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U.S. DEPARTMENT OF COMMERCE / National Bureau of Standards

Integral Equations for Transient Electromagnetic Fields

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Issued February 1982

National Bureau of Standards Technical Note 1157
Nat. Bur. Stand. (U.S.), Tech. Note 1157, 68 pages (Feb. 1982)
CODEN: NBTNAE

U.S. GOVERNMENT PRINTING OFFICE
WASHINGTON: 1982

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Integral equations for the electric and magnetic fields in free space are derived from Maxwell's equations. The fields are expressed in terms of their initial values, boundary values, and sources with the help of a retarded Green function for the scalar wave equation. These equations are then used to derive integral equations for the surface charge and current densities induced by the scattering of a transient electromagnetic field by perfect conductors. An alternative solution of Maxwell's equations with the help of dyadic Green functions is also presented.

Key words: dyadic Green functions; electromagnetic scattering; integral equations; perfect conductors; transient electromagnetic fields; wave equations.

1. Introduction

The Wave Optics program at the National Bureau of Standards is concerned with the scattering of light by small objects and rough surfaces. The most widely used approach to this problem involves (ideally) monochromatic waves, and results are given or data are taken for many different angles. We propose to consider the scattering of pulses finite in duration and spatial extent for one or a few directions. The scattered wave for such a broadband transient field contains information on the shape and the material of the scatterer.

Monochromatic electromagnetic waves are represented by complex vector functions of the space variables for a fixed frequency, and these functions obey the Helmholtz equation. Although an arbitrary field can be decomposed into a superposition of monochromatic waves by Fourier analysis, the equations and solutions are not necessarily equivalent. Thus, we prefer to derive formal solutions and integral equations directly from Maxwell's equations, rather than backing into them starting with monochromatic waves. The expressions we obtain and the method of derivation are not readily available in the literature, and we present them in this note in considerable detail for the interested reader who is not already familiar with the subject.

The basic Maxwell equations have to be complemented by the constitutive relations for the medium or for vacuum, and by initial and boundary conditions. These aspects of the theory are presented and analyzed in Section 2.

A set of linear partial differential equations can be formally solved with the help of the appropriate Green functions. When the Green function with the right boundary conditions can be found, this solution is expressed in terms of integrals of known functions. Otherwise, the formal solution in terms of an arbitrary Green function is in fact an integral equation, which often can serve as a basis for numerical computations.

The numerical solution of equations for monochromatic fields normally involves the inversion of large matrices. In the case of time-varying fields, the consequences of causality allow for a stepping-in-time procedure that does not involve such an operation and can be advantageous under some circumstances.

In Section 3 we derive the formal solutions of Maxwell's equations in terms of Green's functions for the scalar wave equation. We start from the equations for the electromagnetic potentials in a Lorentz gauge, but the final expressions contain only the fields and the sources. We also conclude that unknown fields on the boundary cannot be simply eliminated from the integrals, so that we really arrive at integral equations for the electric and magnetic fields.

The integral equations for the fields can be reduced to simpler equations for the surface charge and current densities in the special case of scattering pulses by perfect conductors. We derive several equations in Section 4, and find that some of these differ from those found elsewhere (Mittra [1]).* We include a detailed discussion of the limiting process that takes the observation point to the boundary surface.

A large part of the mathematical derivations is relegated to four appendices. Appendix A deals with the notation and concepts related to Maxwell's equations in four-dimensional space-time, as well as a discussion of potentials. In Appendix B, we list a number of vector identities we use, and present the derivation of different forms of Green's theorem. In

*Numbers in square brackets indicate the literature references at the end of this report.

Appendix C, we derive a number of properties of Green's functions for the scalar wave equation, as well as the explicit form of the free-space retarded Green function. In Appendix D we develop a solution of Maxwell's equations by means of dyadic Green functions. This approach allows the expression of formal solutions in terms of given boundary conditions alone, although the actual derivation of dyadic Green functions is difficult. In this Appendix we derive an explicit expression for the free-space dyadic Green function for the vector wave equation.

The level of mathematical rigor in this presentation is probably average for the subject. The functions we deal with are assumed to be sufficiently well-behaved and the surfaces smooth enough so that mathematical transformations such as those in Gauss's divergence theorem apply. We recognize that Green's functions are not really well-behaved, and a proper mathematical context for them would be the theory of distributions, but we apply the formalism we develop to them regardless, as is customary in this field. Special care has to be taken when problems appear, as discussed by Yaghjian [2].

2. Maxwell's Equations

Four vector fields are usually defined to describe electromagnetic phenomena; they are the electric field intensity $\vec{E}$, the displacement vector $\vec{D}$, the magnetic induction field $\vec{B}$, and the magnetic field intensity $\vec{H}$. They obey Maxwell's equations

$$\nabla\cdot\vec{D} = \rho, \tag{1}$$

$$\nabla\times\vec{E} = -\partial\vec{B}/\partial t, \tag{2}$$

$$\nabla\cdot\vec{B} = 0, \tag{3}$$

$$\nabla\times\vec{H} = \vec{j}+\partial\vec{D}/\partial t, \tag{4}$$

where the sources ρ and $\vec{j}$ represent the free charge and current densities. These equations do not determine the fields unless additional relations are given, the constitutive relations. They depend on the properties of the medium, and in their simplest form they are

$$\vec{D} = \varepsilon_o\vec{E}, \tag{5}$$

$$\vec{B} = \mu_o\vec{H}, \tag{6}$$

in free space. The same relationships apply to homogeneous, isotropic, nondispersive media, which are characterized by constant permittivity ε and permeability μ. In more general cases, ε and μ can be functions of time and position, functions of frequency for monochromatic fields in dispersive media, or tensors for anisotropic media such as crystals. Even more complicated relationships are possible, such as those for ferromagnetic materials where hysteresis occurs for time-varying fields.

For a conducting medium, it is often possible to relate the current density to the electric field by a simple form of Ohm's law,

$$\vec{j} = \sigma\vec{E}, \tag{7}$$

where σ is the conductivity; for monochromatic fields the permittivity then becomes a complex function of the frequency.

We restrict ourselves to the set of Maxwell's equations in free space, eqs. (1) to (6), but it is only necessary to replace ε_o and μ_o by constants ε and μ to generalize the results to simple dielectrics.

We also assume that the medium and the boundaries are at rest. When this is not the case, Lorentz transformations can often be used to reduce the problem to a rest frame, or a more general four-dimensional formalism may be required, as introduced in Appendix A and used by Marx [3].

Actually, only eqs. (2) and (4) are true equations of motion. The other two Maxwell equations are constraints on the fields and it is sufficient to satisfy them at an initial time t_o. They are then satisfied for all times t, since

$$\begin{aligned}(\partial/\partial t)(\nabla\cdot\vec{D}-\rho) &= \nabla\cdot(\partial\vec{D}/\partial t)-\partial\rho/\partial t = \nabla\cdot(\nabla\times\vec{H}-\vec{j})-\partial\rho/\partial t \\ &= -\nabla\cdot\vec{j}-\partial\rho/\partial t = 0,\end{aligned} \tag{8}$$

$$(\partial/\partial t)\nabla\cdot\vec{B} = \nabla\cdot(\partial\vec{B}/\partial t) = \nabla\cdot(-\nabla\times\vec{E}) = 0, \tag{9}$$

where we have used the conservation of charge, eq. (A11), a requirement that the sources have to satisfy for eqs. (1) and (4) to be compatible.

Also, if ρ is given at the initial time, we can find it at later times from

$$\rho(\vec{x},t) = \rho(\vec{x},t_o) - \int_{t_o}^{t} dt' \nabla \cdot \vec{j}(\vec{x},t'). \tag{10}$$

For free space, eqs. (1) and (4) reduce to

$$\nabla \cdot \vec{E} = \rho/\varepsilon_o, \tag{11}$$

$$\nabla \times \vec{B} = \mu_o \vec{j} + \mu_o \varepsilon_o \partial\vec{E}/\partial t, \tag{12}$$

and the wave solutions propagate with speed c given by

$$c^2 = 1/\varepsilon_o \mu_o \tag{13}$$

When we restrict our attention to a region V of space bounded by a surface S with unit outward normal $\hat{n}$, we have to give boundary conditions in addition to the initial conditions to specify the fields uniquely. We follow Stratton [4] and use eqs. (B9), (2) and (12) to write

$$\nabla \cdot (\vec{E} \times \vec{B}) = -\vec{B} \cdot \partial\vec{B}/\partial t - \varepsilon_o \mu_o \vec{E} \cdot \partial\vec{E}/\partial t - \mu_o \vec{j} \cdot \vec{E}. \tag{14}$$

We integrate over V and use eq. (B15) to obtain

$$\oint_S dS \hat{n} \cdot \vec{E} \times \vec{B} = -(1/2)(d/dt) \int_V dV (\vec{B}^2 + \varepsilon_o \mu_o \vec{E}^2) - \mu_o \int_V dV \vec{j} \cdot \vec{E}. \tag{15}$$

We now show that the fields in the region V are uniquely determined by the current density $\vec{j}$, the initial values of $\vec{E}$ and $\vec{B}$ (subject to the constraints) and either the tangential component of $\vec{E}$ or the tangential component of $\vec{B}$ on S. If the fields are not unique, we could find two solutions that satisfy Maxwell's equations and the other conditions. Since the equations are linear, the difference of these two fields would also satisfy Maxwell's equations with vanishing current density, initial values, and boundary values. When the difference fields are substituted into eq. (15), the last term vanishes because $\vec{j}$ does. If either $\hat{n}\times\vec{E}$ or $\hat{n}\times\vec{B}$ vanishes, the surface integral in this equation is zero and these fields satisfy

$$(d/dt)\int_V dV(\vec{E}^2+\varepsilon_o\mu_o\vec{B}^2) = 0. \tag{16}$$

Thus, the value of the integral is a constant that has to be zero because these fields vanish at the initial time. Since the integrand is non-negative, both $\vec{E}$ and $\vec{B}$ have to be zero and the two solutions of the original problem must be the same, hence unique.

There are two simple relations between the fields on the surface S that can be derived from Maxwell's euations with the help of eq. (B35). The time derivatives of the normal components of the fields are given by

$$\begin{aligned}\partial E_n/\partial t &= \hat{n}\cdot[(1/{}_o\mu_o)\nabla\times\vec{B}-(1/\varepsilon_o)\vec{j}]\\ &= -(1/\varepsilon_o\mu)\nabla_s\cdot(\hat{n}\times\vec{B})-(1/\varepsilon_o)\hat{n}\cdot\vec{j},\end{aligned} \tag{17}$$

$$\partial B_n/\partial t = -\hat{n}\cdot\nabla\times\vec{E}\quad \nabla_s\cdot(\hat{n}\times\vec{E}). \tag{18}$$

The normal component of $\vec{E}$ can then be found if we know its initial value, the tangential component of $\vec{B}$ (or the surface divergence of $\hat{n}\times\vec{B}$) and the normal component of $\vec{j}$; and the normal component of $\vec{B}$ is determined by its initial value and the tangential component of $\vec{E}$.

3. Derivation of Integral Equations for the Fields

To derive integral equations for $\vec{E}$ and $\vec{B}$, we use the appropriate form of Green's theorem from Appendix B, apply it to the potentials in a Lorentz gauge, and select a Green function for the scalar wave equation, as discussed in Appendix C.

We start with eq. (B47) and change the variables of integration to $\vec{x}'$ and t', collectively designated by the four-vector x'. We choose $t+\varepsilon$, where ε is an arbitrary positive number, as the upper limit of integration t_1, and substitute

$$\vec{u}(x') = \vec{A}(x'), \tag{19}$$

$$\vec{v}(x') = \vec{a}G_R(x,x'), \tag{20}$$

where $\vec{A}$ is the vector potential, G_R a retarded Green function that also depends on a four-vector x, and $\vec{a}$ an arbitrary constant vector. Since $\vec{A}$ obeys the wave equation (A20) and G_R satisfies eq. (C8), we obtain

$$\vec{a}\cdot\int_{t_o}^{t} dt'\int_V dV'\mu_o\vec{j}(x')\frac{\partial G_R(x,x')}{\partial t'} + \vec{a}\cdot\frac{\partial\vec{A}(x)}{\partial t} = \vec{a}\cdot\int_V dV'\Big[-\frac{1}{c^2}\frac{\partial\vec{A}(x')}{\partial t'}\frac{\partial G_R(x,x')}{\partial t'}$$

$$-\{\nabla'\times\vec{A}(x')\}\times\nabla' G_R(x,x')-\{\nabla'\cdot\vec{A}(x')\}\nabla' G_R(x,x')\Big]_{t'=t_o}$$

$$-\vec{a}\cdot\int_{t_o}^{t} dt'\Big[\oint_S d\vec{S}'\cdot\{\nabla' G_R(x,x')\}\frac{\partial\vec{A}(x')}{\partial t'} - \oint_S d\vec{S}'\frac{\partial\vec{A}(x')}{\partial t'}\cdot\nabla' G_R(x,x')$$

$$-\oint_S d\vec{S}'\times\{\nabla'\times\vec{A}(x')\}\frac{\partial G_R(x,x')}{\partial t'} + \oint_S d\vec{S}'\cdot\frac{\partial\vec{A}(x')}{\partial t'}\nabla' G_R(x,x') \quad .$$

$$\left. + \oint_S d\vec{S}'\frac{\partial G_R(x,x')}{\partial t'}\nabla'\cdot\vec{A}(x')\right], \tag{21}$$

where the causality condition (C2) implies that there is no contribution to time integrals for t' between t and t+ε, and that the integrand of the volume integral on the right-hand side vanishes for t'=t+ε. Similarly, we substitute

$$u(x') = \Phi(x'), \tag{22}$$

$$v(x') = G_R(x,x'), \tag{23}$$

where Φ is the scalar potential, into eq. (B51) to obtain

$$\nabla\Phi(x) + \frac{1}{\varepsilon_o}\int_{t_o}^{t} dt'\int_V dV'\rho(x')\nabla' G_R(x,x')$$

$$= -\frac{1}{c^2}\int_V dV'\left[\{\nabla'\Phi(x')\}\frac{\partial G_R(x,x')}{\partial t'} + \frac{\partial\Phi(x')}{\partial t'}\nabla' G_R(x,x')\right]_{t'=t_o}$$

$$+\int_{t_o}^{t} dt'\left[\oint_S d\vec{S}'\left\{(\nabla'\Phi(x'))\cdot\nabla' G_R(x,x') - \frac{1}{c^2}\frac{\partial\Phi(x')}{\partial t'}\frac{\partial G_R(x,x')}{\partial t'}\right\}\right.$$

$$\left. -\oint_S d\vec{S}'\cdot\{(\nabla'\Phi(x'))\nabla' G_R(x,x') + (\nabla' G_R(x,x'))\nabla'\Phi(x')\}\right]. \tag{24}$$

We obtain $\partial\vec{A}/\partial t$ from eq. (21), using the property (B16) to eliminate the arbitrary vector $\vec{a}$, and $\nabla\Phi$ from eq. (24). We combine them according to eq. (A14) to find the field

$$\vec{E}(x) = \int_{t_o}^{t} dt'\int_V dV'\left[\mu_o\vec{j}(x')\frac{\partial G_R(x,x')}{\partial t'} + \frac{1}{\varepsilon_o}\rho(x')\nabla' G_R(x,x')\right]$$

$$-\int_V dV'\left[\frac{1}{c^2}\vec{E}(x')\frac{\partial G_R(x,x')}{\partial t'} - \vec{B}(x')\times\nabla' G_R(x,x')\right]_{t'=t_o}$$

$$+\int_{t_o}^{t} dt'\left[\oint_S d\vec{S}'\vec{E}(x')\cdot\nabla' G_R(x,x') - \oint_S d\vec{S}'\cdot\vec{E}(x')\nabla' G_R(x,x')\right.$$

$$\left. -\oint_S d\vec{S}'\cdot\{\nabla' G_R(x,x')\}\vec{E}(x') - \oint_S d\vec{S}'\times\vec{B}(x')\frac{\partial G_R(x,x')}{\partial t'}\right], \qquad (25)$$

where we have introduced $\vec{B}$ from eq. (A15) and used the Lorentz condition (A18).

We now substitute eqs. (19) and (20) into eq. (B57) to find $\nabla\times\vec{A}$; we obtain

$$\mu_o\int_{t_o}^{t} dt'\int_V dV'\vec{j}(x')\cdot\{\nabla' G_R(x,x')\}\times\vec{a} - \vec{a}\cdot\nabla\times\vec{A}(x)$$

$$= -\frac{1}{c^2}\int_V dV'\left[\frac{\partial\vec{A}(x')}{\partial t'}\cdot\{\nabla' G_R(x,x')\}\times\vec{a} - \frac{\partial G_R(x,x')}{\partial t'}\vec{a}\cdot\nabla'\times\vec{A}(x')\right]_{t'=t_o}$$

$$+\int_{t_o}^{t} dt'\oint_S d\vec{S}'\cdot\left[\frac{1}{c^2}\frac{\partial\vec{A}(x')}{\partial t'}\times\vec{a}\frac{\partial G_R(x,x')}{\partial t'} + \{\nabla'\times\vec{A}(x')\}\times\{(\nabla' G_R(x,x'))\times\vec{a}\}\right.$$

$$\left. +\{\nabla'\times\vec{A}(x')\}\vec{a}\cdot\nabla' G_R(x,x') - \{\nabla'\cdot\vec{A}(x')\}\{\nabla' G_R(x,x')\}\times\vec{a}\right]. \qquad (26)$$

We use the Lorentz condition (A18), integration by parts and eq. (B21) to show that

$$\int_V dV'\{\nabla'\Phi(x')\}\times\nabla' G_R(x,x')\Big|_{t'=t_o}$$

$$-\int_{t_o}^{t+\varepsilon} dt'\oint_S d\vec{S}'\times\left[\{\nabla'\Phi(x')\}\frac{\partial G_R(x,x')}{\partial t'} + c^2\{\nabla' G_R(x,x')\}\nabla'\cdot\vec{A}(x')\right]$$

$$= \oint_S d\vec{S}'\times\{\nabla' G_R(x,x')\}\Phi(x')\Big|_{t'=t_o} + \oint_S d\vec{S}'\times\{\nabla'\Phi(x')\}G_R(x,x')\Big|_{t'=t_o}$$

$$+\int_{t_o}^{t+\varepsilon} dt'\oint_S d\vec{S}'\times\left[\left\{\nabla'\frac{\partial\Phi(x')}{\partial t'}\right\}G_R(x,x')+\{\nabla' G_R(x,x')\}\frac{\partial\Phi(x')}{\partial t'}\right]$$

$$= \oint_S d\vec{S}'\times\nabla'\{G_R(x,x')\Phi(x')\}\Big|_{t'=t_o} + \int_{t_o}^{t+\varepsilon} dt'\oint_S d\vec{S}'\times\nabla'\left\{G_R(x,x')\frac{\partial\Phi(x')}{\partial t'}\right\} = 0. \tag{27}$$

We find the field $\vec{B}$ from eq. (26) according to eq. (A15),

$$\vec{B}(x) = \mu_o\int_{t_o}^{t} dt'\int_V dV'\vec{j}(x')\times\nabla' G_R(x,x')$$

$$-\frac{1}{c^2}\int_V dV'\left[\vec{E}(x')\times\nabla' G_R(x,x')+\vec{B}(x')\frac{\partial G_R(x,x')}{\partial t'}\right]_{t'=t_o}$$

$$+\int_{t_o}^{t} dt'\left[\frac{1}{c^2}\oint_S d\vec{S}'\times\vec{E}(x')\frac{\partial G_R(x,x')}{\partial t'} - \oint_S d\vec{S}'\cdot\{\nabla' G_R(x,x')\}\vec{B}(x')\right.$$

$$\left.+\oint_S d\vec{S}'\vec{B}(x')\cdot\nabla' G_R(x,x') - \oint_S d\vec{S}'\cdot\vec{B}(x')\nabla' G_R(x,x')\right]. \tag{28}$$

Equations (25) and (28) give the fields $\vec{E}$ and $\vec{B}$ in terms of the sources, the initial values of the fields, and boundary values. These equations involve a retarded Green function with unspecified boundary conditions, and they are essentially the integral equations we seek. They are not solutions of the set of partial differential equations because both fields appear in the surface integrals, and we have shown in our discussion of the uniqueness of the solutions that only the tangential component of one of the fields needs to be specified. To express the surface integrals in terms of the normal and tangential components of the fields we replace the vector surface element $d\vec{S}'$ by its magnitude of dS' times the normal $\hat{n}'$, as in eq. (B39), and rewrite eqs. (25) and (28) in the form

$$\vec{E}(x) = \int_{t_o}^{t} dt' \int_V dV' \left[\mu_o \vec{j}(x') \frac{\partial G_R(x,x')}{\partial t'} + \frac{1}{\varepsilon_o}\rho(x')\nabla' G_R(x,x')\right]$$

$$-\int_V dV' \left[\frac{1}{c^2}\vec{E}(x') \frac{\partial G_R(x,x')}{\partial t'} - \vec{B}(x')\times\nabla' G_R(x,x')\right]_{t'=t_o}$$

$$-\int_{t_o}^{t} dt' \oint_S dS' \Big[\{\hat{n}'\times\vec{E}(x')\}\times\nabla' G_R(x,x')$$

$$+\hat{n}'\cdot\vec{E}(x')\nabla' G_R(x,x') + \hat{n}'\times\vec{B}(x')\frac{\partial G_R(x,x')}{\partial t'}\Big], \tag{29}$$

$$\vec{B}(x) = \mu_o \int_{t_o}^{t} dt' \int_V dV' \vec{j}(x')\times\nabla' G_R(x,x')$$

$$-\frac{1}{c^2}\int_V dV' \left[\vec{E}(x')\times\nabla' G_R(x,x') + \vec{B}(x')\frac{\partial G_R(x,x')}{\partial t'}\right]_{t'=t_o}$$

$$-\int_{t_o}^{t} dt' \oint_S dS' \Big[\{\hat{n}'\times\vec{B}(x')\}\times\nabla' G_R(x,x')$$

$$+\hat{n}'\cdot\vec{B}(x')\nabla' G_R(x,x') - \frac{1}{c^2}\hat{n}'\times\vec{E}(x')\frac{\partial G_R(x,x')}{\partial t'}\Big]. \tag{30}$$

To see whether we can find actual solutions to the problem we turn to Green functions such as $G_R^{(1)}$ and $G_R^{(2)}$, which satisfy homogeneous boundary conditions on S given by eqs. (C14) and (C15). We examine the solution of the scalar wave equation,

$$\Box\psi(x) = \alpha(x), \tag{31}$$

where the d'Alembertian is defined in eq. (A22) and α is a given source. From eq. (B42) we find

$$\begin{aligned}\psi(x) = &\int_{t_o}^{t} dt'\int_V dV'\alpha(x')G_R(x,x') \\ &- \frac{1}{c^2}\int_V dV'\left[\psi(x')\frac{\partial G_R(x,x')}{\partial t'} - \frac{\partial\psi(x')}{\partial t'}G_R(x,x')\right]_{t'=t_o} \\ &-\int_{t_o}^{t} dt'\oint_S dS'\left[\psi(x')\frac{\partial G_R(x,x')}{\partial n'} - \frac{\partial\psi(x')}{\partial n'}G_R(x,x')\right].\end{aligned} \tag{32}$$

In this case, we need to know the source $\alpha(x)$, the initial values of ψ and $\partial\psi/\partial t$ at time t_o, and either ψ or the normal derivative $\partial\psi/\partial n$ on the surface S. If ψ is given on S, we choose a Green function $G_R^{(1)}$ that vanishes on S and the surface integral has a known integrand. If $\partial\psi/\partial n$ is given, we choose $G_R^{(2)}$, whose normal derivative vanishes on S, and again we know the integrand. There are more complicated boundary conditions that can be satisfied by ψ and G_R, but we will not discuss them here. For eqs. (29) and (30), we find that neither $G_R^{(1)}$ nor $G_R^{(2)}$ will eliminate the unknown terms. For instance, if the tangential component of $\vec{E}$ is given, we might try $G_R^{(1)}$ in eq. (29) and $G_R^{(2)}$ in Eq. (30); the surface term contributions then reduce to

$$\vec{E}_s(x) = -\int_{t_o}^{t} dt' \oint_S dS' \vec{E}(x') \partial G_R^{(1)}(x,x')/\partial n', \tag{33}$$

$$\vec{B}_s(x) = -\int_{t_o}^{t} dt' \oint_S dS' [\{\hat{n}' \times \vec{B}(x')\} \times \nabla_s' G_R^{(2)}(x,x') + \hat{n}' \cdot \vec{B}(x') \nabla_s' G_R^{(2)}(x,x') - \frac{1}{c^2} \hat{n}' \times \vec{E}(x') \partial G_R^{(2)}(x,x')/\partial t']. \tag{34}$$

We can find $\hat{n} \cdot \vec{B}$ by means of eq. (18), but we do not know $\hat{n} \cdot \vec{E}$ in eq. (33) or $\hat{n} \times \vec{B}$ in eq. (34). It is not clear at this point whether further transformations of the surface integrals or other relations derived from Maxwell's equations would allow a determination of the fields through Green's functions for the scalar wave equation by integration alone. The lack of symmetry between source point and field point for the gradient of a Green function, as seen from eqs. (C22) and (C23), limits the transformations we can try.

The scalar Green function is used extensively in the case of monochromatic waves (Helmholtz equation) by Müller [5], where uniqueness is also shown under more general conditions.

Also for the Helmholtz equation, the reduction of a solution to integrations is accomplished by Tai [6] with the help of dyadic Green functions. We examine the use of dyadic Green functions for transient electromagnetic fields in Appendix D.

4. Scattering by Perfect Conductors

We now restrict our problem to the scattering of an incident electromagnetic pulse by one or more perfect conductors.

We use the integral equations (29) and (30) to derive simpler integral equations for the surface charge and current densities.

The initial time t_o is chosen so that the incident pulse has not reached any conductors. The volume V is the exterior of the conductors, and it is bounded by a surface S that has one or more parts just outside the conductors and is closed by a surface at infinity. The incident fields have to be defined so that no contributions to the integrals come from the surface at infinity, and the scattered fields do not reach this surface at a finite time because they propagate with the speed of light.

We assume that there are no sources in V after the initial time, and we separate the fields into incident and scattered parts,

$$\vec{E}(x) = \vec{E}^{in}(x)+\vec{E}^{sc}(x), \tag{35}$$

$$\vec{B}(x) = \vec{B}^{in}(x)+\vec{B}^{sc}(x). \tag{36}$$

The incident fields, by definition, propagate as free fields, and are given by

$$\vec{E}^{in}(x) = -\int_V dV'[(1/c^2)\vec{E}^{in}(x')\partial G_R^{(o)}(x,x')/\partial t'-\vec{B}^{in}(x')\times\nabla' G_R^{(o)}(x,x')]_{t'=t_o}, \tag{37}$$

$$\vec{B}^{in}(x) = -1/c^2\int_V dV'[\vec{E}^{in}(x')\times\nabla' G_R^{(o)}(x,x')+\vec{B}^{in}(x')\partial G_R^{(o)}(x,x')/\partial t']_{t'=t_o}, \tag{38}$$

from eqs. (25) and (28); the fields are expressed in terms of the initial values and the free-space Green function $G_R^{(o)}$.

We introduce the surface charge and current densities on the conductors, ρ_S and $\vec{J}_S$, and relate them to the fields through the boundary conditions

$$\hat{n}\times\vec{E}(x)\big|_{\vec{x}\in S} = 0, \tag{39}$$

$$\hat{n}\cdot\vec{E}(x)\big|_{\vec{x}\in S} = \rho_s(x)/\varepsilon_o, \tag{40}$$

$$\hat{n}\times\vec{B}(x)\big|_{\vec{x}\in S} = \mu_o\vec{J}_s(x), \tag{41}$$

$$\hat{n}\cdot\vec{B}(x)\big|_{\vec{x}\in S} = 0, \tag{42}$$

where $\hat{n}$ is now the outward normal of the conductors, that is, if $\hat{n}'$ is the outward normal as seen from V,

$$\hat{n} = -\hat{n}'. \tag{43}$$

Charge conservation on the surface reduces to

$$\nabla_s\cdot\vec{J}_s+\partial\rho_s/\partial t = 0, \tag{44}$$

a consequence of eqs. (12), (40) and (41) when there is no normal component of the current density $\vec{j}$ and when we use eq. (B35) for $\vec{B}$.

The integral equations for ρ_S and $\vec{J}_S$ are obtained from the boundary conditions (39) to (42) by expressing the fields on the surface in terms of the sources through the integral expressions for the equations.

Since the sources and the initial scattered fields vanish, the integral equations (29) and (30) reduce to

$$\vec{E}(x) = -\int_V dV'[(1/c^2)\vec{E}^{in}(x')\partial G_R^{(o)}/\partial t' - \vec{B}^{in}(x')\times\nabla' G_R^{(o)}]_{t'=t_o}$$
$$+\int_{t_o}^{t} dt' \oint_S dS'[(1/\varepsilon_o)\rho_s(x')\nabla' G_R^{(o)} + \mu_o \vec{J}_s(x')\partial G_R^{(o)}/\partial t'], \tag{45}$$

$$\vec{B}(x) = -(1/c^2)\int_V dV'[\vec{E}^{in}(x')\times\nabla' G_R^{(o)} + \vec{B}^{in}(x')\partial G_R^{(o)}/\partial t']_{t'=t_o}$$
$$+\int_{t_o}^{t} dt' \oint_S dS' \mu_o \vec{J}_s(x')\times\nabla' G_R^{(o)}. \tag{46}$$

We recognize the volume integrals in eqs. (45) and (46) as $\vec{E}^{in}(x)$ and $\vec{B}^{in}(x)$ from eqs. (37) and (38), so that the scattered fields are simply

$$\vec{E}^{sc}(x) = \int_{t_o}^{t} dt' \oint_S dS'[(1/\varepsilon_o)\rho_s(x')\nabla' G_R^{(o)} + \mu_o \vec{J}_s(x')\partial G_R^{(o)}/\partial t'], \tag{47}$$

$$\vec{B}^{sc}(x) = \int_{t_o}^{t} dt' \oint_S dS' \mu_o \vec{J}_s(x')\times\nabla' G_R^{(o)}. \tag{48}$$

We use the explicit form (C34) of the free-space retarded Green function and the properties of the delta function to derive

$$\frac{\partial G_R^{(o)}(x,x')}{\partial t'} = -\frac{\delta'(t-t'-R/c)}{4\pi R}, \tag{49}$$

$$\nabla' G_R^{(o)}(x,x') = \frac{\delta'(t-t'-R/c)R/c+\delta(t-t'-R/c)}{4\pi R^2}\hat{R}, \tag{50}$$

$$\int_{t_o}^{t} dt' \oint_S dS' \rho_s(x') \nabla' G_R^{(o)}(x,x') = \frac{1}{4\pi} \oint_S dS' \left[\frac{\hat{R}}{Rc} \frac{\partial \rho_s(x')}{\partial t'} + \frac{\hat{R}}{R^2} \rho_s(x') \right]_{t'=\tau}, \tag{51}$$

$$\int_{t_o}^{t} dt' \oint_S dS' \vec{J}_s(x') \frac{\partial G_R^{(o)}(x,x')}{\partial t'} = - \frac{1}{4\pi} \oint_S dS' \frac{1}{R} \frac{\partial \vec{J}_s(x')}{\partial t'} \bigg|_{t'=\tau}, \tag{52}$$

$$\int_{t_o}^{t} dt' \oint_S dS' \vec{J}_s(x') \times \nabla' G_R^{(o)}(x,x') = - \frac{1}{4\pi} \oint_S dS' \hat{R} \times \left[\frac{1}{Rc} \frac{\partial \vec{J}_s(x')}{\partial t'} + \frac{1}{R^2} \vec{J}_s(x') \right]_{t'=\tau}, \tag{53}$$

where

$$\hat{R} = (\vec{x}-\vec{x}')/|\vec{x}-\vec{x}'|, \tag{54}$$

$$\tau = t-R/c, \tag{55}$$

and the integrands vanish for times before t_o. The expressions (47) and (48) for the scattered fields thus become

$$\vec{E}^{sc}(x) = \frac{1}{4\pi} \oint_S dS' \left[\frac{\hat{R}}{\varepsilon_o Rc} \frac{\partial \rho_s(x')}{\partial t'} + \frac{\hat{R}}{\varepsilon_o R^2} \rho_s(x') - \frac{\mu_o}{R} \frac{\partial \vec{J}_s(x')}{\partial t'} \right]_{t'=\tau}, \tag{56}$$

$$\vec{B}^{sc}(x) = - \frac{1}{4\pi} \oint_S dS' \hat{R} \times \left[\frac{\mu_o}{Rc} \frac{\partial \vec{J}_s(x')}{\partial t'} + \frac{\mu_o}{R^2} \vec{J}_s(x') \right]_{t'=\tau}. \tag{57}$$

We now substitute eqs. (35) and (36) into the boundary conditions (39) to (42), and we use eqs. (56) and (57) for the scattered fields. When $\vec{x}$ is on the surface S, the integrands become singular as $\vec{x}'$ approaches $\vec{x}$, and we have to determine the contribution to the integral from the region about this singularity. We follow Van Bladel [7] and let $\vec{x}$ approach the surface field point $\vec{x}_o$ along the normal. (Poggio and Miller [8] change the surface about $\vec{x}_o$, but they fail to take into account all the different forms that the integrand can take.)

We separate a small part S_1 of the surface around $\vec{x}_o$ and we call the rest of the surface $\overline{S}$. We assume that ρ_s, $\partial\rho_s/\partial t$, $\vec{J}_s$, and $\partial\vec{J}_s/\partial t$ are slowly varying functions on S_1, and we approximate them by their values at $(\vec{x}_o,t)$. Furthermore, we choose S_1 small enough so that it can be considered flat, and shaped symmetrically about $\vec{x}_o$. If we write

$$\vec{x}-\vec{x}_o = h\hat{n},\ h>0, \tag{58}$$

we take the limit $h\to 0$ to let $\vec{x}$ approach the surface.

We decompose the vector $\vec{x}-\vec{x}'$ into its parts normal and tangential to the surface,

$$\vec{R} = \vec{R}\cdot\hat{n}\hat{n}+\hat{n}\times(\vec{R}\times\hat{n}), \tag{59}$$

and we compute some integrals of singular functions of $\vec{R}$. The first one involves the definition of the solid angle,

$$\int_{S_1}(\hat{R}\cdot\hat{n}\hat{n}/R^2)dS' = \hat{n}\int_{S_1}d\Omega = \hat{n}\Omega, \tag{60}$$

where Ω is the solid angle subtended by S_1 at $\vec{x}$. The tangential component of $\vec{R}$ does not contribute, since

$$\int_{S_1} [\hat{n}\times(\hat{R}\times\hat{n})/R^2]dS' = 0, \tag{61}$$

by the symmetry that makes contributions cancel from elements in S_1 that are symmetric about $\vec{x}_o$. Thus,

$$\lim_{h\to 0}\int_{S_1} (\hat{R}/R^2)dS' = 2\pi\hat{n}. \tag{62}$$

A similar decomposition shows that

$$\lim_{h\to 0}\int_{S_1} (\hat{R}/R)dS' = 0, \tag{63}$$

as would be expected because the integrand is less singular than the previous one. The integral of the tangential part vanishes again by symmetry, and the integral of the normal part is bounded by that over a circular region S_2 of radius $\underline{a}$ enclosing S_1; we compute

$$\int_{S_2} \frac{\hat{R}\cdot\hat{n}}{R}dS' = \int_0^{2\pi} d\phi \int_0^a d\rho \, \frac{\rho}{\sqrt{\rho^2+h^2}} \, \frac{h}{\sqrt{\rho^2+h^2}} = 2\pi h \int_0^a \frac{\rho d\rho}{\rho^2+h^2} = \pi h \log_e \frac{a^2+h^2}{h^2}, \tag{64}$$

which vanishes when h tends to zero. On the other hand, the integral in

$$L = \lim_{h\to 0}\int_{S_1} (1/R)dS' \tag{65}$$

has a finite limit that depends on the size and shape of S_1. For a circular region of radius a,

$$\int_{S_2} \frac{dS'}{R} = 2\pi \int_0^a \frac{\rho d\rho}{\sqrt{h^2+\rho^2}} = 2\pi\left(\sqrt{h^2+a^2}-h\right), \tag{66}$$

$$\lim_{h\to 0} \int_{S_2} (1/R)dS' = 2\pi a \simeq 6.28a, \tag{67}$$

while for a square of side $2a$ we find

$$\int_{S_3} \frac{dS'}{R} = 8\int_0^{\pi/4} d\phi \int_0^{a\sec\phi} \frac{\rho d\rho}{\sqrt{h^2+\rho^2}} = 8\int_0^{\pi/4} d\phi\left(\sqrt{h^2+a^2\sec^2\phi}-h\right), \tag{68}$$

$$\lim_{h\to 0} \int_{S_3} (1/R)dS' = 8a\int_0^{\pi/4} d\phi \sec\phi = 8a \log_e\left(\sqrt{2}+1\right) \simeq 7.05a. \tag{69}$$

If a region S_1 is completely inside a circle of radius a_1 and if a circle of radius a_2 is inside S_1, the value of the integral over S_1 is bounded by

$$2\pi a_1 \geq L \geq 2\pi a_2. \tag{70}$$

We can thus evaluate the integrals over S_1 for the scattered fields in eqs. (56) and (57) when the field point lies also on the surface S.

From the boundary condition (41), we obtain

$$\mu_o \vec{J}_s(x) = \hat{n}\times\vec{B}^{in}(x)\Big|_{\vec{x}\epsilon S} - \frac{\mu_o}{4\pi}\hat{n}\times\left\{\int_{\bar{S}} dS'\vec{R}\times\left[\frac{1}{R^2c}\frac{\partial\vec{J}_s(x')}{\partial t'} + \frac{1}{R^3}\vec{J}_s(x')\right]_{t'=\tau}\right.$$

$$\left. - \frac{1}{c}\frac{\partial\vec{J}_s(x)}{\partial t}\times\int_{S_1} dS'\frac{\vec{R}}{R^2} - \vec{J}_s(x)\times\int_{S_1} dS'\frac{\vec{R}}{R^3}\right\}\Big|_{\vec{x}\epsilon S}, \qquad (71)$$

and substitute the integrals in eqs. (62) and (63) to find the integral equation

$$\vec{J}_s(x) = \frac{2}{\mu_o}\hat{n}\times\vec{B}^{in}(x)\Big|_{\vec{x}\epsilon S} - \frac{1}{2\pi}\hat{n}\times\int_{\bar{S}} dS'\vec{R}\times\left[\frac{1}{R^2c}\frac{\partial\vec{J}_s(x')}{\partial t'} + \frac{1}{R^3}\vec{J}_s(x')\right]_{t'=\tau}\Big|_{\vec{x}\epsilon S}. \qquad (72)$$

Some authors [1,8] define principal value integrals by considering the limiting case of a region S when the excluded part S_1 tends to zero in area, presumably in a symmetric manner. The integral in eq. (62) is independent of S_1, but that in eq. (65) tends to zero, as shown by the inequality (70). Thus eq. (72) retains the same form when the integral over S is replaced by a "principal value" integral, and in this context the integral over a small region S_1 about the observation point vanishes. Since the integral in eq. (72) involves values of $\vec{J}_s$ and $\partial\vec{J}_s/\partial t$ at times earlier than t, a judicious choice of the time steps (small enough not to allow the electromagnetic fields to propagate from the center of one patch to the neighboring ones) allows us to compute the current density by a time-stepping procedure that involves no matrix inversion [1,8]. The integral equation (72) is derived from the expression of the scattered magnetic field, and it is known as a "Magnetic Field Integral Equation" (MFIE).

Another equation, the "Electric Field Integral Equation" (EFIE), can be derived from the boundary condition (39) and the expression for the scattered electric field (56). We have

$$0 = \hat{n}\times\vec{E}^{in}(x)\Bigg|_{\vec{x}\in S} + \frac{1}{4\pi}\hat{n}\times\Bigg\{\int_{\bar{S}} dS'\Bigg[\frac{\vec{R}}{\varepsilon_o R^2 c}\frac{\partial\rho_s(x')}{\partial t'} + \frac{\vec{R}}{\varepsilon_o R^3}\rho_s(x')$$

$$- \frac{\mu_o}{R}\frac{\partial\vec{J}_s(x')}{\partial t'}\Bigg]_{t'=\tau} + \frac{1}{\varepsilon_o c}\frac{\partial\rho_s(x)}{\partial t}\int_{S_1} dS'\frac{\vec{R}}{R^2}$$

$$+ \frac{\rho_s(x)}{\varepsilon_o}\int_{S_1} dS'\frac{\vec{R}}{R^3} - \mu_o\frac{\partial\vec{J}_s(x)}{\partial t}\int_{S_1}\frac{dS'}{R}\Bigg\}\Bigg|_{\vec{x}\in S}. \tag{73}$$

The contribution from the first integral over S_1 vanishes by eq. (63). The second one is given by eq. (62), and the vector product vanishes. The third one is expressed by eq. (65), and it vanishes when a principal value integral is considered. When a small finite region around the observation point is excluded, the integral equation takes the form

$$\frac{\partial\vec{J}_s(x)}{\partial t} = -\frac{4\pi}{\mu_o L}\hat{n}\times\Bigg\{\hat{n}\times\vec{E}^{in}(x)\Bigg|_{\vec{x}\in S}\Bigg\}$$

$$-\hat{n}\times\frac{1}{L}\Bigg\{\hat{n}\times\int_{\bar{S}} dS'\Bigg[\frac{c\vec{R}}{R^2}\frac{\partial\rho_s(x')}{\partial t'} + \frac{c^2\vec{R}}{R^3}\rho_s(x') - \frac{1}{R}\frac{\partial\vec{J}_s(x')}{\partial t'}\Bigg]_{t'=\tau}\Bigg|_{\vec{x}\in S}\Bigg\}, \tag{74}$$

which does not have precisely the form given by Mittra [1].

The surface charge density can be obtained from the divergence of the surface current density from eq. (44), which expresses conservation of charge. Also an integral equation can be obtained from eq. (40), and, proceeding as above, we find

$$\rho_s(x) = 2\varepsilon_o \hat{n}\cdot\vec{E}^{in}(x)\Big|_{\vec{x}\in S} + \frac{1}{2\pi}\hat{n}\cdot\int_{\bar{S}} dS'\left[\frac{\vec{R}}{R^2 c}\frac{\partial\rho_s(x')}{\partial t'} + \frac{\vec{R}}{R^3}\rho_s(x') - \frac{1}{Rc^2}\frac{\partial\vec{J}_s(x')}{\partial t'}\right]_{t'=\tau}\Bigg|_{\vec{x}\in S}. \tag{75}$$

The remaining boundary condition, eq. (42), leads to

$$0 = \hat{n}\cdot\vec{B}^{in}(x)\Big|_{\vec{x}\in S} - \frac{1}{4\pi}\hat{n}\cdot\int_{\bar{S}} dS'\vec{R}\times\left[\frac{\mu_o}{R^2 c}\frac{\partial\vec{J}_s(x')}{\partial t'} + \frac{\mu_o}{R^3}\vec{J}_s(x')\right]_{t'=\tau}\Bigg|_{\vec{x}\in S}, \tag{76}$$

which does not show a contribution from the region S_1. There is no obvious contradiction, as would have been the case for the equation derived from the boundary condition (39), because $\vec{B}^{in}$ should be tangential to the surface when the incident pulse reaches for the first time a region on the surface, so that neither term in eq. (76) differs from zero.

Which equations are best suited to carry out a numerical calculation depends to some extent on the shape of the scatterer. Others could also be used as a check on a solution obtained numerically.

5. Conclusions

In this note we have presented a possibly original derivation of integral equations that relate time-dependent electromagnetic fields to their sources, initial values, and boundary values. The integral equations (25) and (28) involve an arbitrary retarded Green function for the scalar wave equation and its derivatives, but no derivatives of the fields appear. Integrals over unknown boundary values do not disappear completely if the scalar Green function is made to vanish or to have a zero normal derivative on the boundary of the spatial region. To obtain a formal solution of Maxwell's equations, we use dyadic Green functions in a formulation shown in an appendix.

We applied eqs. (25) and (28) to the problem of a pulse scattered by a perfect conductor. We obtained simpler integral equations for the surface change and current densities, mainly the MFIE, eq. (72), and the EFIE, eq. (74). The derivation shows in detail what are the contributions to the fields from sources on a small region of the surface about the observation point. The EFIE has a form different from what is found in the literature. Once the induced sources are found, the scattered fields can be obtained by integration.

Although time-dependent fields can be obtained from monochromatic fields by a Fourier integral, the integral equations in the space-time domain can be advantageous in some cases because causality allows for a numerical solution by a time-step procedure that does not require the inversion of big matrices.

6. Acknowledgements

I thank Drs. George Lieberman and Daniel Maystre for many helpful discussions.

References

1. Mittra, Raj, Integral Equation Methods for Transient Scattering, in Transient Electromagnetic Fields, edited by L. B. Felsen (Springer-Verlag, Berlin, 1976).

2. Yaghjian, Arthur D., A Direct Approach to the Derivation of Electric Dyadic Green's Functions, NBS Technical Note 1000 (December 1977); Electric Dyadic Green's Functions in the Source Region, Proceedings of the IEEE 68, 248 (1980).

3. Marx, Egon, Covariant Formulation of Faraday's Law, Journal of the Franklin Institute 300, 353 (1975).

4. Stratton, Julius A., Electromagnetic Theory, p. 486 (McGraw-Hill, New York, 1941).

5. Müller, Claus, Foundations of the Mathematical Theory of Electromagnetic Waves (Springer-Verlag, Berlin, 1969).

6. Tai, Chen-To, Dyadic Green's Functions in Electromagnetic Theory (Intext, Scranton, 1971).

7. Van Bladel, J., Electromagnetic Fields, p. 392 (McGraw-Hill, New York, 1964).

8. Poggio, A. J. and Miller, E. D., Solutions of Three-Dimensional Scattering Problems, in Computer Techniques for Electromagnetics, edited by Raj Mittra, p. 162 (Pergamon, Oxford, 1973).

Appendix A

The Electromagnetic Field Tensor and Potential Vector

In this appendix we recall the basic equations of electromagnetic theory. We emphasize the underlying unity of space and time, as well as that of the electric and magnetic fields, brought about by the special theory of relativity. We also recall how potentials are introduced and some of the properties related to gauge invariance.

The discovery that Maxwell's equations were invariant not under Galilean transformations but under Lorentz transformations led to new concepts that are basic to the special theory of relativity. Space and time variables are combined into a single four-dimensional space with an indefinite metric. A point in that space is an event, and the coordinates form a four-vector

$$x = (x_\mu) = (ct, \vec{x}), \quad \mu = 0,1,2,3, \tag{A1}$$

where c is the speed of light in vacuum. The Lorentz metric is

$$(g_{\mu\nu}) = (g^{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1), \tag{A2}$$

although the opposite signs can also be chosen. This metric tensor can be used to raise or lower indices of vectors and other tensors; for vectors, this operation changes the sign of the spatial components. Derivatives of a field f(x) are indicated by

$$(\partial_\mu f) = (f_{,\mu}) = (\partial f/\partial x^\mu) = (c^{-1}\partial f/\partial t, -\nabla f). \tag{A3}$$

We use the modified summation convention for repeated lower Greek indices,

$$a_\mu b_\mu = a_o b_o - \vec{a}\cdot\vec{b}. \tag{A4}$$

The electric field $\vec{E}$ and the magnetic flux density $\vec{B}$ can be combined into a second rank antisymmetric tensor

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & cB_3 & -cB_2 \\ -E_2 & -cB_3 & 0 & cB_1 \\ -E_3 & cB_2 & -cB_1 & 0 \end{pmatrix}, \tag{A5}$$

and the charge and current densities form the four-vector

$$(j_\mu) = (c\rho, \vec{j}). \tag{A6}$$

Maxwell's equations can then be written in the form

$$F_{\mu\nu,\nu} = \mu_o c j_\mu, \tag{A7}$$

$$F^D_{\mu\nu,\nu} = 0, \tag{A8}$$

where, in terms of the completely antisymmetric Levi-Civita tensor, the tensor dual to $F_{\mu\nu}$ is given by

$$F^D_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} \quad \text{(A9)}$$

(the result of this operation is to replace $\vec{E}$ by $c\vec{B}$ and $c\vec{B}$ by $-\vec{E}$).

Since the product of a symmetric tensor and an antisymmetric tensor vanishes identically, eq. (A7) leads to

$$\partial_\mu \partial_\nu F_{\mu\nu} = \mu_o c j_{\mu,\mu} = 0, \quad \text{(A10)}$$

which translates into conservation of charge in its differential form,

$$\partial\rho/\partial t + \nabla\cdot\vec{j} = 0. \quad \text{(A11)}$$

The homogeneous Maxwell equations (A8) imply that the fields can be derived from potentials A_μ, that is,

$$F_{\mu\nu} = c(A_{\mu,\nu} - A_{\nu,\mu}). \quad \text{(A12)}$$

In terms of the usual scalar and vector potentials,

$$(A_\mu) = (\Phi/c, \vec{A}), \quad \text{(A13)}$$

$$\vec{E} = -\nabla\Phi - \partial\vec{A}/\partial t, \quad \text{(A14)}$$

$$\vec{B} = \nabla\times\vec{A}. \quad \text{(A15)}$$

These potentials are to some extent arbitrary, and they can be changed by gauge transformations

$$A'_\mu(x) = A_\mu(x)+\Lambda_{,\mu}(x), \tag{A16}$$

where Λ is an arbitary function of $\vec{x}$ and t, without changing the fields $F_{\mu\nu}$.

The relations (A12) imply that the homogeneous Maxwell equations (A8) are automatically satisfied, and we substitute the fields in (A7) to obtain the equations for the potentials,

$$A_{\mu,\nu\nu}-A_{\nu,\nu\mu} = \mu_o j_\mu. \tag{A17}$$

Given a set of potentials, it is possible to choose Λ so that certain additional conditions are satisfied. These conditions can also be imposed from the beginning to simplify the equations that have to be solved. A Lorentz gauge is characterized by the covariant Lorentz condition,

$$A_{\mu,\mu} = \nabla\cdot\vec{A}+(1/c^2)\partial\Phi/\partial t = 0, \tag{A18}$$

and eqs. (A17) reduce to

$$\partial^2 A_\mu = \mu_o j_\mu, \tag{A19}$$

or

$$\Box\vec{A} = \mu_o\vec{j}, \tag{A20}$$

$$\Box\Phi = \rho/\varepsilon_o, \tag{A21}$$

where the d'Alembertian operator is

$$\Box = \partial^2 = (1/c^2)\partial^2/\partial t^2 - \nabla^2. \tag{A22}$$

In Cartesian coordinates, all components of A_μ obey the scalar wave equation, which permits us to find a solution by means of the Green function for that equation.

Another gauge that is widely used is the Coulomb gauge, characterized by

$$\nabla\cdot\vec{A} = 0. \tag{A23}$$

In a Coulomb gauge, the scalar potential obeys Poisson's equation,

$$\nabla^2\Phi = -\rho/\varepsilon_o, \tag{A24}$$

and the equation for the vector potential is

$$\Box\vec{A} = \mu_o\vec{j} - (1/c^2)\nabla\partial\Phi/\partial t. \tag{A25}$$

When the current density is defined for all space, it can be decomposed into longitudinal (irrotational) and transverse (solenoidal) parts by setting

$$\vec{j}_\ell(\vec{x},t) = -\nabla\int dV'[\nabla'\cdot\vec{j}(\vec{x}',t)]/(4\pi|\vec{x}-\vec{x}'|), \qquad (A26)$$

$$\vec{j}_t(\vec{x},t) = \nabla\times[\nabla\times\int dV'[\vec{j}(\vec{x}',t)/(4\pi|\vec{x}-\vec{x}'|)]. \qquad (A27)$$

From the solution of eq. (A21),

$$\Phi(\vec{x},t) = (1/4\pi\varepsilon_o)\int dV'\rho(\vec{x}',t)/|\vec{x}-\vec{x}'|, \qquad (A28)$$

and conservation of charge (A11) we show that

$$\nabla\partial\Phi/\partial t = \vec{j}_\ell/\varepsilon_o, \qquad (A29)$$

so that eq. (A25) becomes

$$\Box\vec{A} = \mu_o\vec{j}_t. \qquad (A30)$$

The Coulomb gauge has the advantage that there are essentially only two components of the potential that obey an equation of motion. The longitudinal part of the vector potential and part of the scalar potential change under gauge transformations; they vanish in the Coulomb gauge, leaving only gauge independent quantities.

A disadvantage of the Coulomb gauge is the nonlocal nature of the expressions (A21) and (A22). Thus, if we have a surface current, the transverse part of that current is no longer confined to the surface. Similarly, the transverse part of the current density is not obtainable from eq. (A22) when the current is not known everywhere in space.

Appendix B

Vector Identities and Green's Theorems

In this appendix we collect or derive vector identities and the generalized versions of Green's theorem used in this note.

We have

$$\nabla(u+v) = \nabla u+\nabla v, \tag{B1}$$

$$\nabla(uv) = v\nabla u+u\nabla v, \tag{B2}$$

$$\nabla\cdot(\vec{u}+\vec{v}) = \nabla\cdot\vec{u}+\nabla\cdot\vec{v}, \tag{B3}$$

$$\nabla\times(\vec{u}+\vec{v}) = \nabla\times\vec{u}+\nabla\times\vec{v}, \tag{B4}$$

$$\nabla\cdot(u\vec{v}) = \vec{v}\cdot\nabla u+u\nabla\cdot\vec{v}, \tag{B5}$$

$$\nabla\times(u\vec{v}) = (\nabla u)\times\vec{v}+u\nabla\times\vec{v}, \tag{B6}$$

$$\nabla(\vec{u}\cdot\vec{v}) = (\nabla\vec{u})\cdot\vec{v}+(\nabla\vec{v})\cdot\vec{u} = \vec{v}\cdot\nabla\vec{u}+\vec{u}\cdot\nabla\vec{v}+\vec{v}\times(\nabla\times\vec{u})+\vec{u}\times(\nabla\times\vec{v}), \tag{B7}$$

$$\nabla\cdot(\vec{u}\vec{v}) = (\nabla\cdot\vec{u})\vec{v}+\vec{u}\cdot\nabla\vec{v}, \tag{B8}$$

$$\nabla\cdot(\vec{u}\times\vec{v}) = (\nabla\times\vec{u})\cdot\vec{v}-\vec{u}\cdot(\nabla\times\vec{v}), \tag{B9}$$

$$\nabla(\vec{u}\times\vec{v}) = (\nabla\vec{u})\times\vec{v}-(\nabla\vec{v})\times\vec{u}, \tag{B10}$$

$$\nabla\times(\vec{u}\times\vec{v}) = \vec{u}\nabla\cdot\vec{v}+\vec{v}\cdot\nabla\vec{u}-\vec{v}\nabla\cdot\vec{u}-\vec{u}\cdot\nabla\vec{v}, \tag{B11}$$

$$\nabla\times\nabla u = 0, \tag{B12}$$

$$\nabla\cdot\nabla\times\vec{u} = 0, \tag{B13}$$

$$\nabla\times(\nabla\times\vec{u}) = \nabla\nabla\cdot\vec{u}-\nabla^2\vec{u}. \tag{B14}$$

There is a mild controversy about this last identity: some authors consider it the definition of the Laplacian of a vector field. When the Laplacian of a vector function is defined as the divergence of the gradient dyadic, eq. (B14) is indeed an identity. The forms of the Laplacians of scalar and vector fields differ in curvilinear coordinates.

Gauss's divergence theorem states that

$$\int_V dV \nabla \cdot \vec{u}(\vec{x}) = \oint_S d\vec{S} \cdot \vec{u}(\vec{x}), \tag{B15}$$

where the volume V is bounded by the closed surface S and the vector surface element points out of V. The conditions under which this theorem is valid depend to some extent on the mathematical context and the definitions of integral and differential operators, and we do not specify them here.

When $\vec{a}$ is a constant but arbitrary vector, we have

$$\vec{a} \cdot \vec{u} = 0 \implies \vec{u} = 0. \tag{B16}$$

If we set successively

$$\vec{u}(\vec{x}) = \vec{a} v(\vec{x}), \tag{B17}$$

$$\vec{u}(\vec{x}) = \vec{a} \times \vec{v}(\vec{x}), \tag{B18}$$

$$\vec{u}(\vec{x}) = \vec{a} \cdot \vec{v}_1(\vec{x}) \vec{v}_2(\vec{x}) + \vec{a} \cdot \vec{v}_2(\vec{x}) \vec{v}_1(\vec{x}) \tag{B19}$$

in eq. (B15), then the property (B16) allows us to derive

$$\int_V dV\nabla v(\vec{x}) = \oint_S d\vec{S}v(\vec{x}), \tag{B20}$$

$$\int_V dV\nabla\times\vec{v}(\vec{x}) = \oint_S d\vec{S}\times\vec{v}(\vec{x}), \tag{B21}$$

$$\int_V dV\nabla\cdot[\vec{v}_1(\vec{x})\vec{v}_2(\vec{x})+\vec{v}_2(\vec{x})\vec{v}_1(\vec{x})] = \oint_S d\vec{S}\cdot[\vec{v}_1(\vec{x})\vec{v}_2(\vec{x})+\vec{v}_2(\vec{x})\vec{v}_1(\vec{x})]. \tag{B22}$$

Sometimes scalar or vector fields are defined only on a surface S. A surface gradient operator ∇_s can be defined for such a field. If the surface is given by parametric equations

$$\vec{x} = \vec{x}(\xi,\eta), \tag{B23}$$

the surface gradient is given by

$$\nabla_s = \vec{a}\frac{\partial}{\partial\xi} + \vec{b}\frac{\partial}{\partial\eta}, \tag{B24}$$

where, in terms of the unit normal $\hat{n}$,

$$\vec{a} = \frac{\partial\vec{x}}{\partial\eta}\times\hat{n}\left|\frac{\partial\vec{x}}{\partial\xi}\times\frac{\partial\vec{x}}{\partial\eta}\right|^{-1}, \tag{B25}$$

$$\vec{b} = \hat{n}\times\frac{\partial\vec{x}}{\partial\xi}\left|\frac{\partial\vec{x}}{\partial\xi}\times\frac{\partial\vec{x}}{\partial\eta}\right|^{-1}. \tag{B26}$$

This operator can be used to define the surface gradient, divergence, and rotation or curl. Some useful relations are

$$\hat{n}\cdot\nabla_s u = 0, \tag{B27}$$

$$\nabla_s \times \hat{n} = 0, \tag{B28}$$

where we use the definition (B24) and

$$\hat{n}^2 = 1, \tag{B29}$$

$$\hat{n}\cdot\partial\hat{n}/\partial\xi = \hat{n}\cdot\partial\hat{n}/\partial\eta = 0, \tag{B30}$$

$$\hat{n}\cdot\partial\vec{x}/\partial\xi = \hat{n}\cdot\partial\vec{x}/\partial\eta = 0, \tag{B31}$$

$$\frac{\partial\vec{x}}{\partial\xi}\cdot\frac{\partial\hat{n}}{\partial\eta} = \frac{\partial\vec{x}}{\partial\eta}\cdot\frac{\partial\hat{n}}{\partial\xi} = -\frac{\partial^2\vec{x}}{\partial\xi\partial\eta}\cdot\hat{n}, \tag{B32}$$

to prove such relations. Eqs. (B1) through (B10) are valid when the operator ∇ is replaced by ∇_s.

When the fields are also defined off the surface, we combine the surface gradient with the normal derivative in

$$\nabla = \nabla_s + \hat{n}\partial/\partial n, \tag{B33}$$

and we have

$$\hat{n}\times\nabla u = \hat{n}\times\nabla_s u = -\nabla_s\times(\hat{n}u), \tag{B34}$$

$$\hat{n}\cdot\nabla\times\vec{u} = \hat{n}\cdot\nabla_s\times\vec{u} = -\nabla_s\cdot(\hat{n}\times\vec{u}), \tag{B35}$$

$$\hat{n}\cdot\nabla_s\times(\nabla_s u) = 0. \tag{B36}$$

Green's theorem is derived from the divergence theorem (B15) when we let

$$\vec{u}(\vec{x}) = v_1(\vec{x})\nabla v_2(\vec{x}) - v_2(\vec{x})\nabla v_1(\vec{x}); \qquad \text{(B37)}$$

we find

$$\int_V dV[v_1(\vec{x})\nabla^2 v_2(\vec{x}) - v_2(\vec{x})\nabla^2 v_1(\vec{x})] = \oint_S dS[v_1(\vec{x})\partial v_2(\vec{x})/\partial n - v_2(\vec{x})\partial v_1(\vec{x})/\partial n], \qquad \text{(B38)}$$

where we have set

$$d\vec{S} = \hat{n}dS, \qquad \text{(B39)}$$

$$\partial v_i/\partial n = \hat{n}\cdot\nabla v_i, \; i = 1,2. \qquad \text{(B40)}$$

This theorem is used to obtain solutions of the Laplace equation by means of Green's functions. Other partial differential equations can be solved in a similar manner with the appropriate Green functions and generalizations of Green's theorem.

In what follows we assume that the fields u, v, $\vec{u}$, and $\vec{v}$ are also functions of the time t, that is, they are functions of the four-vector x. We derive the forms of Green's theorem that give integral equations for the fields when applied to the potentials in a Lorentz gauge. We start from expressions in which the d'Alembertian operator is applied to scalars or vectors, and use the identities (B1) through (B14) to obtain terms that can be partially integrated. The simplest generalization of Green's theorem to wave equations is obtained from

$$u\Box v - v\Box u = (1/c^2)(\partial/\partial t)(u\partial v/\partial t - v\partial u/\partial t) - \nabla\cdot(u\nabla v - v\nabla u). \tag{B41}$$

We integrate each side of this equation over a time interval and over a volume V bounded by a surface S, which may be multiply connected. We carry out an integration over t and use Gauss's theorem (B15) to obtain

$$\int_{t_o}^{t_1} dt \int_V dV(u\Box v - v\Box u) = \frac{1}{c^2}\int_V dV\left[u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right]_{t_o}^{t_1} - \int_{t_o}^{t_1} dt \oint_S dS\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right). \tag{B42}$$

More complicated forms of Green's theorem required in the text are obtained from

$$(\Box\vec{u})\cdot\frac{\partial\vec{v}}{\partial t} + (\Box\vec{v})\cdot\frac{\partial\vec{u}}{\partial t} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial\vec{u}}{\partial t}\cdot\frac{\partial\vec{v}}{\partial t}\right) - (\nabla^2\vec{u})\cdot\frac{\partial\vec{v}}{\partial t} - (\nabla^2\vec{v})\cdot\frac{\partial\vec{u}}{\partial t}, \tag{B43}$$

$$\nabla\cdot[\vec{u}\times(\nabla\times\vec{v}) + \vec{u}\nabla\cdot\vec{v}] = (\nabla\times\vec{u})\cdot\nabla\times\vec{v} + (\nabla\cdot\vec{u})\nabla\cdot\vec{v} + \vec{u}\cdot\nabla^2\vec{v}, \tag{B44}$$

$$\begin{aligned}(\partial/\partial t)[(\nabla\times\vec{u})\cdot\nabla\times\vec{v} + (\nabla\cdot\vec{u})\nabla\cdot\vec{v}] &= \nabla\cdot[(\partial\vec{u}/\partial t)\times(\nabla\times\vec{v}) + (\partial\vec{u}/\partial t)\nabla\cdot\vec{v}] \\ &- (\partial\vec{u}/\partial t)\cdot\nabla^2\vec{v} + \nabla\cdot[(\partial\vec{v}/\partial t)\times(\nabla\times\vec{u}) + (\partial\vec{v}/\partial t)\nabla\cdot\vec{u}] - (\partial\vec{v}/\partial t)\cdot\nabla^2\vec{u},\end{aligned} \tag{B45}$$

where the last equation is obtained by using (B44) once with $\vec{u}$ replaced by $\partial\vec{u}/\partial t$ and once with $\vec{v}$ replaced by $\partial\vec{v}/\partial t$. Replacing the last two terms in eq. (B43) by an expression obtained from (B45), we find

$$(\Box\vec{u})\cdot\frac{\partial\vec{v}}{\partial t}+(\Box\vec{v})\cdot\frac{\partial\vec{u}}{\partial t}=\frac{\partial}{\partial t}\left[\frac{1}{c^2}\frac{\partial\vec{u}}{\partial t}\cdot\frac{\partial\vec{v}}{\partial t}+(\nabla\times\vec{u})\cdot\nabla\times\vec{v}+(\nabla\cdot\vec{u})\nabla\cdot\vec{v}\right]$$

$$-\nabla\cdot\left[\frac{\partial\vec{u}}{\partial t}\times(\nabla\times\vec{v})+\frac{\partial\vec{v}}{\partial t}\times(\nabla\times\vec{u})+\frac{\partial\vec{u}}{\partial t}\nabla\cdot\vec{v}+\frac{\partial\vec{v}}{\partial t}\nabla\cdot\vec{u}\right], \qquad \text{(B46)}$$

$$\int_{t_o}^{t_1}dt\int_V dV\ (\Box\vec{u})\cdot\left[\frac{\partial\vec{v}}{\partial t}+(\Box\vec{v})\cdot\frac{\partial\vec{u}}{\partial t}\right]=\int_V dV\left[\frac{1}{c^2}\frac{\partial\vec{u}}{\partial t}\cdot\frac{\partial\vec{v}}{\partial t}+(\nabla\times\vec{u})\cdot\nabla\times\vec{v}+(\nabla\cdot\vec{u})\nabla\cdot\vec{v}\right]_{t_o}^{t_1}$$

$$-\int_{t_o}^{t_1}dt\oint_S d\vec{S}\cdot\left[\frac{\partial\vec{u}}{\partial t}\times(\nabla\times\vec{v})+\frac{\partial\vec{v}}{\partial t}\times(\nabla\times\vec{u})+\frac{\partial\vec{u}}{\partial t}\nabla\cdot\vec{v}+\frac{\partial\vec{v}}{\partial t}\nabla\cdot\vec{u}\right]. \qquad \text{(B47)}$$

Similarly, we have

$$(\Box u)\nabla v+(\Box v)\nabla u=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\nabla v+\frac{\partial v}{\partial t}\nabla u\right)-\frac{1}{c^2}\nabla\left(\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}\right)-(\nabla^2u)\nabla v-(\nabla^2v)\nabla u, \qquad \text{(B48)}$$

$$\nabla\cdot[(\nabla u)\nabla v+(\nabla v)\nabla u]-\nabla[(\nabla u)\cdot\nabla v]=(\nabla^2u)\nabla v+(\nabla^2v)\nabla u, \qquad \text{(B49)}$$

$$(\Box u)\nabla v+(\Box v)\nabla u=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\nabla v+\frac{\partial v}{\partial t}\nabla u\right)+\nabla\left[(\nabla u)\cdot\nabla v-\frac{1}{c^2}\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}\right]-\nabla\cdot[(\nabla u)\nabla v+(\nabla v)\nabla u] \qquad \text{(B50)}$$

$$\int_{t_o}^{t_1}dt\int_V dV[(\Box u)\nabla v+(\Box v)\nabla u]=\frac{1}{c^2}\int_V dV\left[\frac{\partial u}{\partial t}\nabla v+\frac{\partial v}{\partial t}\nabla u\right]_{t_o}^{t_1}$$

$$+\int_{t_o}^{t_1}dt\left[\oint_S d\vec{S}\left\{(\nabla u)\cdot\nabla v-\frac{1}{c^2}\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}\right\}-\oint_S d\vec{S}\cdot\{(\nabla u)\nabla v+(\nabla v)\nabla u\}\right], \qquad \text{(B51)}$$

where we have used eqs. (B20) and (B22) to obtain the surface integrals. Also

$$(\Box\vec{u})\cdot\nabla\times\vec{v}-(\Box\vec{v})\cdot\nabla\times\vec{u}=(1/c^2)\partial/\partial t(\partial\vec{u}/\partial t\cdot\nabla\times\vec{v}-\partial\vec{v}/\partial t\cdot\nabla\times\vec{u})$$

$$-(1/c^2)(\partial\vec{u}/\partial t\cdot\nabla\times\partial\vec{v}/\partial t-\partial\vec{v}/\partial t\cdot\nabla\times\partial\vec{u}/\partial t)-(\nabla^2\vec{u})\cdot\nabla\times\vec{v}+(\nabla^2\vec{v})\cdot\nabla\times\vec{u}, \qquad \text{(B52)}$$

$$\nabla\cdot(\partial\vec{u}/\partial t\times\partial\vec{v}/\partial t) = \partial\vec{v}/\partial t\cdot\nabla\times\partial\vec{u}/\partial t-\partial\vec{u}/\partial t\cdot\nabla\times\partial\vec{v}/\partial t, \tag{B53}$$

$$\nabla\cdot[(\nabla\times\vec{u})\times(\nabla\times\vec{v})] = [\nabla\times(\nabla\times\vec{u})]\cdot\nabla\times\vec{v}-(\nabla\times\vec{u})\cdot\nabla\times(\nabla\times\vec{v}), \tag{B54}$$

$$\nabla\cdot[(\nabla\cdot\vec{u})\nabla\times\vec{v}-(\nabla\cdot\vec{v})\nabla\times\vec{u}] = (\nabla\times\vec{v})\cdot\nabla\nabla\cdot\vec{u}-(\nabla\times\vec{u})\cdot\nabla\nabla\cdot\vec{v}, \tag{B55}$$

$$(\Box\vec{u})\cdot\nabla\times\vec{v}-(\Box\vec{v})\cdot\nabla\times\vec{u} = (1/c^2)\partial/\partial t(\partial\vec{u}/\partial t\cdot\nabla\times\vec{v}-\partial\vec{v}/\partial t\cdot\nabla\times\vec{u})$$
$$+\nabla\cdot[(1/c^2)\partial\vec{u}/\partial t\times\partial\vec{v}/\partial t+(\nabla\times\vec{u})\times(\nabla\times\vec{v})+(\nabla\times\vec{u})\nabla\cdot\vec{v}-(\nabla\times\vec{v})\nabla\cdot\vec{u}] \tag{B56}$$

$$\int_{t_o}^{t_1}dt\int_V dV[(\Box\vec{u})\cdot\nabla\times\vec{v}-(\Box\vec{v})\cdot\nabla\times\vec{u}] = \frac{1}{c^2}\int_V dV\left[\frac{\partial\vec{u}}{\partial t}\cdot\nabla\times\vec{v}-\frac{\partial\vec{v}}{\partial t}\cdot\nabla\times\vec{u}\right]_{t_o}^{t_1}$$
$$+\int_{t_o}^{t_1}dt\oint_S d\vec{S}\cdot\left[\frac{1}{c^2}\frac{\partial\vec{u}}{\partial t}\times\frac{\partial\vec{v}}{\partial t}+(\nabla\times\vec{u})\times(\nabla\times\vec{v})+(\nabla\times\vec{u})\nabla\cdot\vec{v}-(\nabla\times\vec{v})\nabla\cdot\vec{u}\right]. \tag{B57}$$

Another form of Green's theorem can be derived by starting from

$$\frac{\partial u}{\partial t}\Box v+\frac{\partial v}{\partial t}\Box u = \frac{\partial}{\partial t}\left[\frac{1}{c^2}\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}+(\nabla u)\cdot\nabla v\right]-\nabla\cdot\left(\frac{\partial u}{\partial t}\nabla v+\frac{\partial v}{\partial t}\nabla u\right), \tag{B58}$$

whence

$$\int_{t_o}^{t_1}dt\int_V dV\left(\frac{\partial u}{\partial t}\Box v+\frac{\partial v}{\partial t}\Box u\right) = \int_V dV\left[\frac{1}{c^2}\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}+(\nabla u)\cdot\nabla v\right]_{t_o}^{t_1}-\int_{t_o}^{t_1}dt\oint_S d\vec{S}\cdot\left(\frac{\partial u}{\partial t}\nabla v+\frac{\partial v}{\partial t}\nabla u\right). \tag{B59}$$

We add some relationships that involve a slightly different form of the vector wave equation, which can be used with dyadic Green functions. We recall that, from eq. (B9),

$$\nabla\cdot[\vec{u}\times(\nabla\times\vec{v})] = (\nabla\times\vec{u})\cdot\nabla\times\vec{v}-\vec{u}\cdot\nabla\times(\nabla\times\vec{v}), \tag{B60}$$

whence

$$\nabla\cdot[\vec{u}\times(\nabla\times\vec{v})-\vec{v}\times(\nabla\times\vec{u})] = \vec{v}\cdot\nabla\times(\nabla\times\vec{u})-\vec{u}\cdot\nabla\times(\nabla\times\vec{v}), \tag{B61}$$

$$\begin{aligned}&[(1/c^2)\partial^2\vec{u}/\partial t^2+\nabla\times(\nabla\times\vec{u})]\cdot\vec{v}-[(1/c^2)\partial^2\vec{v}/\partial t^2+\nabla\times(\nabla\times\vec{v})]\cdot\vec{u}\\ &= (1/c^2)(\partial/\partial t)(\vec{v}\cdot\partial\vec{u}/\partial t-\vec{u}\cdot\partial\vec{v}/\partial t)+\nabla\cdot[\vec{u}\times(\nabla\times\vec{v})-\vec{v}\times(\nabla\times\vec{u})],\end{aligned} \tag{B62}$$

$$\begin{aligned}&\int_{t_o}^{t_1}dt\int_V dV\left[\left\{\frac{1}{c^2}\,\frac{\partial^2\vec{u}}{\partial t^2}+\nabla\times(\nabla\times\vec{u})\right\}\cdot\vec{v}-\left\{\frac{1}{c^2}\,\frac{\partial^2\vec{v}}{\partial t^2}+\nabla\times(\nabla\times\vec{v})\right\}\cdot\vec{u}\right]\\ &= \frac{1}{c^2}\int_V dV\left[\vec{v}\cdot\,\frac{\partial\vec{u}}{\partial t}-\vec{u}\cdot\,\frac{\partial\vec{v}}{\partial t}\right]_{t_o}^{t_1}+\int_{t_o}^{t_1}dt\oint_S d\vec{S}\cdot[\vec{u}\times(\nabla\times\vec{v})-\vec{v}\times(\nabla\times\vec{u})].\end{aligned} \tag{B63}$$

Appendix C

Green's Functions for the Scalar Wave Equation

The Green functions for the scalar wave equation are functions of two four-vectors, x and x', that satisfy

$$\Box G(x,x') = \delta^{(4)}(x-x'), \tag{C1}$$

where the source is a Dirac delta function in four variables. To specify the solution further, appropriate boundary and initial conditions have to be added. Two widely used classes of Green's functions are the retarded and advanced ones, which satisfy

$$G_R(x,x') = 0, \ t<t', \tag{C2}$$

$$G_A(x,x') = 0, \ t>t'. \tag{C3}$$

If the region of space is bounded by a surface S, the most common boundary conditions on G are that either G itself or its normal derivative vanishes when the field point is on S. Other initial and boundary conditions are also possible, but we restrict ourselves to the ones above.

In the form (B42) of Green's theorem, we use $\vec{x}''$, t" as variables of integration and set

$$u(x'') = G_R(x'',x), \tag{C4}$$

$$v(x'') = G_A(x'',x'). \tag{C5}$$

We let $t_o \to -\infty$ and obtain

$$\int_{-\infty}^{t_1} dt'' \int_V dV'' \left[G_R(x'',x)\delta^{(4)}(x''-x') - G_A(x'',x')\delta^{(4)}(x''-x) \right]$$

$$= \frac{1}{c^2}\int_V dV'' \left[G_R(x'',x)\frac{\partial G_A(x'',x')}{\partial t''} - \frac{\partial G_R(x'',x)}{\partial t''}G_A(x'',x') \right]_{t''=-\infty}^{t''=t_1}$$

$$-\int_{-\infty}^{t_1} dt'' \oint_S dS'' \left[G_R(x'',x)\frac{\partial G_A(x'',x')}{\partial n''} - \frac{\partial G_R(x'',x)}{\partial n''}G_A(x'',x') \right] . \tag{C6}$$

We choose t_1 greater than both t and t'; the volume integral on the right-hand side vanishes because the integrand is zero at the upper limit since G_A satisfies eq. (C3), and also at the lower limit since G_R satisfies eq. (C2). Furthermore, the surface integral vanishes because either G_R and G_A vanish when $\vec{x}''$ is on S, or $\partial G_R/\partial n''$ and $\partial G_A/\partial n''$ vanish on the surface. Thus, we find that

$$G_R(x',x) = G_A(x,x'), \tag{C7}$$

and, consequently,

$$\Box' G(x,x') = \delta^{(4)}(x-x'). \tag{C8}$$

If we replace G_A in eq. (C5) by $G_R(\vec{x}'',-t'';\vec{x}',-t')$, we prove in the same way the reciprocity relation for G_R (or G_A),

$$G_R(\vec{x}',t';\vec{x},t) = G_R(\vec{x},-t;\vec{x}',-t'). \quad (C9)$$

When we proceed in a similar manner with eq. (B59), we obtain

$$\int_{-\infty}^{t_1} dt'' \int_V dV'' \left[\frac{\partial G_R(\vec{x}'',t'';\vec{x},t)}{\partial t''} \delta^{(4)}(x''-x') + \frac{\partial G_R(\vec{x}'',-t'';\vec{x}',-t')}{\partial t''} \delta^{(4)}(x''-x) \right]$$

$$= \int_V dV'' \left[\frac{1}{c^2} \frac{\partial G_R(\vec{x}'',t'';\vec{x},t)}{\partial t''} \frac{\partial G_R(\vec{x}'',-t'';\vec{x}',-t')}{\partial t''} \right.$$

$$\left. + \{\nabla'' G_R(\vec{x}'',t'';\vec{x},t)\} \cdot \nabla'' G_R(\vec{x}'',-t'';\vec{x}',-t') \right]_{t''=-\infty}^{t''=t_1}$$

$$- \int_{-\infty}^{t_1} dt'' \oint_S dS'' \left[\frac{\partial G_R(\vec{x}'',t'';\vec{x},t)}{\partial t''} \frac{\partial G_R(\vec{x}'',-t'';\vec{x}',-t')}{\partial n''} \right.$$

$$\left. + \frac{\partial G_R(\vec{x}'',t'';\vec{x},t)}{\partial n''} \frac{\partial G_R(\vec{x}'',-t'';\vec{x}',-t')}{\partial t''} \right], \quad (C10)$$

and the same arguments show that the right-hand side vanishes. We then have

$$\frac{\partial G_R(\vec{x}',t';\vec{x},t)}{\partial t'} + \frac{\partial G_R(\vec{x},-t;\vec{x}',-t')}{\partial t} = 0, \quad (C11)$$

whence, from the reciprocity relation,

$$\frac{\partial G_R(\vec{x},t;\vec{x}',t')}{\partial t} = - \frac{\partial G_R(\vec{x},t;\vec{x}',t')}{\partial t'}. \quad (C12)$$

To look for a symmetry for the spatial derivative, we start from eq. (B51) and find

$$\int_{-\infty}^{t_1} dt'' \int_V dV'' \left[\nabla'' G_R(\vec{x}'',t'';\vec{x},t)\,\delta^{(4)}(x''-x') + \nabla'' G_R(\vec{x}'',-t'';\vec{x}',-t')\,\delta^{(4)}(x''-x)\right]$$

$$= \frac{1}{c^2}\int_V dV'' \left[\nabla'' G_R(\vec{x}'',t'';\vec{x},t)\frac{\partial G_R(\vec{x}'',-t'';\vec{x}',-t')}{\partial t''}\right.$$

$$\left. + \frac{\partial G_R(\vec{x}'',t'';\vec{x},t)}{\partial t''}\nabla'' G_R(\vec{x}'',-t'';\vec{x}',-t')\right]_{t''=-\infty}^{t''=t_1}$$

$$+\int_{-\infty}^{t_1} dt'' \oint_S dS'' \left[\hat{n}''\left\{\left(\nabla_s'' G_R(\vec{x}'',t'';\vec{x},t)\right)\cdot\nabla_s'' G_R(\vec{x}'',-t'';\vec{x}',-t')\right.\right.$$

$$\left. - \frac{\partial G_R(\vec{x}'',t'';\vec{x},t)}{\partial n''}\frac{\partial G_R(\vec{x}'',-t'';\vec{x}',-t')}{\partial n''} - \frac{1}{c^2}\frac{\partial G_R(\vec{x}'',t'';\vec{x},t)}{\partial t''}\frac{\partial G_R(\vec{x}'',-t'';\vec{x}',-t')}{\partial t''}\right\}$$

$$\left. - \frac{\partial G_R(\vec{x}'',t'';\vec{x},t)}{\partial n''}\nabla_s'' G_R(\vec{x}'',-t'';\vec{x}',-t') - \frac{\partial G_R(\vec{x}'',-t'';\vec{x}',-t')}{\partial n''}\nabla_s'' G_R(\vec{x}'',t'';\vec{x},t)\right]. \qquad \text{(C13)}$$

We now distinguish between $G_R^{(1)}$, which satisfies

$$G_R^{(1)}(x,x')\Big|_{\vec{x}\in S} = 0, \qquad \text{(C14)}$$

and $G_R^{(2)}$, for which

$$\partial G_R^{(2)}(x,x')/\partial n\Big|_{\vec{x}\in S} = 0. \qquad \text{(C15)}$$

Eq. (C14) also implies that

$$\nabla_s G_R^{(1)}(x,x')\Big|_{\vec{x}\in S} = 0, \qquad \text{(C16)}$$

$$\partial G_R^{(1)}(x,x')/\partial t\Big|_{\vec{x}\in S} = 0. \tag{C17}$$

The reciprocity relation (C9) implies that similar relations are satisfied when the source point $\vec{x}'$ instead of the field point $\vec{x}$ is on the surface. We have

$$G_R^{(1)}(x,x')\Big|_{\vec{x}'\in S} = 0, \tag{C18}$$

$$\partial G_R^{(2)}(x,x')/\partial n'\Big|_{\vec{x}'\in S} = 0, \tag{C19}$$

$$\nabla'_s G_R^{(1)}(x,x')\Big|_{\vec{x}'\in S} = 0, \tag{C20}$$

$$\partial G_R^{(1)}(x,x')/\partial t'\Big|_{\vec{x}'\in S} = 0. \tag{C21}$$

When we substitute eqs. (C14), (C16), and (C17) or eq. (C15) in eq. (C13), we obtain

$$\nabla' G_R^{(1)}(\vec{x}',t';\vec{x},t)+\nabla G_R^{(1)}(\vec{x},-t;\vec{x}',-t')$$
$$= -\int_{-\infty}^{t_1} dt'' \oint_S dS'' \frac{\partial G_R^{(1)}(\vec{x}'',t'';\vec{x},t)}{\partial n''} \frac{\partial G_R^{(1)}(\vec{x}'',-t'';\vec{x}',-t')}{\partial n''}, \tag{C22}$$

$$\nabla' G_R^{(2)}(\vec{x}',t';\vec{x},t)+\nabla G_R^{(2)}(\vec{x},-t;\vec{x}',-t')$$

$$= \int_{-\infty}^{t_1} dt'' \oint_S dS'' \left[\{\nabla_s'' G_R^{(2)}(\vec{x}'',t'';\vec{x},t)\} \cdot \nabla_s'' G_R^{(2)}(\vec{x}'',-t'';\vec{x}',-t') - \frac{1}{c^2} \frac{\partial G_R^{(2)}(\vec{x}'',t'';\vec{x},t)}{\partial t''} \frac{\partial G_R^{(2)}(\vec{x}'',-t'';\vec{x}',-t')}{\partial t''} \right], \tag{C23}$$

and no symmetry can be shown in either case.

The free-space Green function $G_R^{(o)}$ has translational invariance, and consequently it is a function of x-x' alone. In that case,

$$\nabla G_R^{(o)}(x,x') = -\nabla' G_R^{(o)}(x,x'). \tag{C24}$$

This property also means that it is sufficient to evaluate the free space Green function for x'=0. If we express that function in terms of its Fourier transform,

$$G(\vec{x},t) = (2\pi)^{-2} \int d^3k d\omega e^{-i(\omega t-\vec{k}\cdot\vec{x})} \tilde{G}(\vec{k},\omega), \tag{C25}$$

eq. (C1) leads to

$$\tilde{G}(\vec{k},\omega) = -\frac{1}{(2\pi)^2} \frac{1}{\omega^2/c^2-\vec{k}^2}. \tag{C26}$$

To perform the integration over ω, we have to specify the behavior of the path at the poles at $\pm|\vec{k}|c$, which is linked to causality. When $t<0$, the contour can be closed around the upper half-plane with no contribution from the semicircle of infinite radius. For the retarded Green function, the condition (C2) implies that we have to deform the contour so that it lies above the poles. For $t>0$, we close the contour around the lower half-plane, and compute the residues at the poles to obtain

$$G_R(\vec{x},t) = c(2\pi)^{-3}\int d^3k[\sin(kct)/k]e^{i\vec{k}\cdot\vec{x}}, \tag{C27}$$

where k is the magnitude of $\vec{k}$.

To do the angular integrations in momentum space, we choose a spherical coordinate system with the polar axis along $\vec{x}$. For an arbitrary function of k, we find

$$\int d^3k f(k) e^{\pm i\vec{k}\cdot\vec{x}} = (4\pi/r)\int_0^\infty k dk f(k)\sin(kr), \tag{C28}$$

where

$$r = |\vec{x}|. \tag{C29}$$

In particular,

$$G_R(\vec{x},t) = (c/2\pi^2 r)\int_0^\infty dk \sin(kr)\sin(kct). \tag{C30}$$

The integrand is an even function of k, so that we can extend the range of integration to negative values of k and take one half of the result. In terms of exponential functions, we have

$$G_R(\vec{x},t) = -(c/16\pi^2 r)\int_{-\infty}^{\infty} dk\{e^{i(ct+r)k} - e^{i(ct-r)k} - e^{-i(ct-r)k} + e^{-i(ct+r)k}\}, \quad (C31)$$

where the integrals on the right represent Dirac delta functions. Since t is positive and r is non-negative, ct+r cannot be zero and only two terms contribute to

$$G_R(\vec{x},t) = (1/4\pi r)\delta(t-r/c), \quad (C32)$$

which is also valid for t<0 as the delta function vanishes. If we set

$$\vec{R} = \vec{x}-\vec{x}', \ R = |\vec{x}-\vec{x}'|, \quad (C33)$$

the retarded Green function for the scalar wave equation in free space is

$$G_R^{(o)}(x,x') = \frac{\delta(t-t'-R/c)}{4\pi R}. \quad (C34)$$

Similarly, we obtain the advanced Green function if we deform the contour to pass below the poles. The result is

$$G_A^{(o)}(x,x') = \frac{\delta(t-t'+R/c)}{4\pi R}, \quad (C35)$$

and the symmetry properties follow from those of the delta function.

Appendix D

Dyadic Green Functions for Transient Fields

We have seen in the main text of this note that the use of Green's functions for the scalar wave equations has some limitations when we apply them to finite regions of space, which give rise to boundary conditions.

The natural operator that relates what are essentially vector fields to vector sources is a dyadic Green function. These functions have been studied extensively for monochromatic fields that satisfy the vector Helmholtz equations; see, for instace, a book by Tai [6] and an article by Yaghjian [2].

In this appendix, we develop the corresponding basic theory for transient fields that obey Maxwell's equations in free space. We first show the equivalance of Maxwell's equations and the vector wave equation, we then proceed to derive some properties of the dyadic Green functions and we use them to find the solutions for the fields in a finite region of space. Finally, we use the method presented in Appendix C for the scalar Green function to find the free-space dyadic Green function.

Our original problem is to solve Maxwell's equations with given sources,

$$\nabla\cdot\vec{E} = \rho/\varepsilon_o, \tag{D1}$$

$$\nabla\times\vec{E} = -\partial\vec{B}/\partial t, \tag{D2}$$

$$\nabla\cdot\vec{B} = 0, \tag{D3}$$

$$\nabla\times\vec{B} = \mu_o\vec{J}+\mu_o\varepsilon_o\partial\vec{E}/\partial t. \tag{D4}$$

Also given are the initial values of $\vec{E}$ and $\vec{B}$ subject to the constraints (D1) and (D3), and either the tangential component of $\vec{E}$ or the tangential component of $\vec{B}$ on the boundary S.

Taking the curl of each side of eq. (D2) and using eq. (D4), we obtain a differential equation for $\vec{E}$ alone,

$$(1/c^2)\partial^2\vec{E}/\partial t^2+\nabla\times(\nabla\times\vec{E}) = -\mu_o\partial\vec{j}/\partial t. \qquad \text{(D5)}$$

Thus, any solution of Maxwell's equation is a solution of eq. (D5). To solve eq. (D5), we must know the initial value of $\partial\vec{E}/\partial t$ in addition to that of $\vec{E}$; we find it by using eq. (D4) to determine

$$\partial\vec{E}(\vec{x},t_o)/\partial t = c^2\nabla\times\vec{B}(\vec{x},t_o)-\vec{j}(\vec{x},t_o)/\varepsilon_o \qquad \text{(D6)}$$

in terms of the initial values of $\vec{B}$ and $\vec{j}$, both assumed to be known. If the tangential component of $\vec{B}$ is given, eq. (D2) shows that we know the tangential component of $\nabla\times\vec{E}$. Furthermore, once the field $\vec{E}$ is known, the field $\vec{B}$ can be found from its initial value and its time derivative defined by eq. (D2). To show that the field $\vec{E}$ that satisfies eq. (D5) and the field $\vec{B}$ defined by eq. (D2) satisfy eq. (D4) not only initially by eq. (D6) but at all times, it is sufficient to show that the time derivative of the combination of these terms also vanishes. We have

$$(\partial/\partial t)(\partial\vec{E}/\partial t-c^2\nabla\times\vec{B}+\vec{j}/\varepsilon_o) = \partial^2\vec{E}/\partial t^2-c^2\nabla\times\partial\vec{B}/\partial t+(1/\varepsilon_o)\partial\vec{j}/\partial t$$

$$= c^2[(1/c^2)\partial^2\vec{E}/\partial t^2+\nabla\times(\nabla\times\vec{E})+\mu_o\partial\vec{j}/\partial t] = 0, \qquad \text{(D7)}$$

where we used eq. (D2), now part of the definition of $\vec{B}$, and the equation of motion (D5). The constraints (D1) and (D3) are then also satisfied for all times, as shown in Section 2. Equation (D5) is thus equivalent to the set of Maxwell's equations when eq. (D3) is used to define $\vec{B}$, and eq. (D6) is used to obtain the initial value of $\partial\vec{E}/\partial t$.

We demand that the dyadic Green function obey

$$(1/c^2)\partial^2\overleftrightarrow{G}(x,x')/\partial t^2+\nabla\times[\nabla\times\overleftrightarrow{G}(x,x')] = \delta^{(4)}(x-x')\overleftrightarrow{I}\ , \tag{D8}$$

where $\overleftrightarrow{I}$ is the unit 3×3 dyadic, and we further define retarded and advanced Green functions by setting

$$\overleftrightarrow{G}_R(x,x') = 0,\ t<t', \tag{D9}$$

$$\overleftrightarrow{G}_A(x,x') = 0,\ t>t'. \tag{D10}$$

We first demonstrate some symmetry properties of these Green functions. In the form (B63) of Green's theorem, we use $\vec{x}''$ and t'' as variables of integration and set

$$\vec{u}(x'') = \overleftrightarrow{G}_A(x'',x)\cdot\vec{a}, \tag{D11}$$

$$\vec{v}(x'') = \overleftrightarrow{G}_R(x'',x')\cdot\vec{b}, \tag{D12}$$

where $\vec{a}$ and $\vec{b}$ are two arbitrary constant vectors. We proceed as in Appendix

C, and find

$$\int_{-\infty}^{t_1} dt'' \int_V dV'' [\{\overleftrightarrow{G}_R(x'',x')\cdot\vec{b}\}\cdot\vec{a}\delta^{(4)}(x''-x) - \{\overleftrightarrow{G}_A(x'',x)\cdot\vec{a}\}\cdot\vec{b}\delta^{(4)}(x''-x')]$$

$$= \frac{1}{c^2}\int_V dV'' \left[\{\overleftrightarrow{G}_R(x'',x')\cdot\vec{b}\}\cdot\left\{\frac{\partial\overleftrightarrow{G}_A(x'',x)}{\partial t''}\right\}\cdot\vec{a} - \{\overleftrightarrow{G}_A(x'',x)\cdot\vec{a}\}\cdot\left\{\frac{\partial\overleftrightarrow{G}_R(x'',x')}{\partial t''}\cdot\vec{b}\right\}\right]_{t''=-\infty}^{t''=t_1}$$

$$+\int_{-\infty}^{t_1} dt'' \oint_S d\vec{S}''\cdot[\{\overleftrightarrow{G}_A(x'',x)\cdot\vec{a}\}\times\{\nabla''\times\overleftrightarrow{G}_R(x'',x')\}\cdot\vec{b}$$

$$-\{\overleftrightarrow{G}_R(x'',x')\cdot\vec{b}\}\times\{\nabla''\times\overleftrightarrow{G}_A(x'',x)\}\cdot\vec{a}]. \tag{D13}$$

The volume integrals vanish because the integrands vanish at both limits due to eqs. (D9) and (D10), since t_1 is greater than both t' and t". The vanishing of the surface integral depends on the boundary conditions imposed on the Green functions. Two of the possible choices are [6]

$$\hat{n}\cdot\overleftrightarrow{G}^{(1)}(x,x')\Big|_{\vec{x}\in S} = 0, \tag{D14}$$

$$\hat{n}\cdot\nabla\times\overleftrightarrow{G}^{(2)}(x,x')\Big|_{\vec{x}\in S} = 0, \tag{D15}$$

and in either case the surface integral vanishes. Thus, the left-hand side of eq. (D13) is also zero, that is,

$$\vec{a}\cdot\overleftrightarrow{G}_R(x,x')\cdot\vec{b} = \vec{b}\cdot\overleftrightarrow{G}_A(x',x)\cdot\vec{a}, \tag{D16}$$

and, since both $\vec{a}$ and $\vec{b}$ are arbitrary,

$$\overleftrightarrow{G}_R(x,x') = \tilde{\overleftrightarrow{G}}_A(x',x), \tag{D17}$$

where the tilde indicates the transpose of the dyadic $\overleftrightarrow{G}_A$. Thus, if $\overleftrightarrow{G}_A(x',x)$ satisfies eq. (D8) with respect to the variables x'_μ, we have

$$(1/c^2)\partial^2\tilde{\overleftrightarrow{G}}_R(x,x')/\partial t'^2+\nabla'\times[\nabla'\times\tilde{\overleftrightarrow{G}}_R(x,x')] = \delta^{(4)}(x-x')\overleftrightarrow{I}. \tag{D18}$$

Similarly, we set

$$\vec{u}(x'') = \overleftrightarrow{G}_R(\vec{x}'',-t'';\vec{x},-t)\cdot\vec{a} \tag{D19}$$

instead of the field given in eq. (D11), and we show the reciprocity relation

$$\overleftrightarrow{G}_R(\vec{x},t;\vec{x}',t') = \tilde{\overleftrightarrow{G}}_R(\vec{x}',-t';\vec{x},-t). \tag{D20}$$

This relation and eqs. (D14) and (D15) then imply that, when the source point is on the surface,

$$\hat{n}'\cdot\tilde{\overleftrightarrow{G}}^{(1)}(x,x')\Big|_{\vec{x}'\epsilon S} = 0, \tag{D21}$$

$$\hat{n}'\cdot\nabla'\times\tilde{\overleftrightarrow{G}}^{(2)}(x,x')\Big|_{\vec{x}'\epsilon S} = 0. \tag{D22}$$

We now use the x'_μ as variables of integration in eq. (B63), and set

$$\vec{u}(x') = \vec{E}(x'), \tag{D23}$$

$$\vec{v}(x') = \tilde{\overleftrightarrow{G}}_R(x,x')\cdot\vec{a}, \tag{D24}$$

and, since $\vec{E}$ and $\overleftrightarrow{G}_R$ satisfy eqs. (D5) and (D18) respectively, we obtain

$$\begin{aligned}&-\mu_o\int_{t_o}^{t} dt'\int_V dV'\,\partial\vec{j}(x')/\partial t'\cdot\tilde{\overleftrightarrow{G}}_R(x,x')\cdot\vec{a} - \vec{a}\cdot\vec{E}(x)\\ &= -(1/c^2)\int_V dV'[\{\tilde{\overleftrightarrow{G}}_R(x,x')\cdot\vec{a}\}\cdot\partial\vec{E}(x')/\partial t-\vec{E}(x')\cdot\partial\tilde{\overleftrightarrow{G}}_R(x,x')/\partial t'\cdot\vec{a}]_{t'=t_o}\\ &+\int_{t_o}^{t} dt'\oint_S d\vec{S}'\cdot[\vec{E}(x')\times\{\nabla'\times\tilde{\overleftrightarrow{G}}_R(x,x')\cdot\vec{a}\}-\{\tilde{\overleftrightarrow{G}}_R(x,x')\cdot\vec{a}\}\times\{\nabla'\times\vec{E}(x')\}].\end{aligned} \tag{D25}$$

The initial value of $\partial\vec{E}/\partial t$ is given by eq. (D6), and we have assumed that eq. (D2) holds for all t as part of the definition of $\vec{B}$; hence, eliminating the arbitrary vector $\vec{a}$,

$$\begin{aligned}\vec{E}(x) &= -\mu_o\int_{t_o}^{t} dt'\int_V dV'\overleftrightarrow{G}_R(x,x')\cdot\partial\vec{j}(x')/\partial t'\\ &+(1/c^2)\int_V dV'[\overleftrightarrow{G}_R(x,x')\cdot\{c^2\nabla'\times\vec{B}(x')-\vec{j}(x')/\varepsilon_o\}-\partial\overleftrightarrow{G}_R(x,x')/\partial t'\cdot\vec{E}(x')]_{t'=t_o}\\ &-\int_{t_o}^{t} dt'\oint_S dS'\hat{n}'\cdot[\vec{E}(x')\times\{\nabla'\times\tilde{\overleftrightarrow{G}}_R(x,x')\}-\partial\vec{B}(x')/\partial t'\times\tilde{\overleftrightarrow{G}}_R(x,x')].\end{aligned} \tag{D26}$$

When the tangential component of $\vec{E}$ is given, we choose a Green function $\overleftrightarrow{G}_R^{(1)}$; in that case, the first term in the surface integral is known and eq. (D21) makes the second term vanish. If, on the other hand, the tangential component of $\vec{B}$ is given, we can compute $\partial\vec{B}/\partial t$ on S and a choice of $\overleftrightarrow{G}_R^{(2)}$ allows us to express the surface integral in terms of known functions. Thus, eq. (D26) represents a solution of Maxwell's equations in terms of integrals over known functions if the Green functions are available; in practice, these Green functions are seldom known.

We now derive an expression for the dyadic Green function for free space, $\overleftrightarrow{G}_R^{(o)}(x,x')$. We have translational invariance, so that $\overleftrightarrow{G}_R^{(o)}$ is a function of x-x' alone. We proceed as in Appendix C, where the scalar Green function was derived, and we take a four-dimensional Fourier transform. We write

$$\overleftrightarrow{G}_R(\vec{x},t) = (2\pi)^{-2}\int d^3k d\omega e^{-i(\omega t-\vec{k}\cdot\vec{x})}\overleftrightarrow{Q}(\vec{k},\omega), \tag{D27}$$

and eq. (D8) is transformed into

$$-(\omega^2/c^2)\overleftrightarrow{Q}-\vec{k}\times(\vec{k}\times\overleftrightarrow{Q}) = (2\pi)^{-2}\overleftrightarrow{I}. \tag{D28}$$

To solve for $\overleftrightarrow{Q}$, we first multiply from the left by $\vec{k}$ to obtain

$$-(\omega^2/c^2)\vec{k}\cdot\overleftrightarrow{Q} = (2\pi)^{-2}\vec{k}, \tag{D29}$$

and, expanding the triple vector product in (D28) and substituting for $\vec{k}\cdot\overleftrightarrow{Q}$ from eq. (D29) we find

$$-(\omega^2/c^2)\overleftrightarrow{Q}+(c^2/4\pi^2\omega^2)\vec{k}\vec{k}+\vec{k}^2\overleftrightarrow{Q} = (1/4\pi^2)\overleftrightarrow{I}, \tag{D30}$$

$$\overleftrightarrow{Q} = -\frac{1}{4\pi^2}\frac{\overleftrightarrow{I}-(c^2/\omega^2)\vec{k}\vec{k}}{\omega^2/c^2-\vec{k}^2} \tag{D31}$$

We perform first the integration over ω, and we notice that $\overleftrightarrow{Q}$ has simple poles at $\pm|\vec{k}|c$ and a double pole at the origin. To obtain a retarded Green function we deform the contour in the complex ω-plane to pass above the poles. Then, for $t>0$, we close the contour around the lower half-plane and find the value of the integral from the residues at the poles. We find that

$$\int_{C_R} d\omega e^{-i\omega t}\overleftrightarrow{Q}(\vec{k},\omega) = -2\pi i\left[\frac{c}{8\pi^2}\frac{\overleftrightarrow{I}-\vec{k}\vec{k}/k^2}{k}\left(e^{ikct}-e^{-ikct}\right)+\frac{itc^2\vec{k}\vec{k}}{4\pi^2k^2}\right], \tag{D32}$$

and

$$\overleftrightarrow{G}_R(\vec{x},t) = \frac{c}{(2\pi)^3}\int d^3k e^{i\vec{k}\cdot\vec{x}}\left[\left(\overleftrightarrow{I}-\frac{\vec{k}\vec{k}}{k^2}\right)\frac{\sin(kct)}{k}+\frac{ct\vec{k}\vec{k}}{k^2}\right], \tag{D33}$$

$$\overleftrightarrow{G}_R(\vec{x},t) = \frac{c}{(2\pi)^3}\left[\overleftrightarrow{I}\int d^3k\frac{\sin(kct)}{k}e^{i\vec{k}\cdot\vec{x}}+\nabla\nabla\int d^3k\frac{\sin(kct)}{k^3}e^{i\vec{k}\cdot\vec{x}}-ct\nabla\nabla\int d^3k\frac{e^{i\vec{k}\cdot\vec{x}}}{k^2}\right]. \tag{D34}$$

Although the integrands of the last two terms are singular at the origin, we perform the angular integrations in spherical coordinates and use eq. (C28) to obtain

$$\overleftrightarrow{G}_R(\vec{x},t) = \frac{c}{2\pi^2}\left[\frac{1}{r}\overleftrightarrow{I}\int_0^\infty dk\ \sin(kr)\sin(kct)\right.$$

$$\left.+\nabla\nabla\left\{\frac{1}{r}\int_0^\infty dk\frac{\sin(kr)\sin(kct)}{k^2} - \frac{ct}{r}\int_0^\infty dk\frac{\sin(kr)}{k}\right\}\right], \tag{D35}$$

where all integrands are even functions of k and finite at the origin. We extend the ranges of integration to $-\infty$ and deform the contour to pass below the origin before separating the sine functions into exponentials. We then integrate by finding the appropriate residues, and write

$$(1/2\pi)\int_C dke^{i\alpha k} = \delta(\alpha), \tag{D36}$$

$$(1/2\pi)\int_C dke^{i\alpha k}/k = i\theta(\alpha), \tag{D37}$$

$$(1/2\pi)\int_C dke^{i\alpha k}/k^2 = -\alpha\theta(\alpha), \tag{D38}$$

where $\theta(\alpha)$ is the unit step function and C is the deformed contour. We could equally well have chosen the contour to pass above the origin; in that case, the results in eqs. (D37) and (D38) would have been different, but the final result is not changed. We have

$$\overleftrightarrow{G}_R(\vec{x},t) = (c/4\pi r)\overleftrightarrow{I}\{-\delta(ct+r)+\delta(ct-r)\}+(c/2\pi)\nabla\nabla[(1/4r)\{(ct+r)\theta(ct+r)$$

$$-(ct-r)\theta(ct-r)-(-ct+r)\theta(-ct+r)+(-ct-r)\theta(-ct-r)\}-(ct/2r)\{\theta(r)-\theta(-r)\}], \tag{D39}$$

and, remembering that r and ct are non-negative, this expression reduces to

$$\overleftrightarrow{G}_R(\vec{x},t) = \frac{c}{4\pi}\left[\overleftrightarrow{I}\frac{\delta(ct-r)}{r} - \nabla\nabla\frac{(ct-r)\theta(ct-r)}{r}\right]. \tag{D40}$$

The gradients of well-behaved scalar functions of $|\vec{x}|$ are

$$\nabla f(r) = \frac{f'(r)}{r}\vec{x}, \tag{D41}$$

$$\nabla\nabla f(r) = \frac{rf''(r)-f'(r)}{r^3}\vec{x}\vec{x} + \frac{f'(r)}{r}\overleftrightarrow{I}. \tag{D42}$$

Thus far we have dealt with singular functions in a heuristic manner, avoiding the mathematical complications of the theory of distributions. The well-known result

$$\nabla^2(1/r) = -4\pi\delta^{(3)}(\vec{x}) \tag{D43}$$

suggests that special care has to be exercised in evaluating contributions at the origin. One way of finding these contributions consists of evaluating the integral over a vanishingly small sphere of radius ε about the origin using the divergence theorem or related equations such as (B20). We use

$$\int_V dV\nabla f(r) = \oint_S d\vec{S}f(r), \tag{D44}$$

$$\int_V dV\nabla\nabla f(r) = \oint_S d\vec{S}\nabla f(r), \tag{D45}$$

and find, using eq. (D41) for the second relation,

$$\int_V dV \nabla f(r) = \varepsilon^2 f(\varepsilon) \oint d\Omega \hat{n} = 0, \tag{D46}$$

$$\int_V dV \nabla\nabla f(r) = \varepsilon^2 f'(\varepsilon) \oint d\Omega \hat{n}\hat{n} = (4/3)\pi\varepsilon^2 f'(\varepsilon) \overleftrightarrow{I}. \tag{D47}$$

Thus, at least for functions that do not diverge faster than 1/r, eq. (D41) remains unchanged, while eq. (D42) becomes

$$\nabla\nabla f(r) = \frac{rf''(r)-f'(r)}{r^3}\vec{x}\vec{x} + \frac{f'(r)}{r}\overleftrightarrow{I} + \frac{4}{3}\pi \lim_{\varepsilon\to o}\{\varepsilon^2 f'(\varepsilon)\}\delta^{(3)}(\vec{x})\overleftrightarrow{I}. \tag{D48}$$

In particular,

$$\nabla\nabla\frac{1}{r} = \frac{3\vec{x}\vec{x}-r^2\overleftrightarrow{I}}{r^5} - \frac{4\pi}{3}\delta^{(3)}(\vec{x})\overleftrightarrow{I}, \tag{D49}$$

and the scalar invariants of these expressions reduce the equation to (D43). We now use

$$d\theta(\alpha)/d\alpha = \delta(\alpha), \tag{D50}$$

$$\alpha\delta(\alpha) = 0, \tag{D51}$$

$$f(\alpha)\delta(\alpha-\alpha') = f(\alpha')\delta(\alpha-\alpha'), \tag{D52}$$

to derive from

$$f(r) = (ct-r)r^{-1}\theta(ct-r), \tag{D53}$$

$$f'(r) = -ctr^{-2}\theta(ct-r), \tag{D54}$$

$$f''(r) = r^{-1}\delta(ct-r)+2ctr^{-3}\theta(ct-r), \tag{D55}$$

$$\nabla\nabla f(r) = [r^{-3}\delta(ct-r)+3ctr^{-5}\theta(ct-r)]\vec{x}\vec{x}-[ctr^{-3}\theta(ct-r)+(4/3)\pi ct\delta^{(3)}(\vec{x})]\overleftrightarrow{I}, \tag{D56}$$

$$\begin{aligned}\overleftrightarrow{G}_R(\vec{x},t) = (1/4\pi)[&\{r^{-1}\delta(t-r/c)+c^2tr^{-3}\theta(t-r/c)+(4/3)\pi c^2t\delta^{(3)}(\vec{x})\}\overleftrightarrow{I}\\ &-\{r^{-3}\delta(t-r/c)+3c^2tr^{-5}\theta(t-r/c)\}\vec{x}\vec{x}].\end{aligned} \tag{D57}$$

We note that $\overleftrightarrow{G}_R$ vanishes for $t<r/c$, as expected from the special theory of relativity. Finally, the free-space retarded Green function is

$$\begin{aligned}\overleftrightarrow{G}_R^{(o)}(x,x') = (1/4\pi)[&\{R^{-1}\delta(t-t'-R/c)+c^2(t-t')R^{-3}\theta(t-t'-R/c)\\ &+(4/3)\pi c^2(t-t')\theta(t-t')\delta^{(3)}(\vec{R})\}\overleftrightarrow{I}\\ &-\{R^{-3}\delta(t-t'-R/c)+3c^2(t-t')R^{-5}\theta(t-t'-R/c\}\vec{R}\vec{R}].\end{aligned} \tag{D58}$$

where

$$\vec{R} = \vec{x}-\vec{x}'. \tag{D59}$$

NBS-114A (REV. 2-80)

U.S. DEPT. OF COMM. BIBLIOGRAPHIC DATA SHEET (See instructions)	1. PUBLICATION OR REPORT NO. NBS TN 1157	2. Performing Organ. Report No.	3. Publication Date February 1982

4. TITLE AND SUBTITLE

Integral Equations for Transient Electromagnetic Fields

5. AUTHOR(S)

Egon Marx

6. PERFORMING ORGANIZATION (If joint or other than NBS, see instructions)

NATIONAL BUREAU OF STANDARDS
DEPARTMENT OF COMMERCE
WASHINGTON, D C 20234

7. Contract/Grant No.

8. Type of Report & Period Covered

Final

9. SPONSORING ORGANIZATION NAME AND COMPLETE ADDRESS (Street, City, State, ZIP)

Same as above.

10. SUPPLEMENTARY NOTES

☐ Document describes a computer program; SF-185, FIPS Software Summary, is attached.

11. ABSTRACT (A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here)

Integral equations for the electric and magnetic fields in free space are derived from Maxwell's equations. The fields are expressed in terms of their initial values, boundary values, and sources with the help of a retarded Green function for the scalar wave equation. These equations are then used to derive integral equations for the surface charge and current densities induced by the scattering of a transient electromagnetic field by perfect conductors. An alternative solution of Maxwell's equations with the help of dyadic Green functions is also presented.

12. KEY WORDS (Six to twelve entries; alphabetical order; capitalize only proper names; and separate key words by semicolons)

dyadic Green functions; electromagnetic scattering; integral equations; perfect conductors; transient electromagnetic fields; wave equations.

13. AVAILABILITY

☒ Unlimited

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☐ Order From Superintendent of Documents, U.S. Government Printing Office, Washington, D C 20402.

☒ Order From National Technical Information Service (NTIS), Springfield, VA 22161

14. NO. OF PRINTED PAGES

68

15. Price

USCOMM-DC 6043-P80

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NOTE: The Journal was formerly published in two sections: Section A "Physics and Chemistry" and Section B "Mathematical Sciences."

DIMENSIONS/NBS—This monthly magazine is published to inform scientists, engineers, business and industry leaders, teachers, students, and consumers of the latest advances in science and technology, with primary emphasis on work at NBS. The magazine highlights and reviews such issues as energy research, fire protection, building technology, metric conversion, pollution abatement, health and safety, and consumer product performance. In addition, it reports the results of Bureau programs in measurement standards and techniques, properties of matter and materials, engineering standards and services, instrumentation, and automatic data processing. Annual subscription: domestic $11; foreign $13.75.

NONPERIODICALS

Monographs—Major contributions to the technical literature on various subjects related to the Bureau's scientific and technical activities.

Handbooks—Recommended codes of engineering and industrial practice (including safety codes) developed in cooperation with interested industries, professional organizations, and regulatory bodies.

Special Publications—Include proceedings of conferences sponsored by NBS, NBS annual reports, and other special publications appropriate to this grouping such as wall charts, pocket cards, and bibliographies.

Applied Mathematics Series—Mathematical tables, manuals, and studies of special interest to physicists, engineers, chemists, biologists, mathematicians, computer programmers, and others engaged in scientific and technical work.

National Standard Reference Data Series—Provides quantitative data on the physical and chemical properties of materials, compiled from the world's literature and critically evaluated. Developed under a worldwide program coordinated by NBS under the authority of the National Standard Data Act (Public Law 90-396).

NOTE: The principal publication outlet for the foregoing data is the Journal of Physical and Chemical Reference Data (JPCRD) published quarterly for NBS by the American Chemical Society (ACS) and the American Institute of Physics (AIP). Subscriptions, reprints, and supplements available from ACS, 1155 Sixteenth St., NW, Washington, DC 20056.

Building Science Series—Disseminates technical information developed at the Bureau on building materials, components, systems, and whole structures. The series presents research results, test methods, and performance criteria related to the structural and environmental functions and the durability and safety characteristics of building elements and systems.

Technical Notes—Studies or reports which are complete in themselves but restrictive in their treatment of a subject. Analogous to monographs but not so comprehensive in scope or definitive in treatment of the subject area. Often serve as a vehicle for final reports of work performed at NBS under the sponsorship of other government agencies.

Voluntary Product Standards—Developed under procedures published by the Department of Commerce in Part 10, Title 15, of the Code of Federal Regulations. The standards establish nationally recognized requirements for products, and provide all concerned interests with a basis for common understanding of the characteristics of the products. NBS administers this program as a supplement to the activities of the private sector standardizing organizations.

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NBS Interagency Reports (NBSIR)—A special series of interim or final reports on work performed by NBS for outside sponsors (both government and non-government). In general, initial distribution is handled by the sponsor; public distribution is by the National Technical Information Services, Springfield, VA 22161, in paper copy or microfiche form.

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