,7] concerned with approximation techniques which could be applied for calculating, e.g., oblique reflections from thin ionospheric layers or thick ionospheric layers with slowly varying parameters.

The work described above does not appear to be widely known outside the radio physics community although some use of it has been made in the study of non-uniform transmission lines [8,9] and acoustic ducts [10]. The present effort draws on this earlier work and restructures the reflection formulation in a modern, more generally relevant manner. One important aspect of the present formulation is that its application to totally reflecting materials, such as metals, is now included in the theory. An alternative formulation based on the wave impedance and wave admittance functions is also presented. Each of these functions is shown to obey its own Ricatti differential equation. The reflection coefficient formulation and the wave immittance[2] formulations are in fact related by a bilinear algebraic transformation. In a given application the selection of a particular formulation is dictated by the medium. The superiority of the wave immittance formulation over the reflection coefficient formulation is demonstrated for a jellium-metal surface in the final sections of this report.

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[2] Wave immittance refers to the <u>im</u>pedance or ad<u>mittance</u> of the electromagnetic field in a plane parallel to the surface.

2. REFLECTION COEFFICIENT FORMULATION FOR AN ARBITRARY ONE-DIMENSIONAL MATERIAL TRANSITION

Exact solutions of Maxwell's equations, in terms of well-known functions, are available for only a few simple depth-dependent material geometries. In those cases the complex reflection coefficient is determined by solving Maxwell's equations for the fields within the inhomogeneous region, applying boundary conditions to match the interior fields with the exterior plane wave fields, and finally forming the ratio of the appropriate components of the incident and reflected plane wave fields. This procedure may not be easy or practicable to perform. Fortunately, the theory can be reformulated to provide the reflection coefficient directly as the solution of an ordinary differential equation, and thereby completely circumvent the field solutions. This approach is particularly convenient when the material properties are numerically specified.

In the present formulation the entire space ($-\infty \leq z \leq \infty$) is filled with a medium whose properties are permitted to vary with coordinate z (fig. 1). For sufficiently negative z the (exterior) medium is homogeneous and supports the incident and reflected plane waves. In the vicinity of $z = 0$ the medium undergoes a transition which eventually develops, for sufficiently positive z, into a second homogeneous region. The latter, representing the uniform interior of the medium, supports the transmitted plane wave. For the sake of generality both the permittivity (ε) and the permeability (μ) are treated as arbitrary, independent functions which for large positive and negative values of z approach constant values. Losses are included by treating the permittivity and permeability functions as complex entities in the frequency domain.

The reflection process commences with an incident plane wave, which

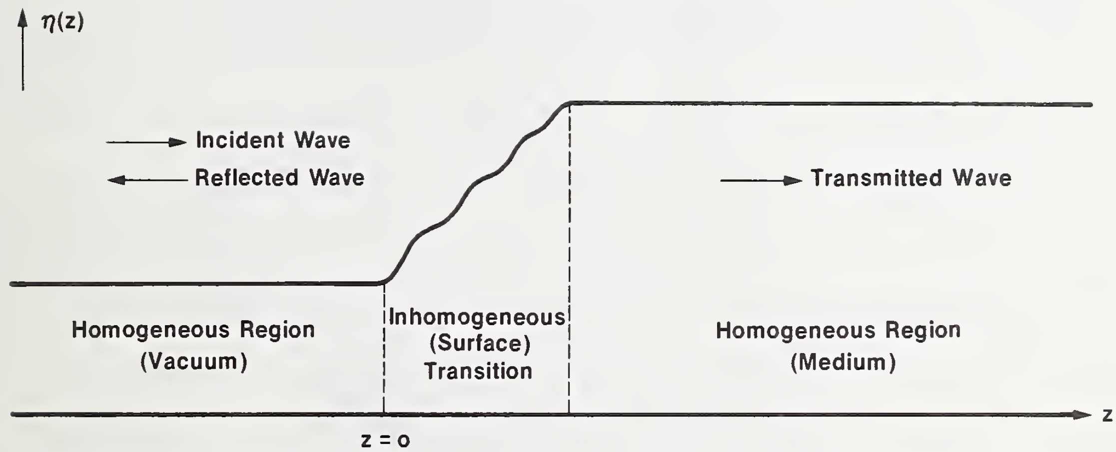


Figure 1. A schematic representation of the medium depicting a possible dependence of the refractive index η upon depth z. To either side of the inhomogeneity (surface transition) the space supports plane wave propagation. However, for a metallic medium the refractive index is imaginary, or complex if losses are present, and the transmitted fields are evanescent.

originates in the $z < 0$ homogeneous region (e.g., vacuum), and impinges upon the transition at $z = 0$ with incidence angle θ_1. The incident fields propagate with diminishing intensity through the transition to emerge in the uniform interior ($z > 0$) as a transmitted plane wave. The reflected energy is visualized as building up, from zero energy in the uniform interior to full value at the surface $z = 0$, as the fields are traced through the inhomogeneity toward the source.

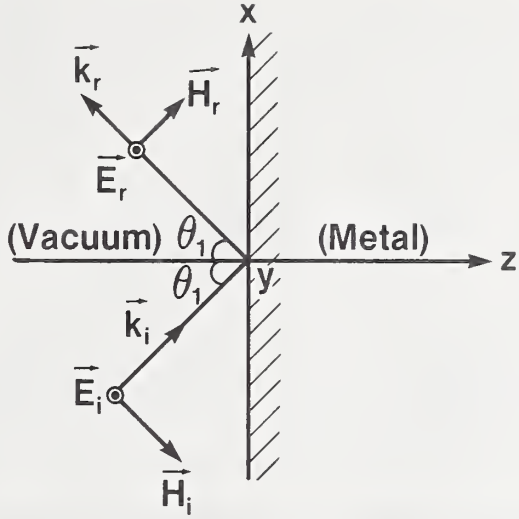
2.1 Reflection Coefficient for an s-Polarized Plane Wave

The exterior, incident and reflected field directions for an s-polarized plane wave are shown in figure 2a. For a medium which varies only with depth, the interior field configuration sufficiently below the surface must be planar and must have the same polarization as the exterior fields. Within the surface transition the fields are not planar and here is where the reflection develops. Throughout the medium, all fields are assumed to vary with time as $\exp(-i\omega t)$, the electric vector is oriented in the y direction perpendicular to the plane of incidence (s-polarization) and the corresponding magnetic vector has only x and z components. Due to the symmetry of the problem the fields are independent of y and Maxwell's equations are simply:

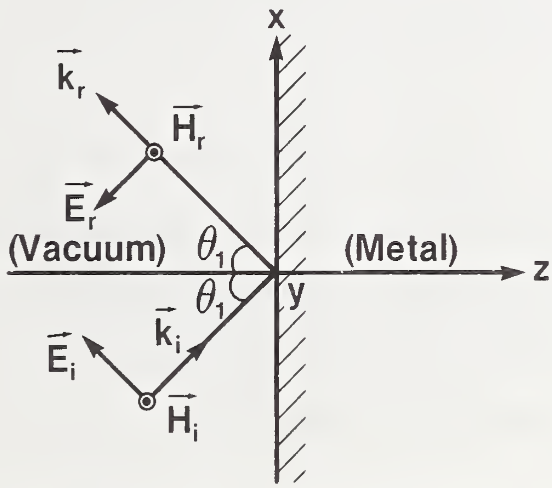
$$\frac{\partial E_y(x,z)}{\partial z} = -i\omega\mu(z)H_x(x,z), \tag{2-1}$$

$$\frac{\partial E_y(x,z)}{\partial x} = +i\omega\mu(z)H_z(x,z), \tag{2-2}$$

$$\frac{\partial H_x(x,z)}{\partial z} - \frac{\partial H_z(x,z)}{\partial x} = -i\omega\varepsilon(z)E_y(x,z). \tag{2-3}$$



(a) Reflection of an s-Polarized Plane Wave



(b) Reflection of a p-Polarized Plane Wave

Figure 2. Field directions for the two principle polarizations of an incident plane wave. The exterior fields are always planar; the interior fields become planar at depths beyond the surface transition.

Substitution of eqs. (2-1) and (2-2) for H_x and H_z into eq. (2-3) gives the following equation for the electric field:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\mu(z)} \frac{d\mu(z)}{dz} \frac{\partial}{\partial z} + k_o^2 n^2(z) \right] E_y(x,z) = 0 \ , \tag{2-4}$$

where $n(z) = \sqrt{\mu(z)\varepsilon(z)}/\sqrt{\mu_0\varepsilon_0}$ is the refractive index and $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$ is the propagation constant of the wave in vacuum. Examination of eq. (2-4), which for a homogeneous region reduces to a Helmholtz equation, suggests the trial solution

$$E_y(x,z) = E(z) \exp\left[ik_o n(z) \sin\theta(z)\ x\right] \quad . \tag{2-5}$$

Within a homogeneous region E, n and θ attain constant values and θ specifies the direction of plane-wave propagation relative to the z axis.

When trial solution (2-5) is substituted into eq. (2-4) and similar terms are grouped together the following expression is obtained after canceling a common exponential factor:

$$\begin{aligned} \Bigg\{ & \left[\frac{d^2E}{dz^2} - \frac{1}{\mu}\frac{d\mu}{dz}\frac{dE}{dz} + k_o^2 n^2 \left(1 - \sin^2\theta\right)E \right] + \\ & \left[2\left(\frac{dE}{dz} - E\,\frac{1}{\mu}\frac{d\mu}{dz}\right)\frac{d}{dz}\left(n\sin\theta\right) + E\,\frac{d^2}{dz^2}\left(n\sin\theta\right) \right](ik_o x) + \\ & E\left[\frac{d}{dz}\left(n\sin\theta\right)\right]^2 (ik_o x)^2 \Bigg\} = 0 \quad . \end{aligned} \tag{2-6}$$

Each term within this expression has been factored into a function only of z (contained by a square bracket) and a power of $(ik_o x)$. Unless the z-dependent factor of each individual term is made to vanish, the sum of terms in eq. (2-6) can not be zero for all values of (ik_0x). Setting the coefficient of $(ik_0x)^2$ to zero immediately provides, for $E \neq 0$,

$$\frac{d}{dz}(\eta \sin\theta) = 0, \quad \text{or} \quad \eta(z)\sin\theta(z) = \text{constant.} \tag{2-7}$$

This relation also satisfies the requirement that the coefficient of (ik_0x) must vanish. The constant appearing in eq. (2-7) is evaluated by means of the boundary condition that within the homogeneous region containing the incident plane wave the angle of incidence is θ_1 and the refractive index has the constant value η_1. This results in a generalization of Snell's law of refraction for a continuous z-dependent medium:[3]

$$\eta(z)\sin\theta(z) = \eta_1 \sin\theta_1 \quad . \tag{2-8}$$

The present significance of eq. (2-8) is that the exponent of the assumed solution, eq. (2-5), is now seen to depend only upon x.

The remaining term in eq. (2-6) yields an equation for the z-dependent electric field amplitude, cf. eq. (2-4),

$$\frac{d^2E}{dz^2} - \frac{1}{\mu}\frac{d\mu}{dz}\frac{dE}{dz} + \kappa^2 E = 0 \quad , \tag{2-9}$$

where

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3 A physical argument supporting eq. (2-8) can be made by dividing the medium into a large number of homogeneous slabs. At each interface Snell's law applies so that $\eta_1\sin\theta_1 = \eta_2\sin\theta_2 = \eta_3\sin\theta_3 = \ldots = \eta_i\sin\theta_i$. In passing to the limit of infinitely many slabs of vanishing thickness, eq. (2-8) is recovered.

$$\kappa = \kappa(z) = k_0 \eta(z) \cos \theta(z) = k_0 \sqrt{\eta^2(z) - \eta_1^2 \sin^2 \theta_1} \quad . \tag{2-10}$$

It will prove convenient to divide E(z) into two parts which in the initial homogeneous region can be interpreted as the amplitudes of the incident and reflected fields, respectively,

$$E(z) = E_i(z) + E_r(z), \tag{2-11}$$

and furthermore to define

$$\frac{dE}{dz} = i\kappa(z)[E_i(z) - E_r(z)]. \tag{2-12}$$

The effect of replacing the two functions E(z) and dE(z)/dz by two new functions $E_i(z)$ and $E_r(z)$ is two-fold. Physically, the electric and magnetic field components now resemble their counterparts in a homogeneous medium. According to eqs. (2-5), (2-8), and (2-11)

$$E_y(x,z) = [E_i(z) + E_r(z)]\exp[ik_0\eta_1 \sin \theta_1 \, x], \tag{2-13}$$

while from eqs. (2-1), (2-2), and (2-13), in view of eqs. (2-7), (2-8), (2-11), and (2-12):

$$H_x(x,z) = \sqrt{\frac{\varepsilon(z)}{\mu(z)}}\,[-E_i(z) + E_r(z)]\cos \theta(z)\exp[ik_0\eta_1 \sin \theta_1 \, x], \tag{2-14}$$

$$H_z(x,z) = \sqrt{\frac{\varepsilon(z)}{\mu(z)}}\,[E_i(z) + E_r(z)]\sin \theta(z)\exp[ik_0\eta_1 \sin \theta_1 \, x]. \tag{2-15}$$

Mathematically, the effect is to reduce the single second order differential equation (2-9) to two first order differential equations. One equation is obtained by substituting (2-11) into (2-12):

$$\frac{dE_i}{dz} + \frac{dE_r}{dz} - i\kappa\,(E_i - E_r) = 0 \tag{2-16}$$

The other equation is obtained by eliminating d^2E/dz^2 from eq. (2-9) using the derivative of eq. (2-12) and then substituting eq. (2-11) for E(z).

$$\frac{dE_i}{dz} - \frac{dE_r}{dz} - i\kappa\,(E_i + E_r) + 2\Gamma_s(E_i - E_r) = 0, \tag{2-17}$$

where

$$\Gamma_s = \frac{1}{2}\,\frac{1}{\kappa/\mu}\,\frac{d}{dz}\,\frac{\kappa}{\mu} = \frac{1}{2}\,\frac{d}{dz}\,\ln\frac{\kappa}{\omega\mu} \tag{2-18}$$

Now, half the difference of eqs. (2-16) and (2-17) is

$$\frac{dE_r}{dz} + i\kappa E_r + \Gamma_s\,(E_r - E_i) = 0 \tag{2-19}$$

whereas half the sum of eqs. (2-16) and (2-17) is

$$\frac{dE_i}{dz} - i\kappa E_i - \Gamma_s\,(E_r - E_i) = 0 \tag{2-20}$$

Multiplying eq. (2-19) by E_i, eq. (2-20) by E_r, and subtracting one equation from the other, gives

$$E_i \frac{dE_r}{dz} - E_r \frac{dE_i}{dz} + 2i\kappa E_i E_r - \Gamma_s (E_i^2 - E_r^2) = 0 \quad . \tag{2-21}$$

Dividing eq. (2-21) by E_i^2, and introducing the definition

$$R_s(z) \equiv \frac{E_r(z)}{E_i(z)} \tag{2-22}$$

allows eq. (2-21) to be written in the form:

$$\frac{dR_s}{dz} + i2\kappa(z)R_s - \Gamma_s(z)[1-R_s^2] = 0 \quad . \tag{2-23}$$

Equation (2-23) is a Ricatti-type ordinary differential equation [11]; it is of first order and non-linear in the dependent variable $R_s(z)$. The following physical interpretation is offered for the function $R_s(z)$:

Within the $z < 0$ homogeneous region both $\kappa(z)$ and $\mu(z)$ are constants and eq. (2-9) reduces to a simple Helmholtz equation. Consequently $E_i(z)$ and $E_r(z)$ can be identified here, respectively, as incident and reflected plane-wave field amplitudes. Furthermore, in accordance with definition (2-18), Γ_s vanishes and eq. (2-23) is readily integrated to give

$$R_s(z) = R_s(0)\exp\,(i2\kappa z) = R_s(0)\exp\,(i2k_0\eta_1\cos\theta_1 z) \quad . \tag{2-24}$$

The function $R_s(z)$ defined by equation (2-22) for $z < 0$ therefore has the properties required of a reflection coefficient.

Deep within the medium, below the transition layer, where the homogeneous properties again prevail $\kappa(z)$ is again a constant and eq. (2-9) again yields plane wave solutions. However, because neither sources nor boundaries appear in this region, $E_r(z)$ must vanish here and $E_i(z)$ must assume the nature of the transmitted plane wave field. As a consequence, the appropriate boundary condition for differential equation (2-23) is that $R_s(z) = 0$ at $z = \infty$, i.e., at depths well within the uniform interior.

Within the transition layer the quantity Γ_s (which is a measure of material inhomogeneity) is no longer zero in eq. (2-23) for $R_s(z)$, while $\kappa(z)$ in eq. (2-9) looses its identity as a plane wave propagation constant. This is evident for a gradual transition (small Γ_s) in a medium of uniform permeability. In that case the WKB approximation [12] should hold and the solution of eq. (2-9) takes the form:

$$E(z) = E_i(z) + E_r(z) = A(z)e^{i\kappa(z)z} + B(z)e^{-i\kappa(z)z} \quad . \tag{2-25}$$

A Fourier spatial decomposition of <u>either</u> term would reveal an infinite spectrum of plane wave components traveling in both positive and negative z directions. That is to say, neither $E_i(z)$ nor $E_r(z)$ has a totally unique velocity of propagation and may even contain stationary wave components.

A closer inspection of eqs. (2-19) and (2-20) reveals, in a more general situation, that within the transition layer $E_i(z)$ and $E_r(z)$ are inseparably coupled by the inhomogeneity (Γ_s) of the medium. If $E_i(z)$ and $E_r(z)$ were truly normal modes of the system (i.e., planar field amplitudes) they would surely satisfy independent equations. Dividing $E(z)$ into components $E_i(z)$ and $E_r(z)$ is merely a convenience, without loss of mathematical rigor, which leads to a physical understanding of the fields whenever the medium is homogeneous and $\Gamma_s(z)$ vanishes.

In view of what has been said it might appear that no physical interpretation can be attached to the ratio $R_s(z) = E_r(z)/E_i(z)$ within the transition. That is not so. Consider the implications of a numerical integration of eq. (2-23) via a step-by-step procedure starting deep within the medium at large positive z, where $R_s = 0$, and progressing through the transition layer to the surface at $z = 0$. The current value of R_s as the numerical integration reaches the surface is indisputably the plane wave reflection coefficient for the material half-space. In the same manner, if the integration were halted at some depth prior to reaching the actual surface, the then current value of R_s should logically be identified with the plane wave reflection coefficient for the given inhomogeneous medium when terminated at this depth by a uniform medium of matching electrical properties. A consequence of this isomorphism, which holds for all z, between the function $R_s(z)$ and the plane wave reflection coefficient is that:

$$|R_s(z)| \leq 1 \quad \text{for all } z \quad . \tag{2-26}$$

This is true because, by energy conservation principles, the plane wave reflection coefficient is prevented from exceeding unit magnitude.

2.2 Reflection Coefficient for a p-Polarized Plane Wave

The p-polarized plane wave is configured as in figure 2b with the magnetic vector in the y direction, perpendicular to the plane of incidence. The electric vector is oriented parallel to this plane and has only x and z components. In a medium which changes only with depth z the electromagnetic field excited by the plane wave must be independent of the y coordinate and consequently Maxwell's equations for an assumed $\exp(-i\omega t)$ time dependence and this polarization reduce to:

$$\frac{\partial H_y(x,z)}{\partial z} = +i\omega\varepsilon(z)E_x(x,z) \quad , \tag{2-27}$$

$$\frac{\partial H_y(x,z)}{\partial x} = -i\omega\varepsilon(z)E_z(x,z) \quad , \tag{2-28}$$

$$\frac{\partial E_x(x,z)}{\partial z} - \frac{\partial E_z(x,z)}{\partial x} = +i\omega\mu(z)H_y(x,z) \ . \tag{2-29}$$

A comparison of these three equations with eqs. (2-1), (2-2), and (2-3) reveals that if in the latter set of equations $\vec{E}$ is replaced by $\vec{H}$, $\vec{H}$ is replaced by $-\vec{E}$, and the material parameters $\varepsilon(z)$ and $\mu(z)$ are interchanged then the above set of equations is obtained. By means of this duality the differential equation for the p-polarized plane wave reflection coefficient can be written down from inspection of eq. (2-23). Namely,

$$\frac{dR_p}{dz} + i2\kappa(z)R_p - \Gamma_p(z)[1-R_p{}^2] = 0 \quad , \tag{2-30}$$

where now

$$R_p(z) = \frac{H_r(z)}{H_i(z)} \tag{2-31}$$

is the reflection coefficient for the magnetic field. According to eq.

(2-10) $\kappa(z)$ is unchanged by the transformation, whereas according to eq. (2-18):

$$\Gamma_p = \frac{1}{2}\,\frac{1}{\kappa/\varepsilon}\,\frac{d}{dz}\,\frac{\kappa}{\varepsilon} = \frac{1}{2}\,\frac{d}{dz}\,\ln\frac{\kappa}{\omega\varepsilon} \quad . \tag{2-32}$$

As in eq. (2-26) the magnitude of $R_p(z)$ obeys the condition:

$$|R_p| \leq 1 \quad \text{for all } z \quad . \tag{2-33}$$

At normal incidence the reflection coefficients for the s- and p-polarized waves must be identical because the plane of incidence degenerates into a line of axial symmetry. This is easily verified because, for $\theta_1 = 0$, $\kappa = \omega\sqrt{\mu\varepsilon}$ and

$$\Gamma_s = \frac{1}{4\mu\varepsilon}\left(\varepsilon\,\frac{d\mu}{dz} - \mu\,\frac{d\varepsilon}{dz}\right) = -\Gamma_p \;. \tag{2-34}$$

Writing Γ for Γ_s or $-\Gamma_p$, and R for R_s or $-R_p$, eqs. (2-23) and (2-30) can be put into a common form for the normally incident plane wave

$$\frac{dR}{dz} + i2\kappa(z)R - \Gamma(z)[1 - R^2] = 0 \tag{2-35}$$

At normal incidence $R_p = H_r/H_i$ is the negative of $R_s = E_r/E_i$, a result of the field directions assigned in figures 2a and 2b.

2.3 Singularities of the Reflection Coefficient Formulation for a Lossless Jellium Metal

For either polarization of the incident plane wave fields the reflection coefficient is determined by solving the following differential equation for $R_i(z)$,

$$\frac{dR_i}{dz} + i2k(z)R_i - \Gamma_i(z)\,[1-R_i^2] = 0, \quad i = s \text{ or } p \quad , \tag{2-36}$$

cf. eqs. (2-23) and (2-30), subjecting the solution to an appropriate boundary condition, such as $R_i(\infty) = 0$, and then evaluating the solution at the surface of the medium. For a metal described as a lossless, non-magnetic, depth-dependent jellium the permeability and permittivity functions [13] are simply:

$$\mu(z) = \mu_0 \tag{2-37}$$

and

$$\varepsilon(z) = \varepsilon_0[1 - (\omega_p/\omega)^2] \quad . \tag{2-38}$$

In eq. (2-38) the plasma frequency ω_p is a function of depth z through its dependence on the local electron density. The coefficients of differential equation (2-36) evaluated by means of eqs. (2-10), (2-18) and (2-32) for an incident wave originating in vacuum, where $\eta_1 = 1$, are explicitly given by:

$$\kappa(z) = \frac{1}{c}\sqrt{\omega^2\cos^2\theta_1 - \omega_p^{\,2}(z)} \quad , \tag{2-39}$$

$$\Gamma_s(z) = -\frac{1}{4}\,[\omega^2\cos^2\theta_1 - \omega_p^{\,2}(z)]^{-1}\,\frac{d}{dz}(\omega_p^{\,2}) \quad , \tag{2-40}$$

$$\Gamma_p(z) = +\frac{1}{4}\,[\omega^2 - \omega_p^{\,2}(z)]^{-1}\,\frac{d}{dz}(\omega_p^{\,2}) \quad , \tag{2-41}$$

where $c = 1/\sqrt{\mu_0\varepsilon_0}$ is the vacuum speed of light.

Differential eq. (2-36) is clearly singular at the pole of Γ_i. For the s-polarized wave this occurs at a depth of z_s satisfying the relation $\omega_p(z_s) = \omega\cos\theta_1$; for the p-polarized wave this occurs at a depth of z_p satisfying the relation $\omega_p(z_p) = \omega$. In order to examine the behavior of R_i at a pole of Γ_i, eq. (2-36) is divided through by Γ_i and Γ_i is allowed to pass to infinity. At the pole of Γ_i it is assumed that R_i is well-behaved and continuous, and $\left.\frac{dR_i}{dz}\right|_{z_i}$ remains finite. Under these conditions equation (2-36) reduces to

$$R_i^{\,2}(z_i) = 1, \quad i = s \text{ or } p \quad . \tag{2-42}$$

Total reflection is seen to occur at a plane z_i where Γ_i is singular.

Examination of the generalized law of refraction (2-8) for $\eta_1 = 1$ and $\eta(z) = \sqrt{\varepsilon/\varepsilon_0} = \sqrt{1 - (\omega_p/\omega)^2}$ reveals, in the plane $z = z_s$ where where $\omega_p(z) = \omega\cos\theta_1$, that $\theta(z_s) = \pi/2$ for <u>either</u> polarization of the field. In other words, the rays of the optical field are turned about at the same plane, $z = z_s$, at which the s-polarized wave is totally reflected. The turning plane of the optical field does not, however, coincide with the reflection plane of the p-polarized wave for reasons that will be made clear later. In any case, for either polarization of the wave and for $z < z_s$, $\theta(z)$ is real, $\kappa(z)$ is real and the fields given by eq. (2-9) or its dual are oscillatory. For $z > z_s$ the law of refraction (2-8) gives an imaginary value for $\theta(z)$ and eq. (2-39) shows that $\kappa(z)$ is imaginary. The fields of either wave polarization exibit an exponential character of decay in this region.

The p-polarized wave has an electric field component in the z direction (not present in the s-polarized wave) which, though rapidly attenuated with depth below the turning plane z_s, is capable of exciting a non-propagating resonance[4] of the jellium plasma at a depth $z_p > z_s$. At this plane, where $\varepsilon(z_p) = 0$, Maxwell's equations (2-26) through (2-28) require the magnetic field to vanish and if an electric field exists that it must be expressed as the gradient of a scalar function. The electric field is in fact an electrostatic oscillation of the medium at the local plasma frequency $\omega_p(z_p) = \omega$. Since the plasma resonance at z_p draws energy from the turning plane of the optical field at z_s the location where $R_s^2 = 1$ differs from the location where $R_p^2 = 1$.

It is useful to know whether $R_i(z_i) = +1$ or -1 so that $R_i(z_i)$ can serve as an alternative to the boundary condition $R_i(\infty) = 0$. This is achieved by examining the field expressions in their respective reflection planes. Since $H_y(x,z)$ for the p-polarized wave is given by an equation dual to eq. (2-13) for the s-polarized wave, viz.,

$$H_y(x,y) = [H_r(z) + H_i(z)] \exp(ik_o n_1 \sin \theta_1 x), \tag{2-43}$$

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[4] In the appendix it is proven that an obliquely incident wave must have an electric field component in the direction of the material gradient in order to excite a plasma oscillation.

the vanishing of H_y in the plane z_p of the plasma resonance leads to the boundary condition

$$R_p(z_p) \equiv \frac{H_r(z_p)}{H_i(z_p)} = -1 \quad . \tag{2-44}$$

For the s-polarized wave consider the possibility of $R_s(z_s) = -1$. This implies $E_r(z_s) = -E_i(z_s)$ and, since $\kappa(z_s) = 0$, according to eqs. (2-11) and (2-12), that both E and $\frac{dE}{dz}$ vanish at z_s. Equation (2-9) and its derivatives further show that all higher derivatives of E with respect to z must also vanish at z_s. According to Taylor's theorem this means $E(z) \equiv 0$ for all z and so there can be no electromagnetic field anywhere within the medium. Since this contradicts experience, $R_s(z_s) \neq -1$.

The alternative possibility, $R_s(z_s) = +1$, leads via eq. (2-14) to $H_x(x,z_s) = 0$ but, by eqs. (2-13) and (2-15) to non-zero $E_y(x,z_s)$ and $H_z(x,z_s)$ field components. The Poynting's vector $(\vec{E} \times \vec{H}^*)$ for these components points, as it should, in the positive x direction, i.e., $\theta(z_s) = \pi/2$. The boundary condition for the s-polarized wave is therefore:

$$R_s(z_s) = +1 \tag{2-45}$$

Singularities of the reflection coefficient formulation of the type encountered with the present lossless-jellium model, can sometimes be avoided by working with the wave immittance formulations described next. A bilinear algebraic transformation links the wave immittance function to the reflection coefficient.

3. WAVE IMMITTANCE FORMULATIONS FOR AN ARBITRARY ONE-DIMENSIONAL MATERIAL TRANSITION

The preceding theory illustrates the relationship which exists between Maxwell's (curl) equations for the fields, the wave equation, and the ordinary (but non-linear) differential equation for the reflection coefficients. The wave impedance and wave admittance formulations offer still another avenue for describing plane wave reflection phenomena. The differential equations governing the wave immittances and the relationship which exists between the immittance and reflection functions are derived next.

3.1 Wave Immittance Formulations for s-Polarized Plane Wave Reflections

The wave impedance, $Z_s(z)$, at some given plane z is defined as the ratio of electric-to-magnetic field components which in that plane leads to positive z propagation. For an incident s-polarized plane wave this ratio is equal to

$$Z_s(z) \equiv \frac{E_y(x,z)}{-H_x(x,z)} \quad . \tag{3-1}$$

From eqs. (2-13) and (2-14) it is evident that Z_s is independent of x and a function only of coordinate z. The derivative of this function,

$$\frac{dZ_s(z)}{dz} = \frac{-1}{H_x}\frac{\partial E_y}{\partial z} + \frac{E_y}{H_x^{\,2}}\frac{\partial H_x}{\partial z} \quad , \tag{3-2}$$

is easily simplified using Maxwell's eqs. (2-1) and (2-3) to eliminate the field derivatives on the right:

$$\frac{dZ_s(z)}{dz} = i\omega\mu(z) + \left[-i\omega\varepsilon(z) + \frac{1}{E_y}\frac{\partial H_z}{\partial x}\right]\left(\frac{E_y}{H_x}\right)^2 . \qquad (3\text{-}3)$$

The x derivative of eq. (2-2), together with definition (3-1), permits this expression to be rewritten as

$$\frac{dZ_s(z)}{dz} = i\omega\mu(z) + \left[-i\omega\varepsilon(z) + \frac{1}{i\omega\mu(z)}\frac{1}{E_y}\frac{\partial^2 E_y}{\partial x^2}\right] Z_s^2(z) . \qquad (3\text{-}4)$$

The remaining field derivative is readily eliminated as a consequence of the exponential x dependence of E_y appearing in eq. (2-5) so that

$$\frac{dZ_s(z)}{dz} = i\omega\mu(z) + \frac{1}{i\omega\mu(z)}\left[\omega^2\mu(z)\varepsilon(z) - k_o^2\eta^2(z)\sin^2\theta(z)\right]Z_s^2(z) . \qquad (3\text{-}5)$$

Introducing the relation $k_o^2\eta^2(z) = \omega^2\mu(z)\varepsilon(z)$ leads to the following following differential equation for $Z_s(z)$:

$$\frac{dZ_s(z)}{dz} - i\omega\mu(z)[1-Z_s^2(z)/Z_{so}^2(z)] = 0 , \qquad (3\text{-}6)$$

where $Z_{so}(z)$, the nominal impedance of the medium at z, is defined by

$$Z_{so}(z) \equiv \sqrt{\frac{\mu(z)}{\varepsilon(z)}}\,\frac{1}{\cos\theta(z)} = \frac{\omega\mu(z)}{\kappa(z)} \qquad (3\text{-}7)$$

and $\kappa(z)$ is given by eq. (2-10).

The wave admittance, Y(s), is the reciprocal of the wave impedance. The differential equation for the wave admittance is most easily derived by placing

$$Z_s(z) = 1/Y_s(z) \qquad \text{and} \qquad Z_{so}(z) = 1/Y_{so}(z) \tag{3-8}$$

in eq. (3-6). This immediately yields the following result:

$$\frac{dY_s(z)}{dz} - i\kappa(z)Y_{so}(z)[1 - Y_s^2(z)/Y_{so}^2(z)] = 0 \tag{3-9}$$

Observe that differential eqs. (3-6) and (3-9) for the wave immittances are both Ricatti-type equations. For a particular medium the location of the poles and zeros of the immittance functions will determine whether it is more convenient to use eq. (3-6) or eq. (3-9).

In the theory that has been developed for uniform transmission lines, waveguides of constant cross-section, and plane waves in homogeneous media, the reflection coefficient is related to the wave impedance via a bilinear transformation such as:

$$R_s(z) = \frac{Z_s(z) - Z_{so}(z)}{Z_s(z) + Z_{so}(z)} \tag{3-10}$$

where, for the cases mentioned, Z_{so} is a constant. It is tempting to postulate that such a relation holds for the present case even though Z_{so} is not a constant. The validity of the bilinear transformation for the present situation is demonstrated by substituting eq. (3-10), with a z-dependent Z_{so}, into eq. (2-23) for R_s; this yields:

$$\frac{dZ_s}{dz} - \left[\frac{1}{Z_{so}}\frac{dZ_{so}}{dz} + 2\Gamma_s\right]Z_s - i\kappa Z_{so}[1 - (Z_s/Z_{so})^2] = 0 \ . \tag{3-11}$$

The second term in this equation vanishes because of eqs. (3-7) and (2-18), and $\kappa Z_{so} = \omega\mu$. Consequently eq. (3-6) is recovered and the applicability of transformation (3-10) for s-polarized plane wave propagation in depth-dependent media is established. The corresponding transformation for the admittance function is obtained by substituting eq. (3-8) into eq. (3-10). This yields in a similar manner:

$$R_s(z) = -\frac{Y_s(z) - Y_{so}(z)}{Y_s(z) + Y_{so}(z)} \quad . \tag{3-12}$$

Below the surface transition, within the uniform volume of the medium, $Z_s(z)$ and $Y_s(z)$ are constants. Consequently both $\frac{dZ_s}{dz}$ and $\frac{dY_s}{dz}$ must vanish here, and eqs. (3-6) and (3-9) show that $Z_s(z)$ and $Y_s(z)$ respectively approach the constant values assumed here by Z_{so} and Y_{so}. The boundary conditions for eqs. (3-6) and (3-9) are therefore explicitly:

$$Z_s(\infty) = Z_{so}(\infty) = \left.\frac{\omega\mu}{\kappa}\right|_{\infty} \tag{3-13}$$

and

$$Y_s(\infty) = Y_{so}(\infty) = \left.\frac{\kappa}{\omega\mu}\right|_{\infty} \tag{3-14}$$

The equivalence of these boundary conditions with the boundary condition $R_s(\infty) = 0$ is readily demonstrated by substitution of eq. (3-13) into eq. (3-10) evaluated at $z \to \infty$, and substitution of eq. (3-14) into eq. (3-12) evaluated at $z \to \infty$.

3.2 Wave Immittance Formulations for p-Polarized Plane Wave Reflections

The equation obeyed by the admittance $Y_p(z)$ of a p-polarized plane wave may be found by differentiating the definition

$$Y_p(z) = \frac{H_y(x,z)}{E_x(x,z)} , \tag{3-15}$$

or by invoking the duality which exists between this function and the impedance function defined by eq. (3-1) for the s-polarized plane wave. The latter course is the more expedient and requires the exchange of μ with ε, Z_s with Y_p, Z_{so} with Y_{po} in eqs. (3-6) and (3-7). This leads to the following Ricatti differential equation for the wave admittance:

$$\frac{dY_p(z)}{dz} - i\omega\varepsilon(z)[1 - Y_p^2(z)/Y_{po}^2(z)] = 0 , \tag{3-16}$$

where $Y_{po}(z)$, the nominal admittance of the medium, is given by:

$$Y_{po}(z) \equiv \sqrt{\frac{\varepsilon(z)}{\mu(z)}} \frac{1}{\cos\theta(z)} = \frac{\omega\varepsilon(z)}{\kappa(z)} = \frac{1}{Z_{po}(z)} . \tag{3-17}$$

The wave impedance satisfies the dual of eq. (5-9), namely,

$$\frac{dZ_p(z)}{dz} - i\kappa(z)Z_{po}(z)[1 - Z_p^2(z)/Z_{po}^2(z)] = 0 \tag{3-18}$$

the reflection coefficient is likewise related to the immittance functions by means of bilinear transformations dual to eqs. (3-10) and (3-12):

$$R_p(z) = \frac{Y_p(z) - Y_{po}(z)}{Y_p(z) + Y_{po}(z)} = - \frac{Z_p(z) - Z_{po}(z)}{Z_p(z) + Z_{po}(z)} \tag{3-19}$$

The boundary conditions, dual to eqs. (3-13) and (3-14), for differential eqs. (3-16) and (3-18) are, respectively:

$$Y_p(\infty) = Y_{po}(\infty) = \left.\frac{\omega\varepsilon}{\kappa}\right|_\infty \tag{3-20}$$

and

$$Z_p(\infty) = Z_{po}(\infty) = \left.\frac{\kappa}{\omega\varepsilon}\right|_\infty \quad . \tag{3-21}$$

These are, of course, equivalent to the boundary condition $R_p(\infty) = 0$, as readily demonstrated using eq. (3-19).

3.3 Singularities of the Wave Immittance Formulations for a Lossless Jellium Metal

The wave immittance formulations are now specialized to a lossless, non-magnetic, depth-dependent jellium metal. Once again eqs. (2-37), (2-38), and (2-39) apply, and for the s-polarized wave eqs. (3-7) and (3-8) give

$$Z_{so}(z) = \sqrt{\frac{\mu_o}{\varepsilon_o}\Big/\left[\cos^2\theta - \frac{\omega_p^{\,2}(z)}{\omega^2}\right]} = \frac{1}{Y_{so}(z)} \tag{3-22}$$

Substitution of this expression together with the definition

$$F_s(z) \equiv i\omega\varepsilon_o\left[\cos^2\theta_1 - \frac{\omega_p^{\,2}(z)}{\omega^2}\right] \tag{3-23}$$

into eqs. (3-6) and (3-9) yields the following differential equations for the s-polarized wave immittance functions:

$$\frac{dZ_s}{dz} + F_s(z)Z_s^2 = i\omega\mu_o \tag{3-24}$$

and

$$\frac{dY_s}{dz} + i\omega\mu_o Y_s^2 = F_s(z) \quad . \tag{3-25}$$

For the p-polarized wave, eq. (5-17) yields

$$Y_{po}(z) = \sqrt{\frac{\varepsilon_o}{\mu_o}\left[1 - \frac{\omega_p^2(z)}{\omega^2}\right]} \Bigg/ \sqrt{\cos^2\theta_1 - \frac{\omega_p^2(z)}{\omega^2}} = \frac{1}{Z_{po}(z)} \quad . \tag{3-26}$$

Substitution of this expression along with the definition

$$F_p(z) = i\omega\mu_o\left[\cos^2\theta_1 - \frac{\omega_p^2(z)}{\omega^2}\right] \Bigg/ \left[1 - \frac{\omega_p^2(z)}{\omega^2}\right] \tag{3-27}$$

into eqs. (3-16) and (3-19) provides the following differential equations for the p-polarized wave immittance functions:

$$\frac{dY_p}{dz} + F_p(z)Y_p^2 = i\omega\varepsilon(z) \quad . \tag{3-28}$$

and

$$\frac{dZ_p}{dz} + i\omega\varepsilon(z)Z_p^2 = F_p(z) \quad . \tag{3-29}$$

Differential equations (3-24), (3-25), (3-28), and (3-29) for the immittance functions are generally more tractable Ricatti equations than those of (2-36) for the reflection coefficient. As a result of transforming to the immittance formulation, the singularity at $\omega_p(z_s) = \omega\cos\theta$ that plagued the differential equation for $R_s(z)$ no longer appears in either eqs. (3-24) or (3-25). The choice between the wave impedance formulation or the wave admittance formulation is straight-forward. For the s-polarization the admittance formulation is preferred because the coefficients on the left of eq. (3-25) are constants. For the p-polarization the impedance formulation is selected to avoid having the plasma resonance pole of F(p) appear as a coefficient of $Y_p^2(z)$.

The boundary conditions for the immittance functions at infinity, as expressed by eqs. (3-13), (3-14), (3-20), and (3-21), remain appropriate for use when, for example, $Y_{so}(\infty)$ and $Y_{po}(\infty)$ are evaluated by means of eqs. (3-22) and (3-26). However, it may at times be more convenient to employ the boundary conditions for the immittance functions at the reflection plane. To obtain these conditions the bilinear transformation of (3-10) is inverted to give

$$Z_s(z_s) = \frac{1 + R(z_s)}{1 - R(z_s)} Z_{os}(z_s) \tag{3-30}$$

At the reflection plane z_s of the s-polarized wave, $\omega_p(z_s) = \omega\cos\theta_1$. Here the nominal impedance of the medium given by eq. (3-22) is infinite, but it is also true that $R_s(z_s) = +1$ according to eq. (2-45). Thus the boundary conditions on the s-polarized wave immittance functions are:

$$Z_s(z_s) = \infty \quad \text{or} \quad Y_s(z_s) = 0 \tag{3-31}$$

The inverse bilinear transformation for the p-polarized wave may be determined from eq. (3-19) but it is also the dual to eq. (3-30), viz.,

$$Y_p(z_p) = \frac{1 + R_p(z_p)}{1 - R_p(z_p)} Y_{po}(z_p) \quad . \tag{3-32}$$

At the reflection plane z_p, $\omega_p(z_p) = \omega$. Here the nominal admittance given by eq. (3-26) is zero, but it is also true that $R_p(z_p) = -1$ according to eq. (2-44). It therefore follows that the boundary conditions as the p-polarized wave immittance functions are

$$Y_p(z_p) = 0 \quad \text{or} \quad Z_p(z_p) = \infty \quad . \tag{3-33}$$

4. CONCLUSIONS

The reflection of an obliquely incident, monochromatic plane wave from the surface of a graded medium has been investigated. Several alternative, but related, formulations were presented in order to provide flexibility in solving a particular problem. Among the advantages of the present formulations are that Maxwell's equations do not need to be solved for the field configurations, the theory is exact, and its application is straightforward. The selection of a particular formulation will, in every case, be guided by the characteristics and the depth-dependence of the medium.

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APPENDIX. INCIDENT FIELD POLARIZATION FOR EXCITING A PLASMA RESONANCE

For an electromagnetic field oscillating according to an $\exp(-i\omega t)$ time dependence, the magnetic induction $\vec{H}$ satisfies the Maxwell equation:

$$\nabla\times\vec{H} = -i\omega\vec{D} \quad . \tag{A-1}$$

The field displacement, $\vec{D}$, is related to the electric intensity, $\vec{E}$, through

$$\vec{D} = \varepsilon(\omega)\vec{E} \quad . \tag{A-2}$$

In writing this expression the conduction current has been lumped together with the displacement current to form the complex permittivity function $\varepsilon(\omega)$. Since the divergence of the curl of any vector must identically vanish, eq. (A-1) leads to

$$\nabla\cdot\vec{D} = 0 \tag{A-3}$$

or, using eq. (A-2), to

$$\varepsilon\ \nabla\cdot\vec{E} + \vec{E}\cdot\nabla\varepsilon = 0 \quad . \tag{A-4}$$

For a homogeneous medium $\nabla\varepsilon = 0$, leaving $\nabla\cdot\vec{E} = 0$. Both the bound charge density $\rho_b = \nabla\cdot\vec{D}$ and the free charge density $\rho_s = \nabla\cdot\vec{E}$ are consequently zero within a homogeneous medium.

For a graded medium the bound charge density remains zero,

$$\rho_b = \nabla\cdot\vec{D} = 0 \quad . \tag{A-5}$$

However, the free charge density,

$$\rho_f = \nabla \cdot \vec{E} = \frac{-\vec{E} \cdot \nabla \varepsilon}{\varepsilon} = -\frac{1}{\varepsilon} E_z \frac{d\varepsilon}{dz} \quad , \tag{A-6}$$

resulting from a displacement of the electronic charge from its equilibrium position, differs from zero if $E_z \neq 0$. Only the p-polarized wave has such an electric field component, and therefore only a wave of this polarization is capable of exciting an oscillation of the free charge. The amplitude of these oscillations increases as the resonance condition $\varepsilon = 0$ is approached. If collision losses are ignored the permittivity function for a jellium model is given by

$$\varepsilon = \varepsilon_0 \, [1-\omega_p{}^2/(z)/\omega^2] \quad . \tag{A-7}$$

According to eq. (A-6) resonance is seen to occur at a depth and plasma frequency satisfying the condition $\varepsilon = 0$, i.e.,

$$\omega_p(z) = \omega \quad . \tag{A-8}$$

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THIRD CLASS
BULK RATE