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Boulder Laboratories

MODE CALCULATIONS FOR VLF PROPAGATION IN THE EARTH-IONOSPHERE WAVEGUIDE

BY
KENNETH P. SPIES AND JAMES R. WAIT

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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July 17, 1961

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by

Kenneth P. Spies and James R. Wait

I. Introduction

The concept that radio waves are channeled between the earth and the ionosphere as in a waveguide has been very useful at VLF [Budden, 1953; Alpert, 1956; Wait, 1957]. Unfortunately, the computational aspects of the problem are quite complicated even when the model is highly idealized. The difficulty stems from the grazing nature of the modes of lowest attenuation. Some progress has been made recently by utilizing higher order approximations for the various spherical wave functions which enter into the problem. In this way the influence of earth curvature has been fully accounted for. The detailed theoretical aspects and essential derivations have been presented elsewhere [Wait, 1960, 1961]. Here the actual computational procedure is outlined and some numerical results are presented. It is believed that the methods used are of general interest and also have possible application to propagation of acoustic and seismic waves in curved layered media.

An essential feature of the techniques used is a simplified representation of the ionospheric reflection coefficient which is valid for highly oblique incidence. This permits a perturbation procedure to be applied to the equation for the mode characteristics. In the examples employed here the ionosphere is represented by a sharply bounded and homogeneous ionized medium with a superimposed magnetic field. Initially the quasi-longitudinal approximation is invoked since it permits a great simplification in the analysis. Its validity has been discussed in detail elsewhere [Budden, 1961]. Generally speaking, it is good when the propagation is in the magnetic meridian or in polar regions for all directions of propagation. These results are then introduced into the mode equation which involves the reflection coefficients ${}_{\parallel}R_{\parallel}$ and ${}_{\perp}R_{\perp}$ and the conversion coefficients ${}_{\parallel}R_{\perp}$ and ${}_{\perp}R_{\parallel}$ evaluated for a complex angle of incidence. Here the earth is regarded as flat in order to simplify the calculation. In the following section the earth's magnetic field is taken to be horizontal and transverse to the direction of propagation. This would correspond to propagation along the magnetic equator. Then the case of an arbitrarily dipping magnetic field is treated in a relatively crude fashion. Most of the calculations mentioned above are then carried out for a curved earth where the modal equation is a great deal more complicated. Nevertheless, by making certain convergent expansions of the Airy functions, the problem becomes tractable and can be treated by perturbation methods.

For the convenience of those engaged in related investigations, a table of inverse tangents of the ratio of two Airy functions is given in Appendix A. A fairly extensive collection of formulas relating to Airy functions is then given in Appendix B. Finally, for sake of completeness, a short note on the concept of conductivity of an ionized gas is contained in Appendix C.

II. Calculations for a Flat Earth-Ionosphere Waveguide

1. Ionospheric Reflection Coefficients in the Quasi-longitudinal (Q-L) Approximation. - In the quasi-longitudinal (Q-L) approximation, ionospheric reflection and conversion coefficients for a sharply bounded homogeneous ionosphere have been given by Budden [1961] in the following form

$$ {}_{\parallel}R_{\parallel} = \frac{(\mu_o+\mu_e)(C^2 - C_o C_e) + (\mu_o \mu_e - 1)(C_o + C_e) C}{(\mu_o+\mu_e)(C^2 + C_o C_e) + (\mu_o \mu_e + 1)(C_o + C_e) C} , \quad (1) $$

$$ {}_{\perp}R_{\perp} = \frac{(\mu_o+\mu_e)(C^2 - C_o C_e) - (\mu_o \mu_e - 1)(C_o + C_e) C}{(\mu_o+\mu_e)(C^2 + C_o C_e) + (\mu_o \mu_e + 1)(C_o + C_e) C} , \quad (2) $$

$$ {}_{\parallel}R_{\perp} = \frac{i 2 C (\mu_o C_o - \mu_e C_e)}{(\mu_o+\mu_e)(C^2 + C_o C_e) + (\mu_o \mu_e + 1)(C_o + C_e) C} , \quad (3) $$

and

$$ {}_{\perp}R_{\parallel} = \frac{i 2 C (\mu_o C_e - \mu_e C_o)}{(\mu_o+\mu_e)(C^2 + C_o C_e) + (\mu_o \mu_e + 1)(C_o + C_e) C} , \quad (4) $$

where C is the cosine of the (complex) angle of incidence

and where

$$ 1 - C^2 = \mu_o^2 (1 - C_o^2) $$
$$ 1 - C^2 = \mu_e^2 (1 - C_e^2) $$

(Snell's Law).

The refractive indices μ_o and μ_e of the ionosphere are defined by the relations

$$ \mu_o^2 = 1 - \frac{i}{L} e^{i \phi_L} \quad \text{and} \quad \mu_e^2 = 1 - \frac{i}{L} e^{-i \phi_L} $$

where $\tan \phi_L = \frac{\omega_L}{\nu} = \frac{\text{longitudinal gyro frequency}}{\text{collisional frequency}}$,

where

$$L = \frac{\omega}{\omega_r} = \frac{\omega(\nu^2 + \omega_L^2)^{\frac{1}{2}}}{\omega_o^2} ,$$

and where ω is the angular frequency and ω_r is an effective conductivity [Wait, 1960] which is defined explicitly above. Note that ${}_{\parallel}R_{\parallel}$ and ${}_{\perp}R_{\perp}$ both approach -1 as C approaches 0 (grazing incidence), whereas ${}_{\parallel}R_{\perp}$ and ${}_{\perp}R_{\parallel}$ both approach 0.

To facilitate solving the mode equation for VLF propagation in a flat earth-ionosphere waveguide, it is convenient to represent the rather complicated reflection coefficients ${}_{\parallel}R_{\parallel}$ and ${}_{\perp}R_{\perp}$ by expressions having the form

$$R = -\exp\left[\alpha_1 C + \alpha_2 C^2 + \alpha_3 C^3 + \ldots\right] \tag{5}$$

where the (complex) parameters α_1, α_2, α_3, ... depend on ionospheric properties, but not on C. The validity of such a procedure depends on the fact that $|C|$ is small for the most important modes at VLF, so that R is near -1.

There is, of course, no unique way of choosing α_1, α_2, α_3, ... The method finally adopted is the following. In

$${}_{\parallel}R_{\parallel} = e^{\log {}_{\parallel}R_{\parallel}} \quad \text{and} \quad {}_{\perp}R_{\perp} = e^{\log {}_{\perp}R_{\perp}} ,$$

the logarithms are expanded in a Taylor series about C = 0 to obtain

$${}_{\parallel}R_{\parallel} = \exp\left[\sum_{k=0}^{\infty} \alpha_k C^k\right] \quad \text{and} \quad {}_{\perp}R_{\perp} = \exp\left[\sum_{k=0}^{\infty} \beta_k C^k\right] \tag{6a)(6b}$$

where

$$\alpha_k = \frac{1}{k!}\left[\frac{\partial^k}{\partial C^k}(\log {}_{\parallel}R_{\parallel})\right]_{C=0} \quad \text{and} \quad \beta_k = \frac{1}{k!}\left[\frac{\partial^k}{\partial C^k}(\log {}_{\perp}R_{\perp})\right]_{C=0} .$$

(7a) (7b)

The first few Taylor series coefficients for log ${}_{\|}R_{\|}$ are

$$a_o = i\pi \quad , \tag{8a}$$

$$a_1 = -\frac{2\mu_o\mu_e}{\mu_o+\mu_e}\left[\frac{\mu_e}{\sqrt{\mu_e^2-1}} + \frac{\mu_o}{\sqrt{\mu_o^2-1}}\right] \quad , \tag{8b}$$

$$a_2 = \frac{a_1^2}{2\mu_o\mu_e} - \frac{2\mu_o\mu_e}{\sqrt{\mu_o^2-1}\sqrt{\mu_e^2-1}} \quad , \tag{8c}$$

$$a_3 = \frac{a_1}{24}\left[a_1^2 + \frac{6a_2}{\mu_o\mu_e} - \frac{6}{\mu_e^2-1}\right] + \frac{1}{2}\frac{\mu_o\mu_e}{\mu_o+\mu_e}\frac{\mu_o}{\sqrt{\mu_o^2-1}}\left[\frac{1}{\mu_o^2-1} - \frac{1}{\mu_e^2-1}\right] , \tag{8d}$$

and those for log ${}_{\perp}R_{\perp}$ are

$$\beta_o = i\pi \quad , \tag{9a}$$

$$\beta_1 = \frac{a_1}{\mu_o\mu_e} \quad , \tag{9b}$$

$$\beta_2 = a_2 \quad , \tag{9c}$$

$$\beta_3 = \frac{\beta_1}{24}(\mu_o^2\mu_e^2 - 1)\left(\frac{6\beta_2}{\mu_o\mu_e} - \beta_1^2\right) + \frac{a_3}{\mu_o\mu_e} \quad . \tag{9d}$$

The ionospheric reflection coefficients ${}_{\|}R_{\|}$ and ${}_{\perp}R_{\perp}$ can now be approximated by using a few terms of the series (6a) and (6b), respectively.

2. An Approximate Solution of the Mode Equation for Perfectly Conducting Ground - When the ground is perfectly conducting, the mode equation for VLF propagation in a flat earth-ionosphere waveguide is given by [Wait, 1960]

$$(e^{i\,2khC} - {}_{\parallel}R_{\parallel})\,(e^{i\,2khC} + {}_{\perp}R_{\perp}) + {}_{\parallel}R_{\perp}\,{}_{\perp}R_{\parallel} = 0. \tag{10}$$

As a first approximation, coupling between the modes is neglected and thus ${}_{\parallel}R_{\perp} = {}_{\perp}R_{\parallel} = 0$. The mode equation (10) separates into the two equations

$$e^{i\,2khC} - {}_{\parallel}R_{\parallel} = 0 \quad \text{(for TM modes)} \tag{11}$$

$$e^{i\,2khC} + {}_{\perp}R_{\perp} = 0 \quad \text{(for TE modes)} \tag{12}$$

If the ionospheric reflection coefficient ${}_{\parallel}R_{\parallel}$ is approximated by using the first two terms of the Taylor series (6a) for $\log {}_{\parallel}R_{\parallel}$, one has

$${}_{\parallel}R_{\parallel} = -\,e^{a_1 C}$$

where a_1 is given by (8b). With this approximation, the solutions of the TM mode equation (11) are

$$C_n = \frac{(2n-1)\pi}{2kh + i\,a_1} \quad (n = 1,\ 2,\ 3,\ \ldots),\; S_n = \sqrt{1 - C_n^2}\,. \tag{13}$$

Curves of $1/\mathrm{Re}(S_n)$ and $-H\,\mathrm{Im}(S_n)$ vs. $H = \frac{kh}{2\pi}$ have been computed when n = 1 for

$B = \frac{1}{H}\left(\frac{\omega}{\omega_r}\right) = 0.02,\ 0.05,\ 0.10,\ 0.20$ and $\phi_L = 0^\circ, 10^\circ, 20^\circ, 30^\circ, 40^\circ, 50^\circ, 60^\circ$. These results are shown in figures 1 through 11. (Only curves for $\phi_L = 0^\circ$ and $\phi_L = 60^\circ$ have been plotted.)

Likewise, if the ionospheric reflection coefficient ${}_{\perp}R_{\perp}$ is approximated by using the first two terms of the Taylor series (6b) for log ${}_{\perp}R_{\perp}$, one has

$$ {}_{\perp}R_{\perp} = - e^{\beta_I C} $$

where β_I is given by (9b). With this approximation, the solutions of the TE mode equation (12) are

$$ C_m \doteq \frac{2m\pi}{2kh + i\beta_I} \quad (m = 1, 2, 3, \ldots) \quad . \tag{14} $$

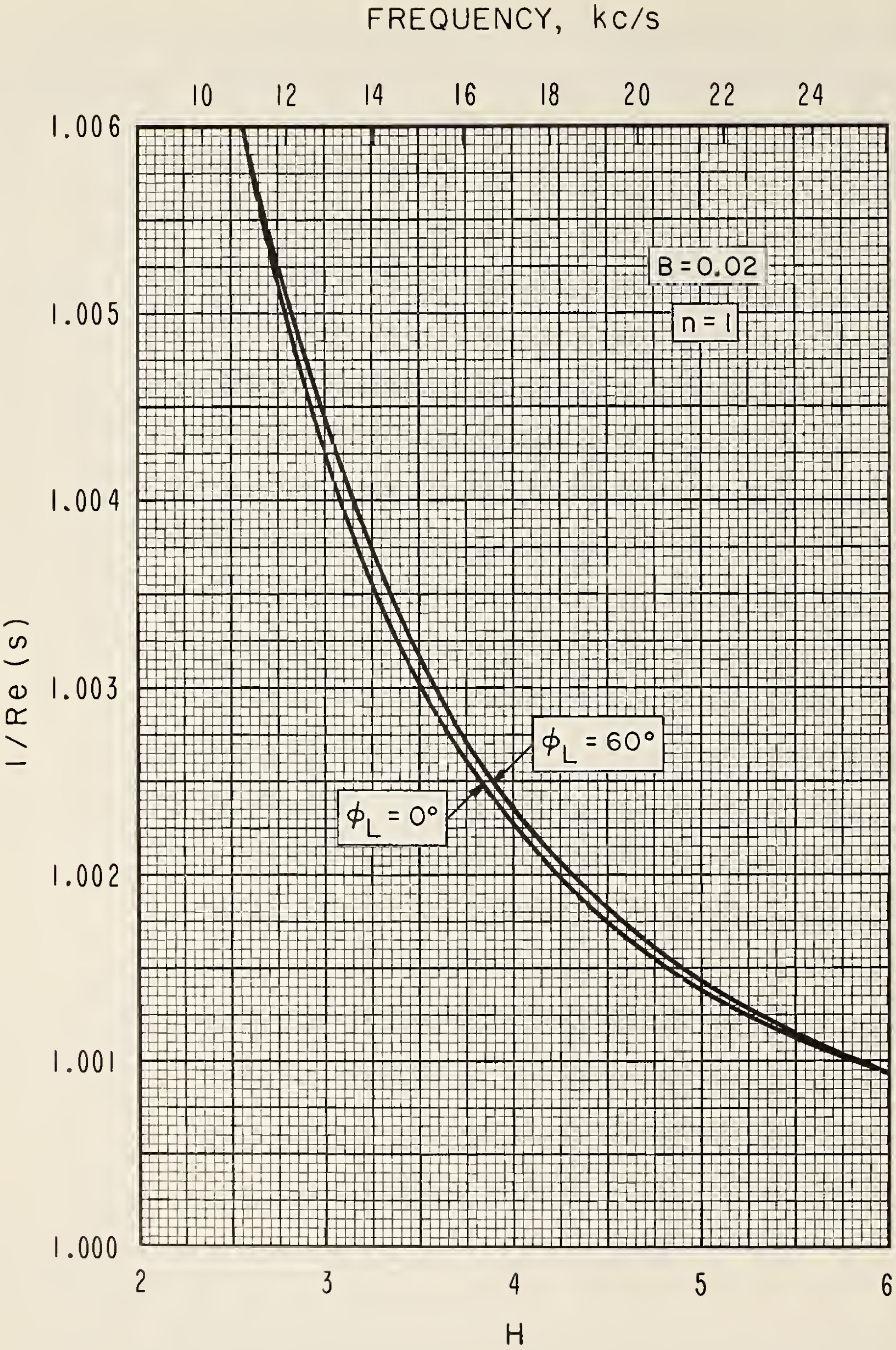
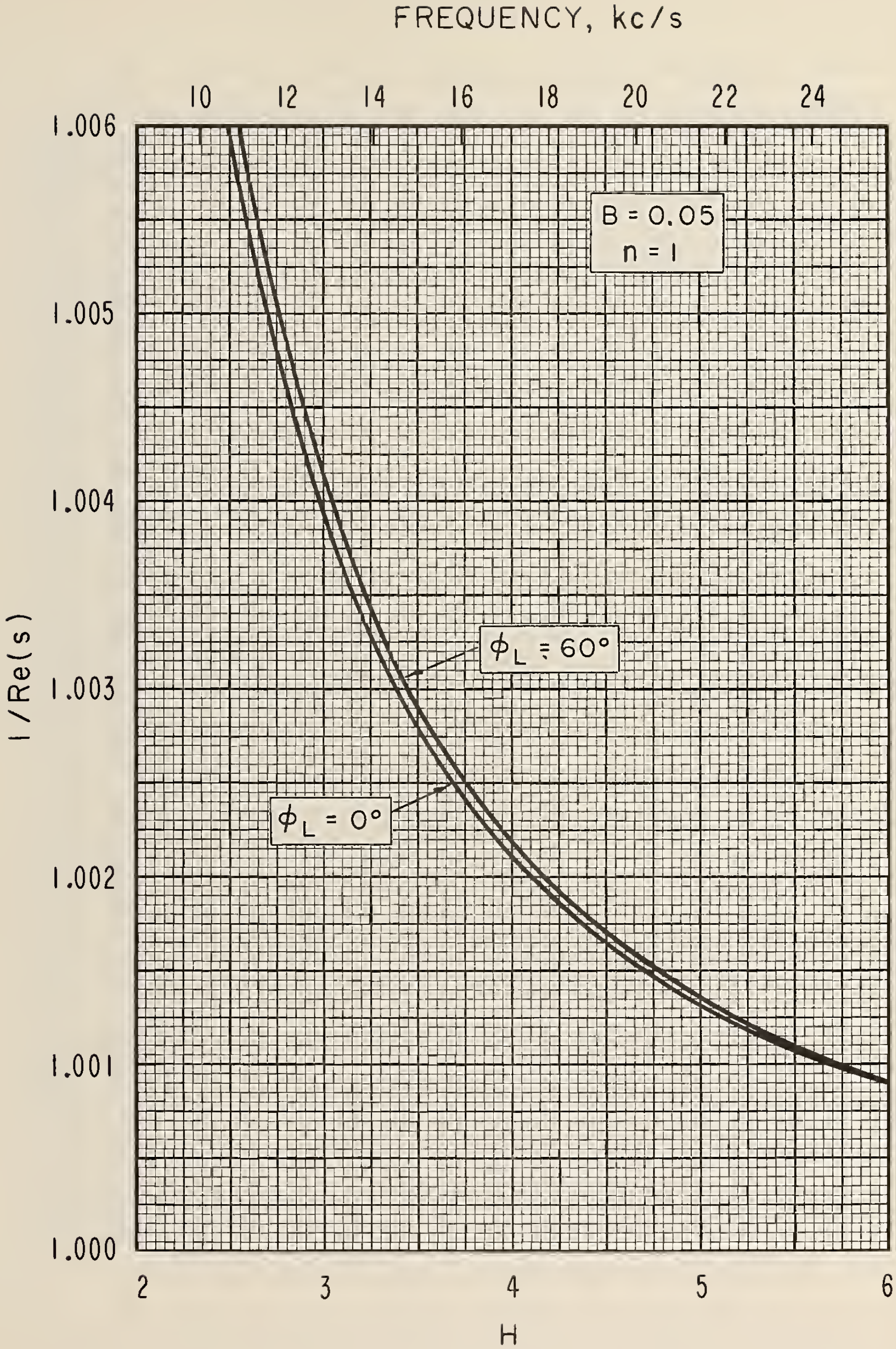

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
1.006
1.005
1.004
1.003
1.002
1.001
1.000
1/Re (s)
B = 0.02
n = 1
ϕ_L = 60°
ϕ_L = 0°
2
3
4
5
6
H

Fig. 1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12
14
16
18
20
22
24
B = 0.05
n = 1
1/Re(s)
1.006
1.005
1.004
1.003
1.002
1.001
1.000
ϕ_L = 60°
ϕ_L = 0°
2
3
4
5
6
H

Fig. 2

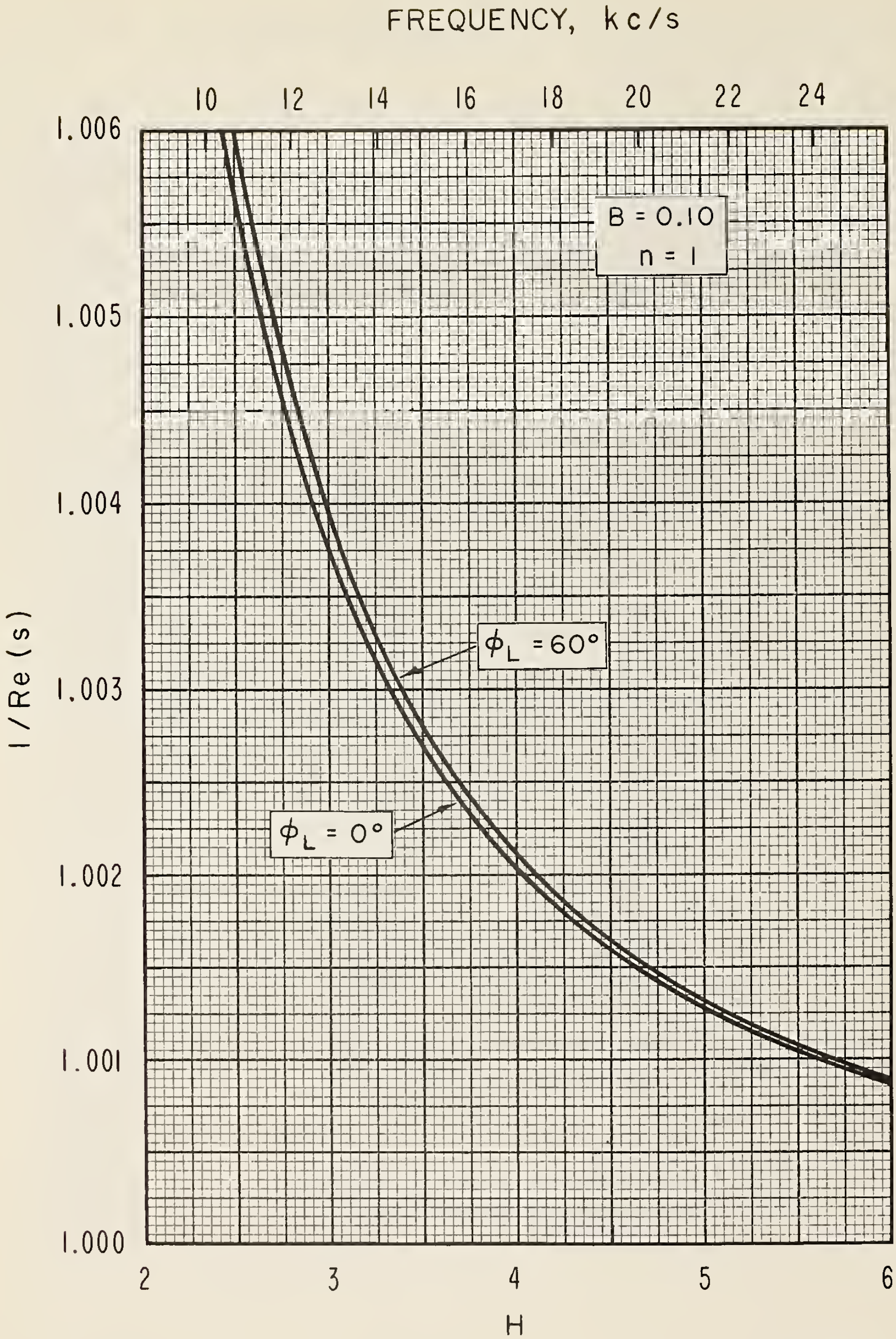

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
1.006
1.005
1.004
1.003
1.002
1.001
1.000
1/Re (s)
B = 0.10
n = 1
ϕ_L = 60°
ϕ_L = 0°
2
3
4
5
6
H

Fig. 3

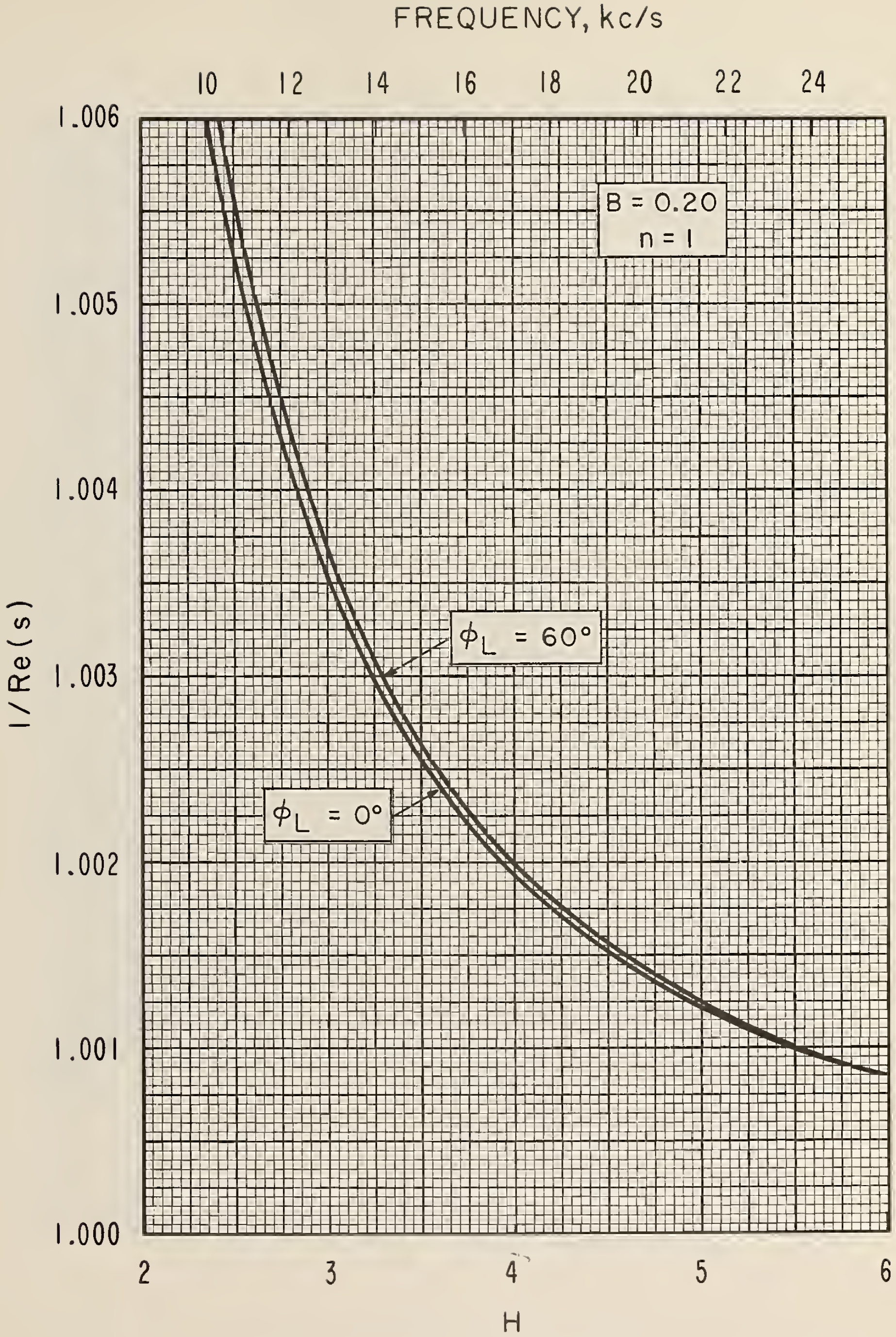

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
1.006
1.005
1.004
1.003
1.002
1.001
1.000
1/Re(s)
B = 0.20
n = 1
ϕ_L = 60°
ϕ_L = 0°
2
3
4
5
6
H

Fig. 4

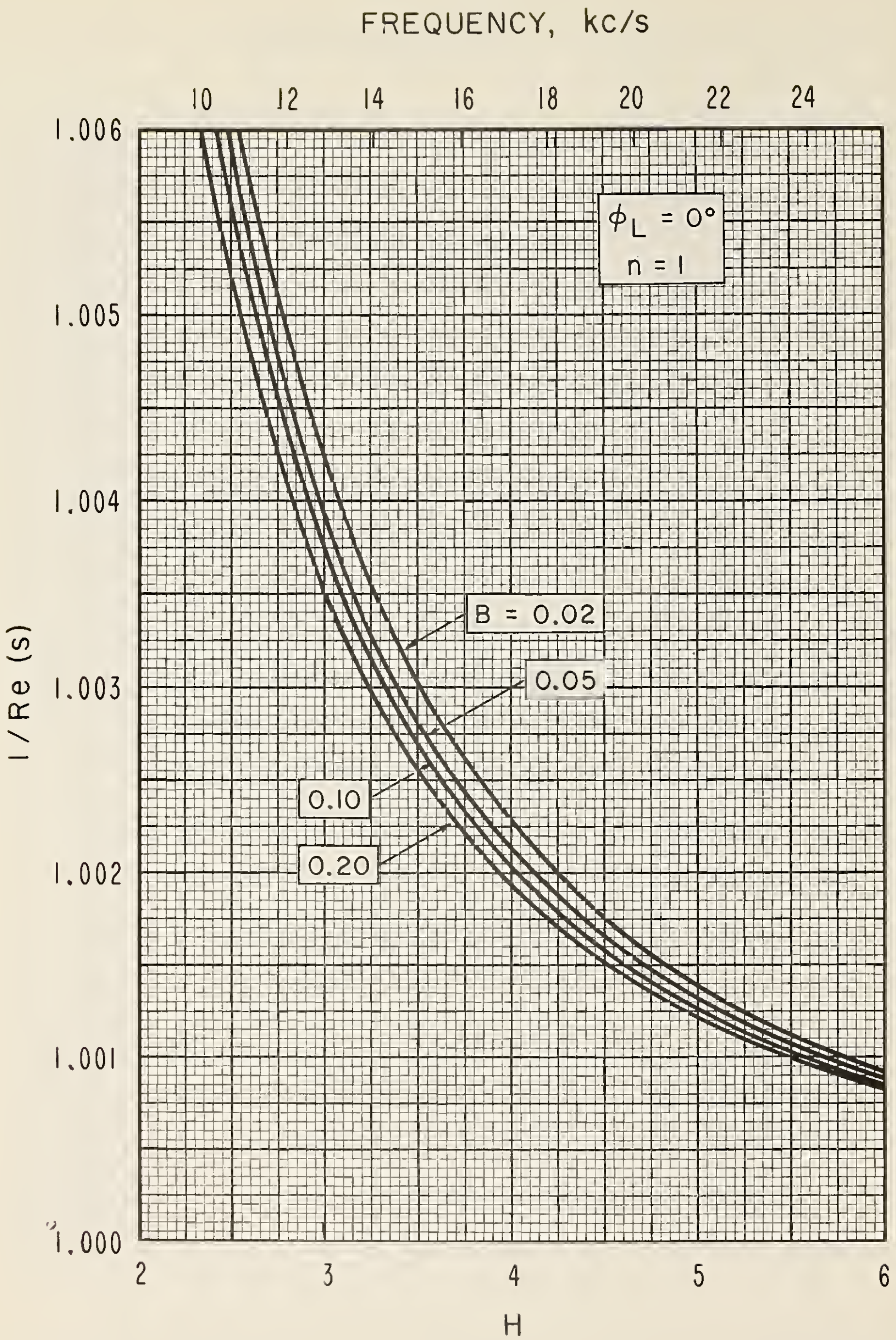
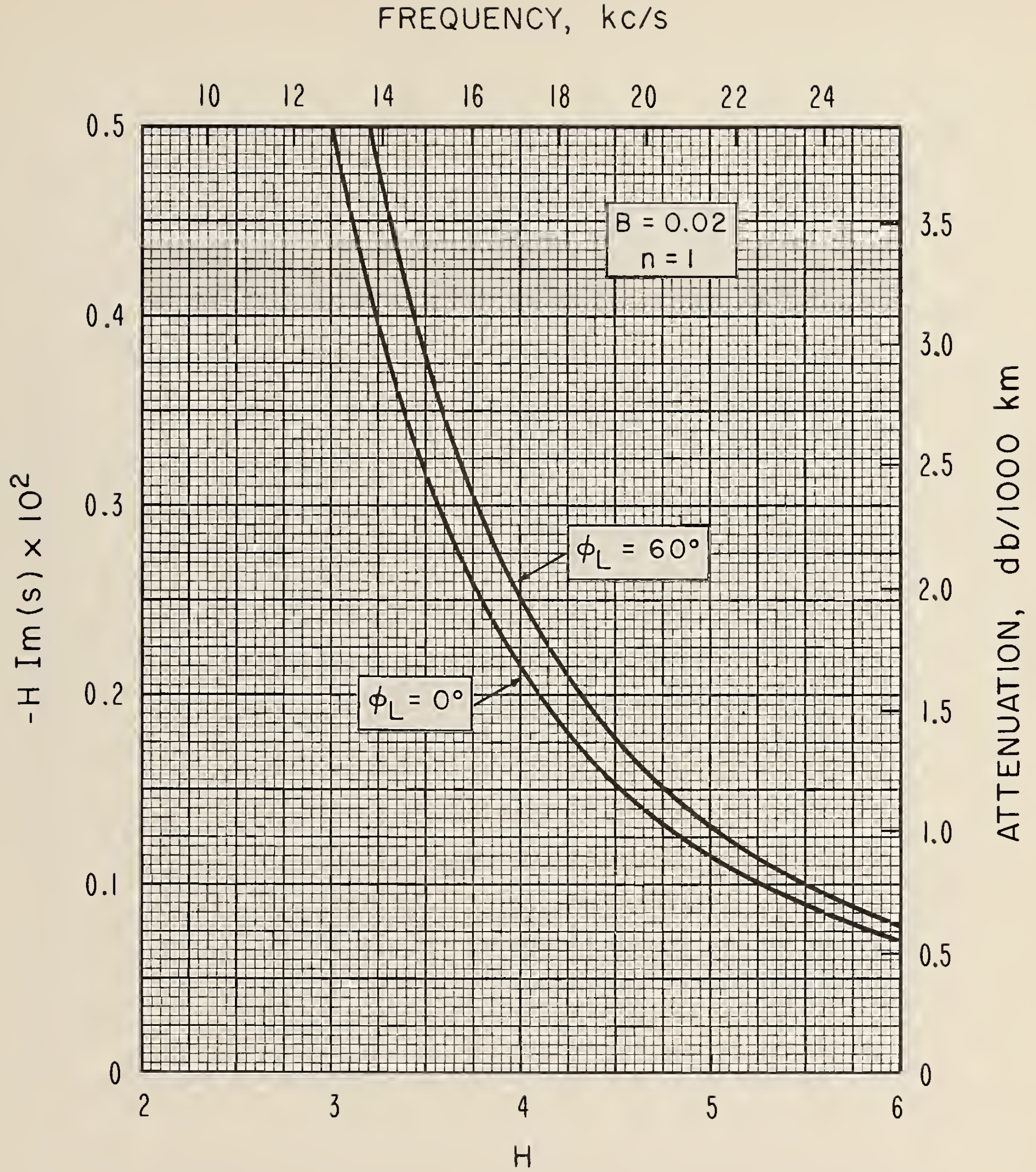

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
1.006
1.005
1.004
1.003
1.002
1.001
1.000
1/Re (s)
ϕ_L = 0°
n = 1
B = 0.02
0.05
0.10
0.20
2
3
4
5
6
H

Fig. 5



FREQUENCY, kc/s
10
12
14
16
18
20
22
24
0.5
0.4
0.3
0.2
0.1
0
-H Im(s) x 10²
B = 0.02
n = 1
φL = 60°
φL = 0°
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
ATTENUATION, db/1000 km
2
3
4
5
6
H

Fig. 6

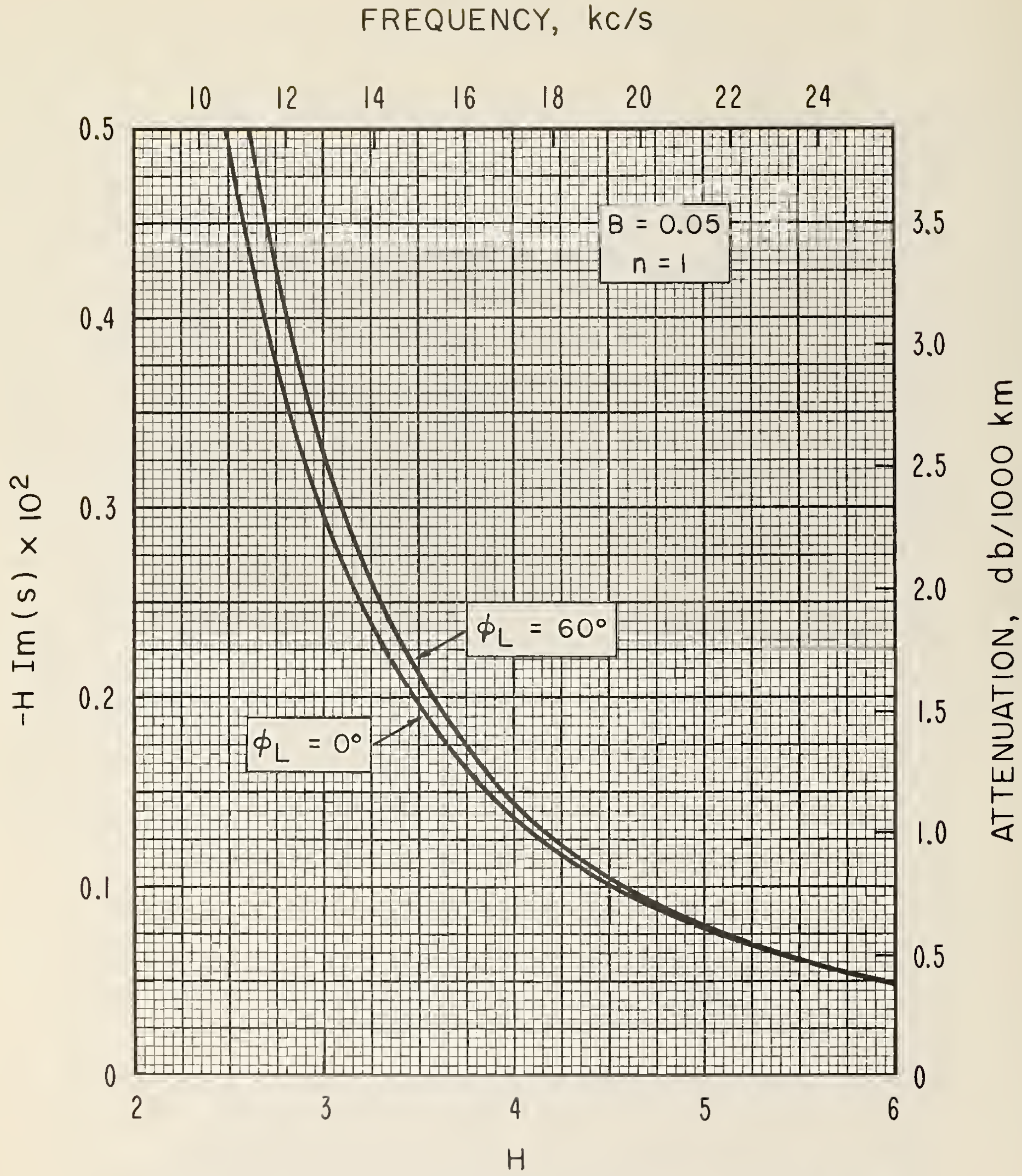

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
0.5
0.4
0.3
0.2
0.1
0
-H Im(s) × 10²
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
ATTENUATION, db/1000 km
B = 0.05
n = 1
φL = 60°
φL = 0°
2
3
4
5
6
H

Fig. 7

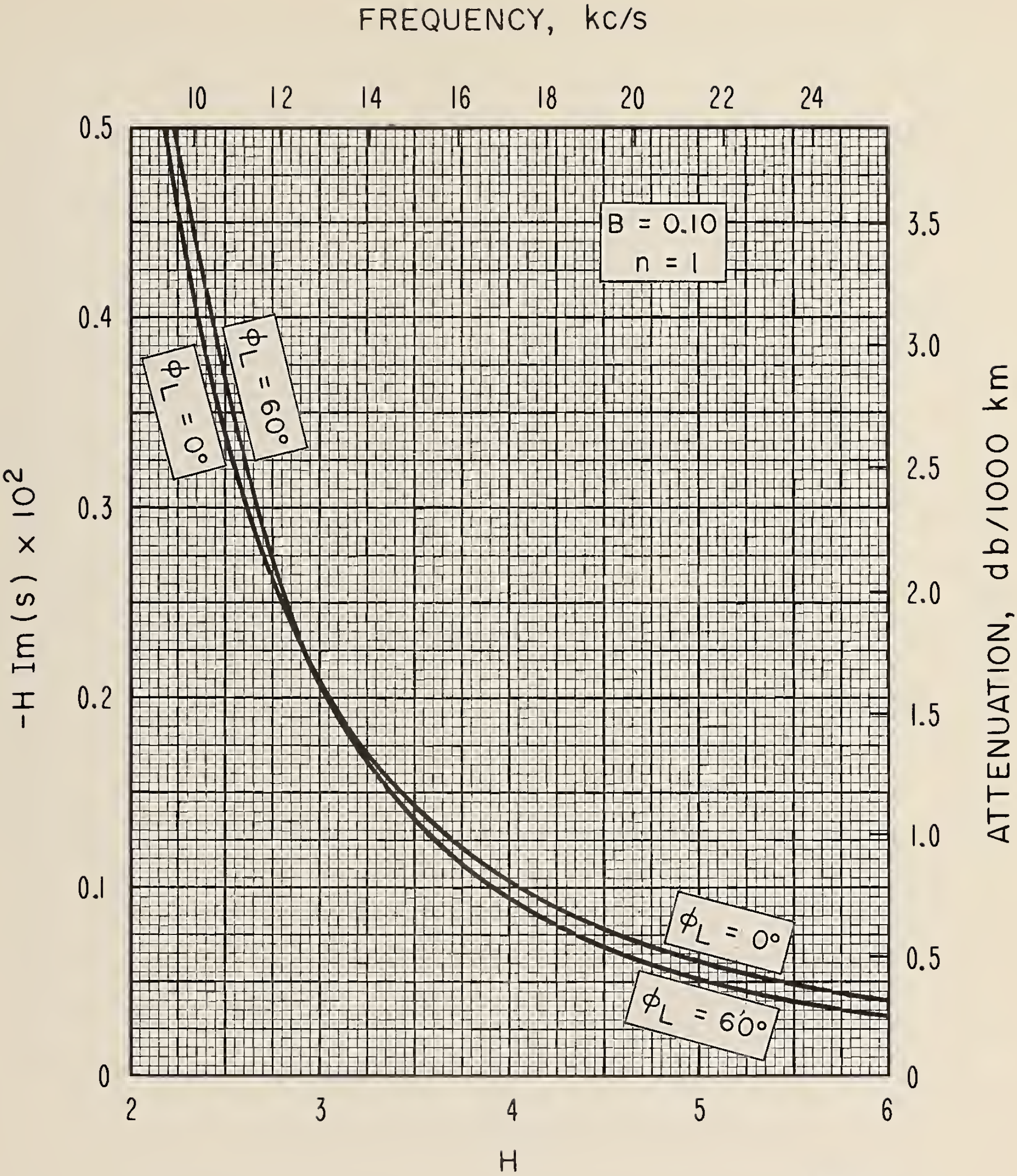

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
0.5
0.4
0.3
0.2
0.1
0
2
3
4
5
6
H
-H Im(s) x 10^2
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
ATTENUATION, db/1000 km
B = 0.10
n = 1
φL = 0°
φL = 60°
φL = 0°
φL = 60°

Fig. 8

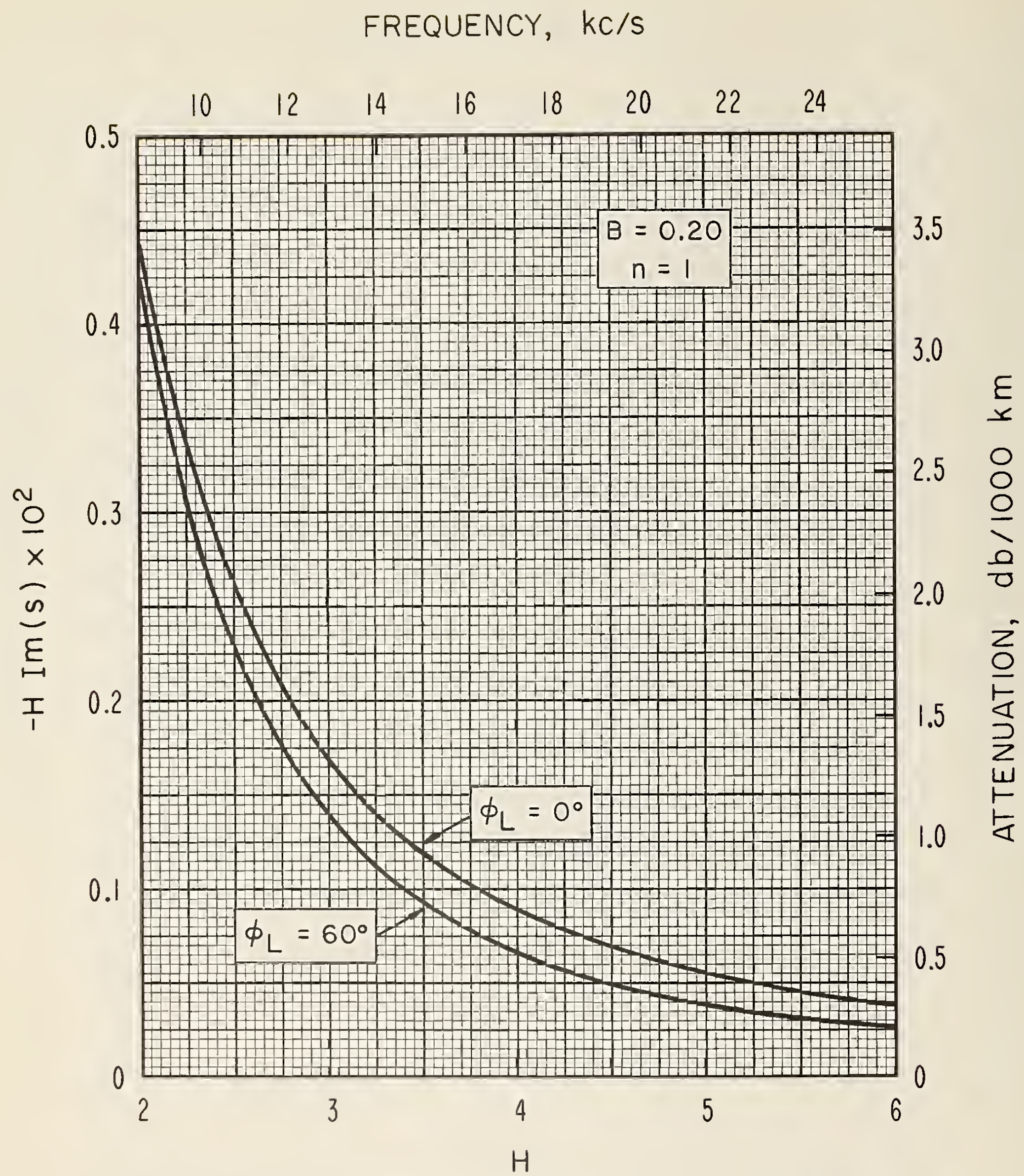
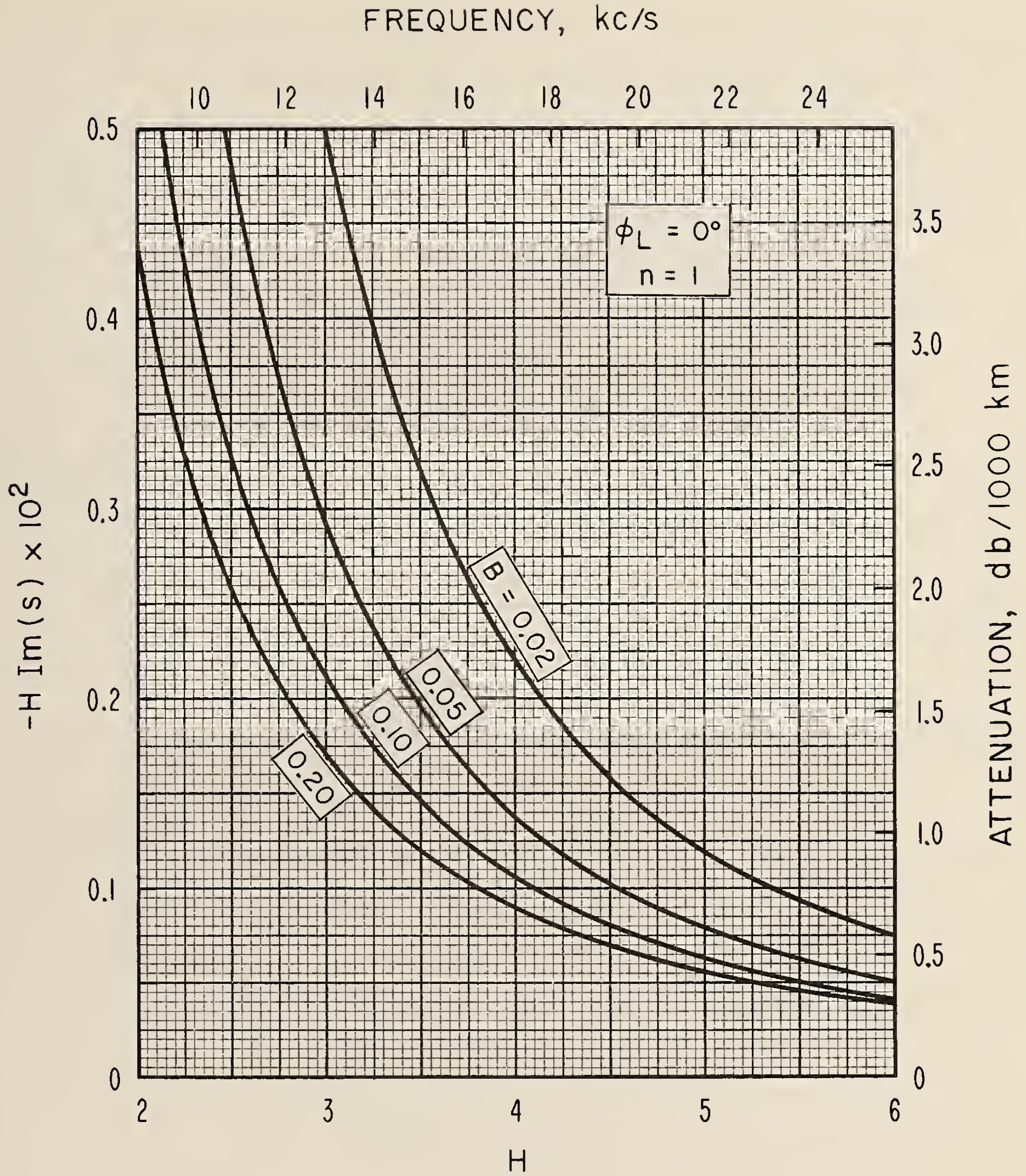

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
0.5
0.4
0.3
0.2
0.1
0
-H Im(s) x 10^2
B = 0.20
n = 1
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
ATTENUATION, db/1000 km
ϕL = 0°
ϕL = 60°
2
3
4
5
6
H

Fig. 9



FREQUENCY, kc/s
10
12
14
16
18
20
22
24
ϕ_L = 0°
n = 1
0.5
0.4
0.3
0.2
0.1
0
$-H \, Im(s) \times 10^2$
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
ATTENUATION, db/1000 km
B = 0.02
0.05
0.10
0.20
2
3
4
5
6
H

Fig. 10

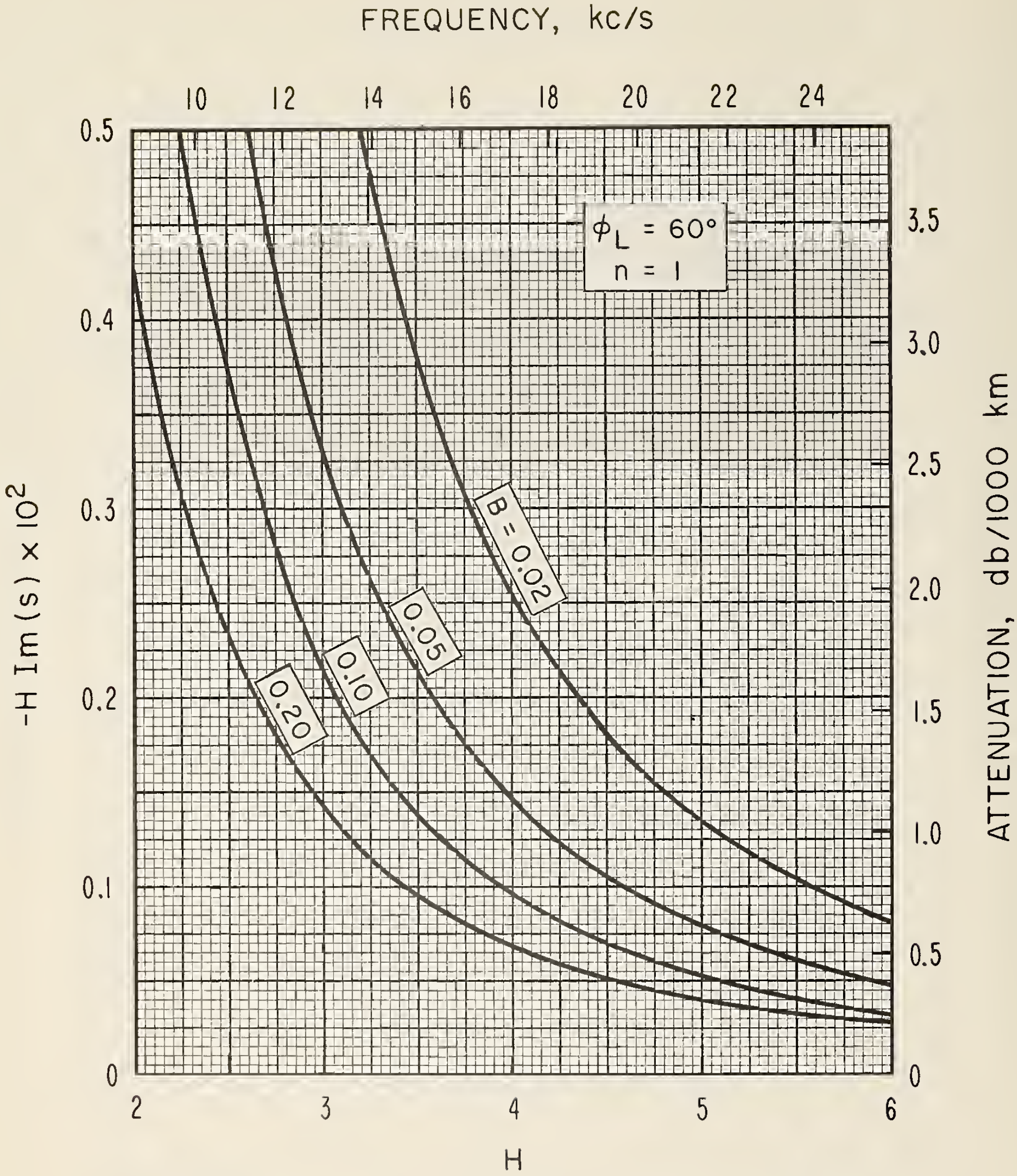

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
ϕ_L = 60°
n = 1
B = 0.02
0.05
0.10
0.20
-H Im(s) x 10^2
0.5
0.4
0.3
0.2
0.1
0
ATTENUATION, db/1000 km
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
2
3
4
5
6
H

Fig. 11

3. Ionospheric Reflection Coefficient for Transverse Magnetic Field -

The ionospheric reflection coefficient ${}_{||}R_{||}$ given by Barber and Crombie [1959] for a transverse terrestrial magnetic field may be written as

$$ {}_{||}R_{||} = \frac{C - \Delta}{C + \Delta} \tag{15} $$

where

$$ \Delta = \frac{C^2 + \left[\frac{1 + \delta}{\delta + \delta^2 - \gamma^2}\right]^{\frac{1}{2}} (\delta + \delta^2 - \gamma^2) - i\gamma\,[1 - C^2]^{\frac{1}{2}}}{(1 + \delta)^2 - \gamma^2} \tag{16a} $$

and

$$ \delta = i\,\frac{\omega}{\omega_r}\ , \quad \gamma = \begin{cases} +\,\dfrac{\omega}{\omega_r}\,|\tan\phi_T| & \text{(East-to-West Propagation)} \\ -\,\dfrac{\omega}{\omega_r}\,|\tan\phi_T| & \text{(West-to-East Propagation)} \end{cases} \tag{16b} $$

Just as in the preceding section, expand log ${}_{||}R_{||}$ in a Taylor series about $C = 0$ to get

$$ {}_{||}R_{||} = \exp\left[\sum_{k=0}^{\infty} a_k\, C^k\right] \tag{17} $$

where

$$ a_k = \frac{1}{k!}\left[\frac{\partial^k}{\partial C^k}(\log {}_{||}R_{||})\right]_{C=0} \ . \tag{18} $$

The ionospheric reflection coefficient ${}_{||}R_{||}$ can now be approximated by using a few terms of the series (17). The first few Taylor series coefficients for log ${}_{||}R_{||}$ are

$$ a_o = i\pi \tag{19a} $$

$$a_1 = - \frac{2[(1+\delta)^2 - \gamma^2]}{[(1+\delta)(\delta+\delta^2-\gamma^2)]^{\frac{1}{2}} - i\gamma} \tag{19b}$$

$$a_2 = 0 \tag{19c}$$

$$a_3 = \frac{1}{12} a_1^2 \left\{ \frac{3}{(1+\delta)^2 - \gamma^2} \left[\frac{(\delta+\delta^2-\gamma^2)^{\frac{3}{2}}}{(1+\delta)^{\frac{1}{2}}} + i\gamma \right] + a_1 \right\} \tag{19d}$$

$$a_4 = 0 \quad . \tag{19e}$$

4. An Approximate Solution of the TM Mode Equation for Perfectly Conducting Ground in the Transverse Case - For a purely transverse and horizontal terrestrial magnetic field, the ionospheric reflection coefficients ${}_{\parallel}R_{\perp}$ and ${}_{\perp}R_{\parallel}$ vanish, so that TE and TM modes are not coupled when the ground is perfectly conducting. The TM mode equation for VLF propagation in a flat earth-ionosphere waveguide is then given by

$$e^{i\,2kh\,C} - {}_{\parallel}R_{\parallel} = 0 \quad . \qquad (20)$$

If the ionospheric reflection coefficient ${}_{\parallel}R_{\parallel}$ is approximated by using the first two terms of the Taylor series (17) for log ${}_{\parallel}R_{\parallel}$, one has

$${}_{\parallel}R_{\parallel} = - e^{a_1 C}$$

where a_1 is given by (19b).

With this approximation, the solutions of (20) are

$$C_n = \frac{(2n-1)\pi}{2kh + i\,a_1} \quad (n = 1, 2, 3, \ldots) \quad . \qquad (21)$$

Curves of $1/\mathrm{Re}(S_n)$ and $- H\,\mathrm{Im}(S_n)$ vs. $H = \frac{kh}{2\pi}$ have been computed when n = 1 for

$$B = \frac{1}{H}\left(\frac{\omega}{\omega_r}\right) = 0.02,\ 0.05,\ 0.10,\ 0.20 \ \text{ and } \ \phi_T = 0^{\circ},\ 30^{\circ},\ 60^{\circ}.$$

These results are shown in figures 12 - 19. Frequency and attenuation scales corresponding to h = 70 km have been appended.

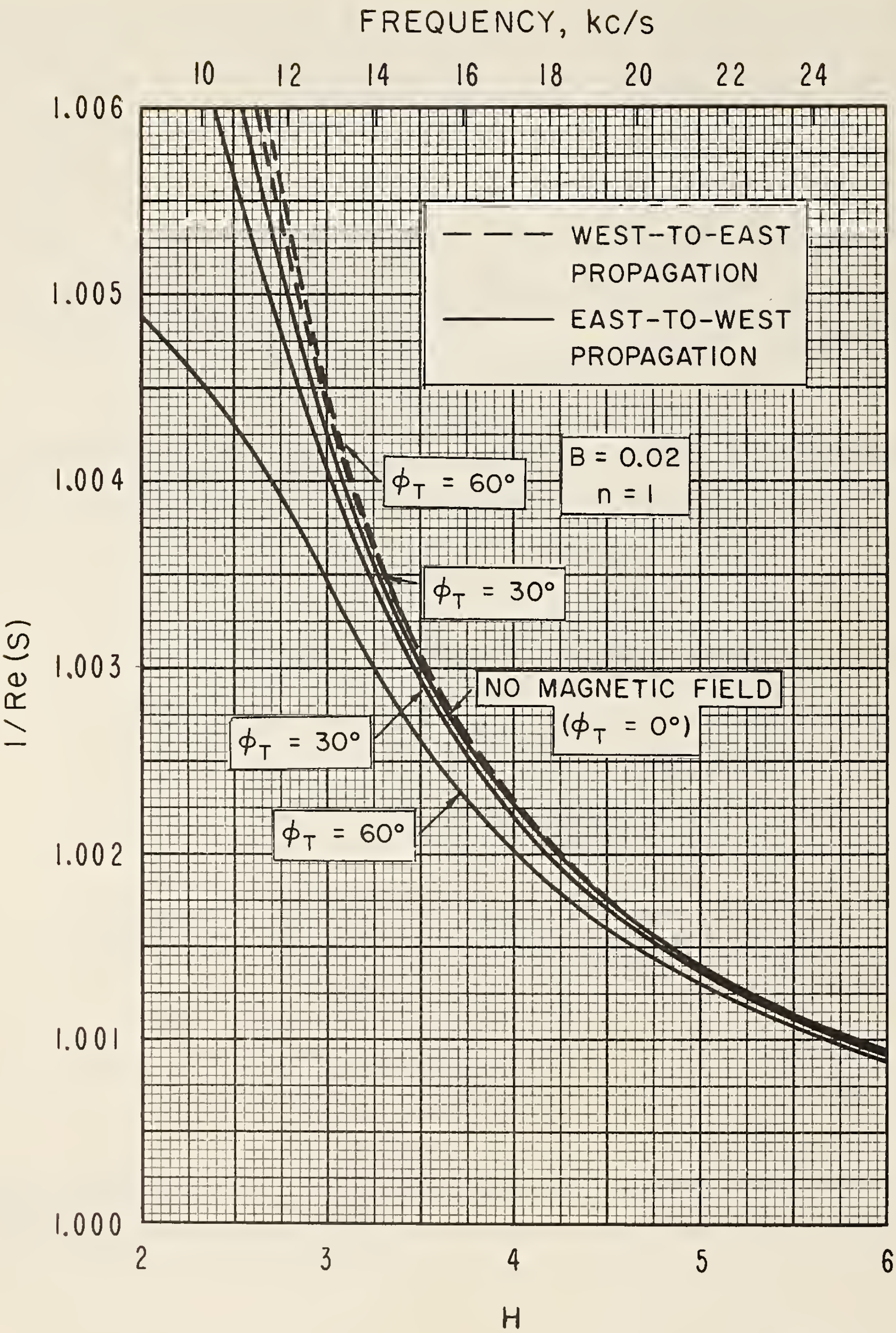

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
1.006
1.005
1.004
1.003
1.002
1.001
1.000
1/Re(S)
WEST-TO-EAST PROPAGATION
EAST-TO-WEST PROPAGATION
B = 0.02
n = 1
φT = 60°
φT = 30°
NO MAGNETIC FIELD
(φT = 0°)
φT = 30°
φT = 60°
2
3
4
5
6
H

Fig. 12

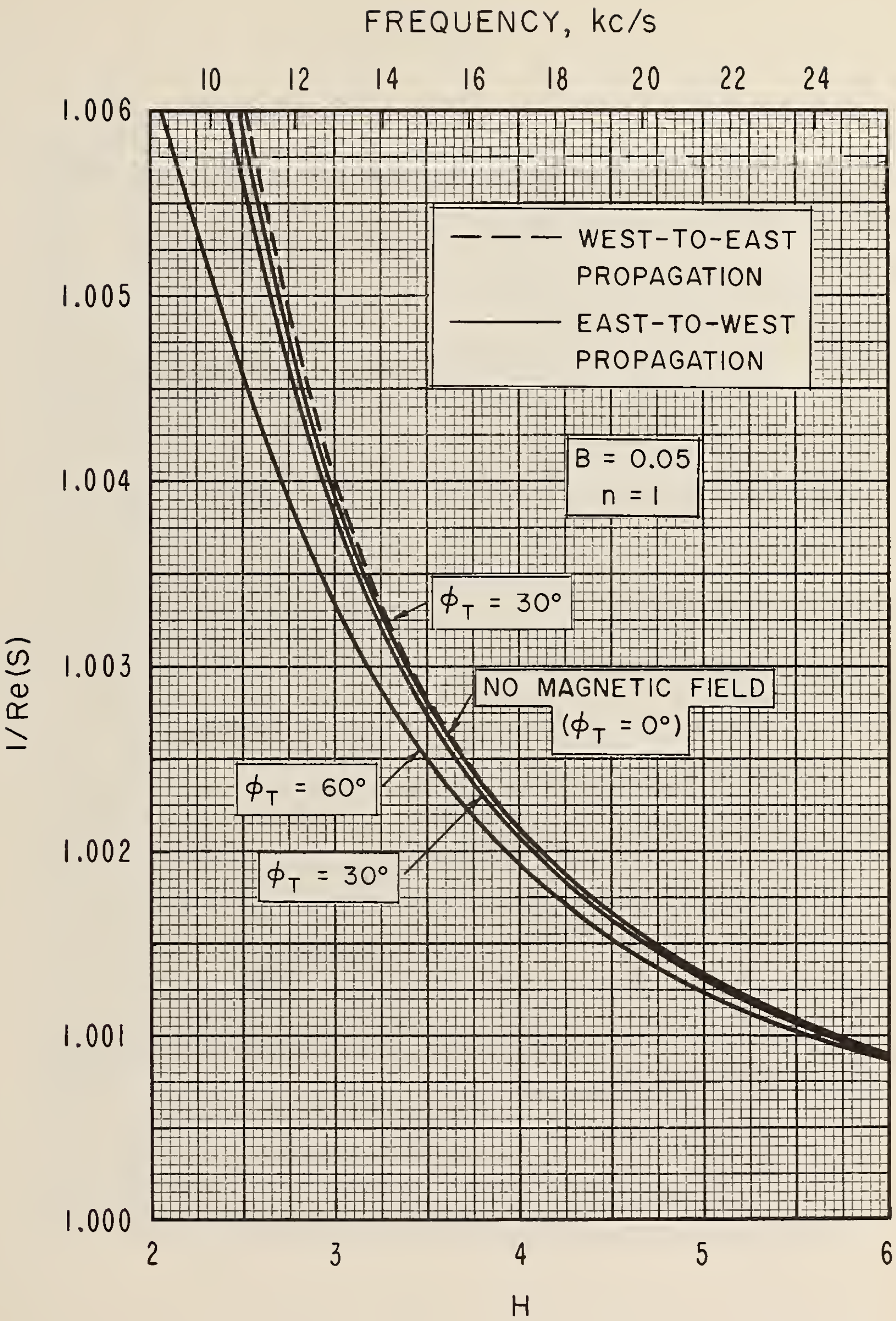

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
1.006
1.005
1.004
1.003
1.002
1.001
1.000
WEST-TO-EAST PROPAGATION
EAST-TO-WEST PROPAGATION
B = 0.05
n = 1
ϕ_T = 30°
NO MAGNETIC FIELD (ϕ_T = 0°)
ϕ_T = 60°
ϕ_T = 30°
1/Re(S)
2
3
4
5
6
H

Fig. 13

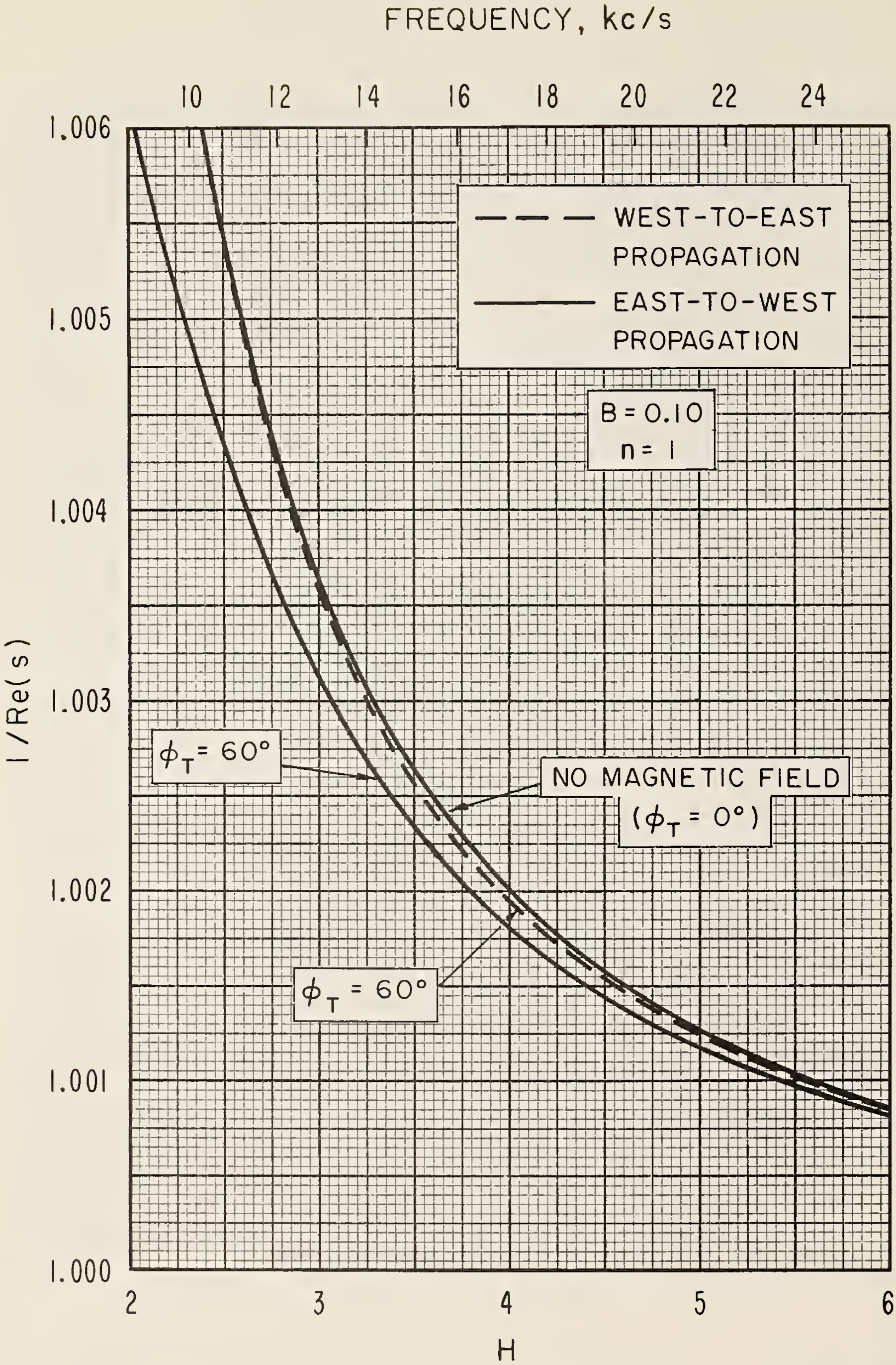

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
1.006
1.005
1.004
1.003
1.002
1.001
1.000
WEST-TO-EAST PROPAGATION
EAST-TO-WEST PROPAGATION
B = 0.10
n = 1
1/Re(s)
φT = 60°
NO MAGNETIC FIELD (φT = 0°)
φT = 60°
2
3
4
5
6
H

Fig. 14

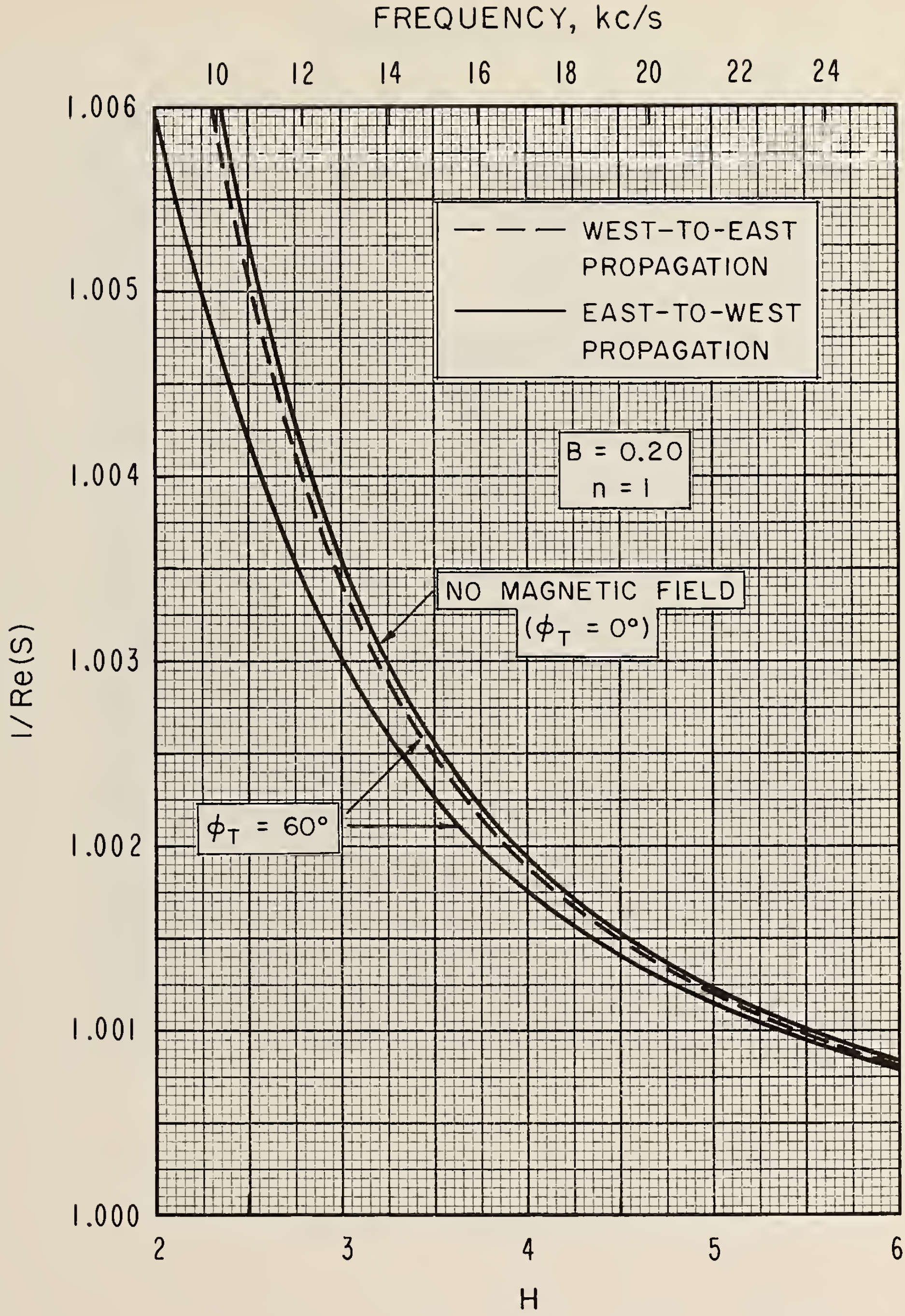

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
1.006
1.005
1.004
1.003
1.002
1.001
1.000
1/Re(S)
WEST-TO-EAST PROPAGATION
EAST-TO-WEST PROPAGATION
B = 0.20
n = 1
NO MAGNETIC FIELD
(ϕ_T = 0°)
ϕ_T = 60°
2
3
4
5
6
H

Fig. 15

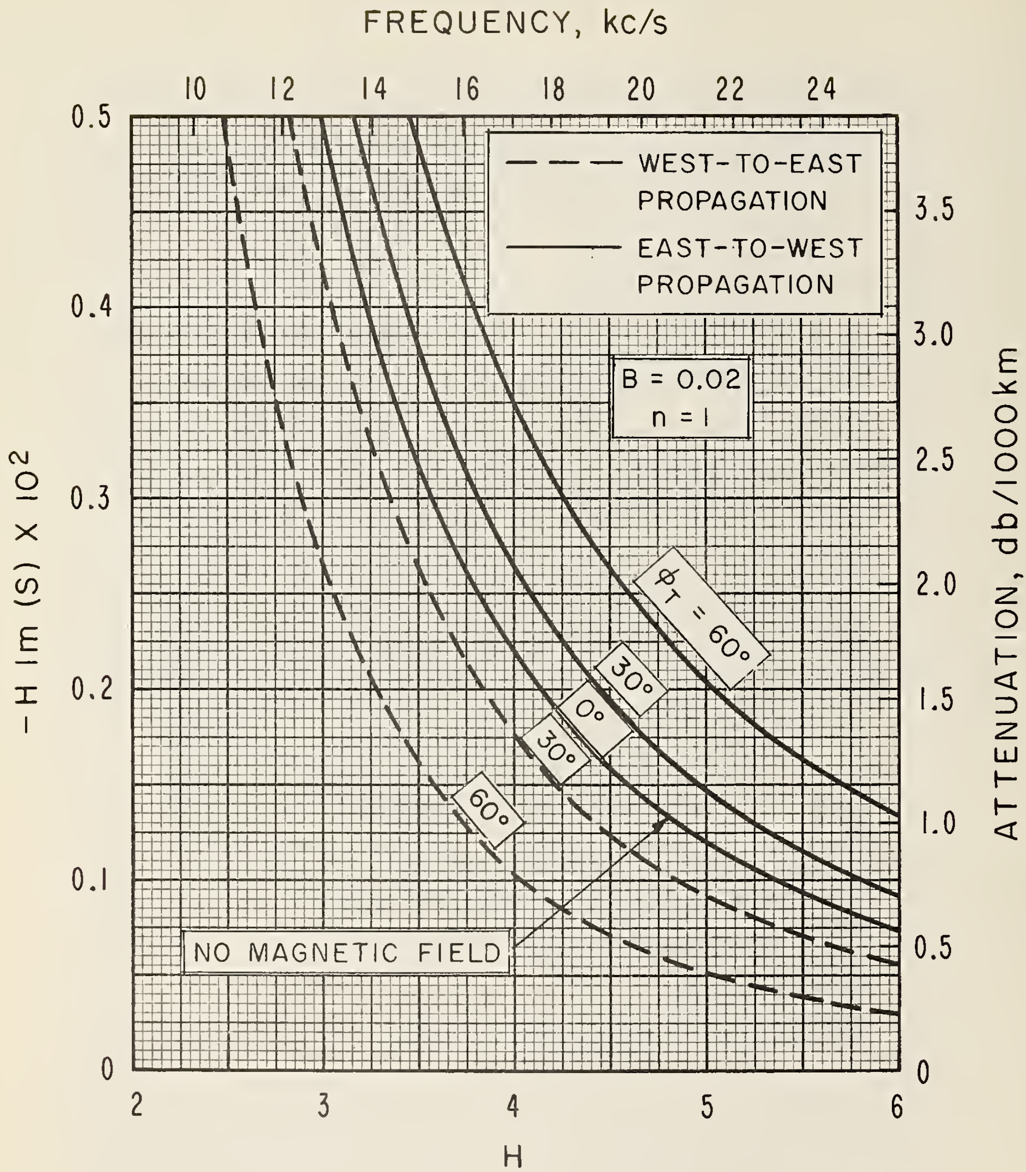

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
0.5
0.4
0.3
0.2
0.1
0
-H Im (S) X 10²
WEST-TO-EAST PROPAGATION
EAST-TO-WEST PROPAGATION
B = 0.02
n = 1
φT = 60°
30°
0°
30°
60°
NO MAGNETIC FIELD
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
ATTENUATION, db/1000km
2
3
4
5
6
H

Fig. 16

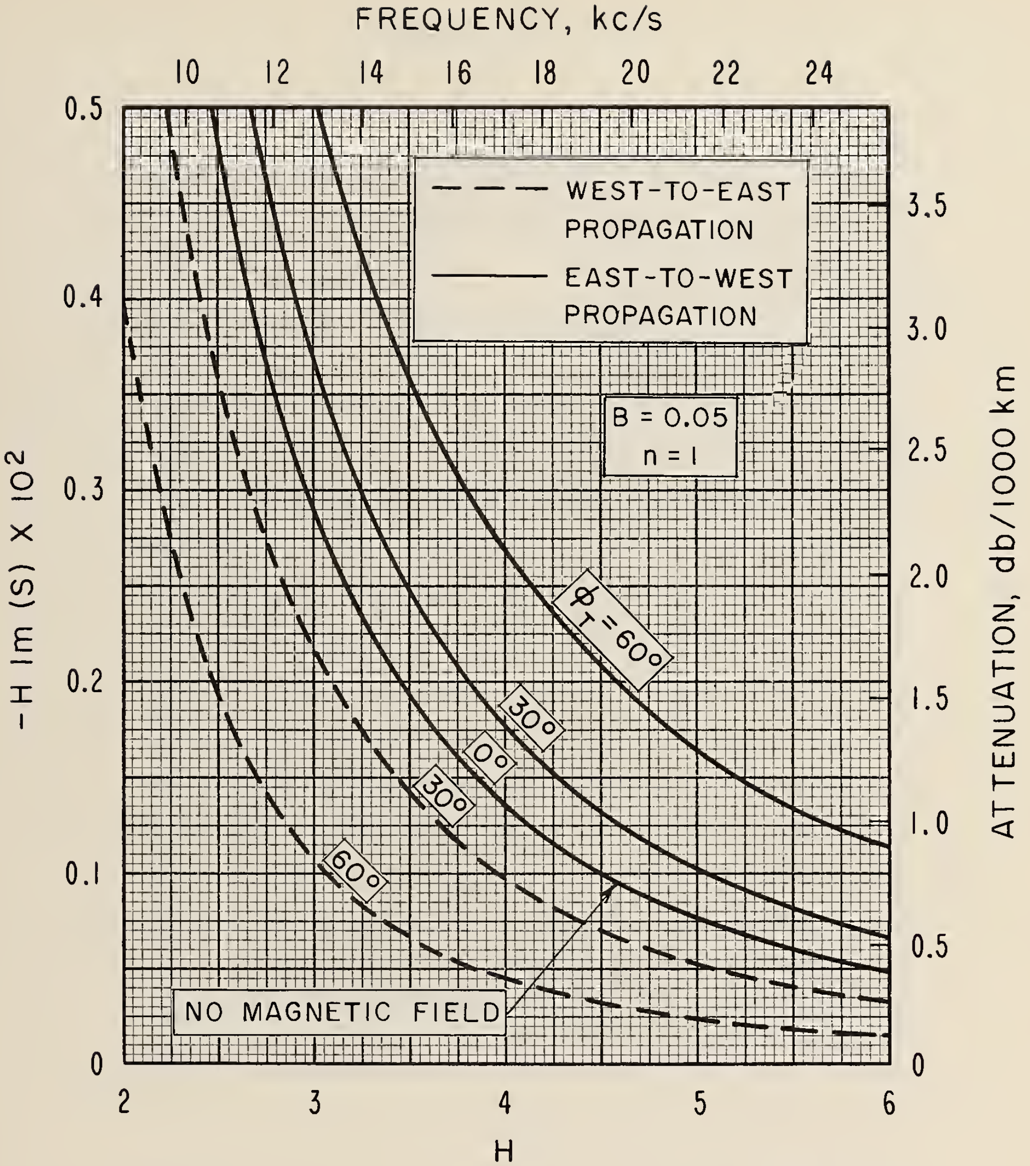

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
0.5
0.4
0.3
0.2
0.1
0
-H Im (S) X 10²
WEST-TO-EAST PROPAGATION
EAST-TO-WEST PROPAGATION
B = 0.05
n = 1
φT = 60°
30°
0°
30°
60°
NO MAGNETIC FIELD
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
ATTENUATION, db/1000 km
2
3
4
5
6
H

Fig. 17

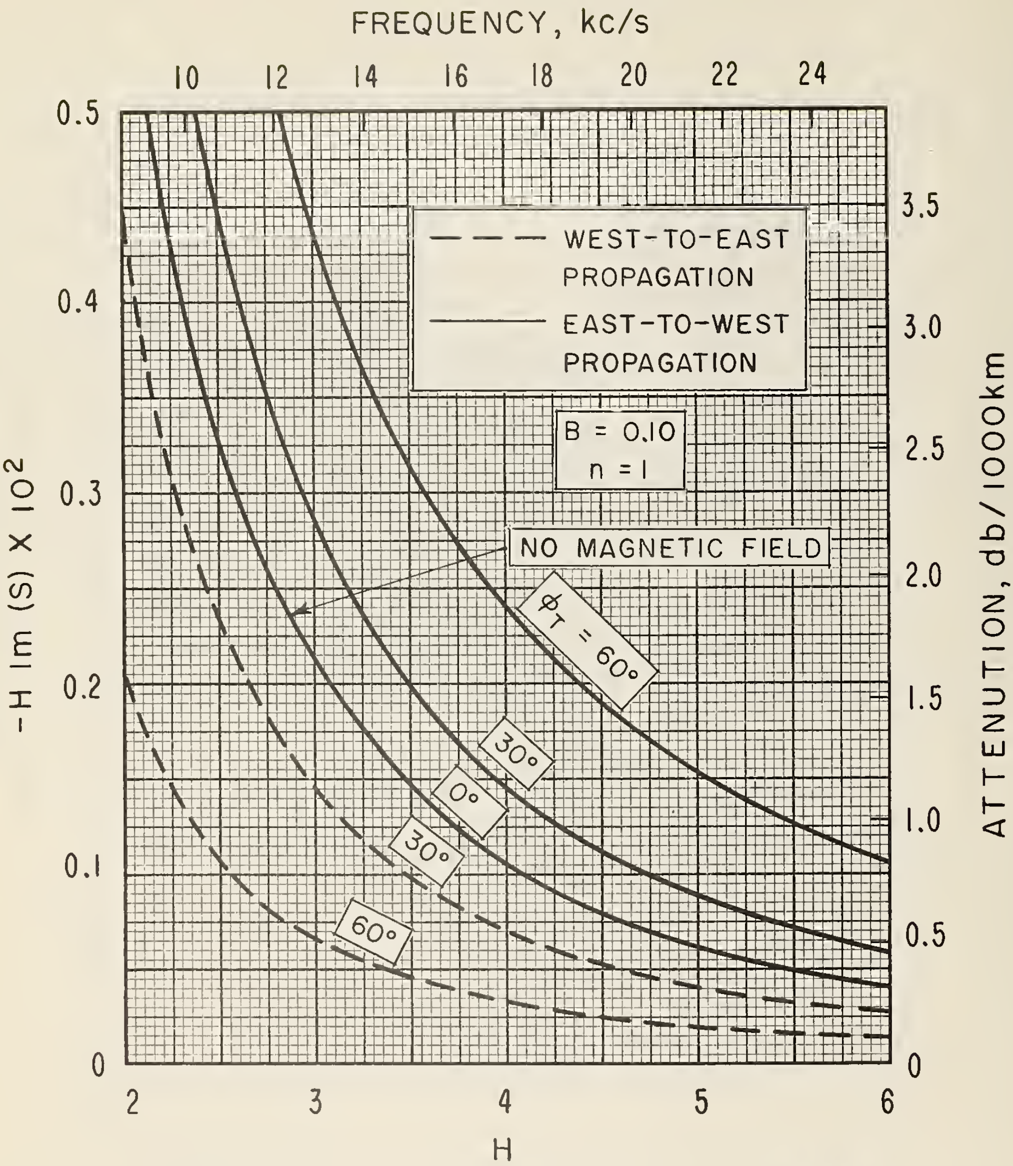
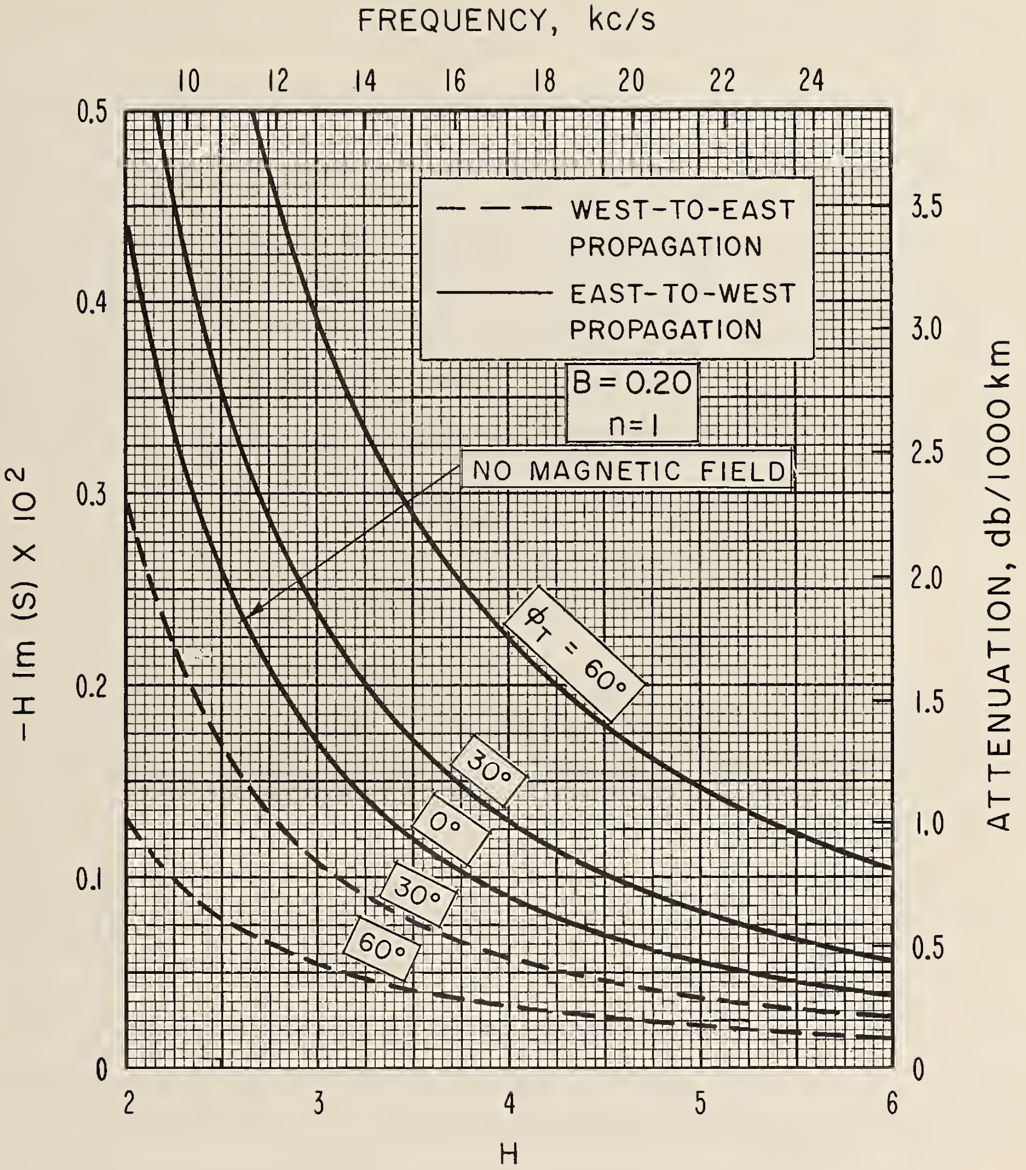

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
0.5
0.4
0.3
0.2
0.1
0
-H Im (S) X 10^2
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
ATTENUATION, db/1000km
WEST-TO-EAST PROPAGATION
EAST-TO-WEST PROPAGATION
B = 0.10
n = 1
NO MAGNETIC FIELD
φT = 60°
30°
0°
30°
60°
2
3
4
5
6
H

Fig. 18



FREQUENCY, kc/s
10
12
14
16
18
20
22
24
0.5
0.4
0.3
0.2
0.1
0
−H Im (S) X 10²
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
ATTENUATION, db/1000 km
WEST-TO-EAST PROPAGATION
EAST-TO-WEST PROPAGATION
B = 0.20
n = 1
NO MAGNETIC FIELD
φT = 60°
30°
0°
30°
60°
2
3
4
5
6
H

Fig. 19

5. Arbitrarily Dipping Magnetic Field -

In general, the ionospheric reflection coefficients depend on the direction of propagation with respect to the terrestrial magnetic field. Johler [1961] has evaluated ${}_{\parallel}R_{\parallel}$, ${}_{\perp}R_{\perp}$, ${}_{\parallel}R_{\perp}$, ${}_{\perp}R_{\parallel}$ for a (real) angle of incidence of 82°, various magnetic dip angles I, and directions of propagation ϕ_a (measured clockwise from north). These results have been used to estimate the effect of variations in I and ϕ_a on the attenuation and phase of TM waveguide modes. This was done in the following way. First, the ionospheric reflection coefficient ${}_{\parallel}R_{\parallel}$ was approximated by an expression of the form

$$ {}_{\parallel}R_{\parallel} = -e^{\alpha C} \tag{22} $$

where the parameter α was chosen so that (22) agrees with the exact value computed by Johler when $C = \cos 82^{\circ}$ [note that when $C = \cos 90^{\circ}$, (22) automatically reduces to the exact value]. Since ${}_{\parallel}R_{\parallel} \, e^{i2\pi}$ is equal to Johler's T_{ee}, this requires that

$$ \operatorname{Re}(\alpha) = \frac{\log_e |T_{ee}|}{\cos 82^{\circ}} \quad \text{and} \quad \operatorname{Im}(\alpha) = \frac{\arg(T_{ee}) - \pi}{\cos 82^{\circ}} \, . \qquad (23a)\ (23b) $$

If coupling between TM and TE modes is neglected (equivalent to setting ${}_{\parallel}R_{\perp} = {}_{\perp}R_{\parallel} = 0$), the TM mode equation for VLF propagation in a flat earth-ionosphere waveguide becomes

$$ e^{i2khC} - {}_{\parallel}R_{\parallel} = 0 \, . \tag{24} $$

Using the expression (22) for ${}_{\parallel}R_{\parallel}$, the solutions of (24) are

$$ C_n = \frac{(2n-1)\pi}{2kh + i\alpha} \qquad (n = 1, 2, 3, \ldots). $$

Now let

$$P = \frac{\text{Attenuation with Magnetic Field}}{\text{Attenuation without Magnetic Field}}$$

$$= \frac{[\mathrm{Im}(S)] \text{ with Magnetic Field}}{[\mathrm{Im}(S)] \text{ without Magnetic Field}} \tag{25a}$$

and

$$Q = \frac{\text{Phase Velocity Deviation with Magnetic Field}}{\text{Phase Velocity Deviation without Magnetic Field}}$$

$$= \frac{[\frac{v}{c} - 1] \text{ with Magnetic Field}}{[\frac{v}{c} - 1] \text{ without Magnetic Field}}$$

$$= \frac{[\frac{1}{\mathrm{Re}(S)} - 1] \text{ with Magnetic Field}}{[\frac{1}{\mathrm{Re}(S)} - 1] \text{ without Magnetic Field}} \; . \tag{25b}$$

For the dominant mode (n = 1), curves of P and Q vs. direction of propagation (ϕ_a) have been computed for:

Magnetic dip angles , $I = 0^\circ,\ 45^\circ,\ 84.3^\circ$,

Electron densities , $N = 3(10)^3,\ (10)^3$ electrons/cm^3 ,

Frequencies, $f = 10,\ 22$ kc/s .

The results are shown in figures 20 through 24.

In the case of $I = 0^{\circ}$, P and Q were also computed for $\phi_a = 90^{\circ}$ and $\phi_a = 270^{\circ}$ -- that is, for a (horizontal) transverse terrestrial magnetic field. In order to make the methods of computation uniform, the exact formula (15) for the ionospheric reflection coefficient $_{\parallel}R_{\parallel}$ was first evaluated for $C = \cos 82^{\circ}$. The determination of α and subsequent calculations were then carried out according to the procedure outlined above.

For $I = 90^{\circ}$ (vertical magnetic field), P and Q are independent of the direction of propagation ϕ_a. In this case, P and Q have been plotted as a function of frequency (figure 25).

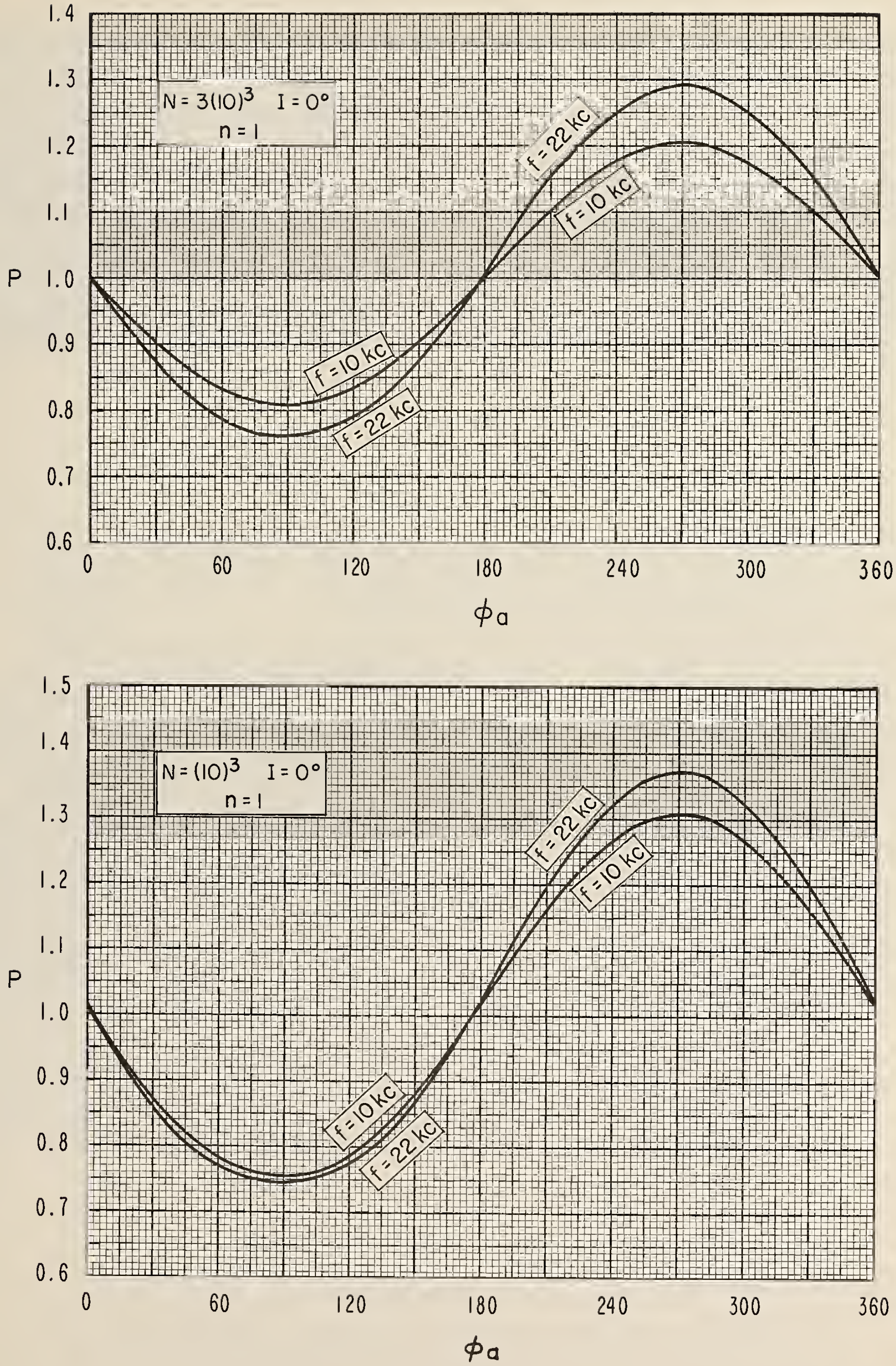
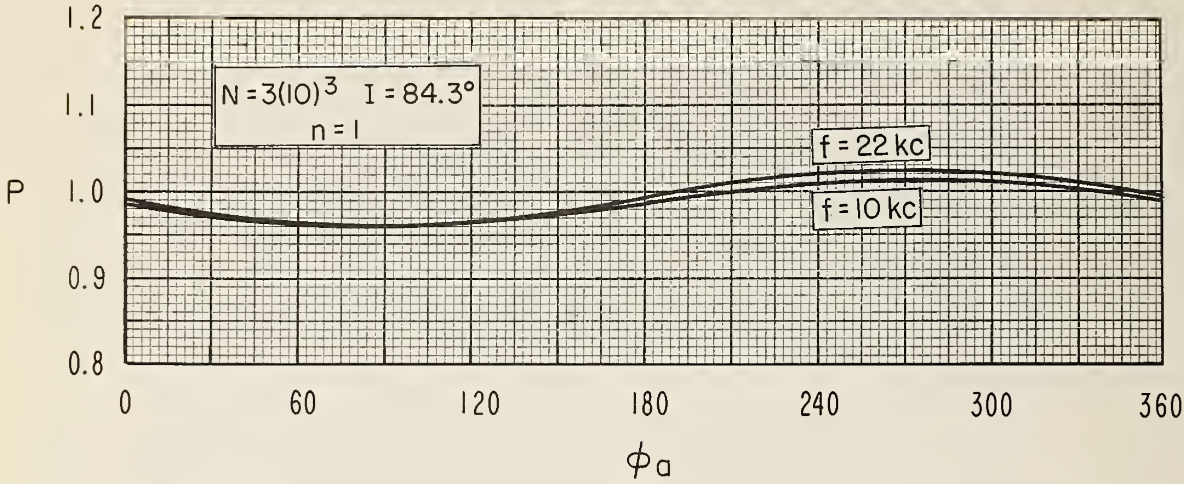

N = 3(10)^3 I = 0°
n = 1
f = 22 kc
f = 10 kc
f = 10 kc
f = 22 kc
P
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0
60
120
180
240
300
360
φa
N = (10)^3 I = 0°
n = 1
f = 22 kc
f = 10 kc
f = 10 kc
f = 22 kc
P
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0
60
120
180
240
300
360
φa

Fig. 20


N = 3(10)^3 I = 84.3°
n = 1
f = 22 kc
f = 10 kc
P
1.2
1.1
1.0
0.9
0.8
0
60
120
180
240
300
360
ϕ_a

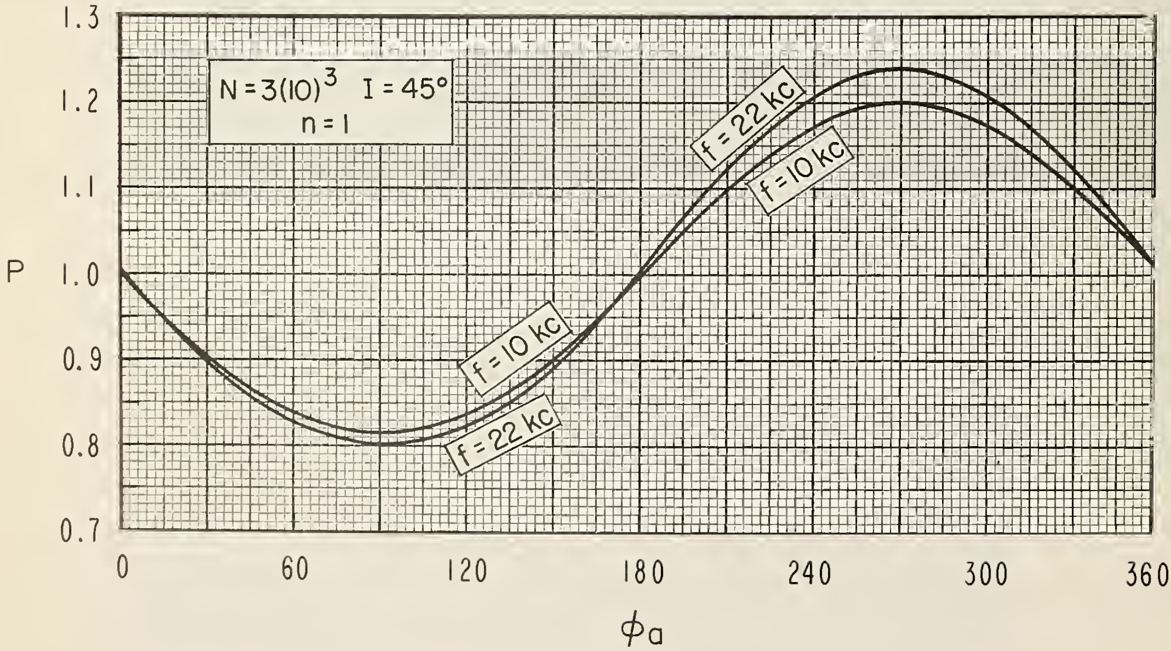
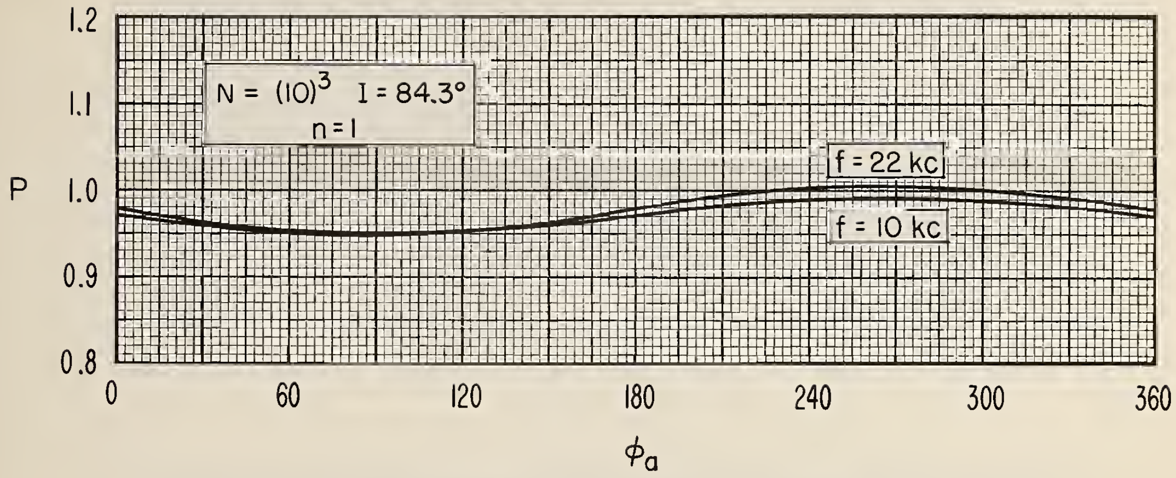
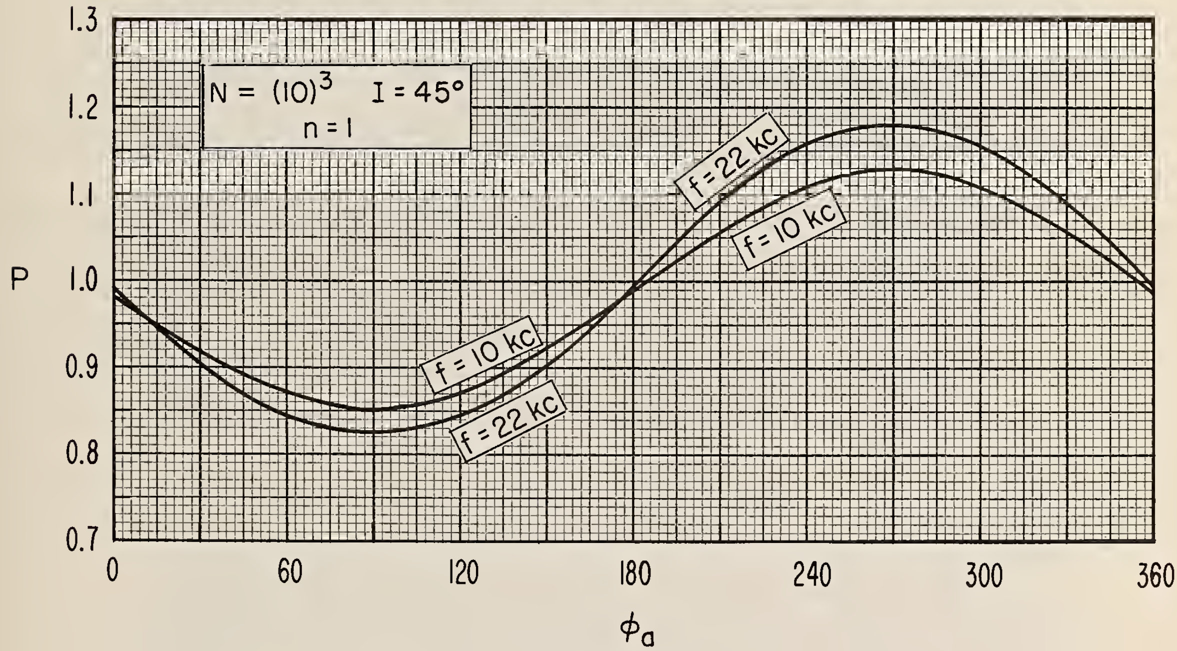

N = 3(10)^3 I = 45°
n = 1
f = 22 kc
f = 10 kc
f = 10 kc
f = 22 kc
P
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0
60
120
180
240
300
360
ϕ_a

Fig. 21

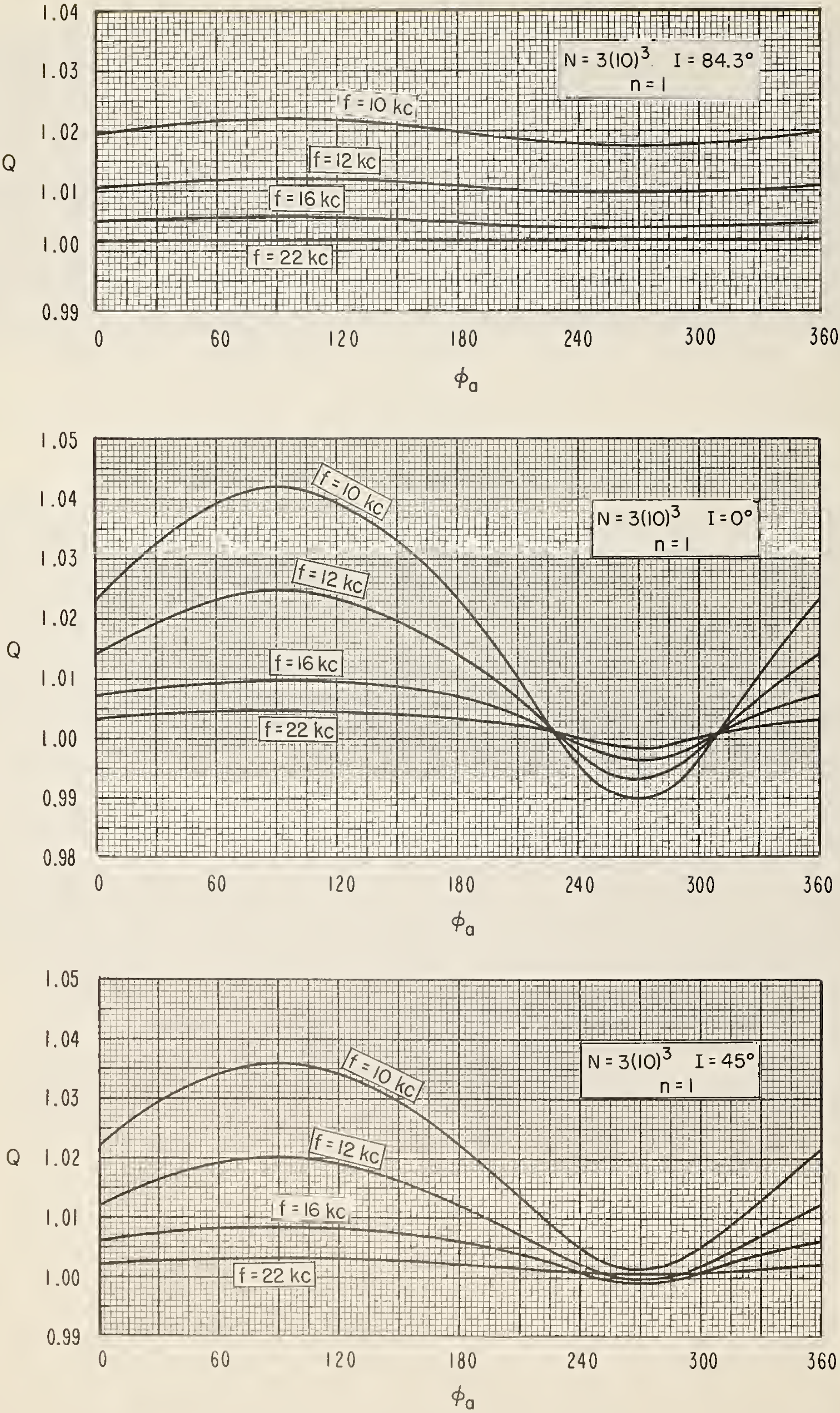


N = (10)3 I = 84.3°
n = 1
f = 22 kc
f = 10 kc
P
1.2
1.1
1.0
0.9
0.8
0
60
120
180
240
300
360
ϕa



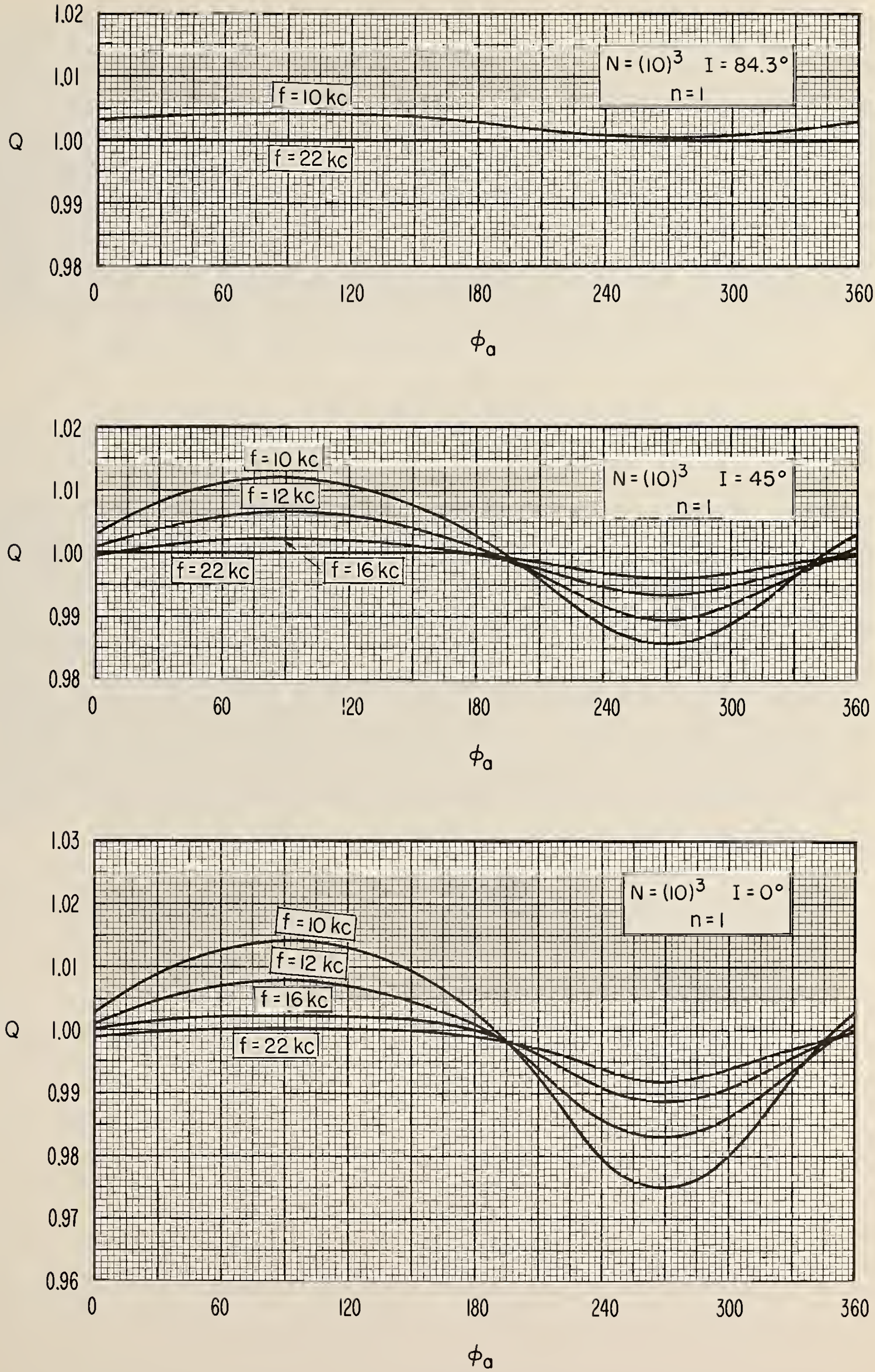
N = (10)3 I = 45°
n = 1
f = 22 kc
f = 10 kc
f = 10 kc
f = 22 kc
P
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0
60
120
180
240
300
360
ϕa

Fig. 22



1.04
1.03
1.02
1.01
1.00
0.99
Q
N = 3(10)^3 I = 84.3°
n = 1
f = 10 kc
f = 12 kc
f = 16 kc
f = 22 kc
0
60
120
180
240
300
360
φa
1.05
1.04
1.03
1.02
1.01
1.00
0.99
0.98
Q
N = 3(10)^3 I = 0°
n = 1
f = 10 kc
f = 12 kc
f = 16 kc
f = 22 kc
0
60
120
180
240
300
360
φa
1.05
1.04
1.03
1.02
1.01
1.00
0.99
Q
N = 3(10)^3 I = 45°
n = 1
f = 10 kc
f = 12 kc
f = 16 kc
f = 22 kc
0
60
120
180
240
300
360
φa

Fig. 23



1.02
1.01
1.00
0.99
0.98
Q
f = 10 kc
f = 22 kc
N = (10)^3 I = 84.3°
n = 1
0
60
120
180
240
300
360
ϕ_a
1.02
1.01
1.00
0.99
0.98
Q
f = 10 kc
f = 12 kc
f = 22 kc
f = 16 kc
N = (10)^3 I = 45°
n = 1
0
60
120
180
240
300
360
ϕ_a
1.03
1.02
1.01
1.00
0.99
0.98
0.97
0.96
Q
f = 10 kc
f = 12 kc
f = 16 kc
f = 22 kc
N = (10)^3 I = 0°
n = 1
0
60
120
180
240
300
360
ϕ_a

Fig. 24

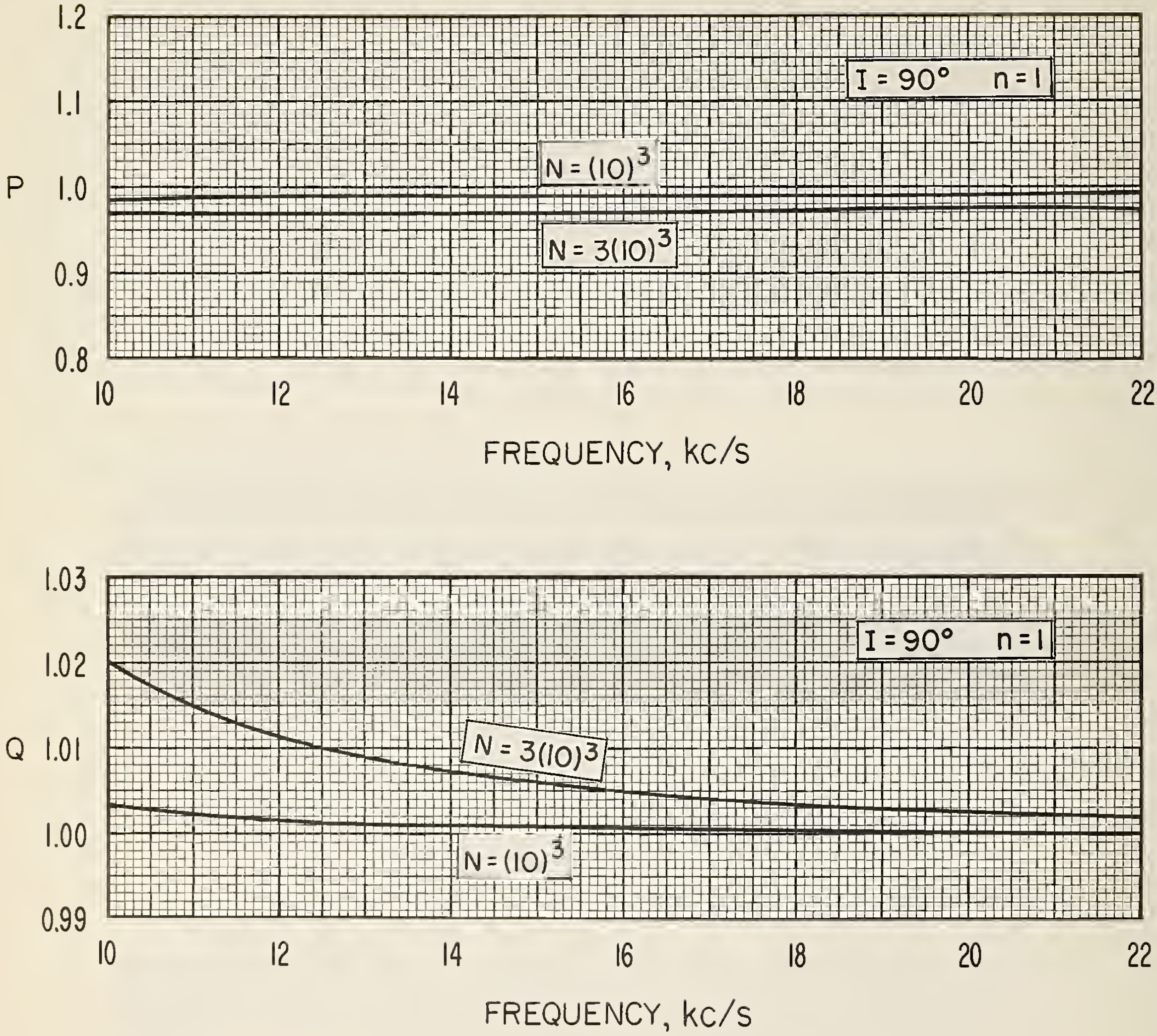

I = 90° n = 1
N = (10)^3
N = 3(10)^3
P
1.2
1.1
1.0
0.9
0.8
10
12
14
16
18
20
22
FREQUENCY, kc/s
I = 90° n = 1
N = 3(10)^3
N = (10)^3
Q
1.03
1.02
1.01
1.00
0.99
10
12
14
16
18
20
22
FREQUENCY, kc/s

Fig. 25

III. Calculations for a Spherical Earth-Ionosphere Waveguide Using the Quasi-longitudinal Approximation

1. <u>The Mode Equation</u> - A simplified form of the mode equation for VLF propagation in a spherical earth-ionosphere waveguide has been given by Wait and Spies [1960]. If the ionospheric reflection coefficient R_i is expressed as

$$R_i = - \exp \left[\alpha_1 \left(C^2 + \frac{2h}{a}\right)^{\frac{1}{2}} \right] , \qquad (26)$$

where the parameter α_1 is that computed previously (in section II) for the quasi-longitudinal approximation, this mode equation may be written as

$$\frac{2}{3} ka \left(C^2 + \frac{2h}{a}\right)^{3/2} + i \alpha_1 \left(C^2 + \frac{2h}{a}\right)^{1/2} + i \log \left[\frac{w_2'(t) - q\, w_2(t)}{w_1'(t) - q\, w_1(t)} \right]$$

$$- (4n - 1) \frac{\pi}{2} = 0 \qquad (27)$$

where n is an integer and $t = - \left(\frac{ka}{2}\right)^{2/3} C^2$.

A restriction on this mode equation is that $h/a \ll 1$.

2. A Crude Solution for Perfectly Conducting Ground - When the ground is perfectly conducting (q = 0), $i \log [w_2'(t)/w_1'(t)]$ may be approximated by

$$i \log \left[\frac{w_2'(0)}{w_1'(0)}\right] = -\arg[w_2'(0)] + \arg[w_1'(0)] = \frac{\pi}{3}$$

and if $|C^2| << \frac{2h}{a}$, if follows that

$$\left(C^2 + \frac{2h}{a}\right)^{3/2} \approx \left(\frac{2h}{a}\right)^{3/2}\left[1 + \frac{3}{2}\frac{C^2}{\frac{2h}{a}}\right] \text{ and } \left(C^2 + \frac{2h}{a}\right)^{1/2} \approx \left(\frac{2h}{a}\right)^{1/2}\left[1 + \frac{1}{2}\frac{C^2}{\frac{2h}{a}}\right].$$

The resulting mode equation can be solved at once for C^2 to give

$$C^2 \approx \frac{(12n-5)\frac{\pi}{6} - \frac{2}{3} ka \left(\frac{2h}{a}\right)^{3/2} - i\alpha_1 \left(\frac{2h}{a}\right)^{1/2}}{ka\left(\frac{2h}{a}\right)^{1/2} + i\frac{\alpha_1}{2}\left(\frac{2h}{a}\right)^{-1/2}} . \qquad (28)$$

Using a value h/a = 0.01, curves of $1/\text{Re}(S_n)$ and $-H\,\text{Im}(S_n)$ vs. $H = \frac{kh}{2\pi}$ have been computed when n = 1 for

$$B = \frac{1}{H}\left(\frac{\omega}{\omega_r}\right) = 0.02,\ 0.05,\ 0.10,\ 0.20 \text{ and } \phi_L = 0^\circ,\ 60^\circ.$$

These are shown in figures 26 through 30. Frequency and attenuation scales corresponding to h = 70 km have been appended. Some curves have also been drawn to show comparisons between solutions for flat and spherical earth-ionosphere waveguides (Fig. 31). The particular curve labelled [1 - h/2a)] × Flat Earth is a semi-empirical result which has been used previously for interpreting diurnal change of ionospheric reflection heights [Wait, 1959]. For such purposes, it is a good approximation (here h/a = 0.01).

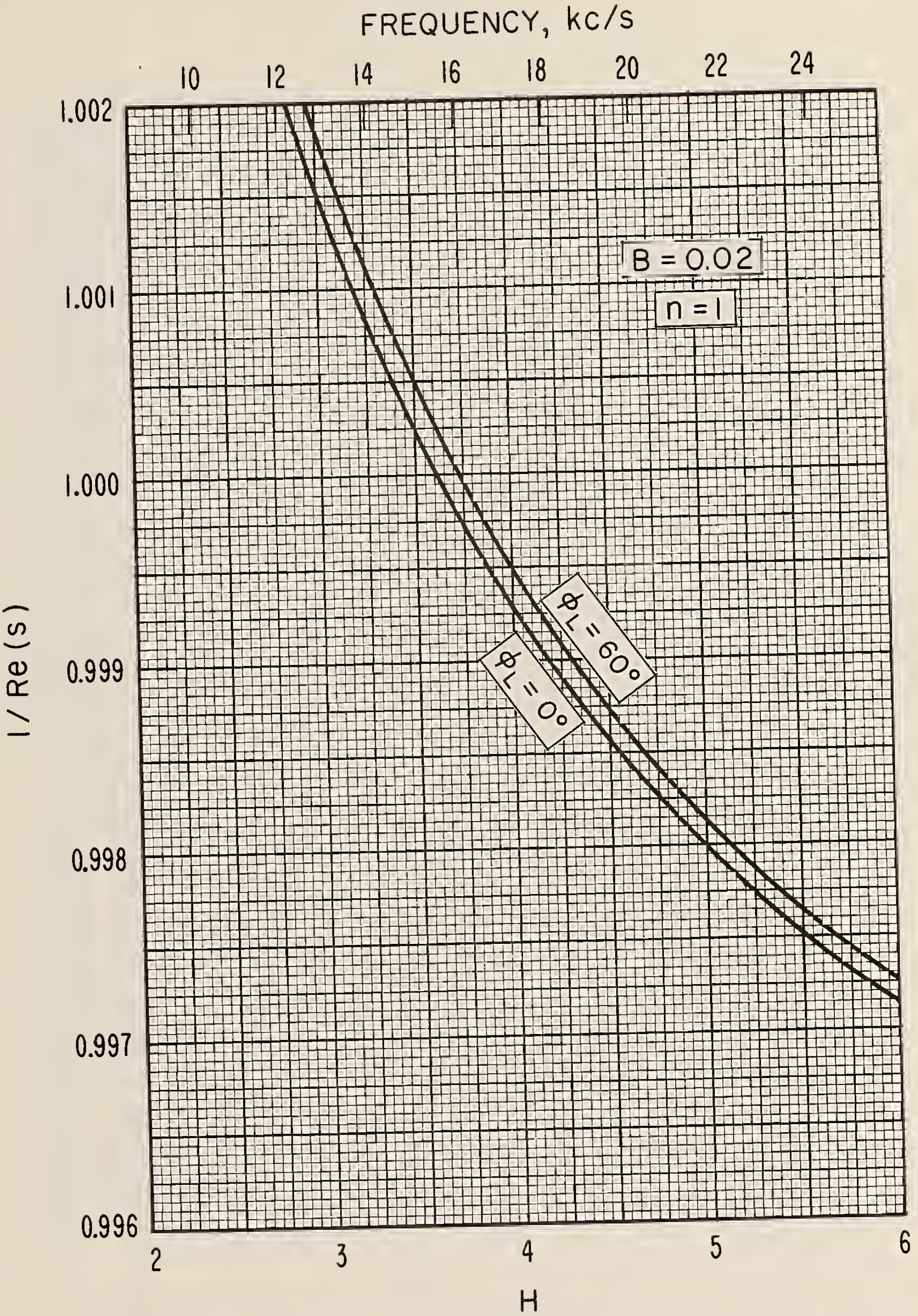

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
1.002
1.001
1.000
0.999
0.998
0.997
0.996
1 / Re (s)
B = 0.02
n = 1
$\phi_L = 60°$
$\phi_L = 0°$
2
3
4
5
6
H

Fig. 26

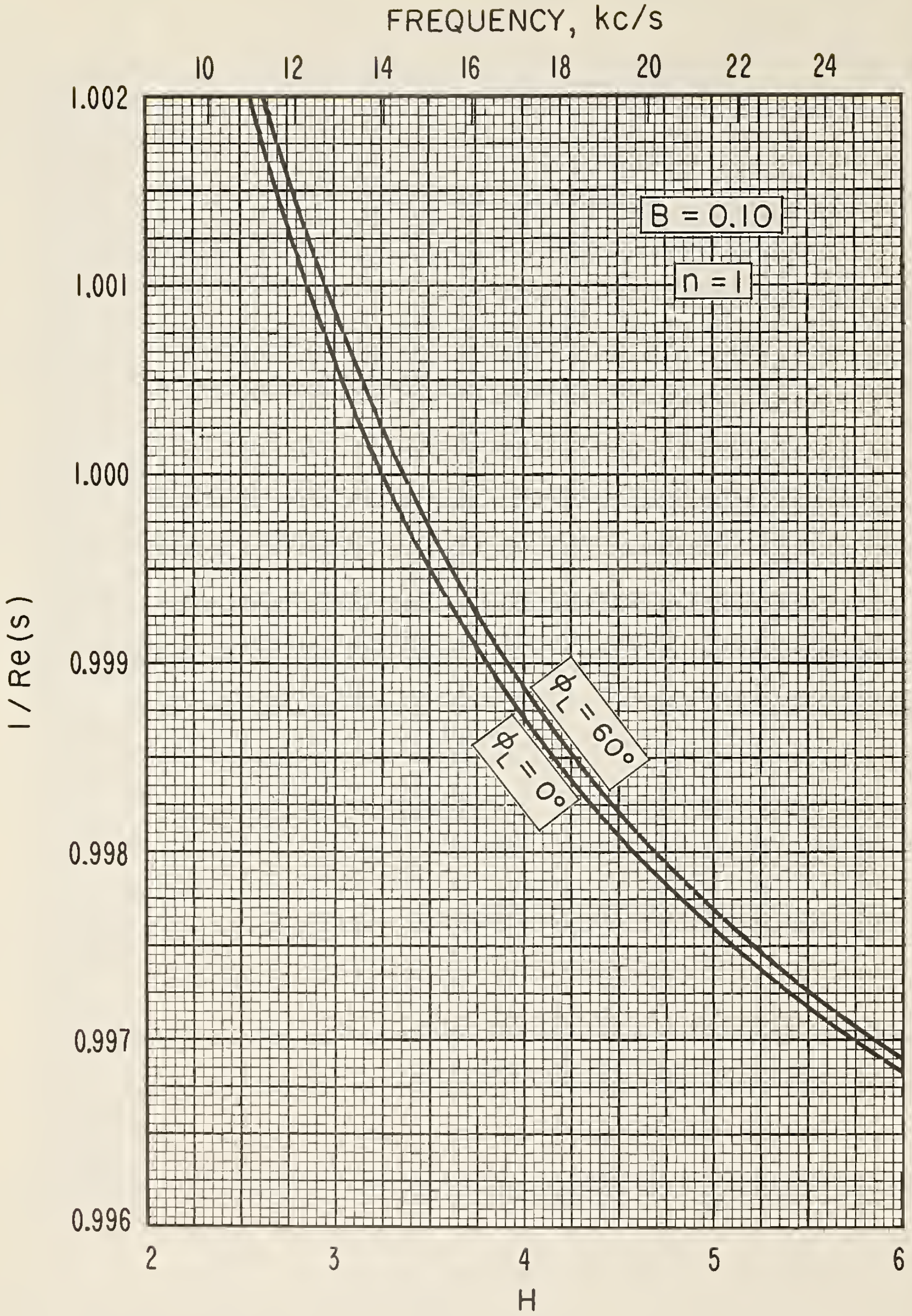

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
1.002
1.001
1.000
0.999
0.998
0.997
0.996
1 / Re(s)
B = 0.10
n = 1
$\phi_L = 60°$
$\phi_L = 0°$
2
3
4
5
6
H

Fig. 27

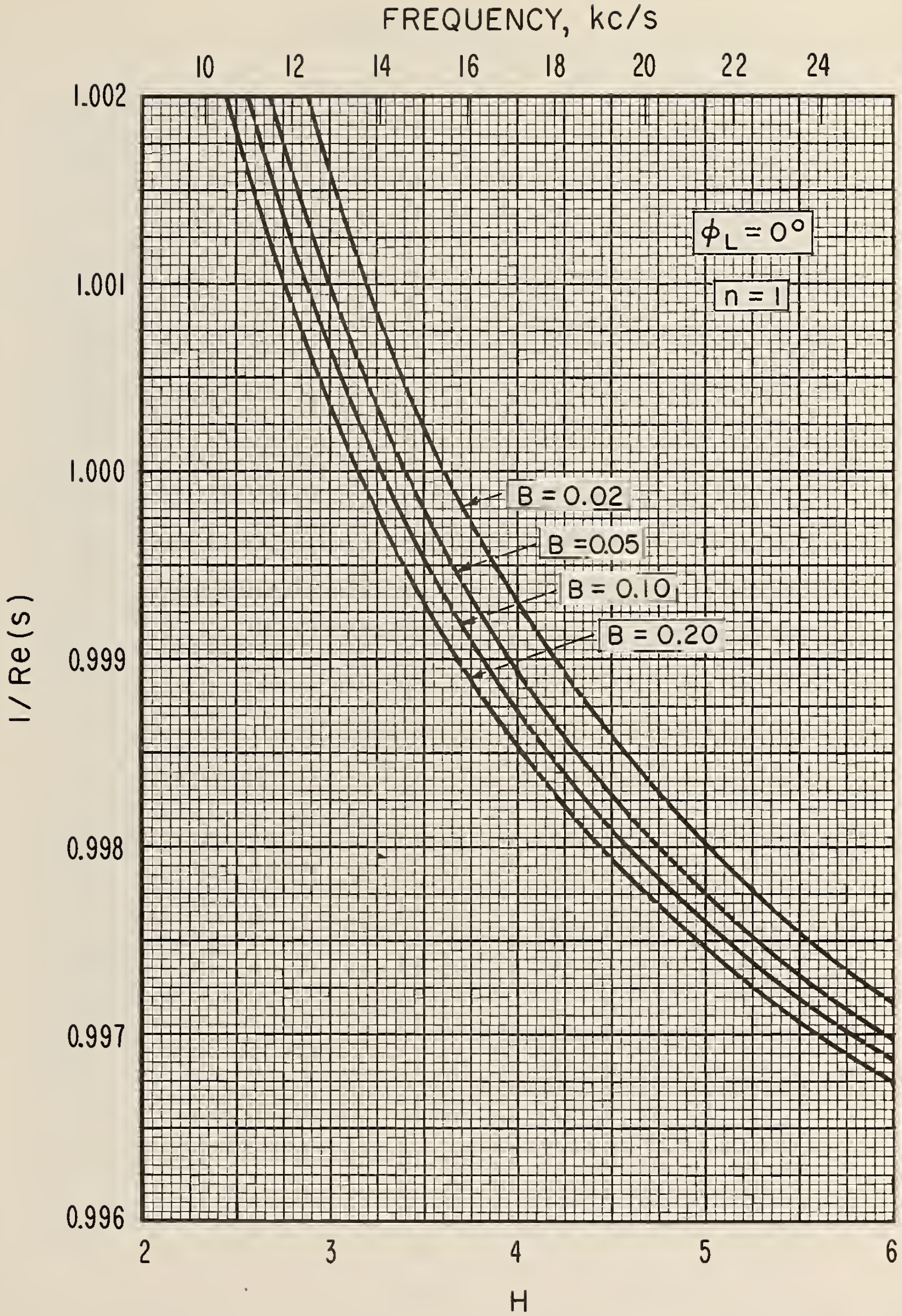

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
1.002
1.001
1.000
0.999
0.998
0.997
0.996
1/Re(s)
$\phi_L = 0°$
n = 1
B = 0.02
B = 0.05
B = 0.10
B = 0.20
2
3
4
5
6
H

Fig. 28

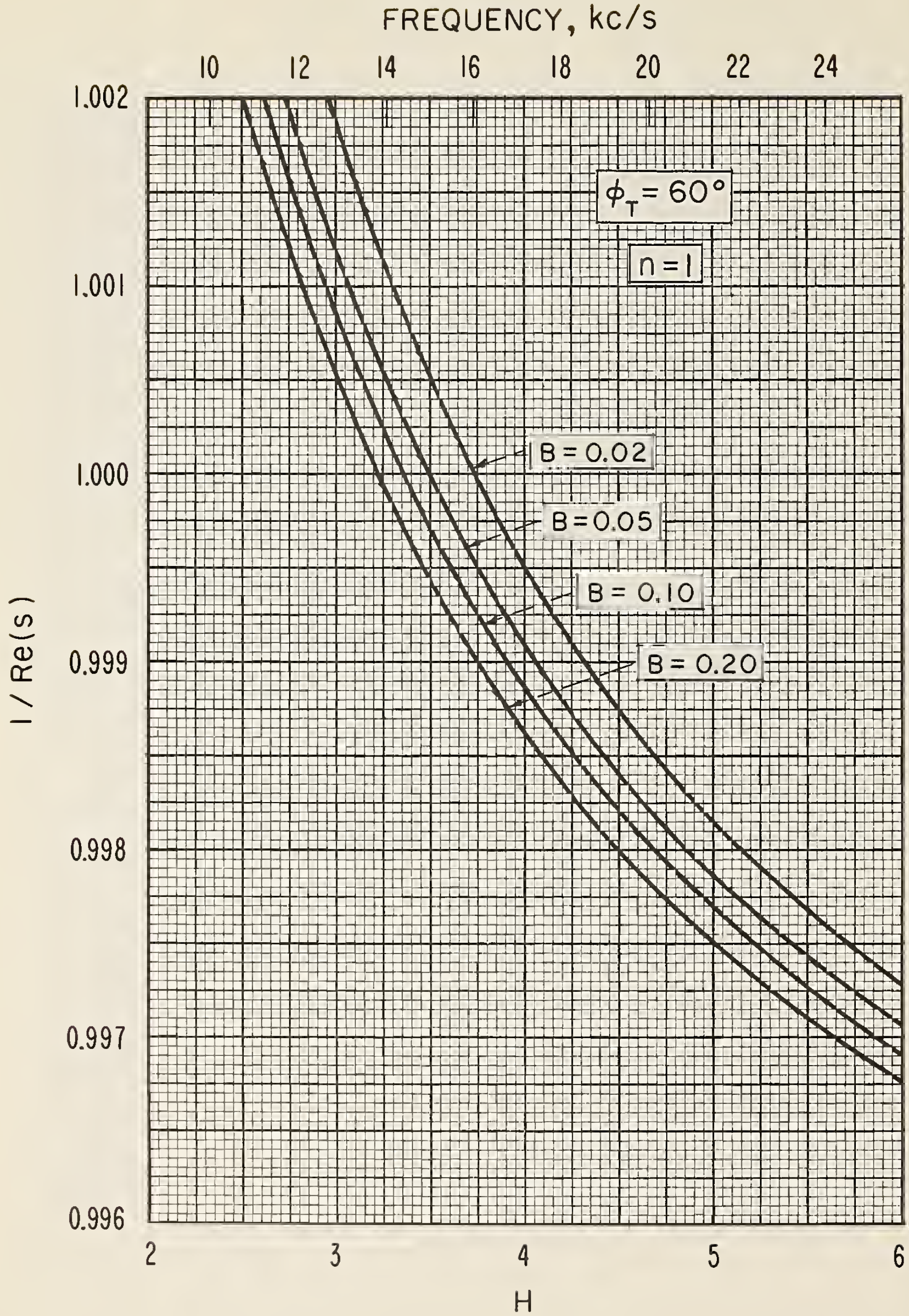

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
1.002
1.001
1.000
0.999
0.998
0.997
0.996
1 / Re(s)
$\phi_T = 60°$
n = 1
B = 0.02
B = 0.05
B = 0.10
B = 0.20
2
3
4
5
6
H

Fig. 29

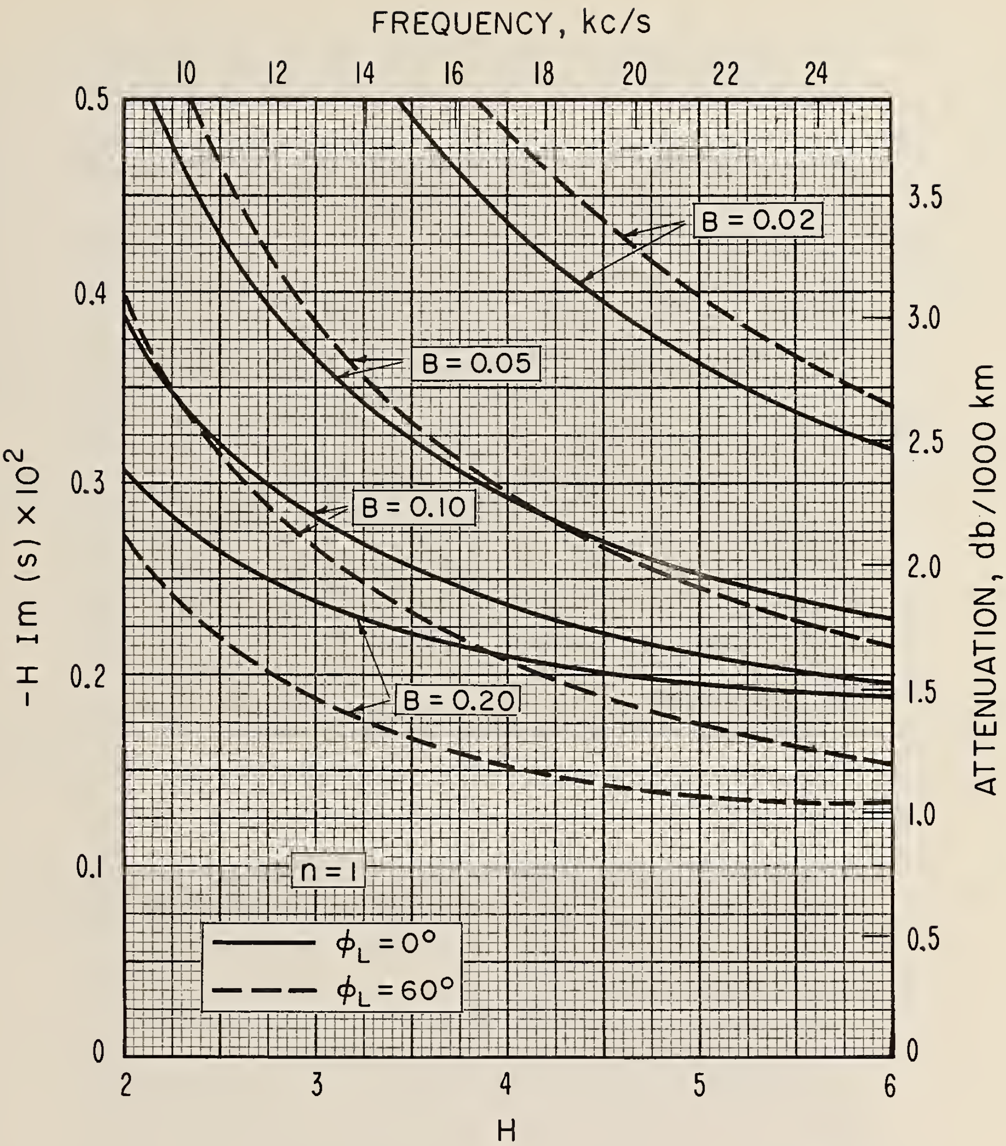

FREQUENCY, kc/s
10
12
14
16
18
20
22
24
0.5
0.4
0.3
0.2
0.1
0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
B = 0.02
B = 0.05
B = 0.10
B = 0.20
n = 1
$\phi_L = 0°$
$\phi_L = 60°$
$-H \, Im \, (s) \times 10^2$
ATTENUATION, db/1000 km
2
3
4
5
6
H

Fig. 30

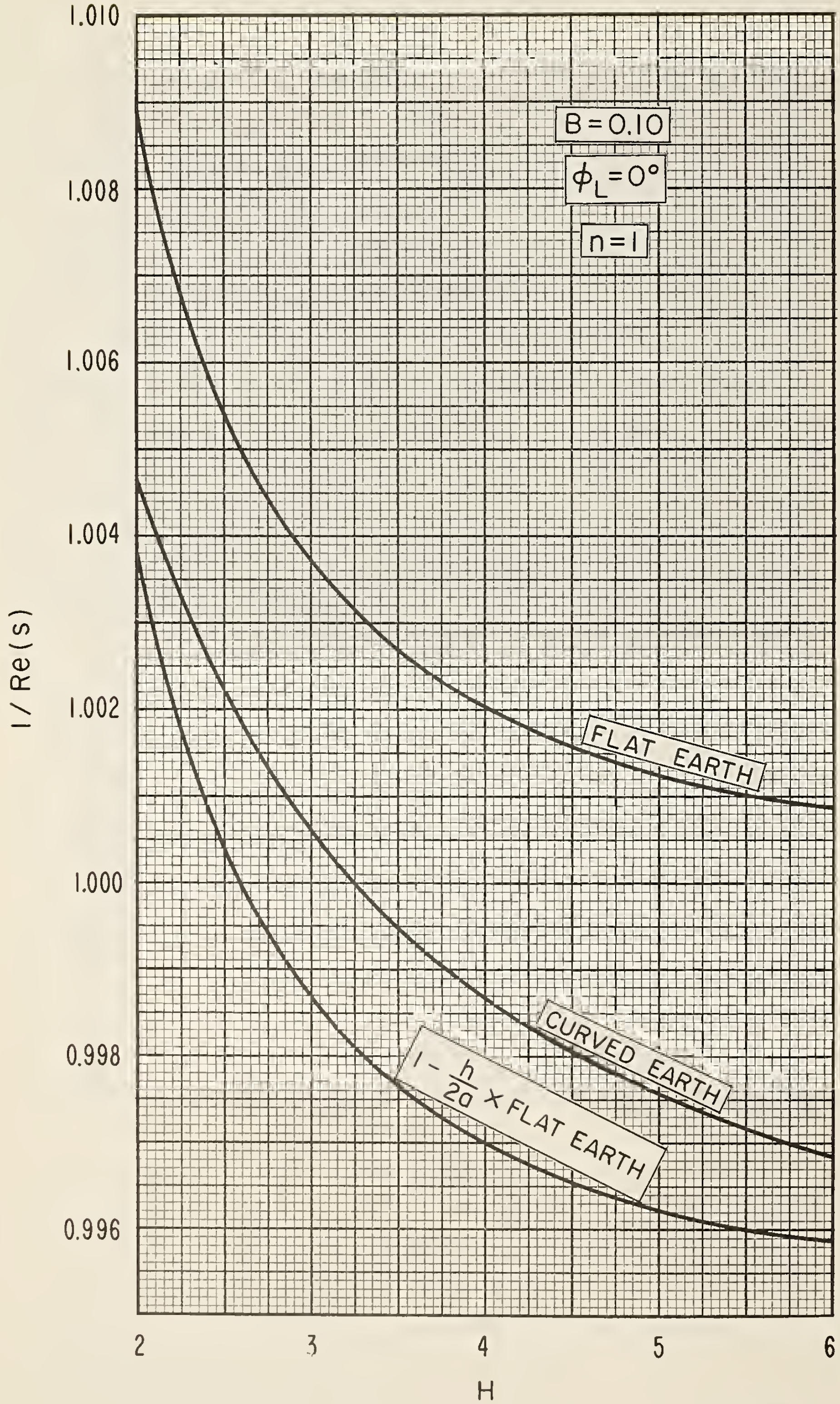

B=0.10
ϕ_L=0°
n=1
FLAT EARTH
CURVED EARTH
$1-\frac{h}{2a}$ × FLAT EARTH
1.010
1.008
1.006
1.004
1.002
1.000
0.998
0.996
1/Re(s)
2
3
4
5
6
H

Fig. 31

An exact solution of the mode equation (27) for perfectly reflecting boundaries (to be described in the following section) indicates that the approximate method described above must be used with considerable caution. For one thing, the assumption that $|C^2| \ll 2\,h/a$ is not always justified for n=1. In fact, for n = 2, the reverse is often true, especially for the lower frequencies (around 10 kc/s). Also when the condition $|C^2| \ll 2\,h/a$ fails, the approximation $i \log[w_2'(t)/w_1'(t)] \approx \pi/3$ is also very poor.

To obtain the corresponding "crude solution" for finitely conducting ground, one may approximate

$$i \log \left[\frac{w_2'(t) - q\, w_2(t)}{w_1'(t) - q\, w_1(t)} \right]$$

by expanding the logarithm in a Taylor series about t = 0, then neglecting terms involving t^2 and higher powers to get

$$i \log \left[\frac{w_2'(t) - q\, w_2(t)}{w_1'(t) - q\, w_1(t)} \right] \approx \frac{\pi}{3} + i \log \left[\frac{1 - q \frac{w_2(0)}{w_2'(0)}}{1 - q \frac{w_1(0)}{w_1'(0)}} \right]$$

$$-i \left(\frac{ka}{2}\right)^{2/3} \frac{q^2 \left(\frac{w_1(0)}{w_1'(0)} - \frac{w_2(0)}{w_2'(0)}\right) C^2}{\left(1 - q \frac{w_1(0)}{w_1'(0)}\right)\left(1 - q \frac{w_2(0)}{w_2'(0)}\right)} . \qquad (29)$$

Using this approximation, the method of solution described above gives

$$C^2 \approx \frac{(12n-5)\frac{\pi}{6} - \frac{2}{3}\, ka \left(\frac{2h}{a}\right)^{3/2} - i\, \alpha_1 \left(\frac{2h}{a}\right)^{1/2} - i\, \delta_o}{ka \left(\frac{2h}{a}\right)^{1/2} + i\, \frac{\alpha_1}{2} \left(\frac{2h}{a}\right)^{-1/2} - i \left(\frac{ka}{2}\right)^{2/3} \delta_1} \qquad (30)$$

$$\text{where } \delta_o = \log\left[\frac{1 - q\,\frac{w_2(0)}{w_2'(0)}}{1 - q\,\frac{w_1(0)}{w_1'(0)}}\right] \text{ and } \delta_1 = \frac{q^2\left(\frac{w_1(0)}{w_1'(0)} - \frac{w_2(0)}{w_2'(0)}\right)}{\left(1 - q\,\frac{w_1(0)}{w_1'(0)}\right)\left(1 - q\,\frac{w_2(0)}{w_2'(0)}\right)} \quad . \tag{31}$$

However, further effort along this line does not seem warranted, since there still exists the difficulty that $|C^2| \ll 2\,h/a$ is not always true (nor is the approximation (29) always realistic).

3. Solution for Perfectly Reflecting Boundaries - When $R_g = +1$ ($\sigma = \infty$ or $q = 0$) and $R_i = -1$ ($\alpha_1 = 0$), the mode equation (27) may be written as

$$\left(C^2 + \frac{2h}{a}\right)^{3/2} = \frac{1}{\frac{2}{3}ka}\left[(4n-1)\frac{\pi}{2} + 2\tan^{-1}\left(\frac{v'(t)}{u'(t)}\right)\right] \quad . \tag{32}$$

For perfectly reflecting boundaries, the modes propagate without attenuation, so that real solutions of (32) are desired. For real t,

$$i\log\left[\frac{w_2(t)}{w_1'(t)}\right] = -\,2\tan^{-1}\left(\frac{v'(t)}{u'(t)}\right) \quad , \tag{33}$$

where the inverse tangent is a continuous function[1] of t such that

$$\tan^{-1}\left(\frac{v'(0)}{u'(0)}\right) = -\,\frac{\pi}{6} \quad .$$

First approximations to the solutions of (32) were obtained for n = 1 and n = 2 by graphical means -- plotting the right- and left-hand members of (32) vs. C^2, then reading off the abscissae of the points of intersection. Two further approximations were then obtained by using the "method of false position."

---

[1] In appendix A, this function is tabulated to six decimal places at intervals of 0.1 from t = - 10.0 to t = + 5.0.

Curves of $(\frac{v}{c} - 1)$ vs. frequency have been computed for h = 60, 70, 80, 90, 100 km. The frequency range covered extends from 8 kc/s to 30 kc/s. [Using graphical means, curves have been interpolated for h = 65, 75, 85, 95 km.] These results are shown in figures 32 and 33.

For h = 60, 80, 100 km, curves of $(\frac{v}{c} - 1)$ vs. frequency were also computed using the crude method of section 2 [set $a_1 = 0$ in (28)] when n = 1. A comparison of the crude solution with the exact solution of this section is shown in figure 34.

4. <u>Newton's Method for Solving the Mode Equation</u> - Let $z = C^2$ and write the mode equation (27) as

$$F(z) = 0 \tag{34}$$

where
$$F(z) = \frac{2}{3} k a \left(z + \frac{2h}{a}\right)^{3/2} + i\, a_1 \left(z + \frac{2h}{a}\right)^{1/2} + i \log \left| \frac{w_2'(t) - q\, w_2(t)}{w_1'(t) - q\, w_1(t)} \right| - (4n - 1)\frac{\pi}{2} \tag{35}$$

and

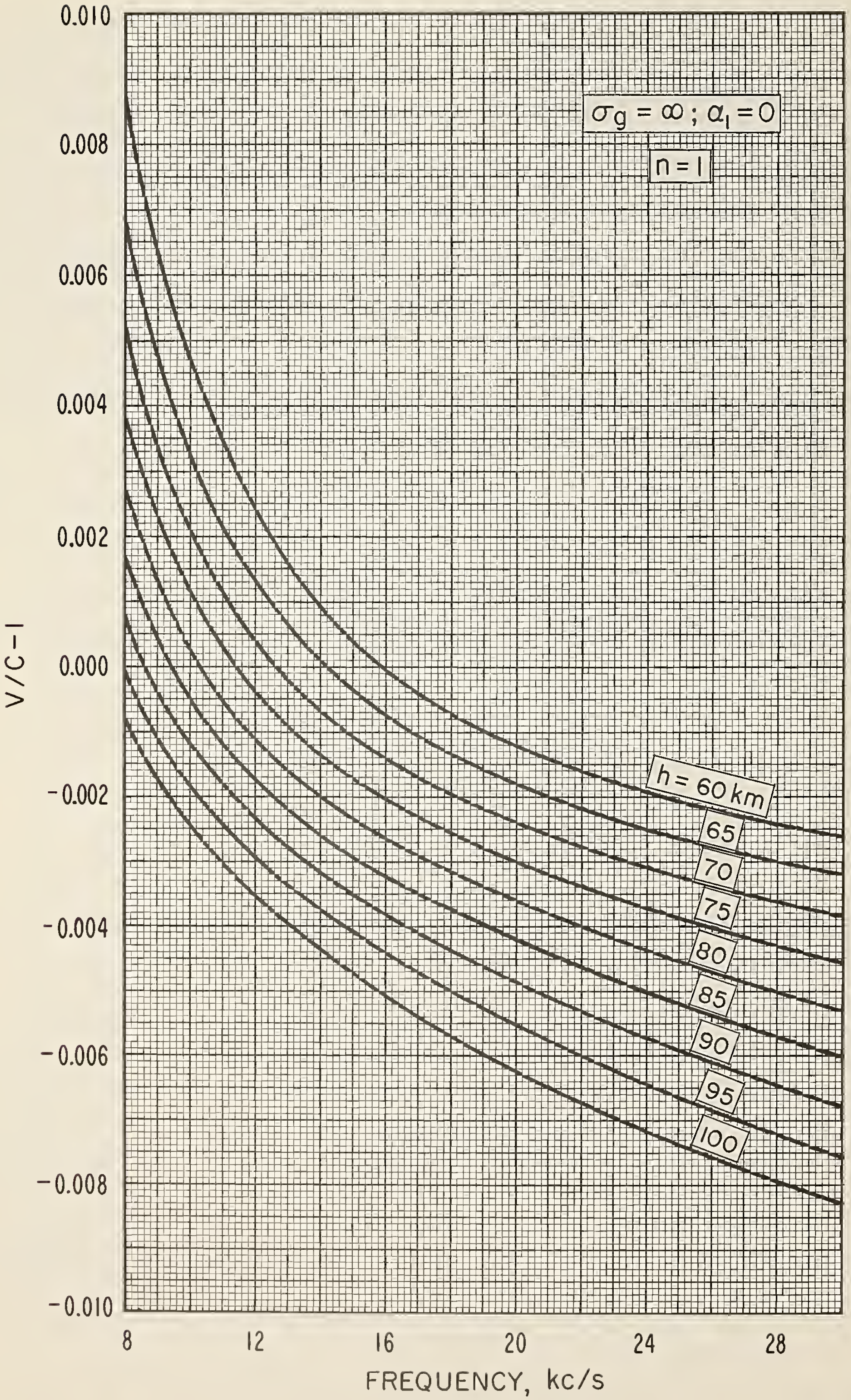
$$t = -\left(\frac{ka}{2}\right)^{2/3} z .$$

According to Newton's method, if z_o is an approximate root of (34), a next approximation z_1 is given by

$$z_1 = z_o + \Delta z$$

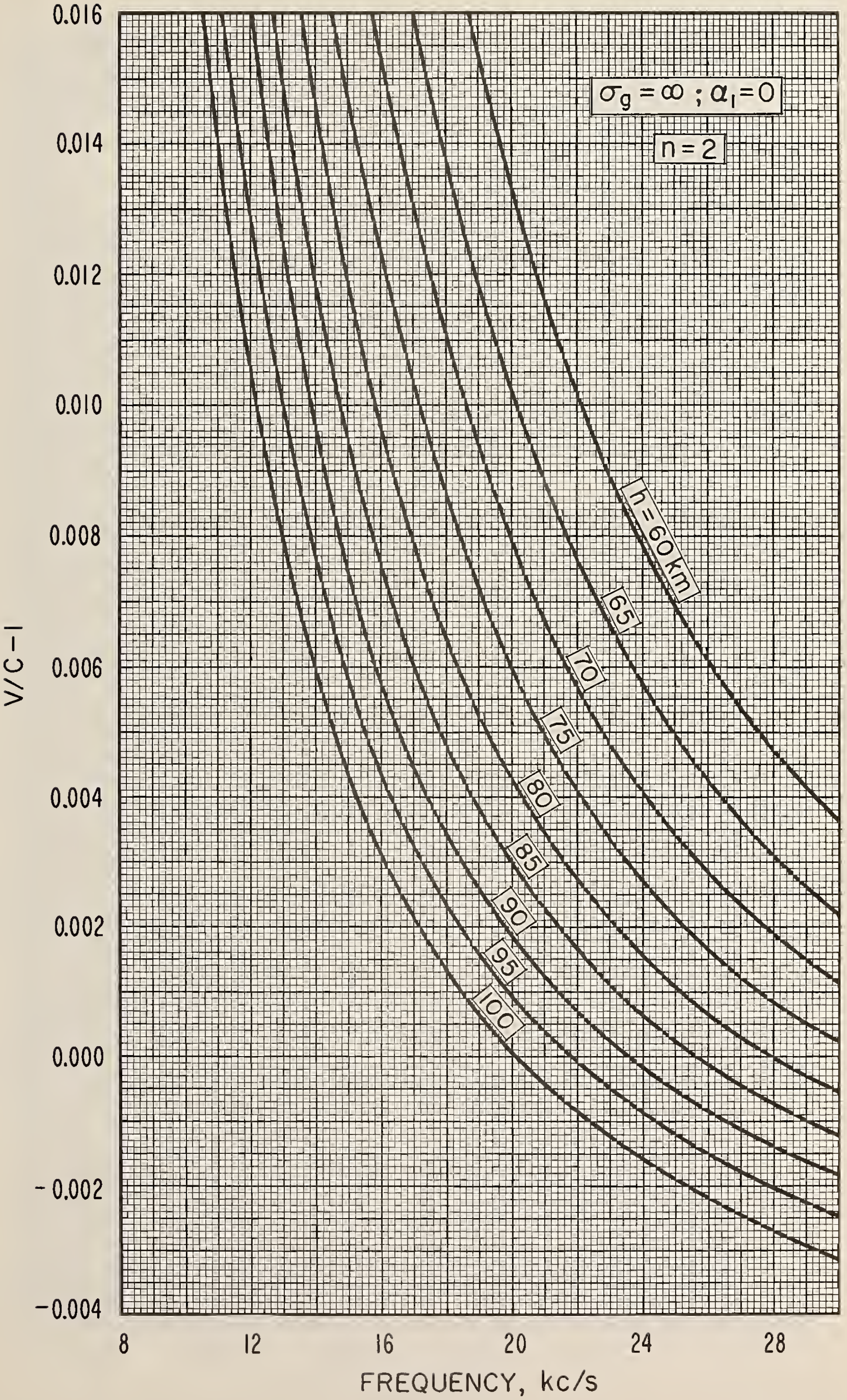
where

$$\Delta z = - \frac{F(z_o)}{F'(z_o)} .$$



$\sigma_g = \infty$; $\alpha_1 = 0$
n = 1
h = 60 km
65
70
75
80
85
90
95
100
0.010
0.008
0.006
0.004
0.002
0.000
-0.002
-0.004
-0.006
-0.008
-0.010
V/C-1
8
12
16
20
24
28
FREQUENCY, kc/s

Fig. 32



$\sigma_g = \infty$; $\alpha_1 = 0$
n = 2
h = 60 km
65
70
75
80
85
90
95
100
0.016
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000
- 0.002
-0.004
V/C - I
8
12
16
20
24
28
FREQUENCY, kc/s

Fig. 33

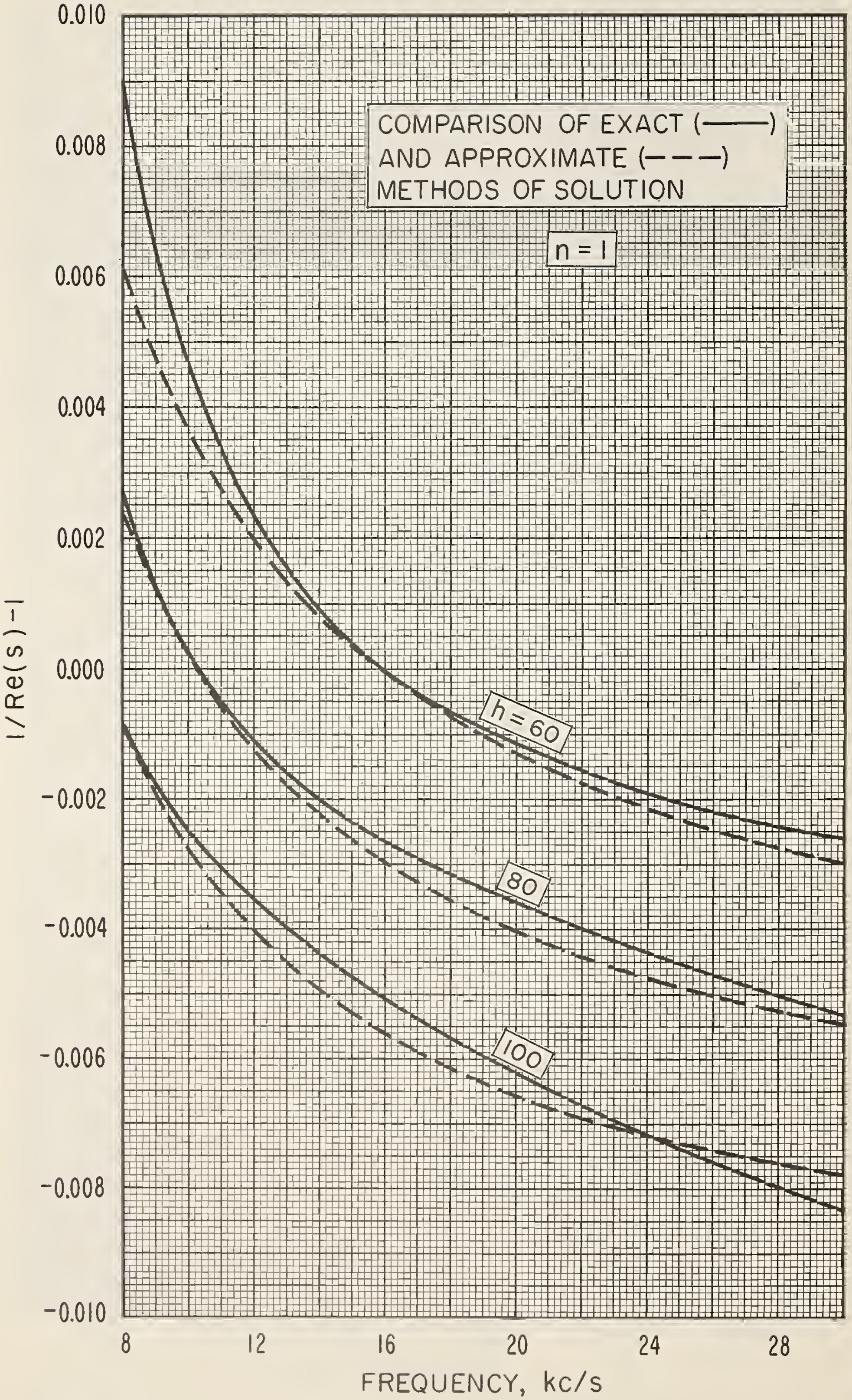

COMPARISON OF EXACT (——)
AND APPROXIMATE (– – –)
METHODS OF SOLUTION
n = I
h = 60
80
100
0.010
0.008
0.006
0.004
0.002
0.000
-0.002
-0.004
-0.006
-0.008
-0.010
I/Re(s)-I
8
12
16
20
24
28
FREQUENCY, kc/s

Fig. 34

Differentiating (35) with respect to z and using the relation

$$w_1'(t)\, w_2(t) - w_1(t)\, w_2'(t) = 2i$$

gives $$F'(z) = ka\left(z + \frac{2h}{a}\right)^{1/2} + i\,\frac{\alpha_1}{2}\left(z + \frac{2h}{a}\right)^{-1/2}$$

$$+ \left(\frac{ka}{2}\right)^{2/3} \frac{2\,(t - q^2)}{[w_1'(t) - q\, w_1(t)]\,[w_2'(t) - q\, w_2(t)]} ,$$

so that $\Delta z =$

$$- \frac{\frac{2}{3} ka\left(z_o + \frac{2h}{a}\right)^{3/2} + i\,\alpha_1\left(z_o + \frac{2h}{a}\right)^{\frac{1}{2}} + i \log\left[\frac{w_2'(t_o) - q\, w_2(t_o)}{w_1(t_o) - q\, w_1(t_o)}\right] - (4n-1)\frac{\pi}{2}}{ka\left(z_o + \frac{2h}{a}\right)^{\frac{1}{2}} + i\,\frac{\alpha_1}{2}\left(z_o + \frac{2h}{a}\right)^{-\frac{1}{2}} + \left(\frac{ka}{2}\right)^{2/3} \frac{2(t_o - q^2)}{[w_1'(t_o) - q\, w_1(t_o)][w_2'(t_o) - q\, w_2(t_o)]}} \tag{36}$$

where $$t_o = -\left(\frac{ka}{2}\right)^{2/3} z_o .$$

Further approximations can be obtained by successive applications of Newton's method.

5. <u>Solution for Perfectly Reflecting Ionosphere and Finitely Conducting Ground</u> - For a perfectly reflecting ionosphere, $\alpha_1 = 0$, so that the mode equation (27) becomes

$$\frac{2}{3} ka\left(z + \frac{2h}{a}\right)^{3/2} + i \log\left[\frac{w_2'(t) - q\, w_2(t)}{w_1'(t) - q\, w_1(t)}\right] - (4n-1)\frac{\pi}{2} = 0 \tag{37}$$

where $z = C^2$. If z_o is an approximate root of (37), Newton's method then gives a next approximation, $z_1 = z_o + \Delta z$

where

$$\Delta z = - \frac{\frac{2}{3} ka\left(z_o + \frac{2h}{a}\right)^{3/2} + i \log\left[\frac{w_2'(t_o) - q\, w_2(t_o)}{w_1'(t_o) - q\, w_1(t_o)}\right] - (4n-1)\frac{\pi}{2}}{ka\left(z_o + \frac{2h}{a}\right)^{\frac{1}{2}} + \left(\frac{ka}{2}\right)^{2/3} \frac{2(t_o - q^2)}{\left[w_1'(t_o) - q\, w_1(t_o)\right]\left[w_2'(t_o) - q\, w_2(t_o)\right]}} . \tag{38}$$

If, for the first approximation, one chooses z_o to be the solution of the mode equation (32) for perfectly reflecting boundaries, so that

$$\frac{2}{3} ka\left(z_o + \frac{2h}{a}\right)^{3/2} + i \log\left[\frac{w_2'(t_o)}{w_1'(t_o)}\right] - (4n-1)\frac{\pi}{2} = 0 ,$$

then in the expression for the second approximation $z_1 = z_o + \Delta z$, Δz is given by

$$\Delta z = \frac{i \log\left[\frac{w_2'(t_o)}{w_1'(t_o)}\right] - i \log\left[\frac{w_2'(t_o) - q\, w_2(t_o)}{w_1'(t_o) - q\, w_1(t_o)}\right]}{ka\left(z_o + \frac{2h}{a}\right)^{\frac{1}{2}} + \left(\frac{ka}{2}\right)^{2/3} \frac{2(t_o - q^2)}{\left[w_1'(t_o) - q\, w_1(t_o)\right]\left[w_2'(t_o) - q\, w_2(t_o)\right]}} . \tag{39}$$

For any further approximations, however, one must use (38) to evaluate Δz.

Choosing z_o to be the solution of the mode equation for perfectly reflecting boundaries, a single application of Newton's method has been used to compute curves of $(\frac{v}{c} - 1)$ vs. frequency and attenuation (db/1000 km) vs. frequency for n = 1 when $\sigma_g = 5 \times 10^{-3}$ mho/m and h = 60, 70, 80, 90, 100 km. The frequency range covered extends from 8 kc/s to 30 kc/s. Using graphical means, curves have been interpolated for h = 65, 75, 85, 95 km. These results are shown in figures 35 and 36.

6. Solution for Imperfectly Reflecting Ionosphere and Perfectly Conducting Ground - For a perfectly conducting ground $q = 0$ ($\sigma_g = \infty$), so that the mode equation (27) becomes

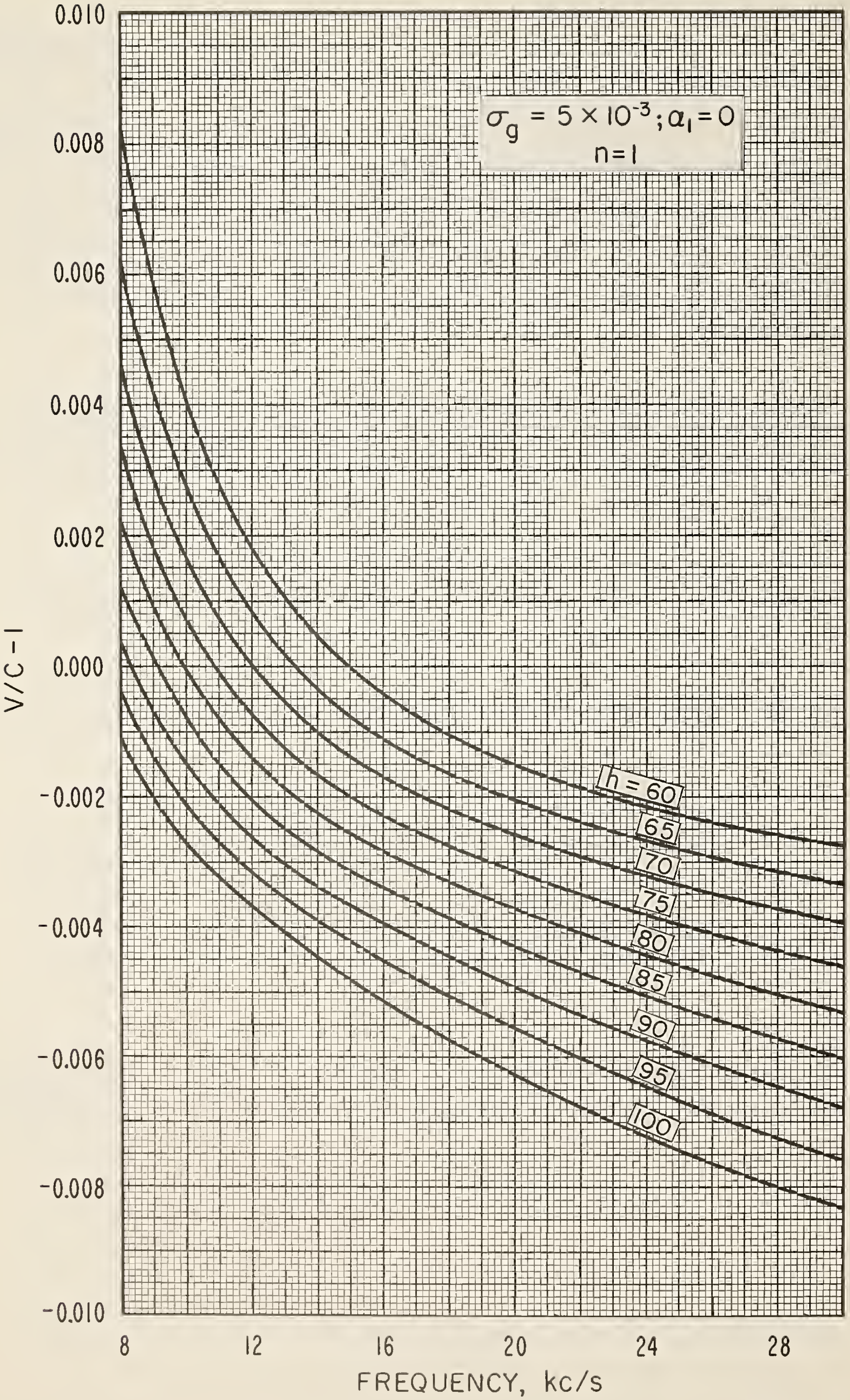
$$\frac{2}{3} ka \left(z + \frac{2h}{a}\right)^{3/2} + i\, \alpha_1 \left(z + \frac{2h}{a}\right)^{1/2} + i \log\left[\frac{w_2'(t)}{w_1'(t)}\right] - (4n-1)\,\frac{\pi}{2} = 0 \qquad (40)$$

where $z = C^2$. If z_o is an approximate root of (40), Newton's method then gives a next approximation $z_1 = z_o + \Delta z$

$$\text{where} \quad \Delta z = - \frac{\frac{2}{3} ka \left(z_o + \frac{2h}{a}\right)^{3/2} + i\, \alpha_1 \left(z_o + \frac{2h}{a}\right)^{\frac{1}{2}} + i \log\left[\frac{w_2'(t_o)}{w_1'(t_o)}\right] - (4n-1)\,\frac{\pi}{2}}{ka \left(z_o + \frac{2h}{a}\right)^{\frac{1}{2}} + i\, \frac{\alpha_1}{2} \left(z_o + \frac{2h}{a}\right)^{-\frac{1}{2}} + \left(\frac{ka}{2}\right)^{2/3} \frac{2\, t_o}{w_1'(t_o)\, w_2'(t_o)}} \; . \qquad (41)$$

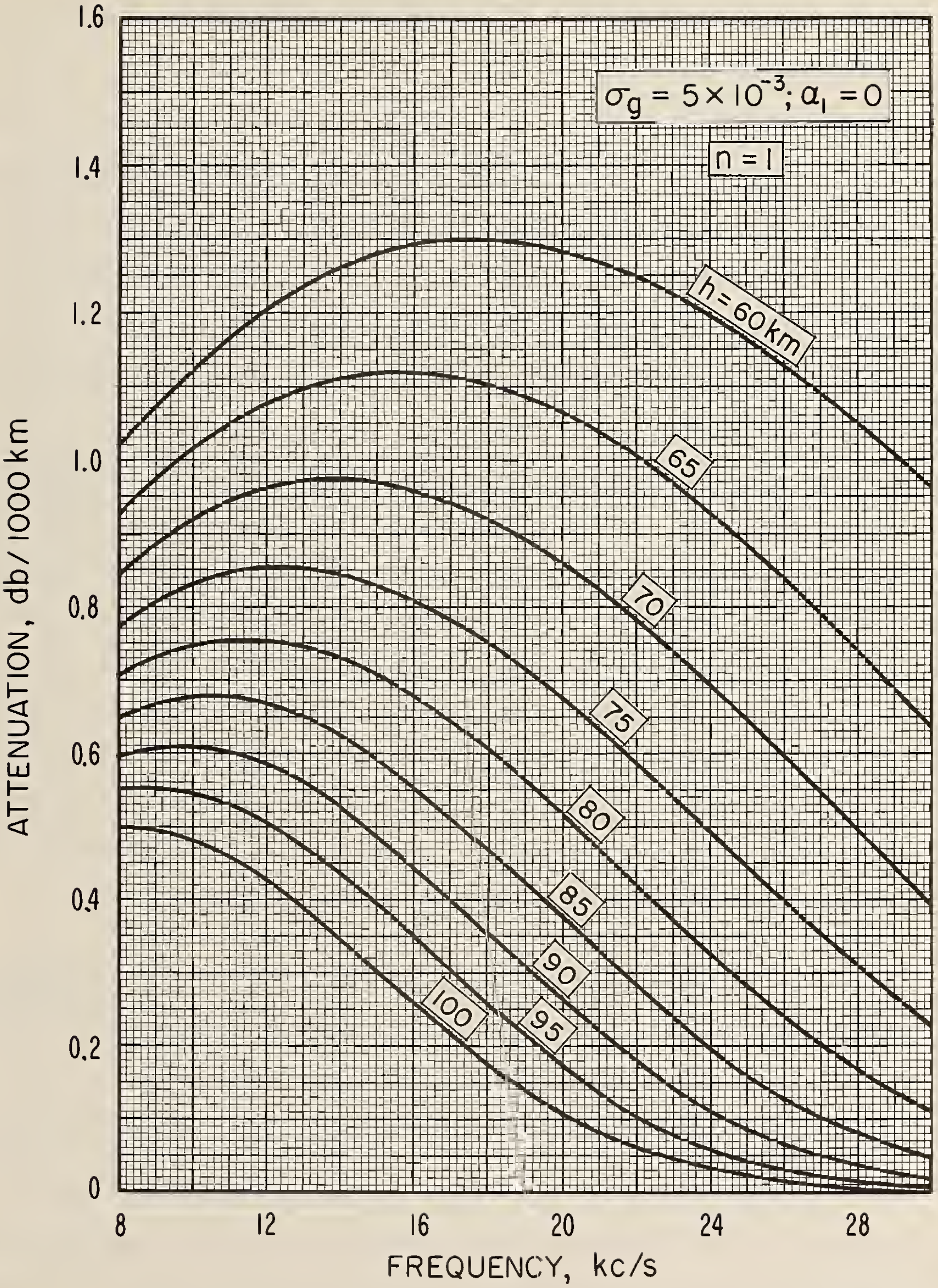
If, for the first approximation, one chooses z_o to be the solution of the mode equation (32) for perfectly reflecting boundaries, so that

$$\frac{2}{3} ka \left(z_o + \frac{2h}{a}\right)^{3/2} + i \log\left[\frac{w_2'(t_o)}{w_1'(t_o)}\right] - (4n-1)\,\frac{\pi}{2} = 0$$



0.010
0.008
0.006
0.004
0.002
0.000
-0.002
-0.004
-0.006
-0.008
-0.010
V/C - 1
$\sigma_g = 5 \times 10^{-3}; \alpha_1 = 0$
n=1
h = 60
65
70
75
80
85
90
95
100
8
12
16
20
24
28
FREQUENCY, kc/s

Fig. 35



$\sigma_g = 5 \times 10^{-3}; a_1 = 0$
n = 1
h = 60 km
65
70
75
80
85
90
95
100
ATTENUATION, db/1000 km
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
8
12
16
20
24
28
FREQUENCY, kc/s

Fig. 36

then in the expression for the <u>second</u> approximation $z_1 = z_o + \Delta z$, Δz is given by

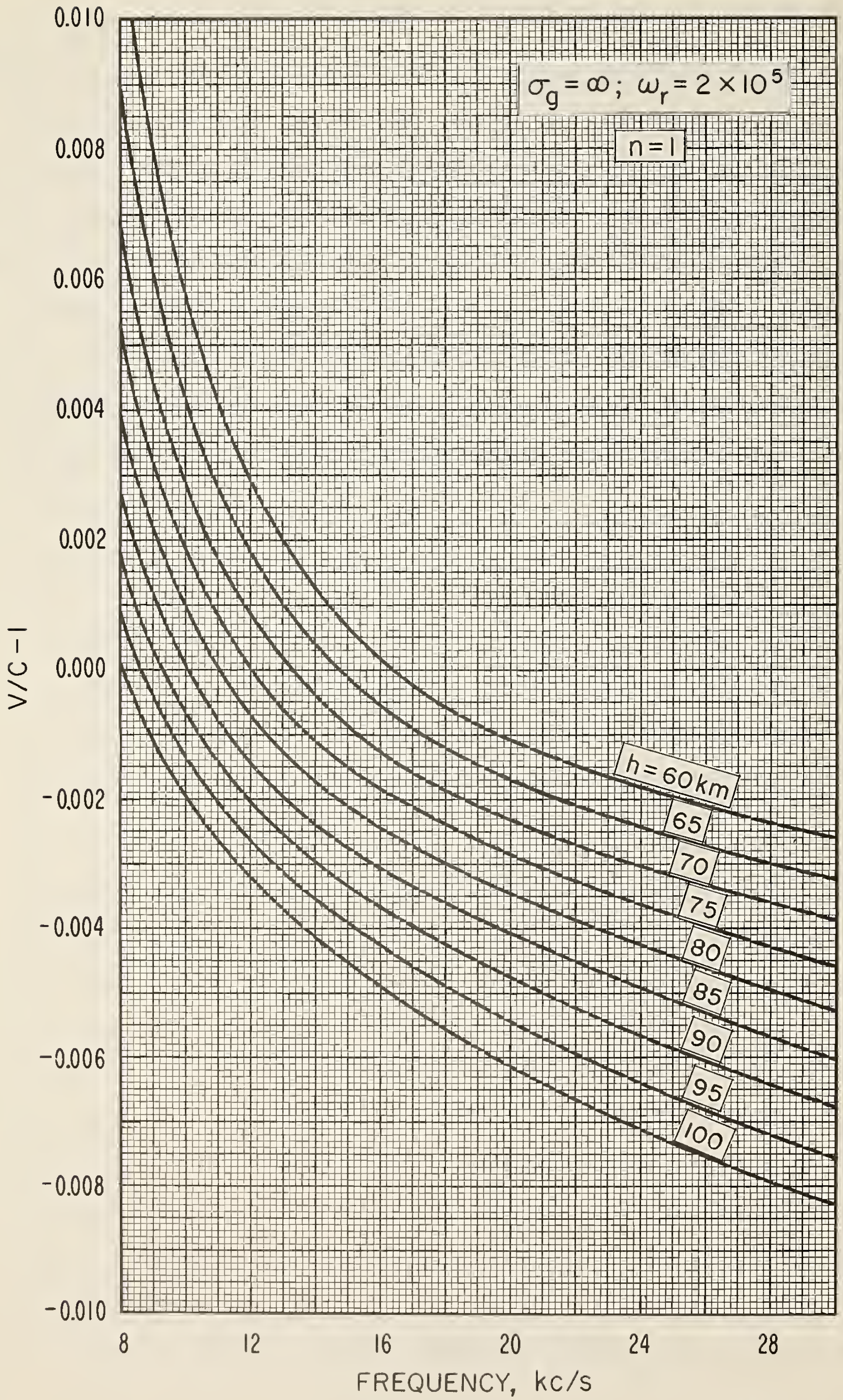
$$\Delta z = - \frac{i\, a_i \left(z_o + \frac{2h}{a}\right)^{\frac{1}{2}}}{ka\left(z_o + \frac{2h}{a}\right)^{\frac{1}{2}} + i\,\frac{a_1}{2}\left(z_o + \frac{2h}{a}\right)^{-\frac{1}{2}} + \left(\frac{ka}{2}\right)^{2/3} \frac{2\, t_o}{[u'(t_o)]^2 + [v'(t_o)]^2}} . \qquad (42)$$

For any further approximations, however, one must use (41) to evaluate Δz.

Choosing z_o to be the solution of the mode equation for perfectly reflecting boundaries, a single application of Newton's method has been used to compute curves of $(\frac{v}{c} - 1)$ vs. frequency and attenuation (db/1000 km) vs. frequency for n = 1 and n = 2 in the absence of a terrestrial magnetic field when $\omega_r = 2 \times 10^5$ and h = 60, 70, 80, 90, 100 km. Again, the frequency range covered extends from 8 kc/s to 30 kc/s. Using graphical means, $(\frac{v}{c} - 1)$ vs. frequency curves have been interpolated for h = 65, 75, 85, 95 km for n = 2. Attenuation vs. frequency curves have been interpolated for these same values of h. These results are shown in figures 37 through 40.

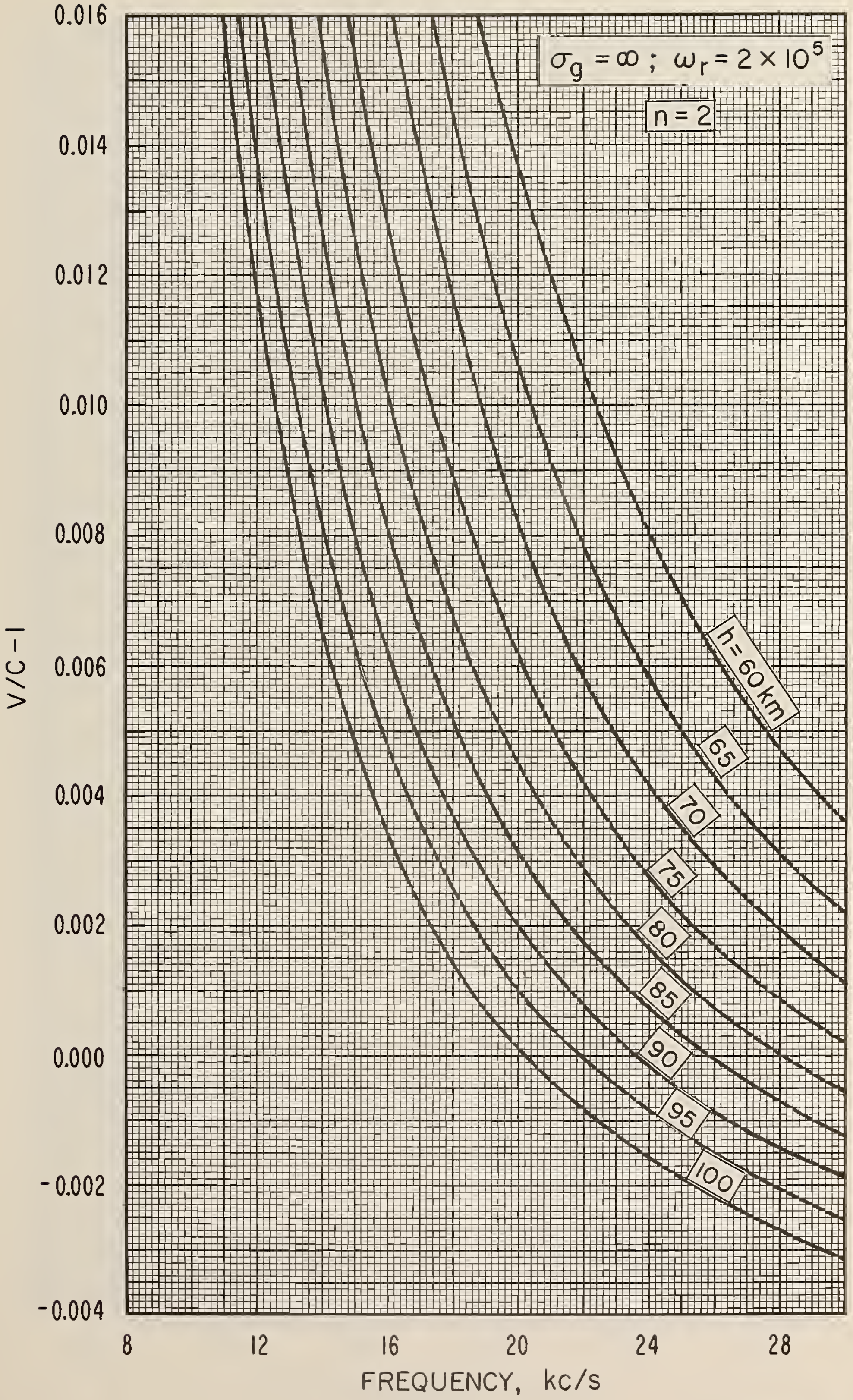
7. <u>Solution for Imperfectly Reflecting Ionosphere and Finitely Conducting Ground</u> - Using Newton's method, solutions of mode equation (27) have been obtained for n = 1 when h = 70 km, $\omega_r = 2 \times 10^5$, and σ_g = 1, 2, 5, 10, 20 millimhos/meter. Again, the frequency range extended from 8 to 30 kc/s. Initially, z_o was chosen to be the solution of (27) when $\omega_r = 2 \times 10^5$ and $\sigma_g = \infty$. However, for the smaller conductivities at the lower frequencies, it was found that such a choice of starting values did not result in convergence, so the computing technique was modified in the following manner. The solutions for σ_g = 20 millimhos/meter were obtained first by choosing z_o to be the solution of mode equation (27) for perfectly conducting ground and

$\omega_r = 2 \times 10^5$. The solutions for $\sigma_g = 20$ were then used as starting values to obtain solutions for $\sigma_g = 10$ and these solutions were in turn used as starting values for $\sigma_g = 5$, and so on. In every case, one application of Newton's method was sufficient to produce adequate convergence, though a second application was carried out as a check on the computations. The results of these calculations, in the form of curves of $(\frac{v}{c} - 1)$ vs. frequency and attenuation (db/1000 km) vs. frequency, are shown in figures 41 and 42. Calculations for $\sigma_g = 4$ mhos/meter (corresponding to sea water) were also carried out, but the results were practically identical to those for $\sigma_g = \infty$.



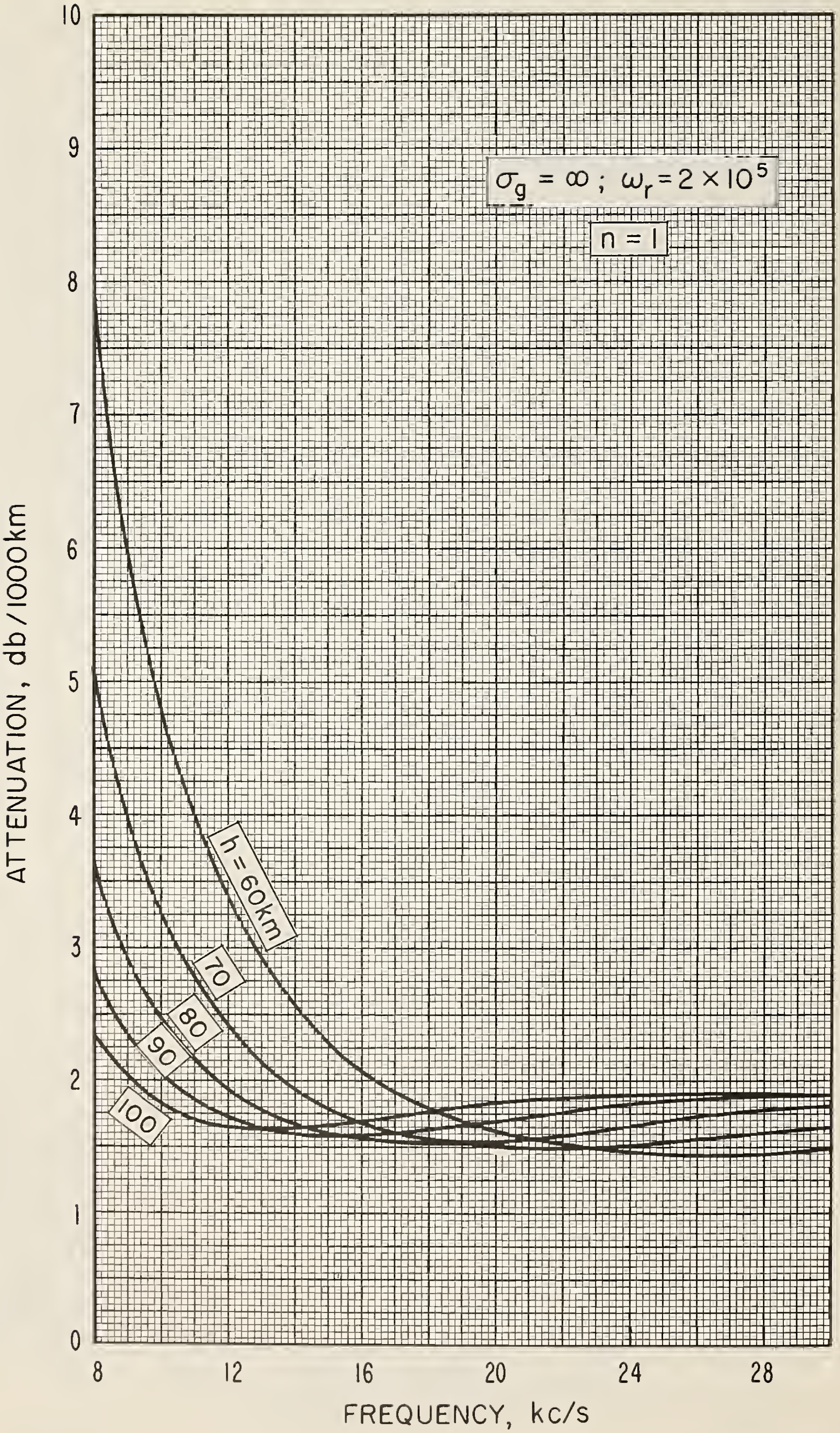
0.010
0.008
0.006
0.004
0.002
0.000
-0.002
-0.004
-0.006
-0.008
-0.010
V/C − I
8
12
16
20
24
28
FREQUENCY, kc/s
$\sigma_g = \infty$; $\omega_r = 2 \times 10^5$
n = I
h = 60 km
65
70
75
80
85
90
95
100

Fig. 37



$\sigma_g = \infty$; $\omega_r = 2 \times 10^5$
n = 2
h= 60 km
65
70
75
80
85
90
95
100
0.016
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000
-0.002
-0.004
V/C-1
8
12
16
20
24
28
FREQUENCY, kc/s

Fig. 38



$\sigma_g = \infty$; $\omega_r = 2 \times 10^5$
n = 1
h = 60km
70
80
90
100
ATTENUATION, db/1000km
0
1
2
3
4
5
6
7
8
9
10
FREQUENCY, kc/s
8
12
16
20
24
28

Fig. 39

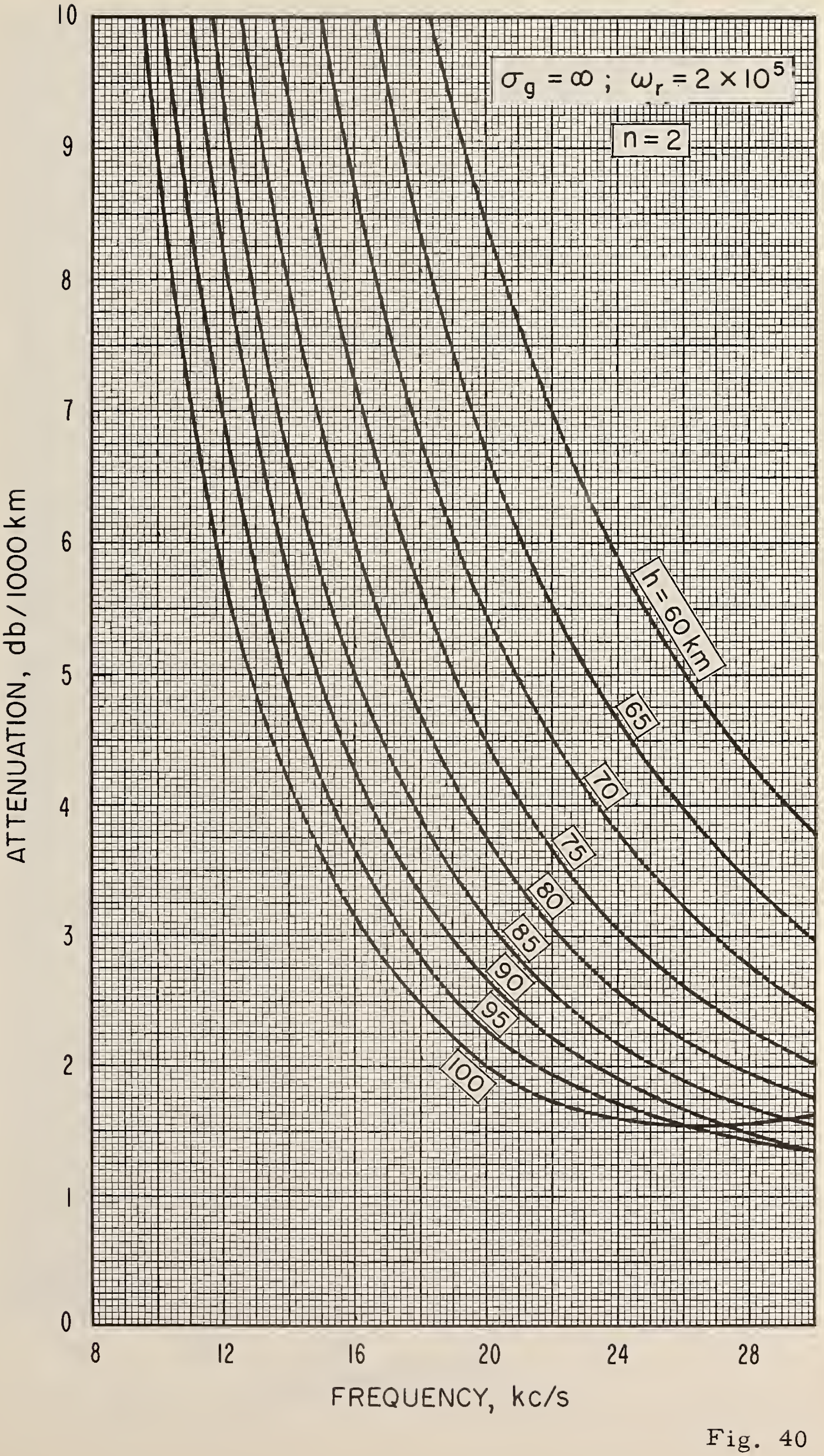

$\sigma_g = \infty$; $\omega_r = 2 \times 10^5$
n = 2
h = 60 km
65
70
75
80
85
90
95
100
ATTENUATION, db/1000 km
FREQUENCY, kc/s
0
1
2
3
4
5
6
7
8
9
10
8
12
16
20
24
28

Fig. 40

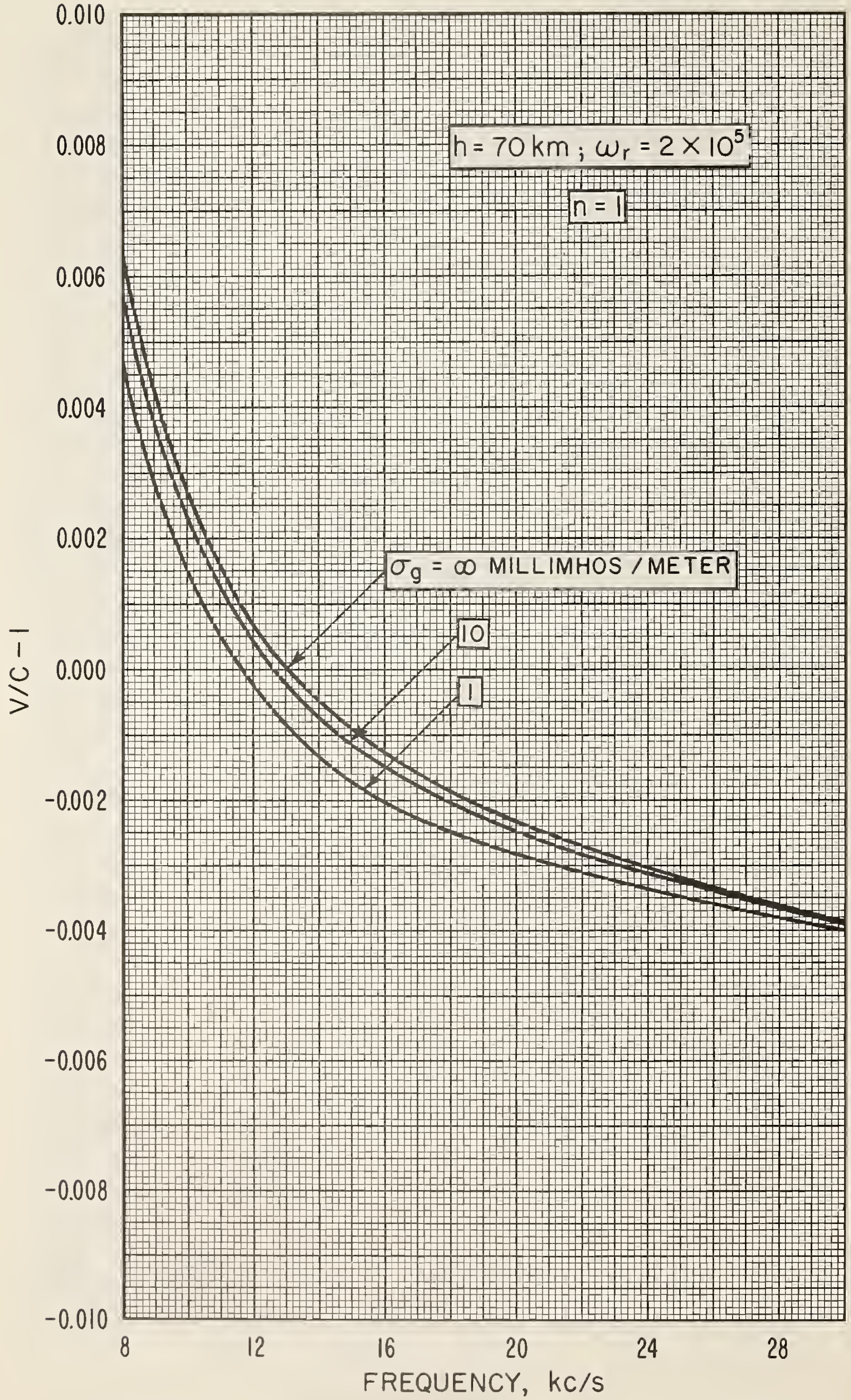

0.010
0.008
0.006
0.004
0.002
0.000
-0.002
-0.004
-0.006
-0.008
-0.010
V/C − 1
h = 70 km ; ω_r = 2 × 10[5]
n = 1
σ_g = ∞ MILLIMHOS / METER
10
1
8
12
16
20
24
28
FREQUENCY, kc/s

Fig. 41

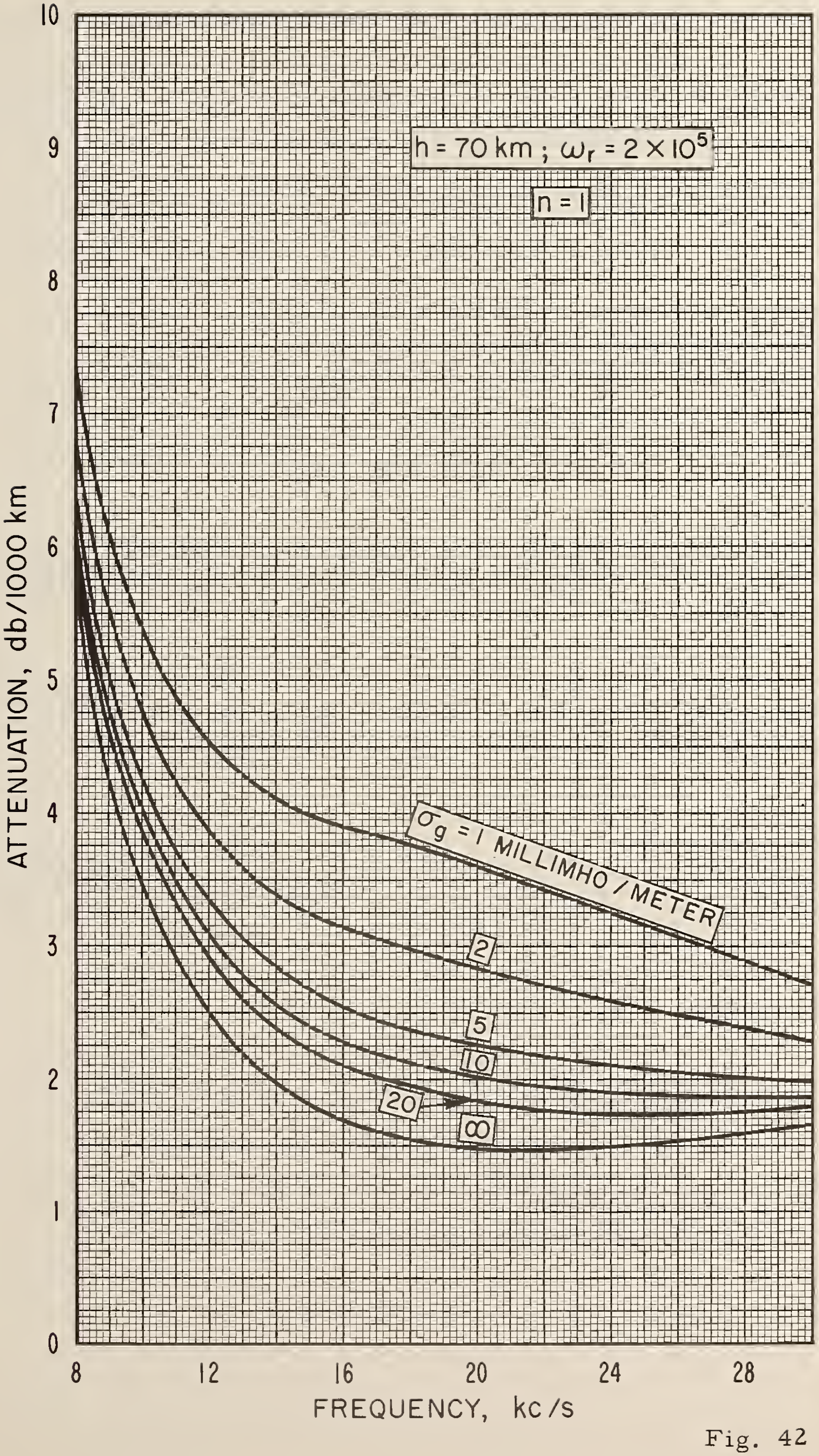

h = 70 km ; ω_r = 2 × 10^5
n = 1
σ_g = 1 MILLIMHO / METER
2
5
10
20
∞
ATTENUATION, db/1000 km
FREQUENCY, kc/s
0
1
2
3
4
5
6
7
8
9
10
8
12
16
20
24
28

Fig. 42

IV. Further Calculations for a Spherical Earth-Ionosphere Waveguide

1. <u>A More Accurate Form of the Mode Equation</u> - A more accurate form of the mode equation for VLF propagation in a spherical earth-ionosphere waveguide is given by [Wait, 1960]

$$\left[\frac{w_2'(t_a) - q\, w_2(t_a)}{w_1'(t_a) - q\, w_1(t_a)}\right]\left[\frac{w_1'(t_c) + q\, w_1(t_c)}{w_2'(t_c) + q\, w_2(t_c)}\right] e^{-i\Phi} = e^{-i2n\pi} \qquad (43)$$

where

$$\Phi = 2(\gamma_c - \gamma_a) - 2(\rho_c - \rho_a) \;, \qquad (\mathrm{Re}\,(S) < 1)$$

$$= 2\gamma_c - 2\rho_c \;, \qquad (1 < \mathrm{Re}\,(S) < 1 + \frac{h}{a})$$

$$\rho_a = \frac{ka\,S}{3}\left[\frac{(ka)^2}{(ka\,S)^2} - 1\right]^{3/2} \;, \qquad \gamma_a = \int_{ka\,S}^{ka}\left[1 - \frac{(ka\,S)^2}{x^2}\right]^{1/2} dx$$

$$\rho_c = \frac{ka\,S}{3}\left[\frac{(kc)^2}{(ka\,S)^2} - 1\right]^{3/2} \;, \qquad \gamma_c = \int_{ka\,S}^{kc}\left[1 - \frac{(ka\,S)^2}{x^2}\right]^{1/2} dx$$

and

$$\rho_a = \frac{2}{3}(-t_a)^{3/2} \;, \qquad t_a = -(\frac{3}{2}\rho_a)^{2/3} \;,$$

$$\rho_c = \frac{2}{3}(-t_c)^{3/2} \;, \qquad t_c = -(\frac{3}{2}\rho_c)^{2/3} \;,$$

$$c = h + a \,.$$

Equation (43) is not restricted by the condition $h/a \ll 1$. By carrying out the above integrations and a certain amount of algebra, this mode equation may be written (if $z = C^2$)

$$i \log \left[\frac{w_2'(t_a) - q\, w_2(t_a)}{w_1'(t_a) - q\, w_1(t_a)} \right] + i \log \left[\frac{w_1'(t_c) + q_i\, w_1(t_c)}{w_2'(t_c) + q_i\, w_2(t_c)} \right] + \Phi - 2n\pi = 0 \tag{44}$$

where $t_a = -\left(\frac{ka}{2}\right)^{2/3} \frac{z}{(1-z)^{2/3}}$, $t_c = -\left(\frac{ka}{2}\right)^{2/3} \frac{z + \frac{2h}{a} + \frac{h^2}{a^2}}{(1-z)^{2/3}}$,

and

$$\Phi = 2\,ka \left\{ \sqrt{z + \frac{2h}{a} + \frac{h^2}{a^2}} - \sqrt{z} + \sqrt{1-z} \left[\cos^{-1}(\sqrt{1-z}) - \cos^{-1}\left(\frac{\sqrt{1-z}}{1 + \frac{h}{a}} \right) \right] \right\}$$

$$- \frac{2\,ka}{3} \frac{1}{1-z} \left[\left(z + \frac{2h}{a} + \frac{h^2}{a^2} \right)^{3/2} - z^{3/2} \right]. \quad (\mathrm{Re}(z) > 0)$$

$$= 2\,ka \left[\sqrt{z + \frac{2h}{a} + \frac{h^2}{a^2}} - \sqrt{1-z} \cos^{-1}\left(\frac{\sqrt{1-z}}{1 + \frac{h}{a}} \right) \right] - \frac{2\,ka}{3} \frac{\left(z + \frac{2h}{a} + \frac{h^2}{a^2} \right)^{3/2}}{1-z}$$

$$(\mathrm{Re}(z) < 0).$$

The (complex) inverse cosines above are made unique by requiring them to vanish as their arguments approach unity. Since the arguments are in fact close to one, an infinite series representation convenient for this range is obtained by considering $\cos^{-1}(1-u)$ where $|u|$ is small. Now

$$\cos^{-1}(1-u) = \frac{\pi}{2} - \sin^{-1}(1-u) = \frac{\pi}{2} - \int_{o}^{1-u} \frac{dt}{\sqrt{1-t^2}} ,$$

so

$$\cos^{-1}(1-u) = \frac{\pi}{2} - \left[\int_{o}^{1} \frac{dt}{\sqrt{1-t^2}} - \int_{1-u}^{1} \frac{dt}{\sqrt{1-t^2}} \right] = \int_{1-u}^{1} \frac{dt}{\sqrt{1-t^2}} .$$

Introducing a new variable of integration by means of the relation $w = 1 - t$, one gets

$$\cos^{-1}(1-u) = - \int_{u}^{o} \frac{dw}{\sqrt{2w - w^2}} = \frac{1}{\sqrt{2}} \int_{o}^{u} w^{-\frac{1}{2}} (1 - \frac{w}{2})^{-\frac{1}{2}} dw .$$

Expanding the right-hand integrand into an infinite series (using the binomial theorem) and integrating term-by-term gives the desired result:

$$\cos^{-1}(1-u) = \sqrt{2u} \left[1 + \frac{1}{1!(3)(2)^2} u + \frac{(1)(3)}{2!(5)(2)^4} u^2 + \frac{(1)(3)(5)}{3!(7)(2)^6} u^3 + \ldots \right] . \qquad (45)$$

This result is now used to obtain infinite series for

$$\cos^{-1}(\sqrt{1-z}) \qquad \text{and} \qquad \cos^{-1}\left(\frac{\sqrt{1-z}}{1+\frac{h}{a}} \right) .$$

Expanding $\sqrt{1-z}$ into an infinite series (again using the binomial theorem), one has

$$\cos^{-1}(\sqrt{1-z}) = \cos^{-1}(1-u_1) \tag{46a}$$

where
$$u_1 = \frac{1}{1!(2)} z + \frac{1}{2!(2)^2} z^2 + \frac{(1)(3)}{3!(2)^3} z^3 + \frac{(1)(3)(5)}{4!(2)^4} z^4 + \ldots . \tag{46b}$$

Now
$$\frac{\sqrt{1-z}}{1+\frac{h}{a}} = \left(1 - \frac{\frac{h}{a}}{1+\frac{h}{a}}\right)(1-u_1) = 1 - \frac{1}{1+\frac{h}{a}}\left[\frac{h}{a} - \frac{h}{a} u_1 + u_1 \left(1+\frac{h}{a}\right)\right] .$$

Thus
$$\cos^{-1}\left(\frac{\sqrt{1-z}}{1+\frac{h}{a}}\right) = \cos^{-1}(1-u_2) \tag{47a}$$

where
$$u_2 = \left(1+\frac{h}{a}\right)^{-1}\left(\frac{h}{a} + u_1\right). \tag{47b}$$

2. <u>Solution for Perfectly Reflecting Boundaries</u> - When the ground is perfectly reflecting, $q = 0$; when the ionosphere is perfectly reflecting, $q_i = \infty$. The mode equation (44) then becomes

$$i \log\left[\frac{w_2'(t_a)}{w_1'(t_a)}\right] + i \log\left[\frac{w_1(t_c)}{w_2(t_c)}\right] + \Phi - 2n\pi = 0. \tag{48}$$

Again, real solutions of (48) are desired since the modes propagate without attenuation when the boundaries are perfectly reflecting. For real arguments,

$$i \log \left[\frac{w_2'(t_a)}{w_1'(t_a)} \right] = -2 \tan^{-1} \left[\frac{v'(t_a)}{u'(t_a)} \right] \qquad (49)$$

and

$$i \log \left[\frac{w_1(t_c)}{w_2(t_c)} \right] = 2 \tan^{-1} \left[\frac{v(t_c)}{u(t_c)} \right] \qquad (50)$$

where the inverse tangents are continuous functions[2] of t such that

$$\tan^{-1} \left[\frac{v'(0)}{u'(0)} \right] = -\frac{\pi}{6} \qquad \text{and} \qquad \tan^{-1} \left[\frac{v(0)}{u(0)} \right] = +\frac{\pi}{6} .$$

The mode equation for perfectly reflecting boundaries can now be written as

$$\tan^{-1} \left[\frac{v(t_c)}{u(t_c)} \right] - \tan^{-1} \left[\frac{v'(t_a)}{u'(t_a)} \right] + \frac{1}{2} \Phi - n\pi = 0. \qquad (51)$$

where t_a, t_c and Φ are defined in the previous section [following (44)].

---

2 In appendix A, these functions are tabulated to six decimal places at intervals of 0.1 from t = -10.0 to t = + 5.0.

The solutions of the mode equation (32) should be a fair approximation to the solutions of the mode equation (48) above. With these solutions as a guide, a single application of the "method of false position" has been used to obtain a further approximation for $n = 1$ when $h = 60$, 100 km. These results are shown in figure 43. As expected, they differ but little from those obtained by solving the mode equation (32).

3. Newton's Method for Solving the Mode Equation - Write the mode equation (44) as

$$F(z) = 0 \qquad (52)$$

where
$$F(z) = i \log \left[\frac{w_2'(t_a) - q\, w_2(t_a)}{w_1'(t_a) - q\, w_1(t_a)}\right] + i \log \left[\frac{w_1'(t_c) + q_i w_1(t_c)}{w_2'(t_c) + q_i\, w_2(t_c)}\right] + \Phi(z) - 2n\pi \qquad (53)$$

and t_a, t_c, $\Phi(z)$ are defined following (44). Then according to Newton's method, if z_o is an approximate root of (53), a next approximation z_1 is given by

$$z_1 = z_o + \Delta z$$

where

$$\Delta z = -\frac{F(z_o)}{F'(z_o)} .$$

Differentiating (53) with respect to z and using the relation

$$w_1'(t)\, w_2(t) - w_1(t)\, w_2'(t) = 2i$$

gives

$$F'(z) = \frac{2\left(\frac{ka}{2}\right)^{2/3}}{(1-z)\,(1-z)^{2/3}} \left\{ \frac{(t_a - q^2)(1 - \frac{1}{3} z)}{[w_1'(t_a) - q\, w_1(t_a)][w_2'(t_a) - q\, w_2(t_a)]} \right.$$

$$\left. - \frac{(t_c - q_i^2)(1 - z) + \frac{2}{3}\left(z + \frac{2h}{a} + \frac{h^2}{a^2}\right)}{[w_1'(t_c) + q_i\, w_1(t_c)]\,[w_2'(t_c) + q_i\, w_2(t_c)]} \right\} + \frac{d\Phi}{dz}$$

where

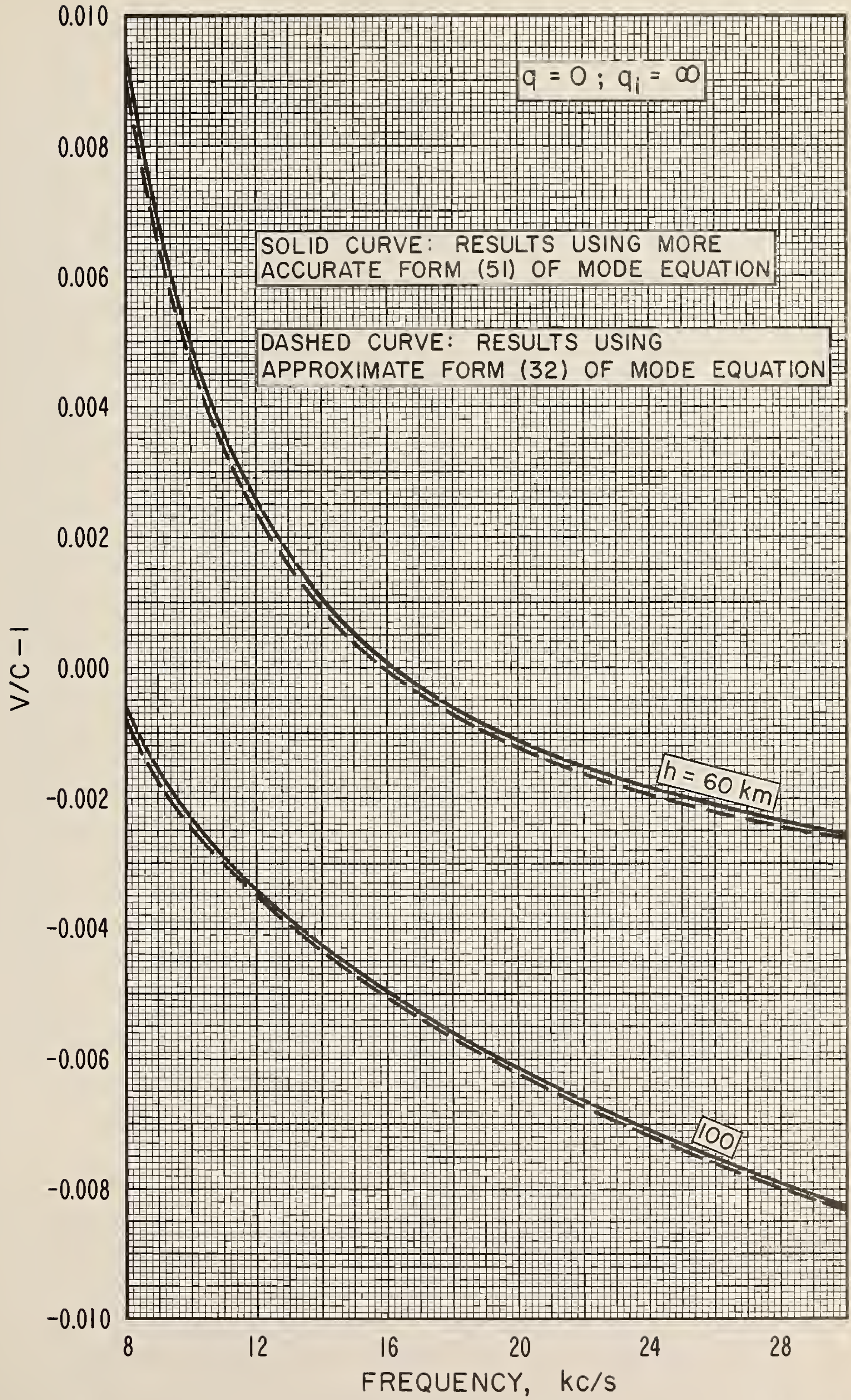
$$\frac{d\Phi}{dz} = ka \left\{ \frac{1}{\sqrt{1-z}} \left[\cos^{-1}\left(\frac{\sqrt{1-z}}{1 + \frac{h}{a}} \right) - \cos^{-1}\left(\sqrt{1-z}\right) \right] \right.$$

$$\left. - \frac{1}{1-z} \left[\sqrt{z + \frac{2h}{a} + \frac{h^2}{a^2}} - \sqrt{z} \right] \right\} + \frac{2}{3}\, ka\, \frac{1}{(1-z)^2} \left[\left(z + \frac{2h}{a} + \frac{h^2}{a^2} \right)^{3/2} - z^{3/2} \right]$$

$$(\mathrm{Re}\,(z) > 0)$$

$$= ka \left\{ \frac{1}{\sqrt{1-z}} \cos^{-1}\left(\frac{\sqrt{1-z}}{1 + \frac{h}{a}} \right) - \frac{1}{1-z} \sqrt{2 + \frac{2h}{a} + \frac{h^2}{a^2}} \right\}$$

$$+ \frac{2}{3}\, ka\, \frac{1}{(1-z)^2} \left(z + \frac{2h}{a} + \frac{h^2}{a^2} \right)^{3/2} . \qquad (\mathrm{Re}\,(z) < 0)$$



0.010
0.008
0.006
0.004
0.002
0.000
-0.002
-0.004
-0.006
-0.008
-0.010
V/C − I
8
12
16
20
24
28
FREQUENCY, kc/s
q = 0 ; qi = ∞
SOLID CURVE: RESULTS USING MORE ACCURATE FORM (51) OF MODE EQUATION
DASHED CURVE: RESULTS USING APPROXIMATE FORM (32) OF MODE EQUATION
h = 60 km
100

Fig. 43

Appendix A

Table of Inverse Tangents of Airy Functions†

t	$\tan^{-1}\left[\frac{v(t)}{u(t)}\right]$	δ^2_m	$\tan^{-1}\left[\frac{v'(t)}{u'(t)}\right]$	δ^2_m
0.0	0.523 599	+4 603	-0.523 599	+ 11 929
0.1	0.462 765	4 595	-0.517 681	+ 11 671
0.2	0.406 521	4 560	-0.500 215	+ 10 745
0.3	0.354 832	4 500	-0.472 147	+ 9 044
0.4	0.307 637	4 409	-0.435 170	+ 6 612
0.5	0.264 845	4 288	-0.391 666	+ 3 720
0.6	0.226 335	4 134	-0.344 452	+ 771
0.7	0.191 953	3 950	-0.296 397	- 1 797
0.8	0.161 515	3 734	-0.250 015	- 3 690
0.9	0.134 806	3 492	-0.207 183	- 4 825
1.0	0.111 585	3 228	-0.169 050	- 5 276
1.1	0.091 589	2 946	-0.136 099	- 5 215
1.2	0.074 538	2 659	-0.108 305	- 4 839
1.3	0.060 145	2 364	-0.085 318	- 4 289
1.4	0.048 117	2 077	-0.066 609	- 3 679
1.5	0.038 168	1 799	-0.051 582	- 3 086
1.6	0.030 021	1 541	-0.039 649	- 2 535
1.7	0.023 418	1 298	-0.030 264	- 2 056
1.8	0.018 118	1 082	-0.022 948	- 1 646
1.9	0.013 905	893	-0.017 290	- 1 302
2.0	0.010 589	724	-0.012 946	- 1 023
2.1	0.008 002	583	-0.009 635	- 796
2.2	0.006 002	463	-0.007 128	- 614
2.3	0.004 469	365	-0.005 242	- 472
2.4	0.003 304	284	-0.003 833	- 357
2.5	0.002 426	220	-0.002 786	- 272

† These functions, expressed in degrees, have also been extensively tabulated by Miller [1946]. Note that

$$\tan^{-1}\left[\frac{v(t)}{u(t)}\right] = \chi(t) \quad \text{and} \quad \tan^{-1}\left[\frac{v'(t)}{u'(t)}\right] = \psi(t),$$

where $\chi(t)$ and $\psi(t)$ are the functions given in Miller's table. (The authors are grateful to Nelson A. Logan for pointing out this relationship.) (Also, see footnote on page 76.)

t	$\tan^{-1}\left[\frac{v(t)}{u(t)}\right]$	δ^2	$\tan^{-1}\left[\frac{v'(t)}{u'(t)}\right]$	δ^2
2.5	0.002 426	+222	- 0.002 786	- 275
2.6	0.001 770	168	- 0.002 014	- 206
2.7	0.001 282	129	- 0.001 448	- 153
2.8	0.000 923	096	- 0.001 035	- 113
2.9	0.000 660	073	- 0.000 735	- 085
3.0	0.000 470	052	- 0.000 520	- 060
3.1	0.000 332	039	- 0.000 365	- 045
3.2	0.000 233	029	- 0.000 255	- 033
3.3	0.000 163	020	- 0.000 178	- 022
3.4	0.000 113	015	- 0.000 123	- 017
3.5	0.000 078	011	- 0.000 085	- 011
3.6	0.000 054	007	- 0.000 058	- 008
3.7	0.000 037	005	- 0.000 039	- 007
3.8	0.000 025	004	- 0.000 027	- 003
3.9	0.000 017	002	- 0.000 018	- 003
4.0	0.000 011	003	- 0.000 012	- 002
4.1	0.000 008	000	- 0.000 008	- 001
4.2	0.000 005	001	- 0.000 005	- 002
4.3	0.000 003	001	- 0.000 004	+ 001
4.4	0.000 002	000	- 0.000 002	- 002
4.5	0.000 001	001	- 0.000 002	+ 001
4.6	0.000 001	000	- 0.000 001	- 001
4.7	0.000 001	- 001	- 0.000 001	000
4.8	0.000 000	+ 001	0.000 000	- 001
4.9	0.000 000	000	0.000 000	000
5.0	0.000 000	000	0.000 000	000

t	$\tan^{-1}\left[\frac{v(t)}{u(t)}\right]$	δ^2_m	$\tan^{-1}\left[\frac{v'(t)}{u'(t)}\right]$	δ^2_m
0.0	0.523 599	+ 4 603	-0.523 599	+ 11 929
-0.1	0.589 033	4 595	-0.517 678	11 698
-0.2	0.659 059	4 568	-0.500 116	11 159
-0.3	0.733 650	4 525	-0.471 424	10 463
-0.4	0.812 764	4 474	-0.432 278	9 716
-0.5	0.896 350	4 410	-0.383 414	8 977
-0.6	0.984 345	4 343	-0.325 564	8 286
-0.7	1.076 682	4 270	-0.259 417	7 655
-0.8	1.173 288	4 192	-0.185 602	7 096
-0.9	1.274 086	4 115	-0.104 680	6 594
-1.0	1.378 999	4 036	-0.017 151	6 163
-1.1	1.487 948	3 958	+0.076 550	5 778
-1.2	1.660 855	3 879	+0.176 038	5 442
-1.3	1.717 641	3 801	+0.280 976	5 147
-1.4	1.838 229	3 728	+0.391 068	4 888
-1.5	1.962 545	3 655	0.506 054	4 659
-1.6	2.090 516	3 582	0.625 704	4 455
-1.7	2.222 070	3 515	0.749 813	4 274
-1.8	2.357 139	3 447	0.878 199	4 110
-1.9	2.495 656	3 384	1.010 698	3 964
-2.0	2.637 557	3 320	1.147 164	3 833
-2.1	2.782 779	3 265	1.287 465	3 711
-2.2	2.931 265	3 203	1.431 479	3 602
-2.3	3.082 955	3 151	1.579 097	3 503
-2.4	3.237 796	3 097	1.730 219	3 409
-2.5	3.395 734	3 044	1.884 752	3 326

Note: Where fourth and higher differences are not negligible, their effect on interpolated values may be taken into account by using "modified second differences," δ^2_m, instead of the usual second difference, δ^2. (A good discussion of this technique, known as "throwback," can be found in Kopal, Numerical Analysis (John Wiley, New York, 1955). The appropriate second differences, when used in Everett's interpolation formula, should give results correct to within at least one or two units in the last figure tabulated.

t	$\tan^{-1}\left[\frac{v(t)}{u(t)}\right]$	δ^2	$\tan^{-1}\left[\frac{v'(t)}{u'(t)}\right]$	δ^2
-2.5	3.395 734	+ 3 045	1.884 752	+ 3 327
-2.6	3.556 717	2 998	2.042 612	3 247
-2.7	3.720 698	2 950	2.203 719	3 175
-2.8	3.887 629	2 904	2.368 001	3 105
-2.9	4.057 464	2 860	2.535 388	3 044
-3.0	4.230 159	2 820	2.705 819	2 982
-3.1	4.405 674	2 778	2.879 232	2 926
-3.2	4.583 967	2 739	3.055 571	2 875
-3.3	4.764 999	2 701	3.234 785	2 825
-3.4	4.948 732	2 666	3.416 824	2 776
-3.5	5.135 131	2 631	3.601 639	2 732
-3.6	5.324 161	2 596	3.789 186	2 690
-3.7	5.515 787	2 564	3.979 423	2 650
-3.8	5.709 977	2 533	4.172 310	2 611
-3.9	5.906 700	2 501	4.367 808	2 573
-4.0	6.105 924	2 473	4.565 879	2 540
-4.1	6.307 621	2 444	4.766 490	2 505
-4.2	6.511 762	2 416	4.969 606	2 473
-4.3	6.718 319	2 390	5.175 195	2 441
-4.4	6.927 266	2 363	5.383 225	2 413
-4.5	7.138 576	2 338	5.593 668	2 384
-4.6	7.352 224	2 315	5.806 495	2 355
-4.7	7.568 187	2 289	6.021 677	2 329
-4.8	7.786 439	2 267	6.239 188	2 304
-4.9	8.006 958	2 246	6.459 003	2 279
-5.0	8.229 723	2 222	6.681 097	2 254

t	$\tan^{-1}\left[\frac{v(t)}{u(t)}\right]$	δ^2	$\tan^{-1}\left[\frac{v'(t)}{u'(t)}\right]$	δ^2
-5.0	8.229 723	+2 222	6.681 097	+ 2 254
-5.1	8.454 710	2 202	6.905 445	2 232
-5.2	8.681 899	2 180	7.132 025	2 209
-5.3	8.911 268	2 162	7.360 814	2 187
-5.4	9.142 799	2 142	7.591 790	2 166
-5.5	9.376 472	2 121	7.824 932	2 146
-5.6	9.612 266	2 105	8.060 220	2 125
-5.7	9.850 165	2 086	8.297 633	2 106
-5.8	10.090 150	2 067	8.537 152	2 088
-5.9	10.332 202	2 052	8.778 759	2 069
-6.0	10.576 306	2 034	9.022 435	2 051
-6.1	10.822 444	2 017	9.268 162	2 034
-6.2	11.070 599	2 003	9.515 923	2 017
-6.3	11.320 757	1 985	9.765 701	2 001
-6.4	11.572 900	1 971	10.017 480	1 984
-6.5	11.827 014	1 955	10.271 243	1 969
-6.6	12.083 083	1 942	10.526 975	1 954
-6.7	12.341 094	1 927	10.784 661	1 938
-6.8	12.601 032	1 912	11.044 285	1 924
-6.9	12.862 882	1 900	11.305 833	1 910
-7.0	13.126 632	1 885	11.569 291	1 896
-7.1	13.392 267	1 872	11.834 645	1 881
-7.2	13.659 774	1 860	12.101 880	1 870
-7.3	13.929 141	1 848	12.370 985	1 855
-7.4	14.200 356	1 833	12.641 945	1 843
-7.5	14.473 404	1 823	12.914 748	1 830

t	$\tan^{-1}\left[\frac{v(t)}{u(t)}\right]$	δ^2	$\tan^{-1}\left[\frac{v'(t)}{u'(t)}\right]$	δ^2
-7.5	14.473 404	+ 1 823	12.914 748	+ 1 830
-7.6	14.748 275	1 811	13.189 381	1 819
-7.7	15.024 957	1 799	13.465 833	1 806
-7.8	15.303 438	1 786	13.744 091	1 794
-7.9	15.583 705	1 777	14.024 143	1 783
-8.0	15.865 749	1 766	14.305 978	1 771
-8.1	16.149 559	1 753	14.589 584	1 761
-8.2	16.435 122	1 744	14.874 951	1 750
-8.3	16.722 429	1 734	15.162 068	1 738
-8.4	17.011 470	1 722	15.450 923	1 729
-8.5	17.302 233	1 713	15.741 507	1 717
-8.6	17.594 709	1 703	16.033 808	1 709
-8.7	17.888 888	1 693	16.327 818	1 697
-8.8	18.184 760	1 684	16.623 525	1 689
-8.9	18.482 316	1 674	16.920 921	1 678
-9.0	18.781 546	1 666	17.219 995	1 669
-9.1	19.082 442	1 654	17.520 738	1 661
-9.2	19.384 992	1 647	17.823 142	1 650
-9.3	19.689 189	1 639	18.127 196	1 642
-9.4	19.995 025	1 629	18.432 892	1 632
-9.5	20.302 490	1 620	18.740 220	1 625
-9.6	20.611 575	1 613	19.049 173	1 616
-9.7	20.922 273	1 604	19.359 742	1 607
-9.8	21.234 575	1 596	19.671 918	1 600
-9.9	21.548 473	1 588	19.985 694	1 590
-10.0	21.863 959		20.301 060	

Appendix B

Some Formulae Involving Airy Functions

The Airy functions u(t) and v(t) are linearly independent solutions of the differential equation [Miller, 1946; Fock, 1946]

$$\frac{d^2 f}{dt^2} - tf = 0$$

i.e., $u''(t) = t\,u(t)$ and $v''(t) = t\,v(t)$.

<u>Infinite Series Representation:</u>

$$u(t) = \frac{\sqrt{3\pi}}{\sqrt[3]{9}\,\Gamma(2/3)}\, y_1(t) + \frac{\sqrt{3\pi}}{\sqrt[3]{3}\,\Gamma(1/3)}\, y_2(t)$$

$$= 1.089\ 929\ 069\ y_1(t) + 0.794\ 570\ 425\ y_2(t)$$

$$v(t) = \frac{\sqrt{\pi}}{\sqrt[3]{9}\,\Gamma(2/3)}\, y_1(t) - \frac{\sqrt{\pi}}{\sqrt[3]{3}\,\Gamma(1/3)}\, y_2(t)$$

$$= 0.629\ 270\ 841\ y_1(t) - 0.458\ 745\ 449\ y_2(t)$$

where

$$y_1(t) = 1 + \frac{1}{3!}t^3 + \frac{(1)(4)}{6!}t^6 + \frac{(1)(4)(7)}{9!}t^9 + \frac{(1)(4)(7)(10)}{12!}t^{12} + \ldots$$

$$y_2(t) = t + \frac{2}{4!}t^4 + \frac{(2)(5)}{7!}t^7 + \frac{(2)(5)(8)}{10!}t^{10} + \frac{(2)(5)(8)(11)}{13!}t^{13} + \ldots$$

These are valid for all t.

Of course, $y_1(t)$ and $y_2(t)$ are also linearly independent solutions of the differential equation satisfied by $u(t)$ and $v(t)$. In fact, $y_1(t)$ and $y_2(t)$ are obtained when the differential equation is solved by the method of Frobenius.

Wronskian Identity:

$$u'(t)\, v(t) - u(t)\, v'(t) = 1$$

Definite Integral Representation:

$$u(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left\{ \exp\left(-\frac{1}{3} x^3 + tx\right) + \sin\left(\frac{1}{3} x^3 + tx\right) \right\} dx$$

$$v(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos\left(\frac{1}{3} x^3 + tx\right) dx$$

Representation in Terms of Bessel Functions:

If principal values are taken

$$u(t) = \sqrt{\frac{\pi t}{3}} \left[I_{-1/3}\left(\frac{2}{3} t^{3/2}\right) + I_{1/3}\left(\frac{2}{3} t^{3/2}\right) \right]$$

$$v(t) = \frac{\sqrt{\pi t}}{3} \left[I_{-1/3}\left(\frac{2}{3} t^{3/2}\right) - I_{1/3}\left(\frac{2}{3} t^{3/2}\right) \right] = \sqrt{\frac{t}{3\pi}}\, K_{1/3}\left(\frac{2}{3} t^{3/2}\right)$$

$$u(-t) = \sqrt{\frac{\pi t}{3}} \left[J_{-1/3}\left(\frac{2}{3} t^{3/2}\right) - J_{1/3}\left(\frac{2}{3} t^{3/2}\right) \right]$$

$$v(-t) = \frac{\sqrt{\pi t}}{3}\left[J_{-1/3}\left(\frac{2}{3}t^{3/2}\right) + J_{1/3}\left(\frac{2}{3}t^{3/2}\right)\right]$$

$$u'(t) = \sqrt{\frac{\pi}{3}}\, t\left[I_{2/3}\left(\frac{2}{3}t^{3/2}\right) + I_{-2/3}\left(\frac{2}{3}t^{3/2}\right)\right]$$

$$v'(t) = \frac{\sqrt{\pi}}{3}\, t\left[I_{2/3}\left(\frac{2}{3}t^{3/2}\right) - I_{-2/3}\left(\frac{2}{3}t^{3/2}\right)\right]$$

$$u'(-t) = \sqrt{\frac{\pi}{3}}\, t\left[J_{2/3}\left(\frac{2}{3}t^{3/2}\right) + J_{-2/3}\left(\frac{2}{3}t^{3/2}\right)\right]$$

$$v'(-t) = \frac{\sqrt{\pi}}{3}\, t\left[J_{2/3}\left(\frac{2}{3}t^{3/2}\right) - J_{-2/3}\left(\frac{2}{3}t^{3/2}\right)\right]$$

Asymptotic Expansions:

Large positive numbers: $\left[|t| \to \infty, |\arg t| < \frac{\pi}{3}\right]$

$$u(t) \approx \frac{1}{t^{1/4}} \exp\left(\frac{2}{3}t^{3/2}\right)$$

$$v(t) \approx \frac{1}{2\, t^{1/4}} \exp\left(-\frac{2}{3}t^{3/2}\right)$$

$$u'(t) \approx t^{1/4} \exp\left(\frac{2}{3}t^{3/2}\right)$$

$$v'(t) \approx -\frac{t^{1/4}}{2} \exp\left(-\frac{2}{3}t^{3/2}\right)$$

Large Negative Numbers: $\left[|t| \to \infty, \ \arg(-t) < \frac{2\pi}{3}\right]$

$$u(t) \approx \frac{1}{(-t)^{1/4}} \cos\left[\frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right] ,$$

$$v(t) \approx \frac{1}{(-t)^{1/4}} \sin\left[\frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right] ,$$

$$u'(t) \approx (-t)^{1/4} \sin\left[\frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right] ,$$

$$v'(t) \approx -(-t)^{1/4} \cos\left[\frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right] ,$$

In general,

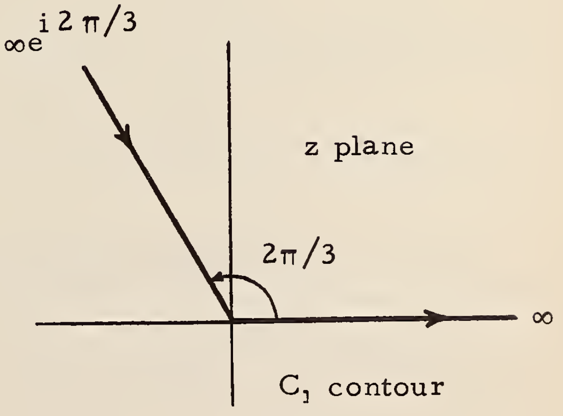
$$w_1(t) = u(t) - i\, v(t) \quad \dagger ,$$

$$w_2(t) = u(t) + i\, v(t) \quad \dagger .$$

<u>Contour Integral Representation:</u>

$$w_1(t) = \frac{1}{\sqrt{\pi}} \int_{C_1} \exp(tz - \frac{1}{3} z^3)\, dz$$

$$w_1'(t) = \frac{1}{\sqrt{\pi}} \int_{C_1} z \exp(tz - \frac{1}{3} z^3)\, dz$$



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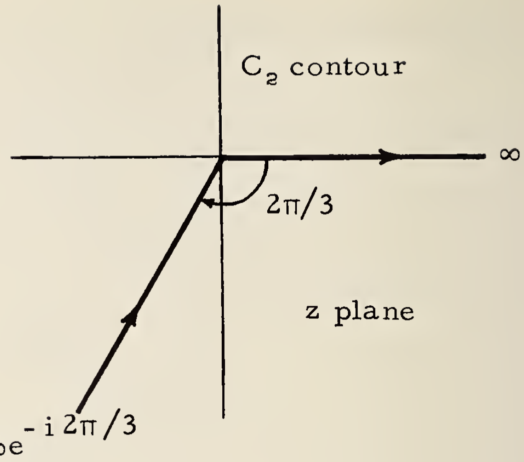
† It should be noted here that V. A. Fock [1946] defines $w_1(t) = u(t) + i\, v(t)$ and $w_2(t) = u(t) - i\, v(t)$.

$$w_2(t) = \frac{1}{\sqrt{\pi}} \int_{C_2} \exp(tz - \frac{1}{3} z^3)\,dz$$



$$w_2'(t) = \frac{1}{\sqrt{\pi}} \int_{C_2} z \exp(tz - \frac{1}{3} z^3)\,dz$$

Wronskian Identity:

$$w_1'(t)\, w_2(t) - w_1(t)\, w_2'(t) = i\,2$$

Representation in Terms of Hankel Functions:

$$w_1(t) = e^{-i\,2\pi/3} \sqrt{-\frac{\pi t}{3}}\; H^{(2)}_{1/3}\left[\frac{2}{3}(-t)^{3/2}\right]$$

$$w_2(t) = e^{+i\,2\pi/3} \sqrt{-\frac{\pi t}{3}}\; H^{(1)}_{1/3}\left[\frac{2}{3}(-t)^{3/2}\right]$$

$$w_1'(t) = e^{-i\,\pi/3} \sqrt{-\frac{\pi t}{3}}\; H^{(2)}_{2/3}\left[\frac{2}{3}(-t)^{3/2}\right]$$

$$w_2'(t) = e^{+i\,\pi/3} \sqrt{-\frac{\pi t}{3}}\; H^{(1)}_{2/3}\left[\frac{2}{3}(-t)^{3/2}\right]$$

Asymptotic Forms:

$|t| \to \infty,$
$|\arg t| < \frac{\pi}{3}$: $$w_{\frac{1}{2}}(t) \approx \frac{1}{t^{1/4}} \exp\left(\frac{2}{3} t^{3/2}\right)$$

$$w'_{\frac{1}{2}}(t) \approx t^{1/4} \exp\left(\frac{2}{3} t^{3/2}\right)$$

$|t| \to \infty,$
$|\arg(-t)| < \frac{2\pi}{3}$: $$w_{\frac{1}{2}}(t) \approx \frac{e^{\mp i\pi/4}}{(-t)^{1/4}} \exp\left[\mp i \frac{2}{3}(-t)^{3/2}\right]$$

$$w'_{\frac{1}{2}}(t) \approx \pm i(-t)^{1/4} e^{\mp i\pi/4} \exp\left[\mp i\frac{2}{3}(-t)^{3/2}\right]$$

$$\approx \pm i(-t)^{1/2} w_{\frac{1}{2}}(t)$$

Values on Special Rays:

$$w_1(r e^{+i\pi/3}) = e^{-i\pi/3} w_2(-r) \qquad w_2(r e^{+i\pi/3}) = 2 e^{+i\pi/6} v(-r)$$

$$w_1(r e^{+i2\pi/3}) = 2 e^{-i\pi/6} v(r) \qquad w_2(r e^{+i2\pi/3}) = e^{+i\pi/3} w_1(r)$$

$$w_1(r e^{-i\pi/3}) = 2 e^{-i\pi/6} v(-r) \qquad w_2(r e^{-i\pi/3}) = e^{+i\pi/3} w_1(-r)$$

$$w_1(r e^{-i2\pi/3}) = e^{-i\pi/3} w_2(r) \qquad w_2(r e^{-i2\pi/3}) = 2 e^{+i\pi/6} v(r)$$

$$w_1'(r e^{+i\pi/3}) = e^{+i\pi/3} w_2'(-r) \qquad w_2'(r e^{+i\pi/3}) = 2 e^{-i\pi/6} v'(-r)$$

$$w_1'(re^{+i2\pi/3}) = 2e^{+i\pi/6}\, v'(r) \qquad w_2'(re^{+i2\pi/3}) = e^{-i\pi/3}\, w_1'(r)$$

$$w_1'(re^{-i\pi/3}) = 2e^{+i\pi/6}\, v'(-r) \qquad w_2'(re^{-i\pi/3}) = e^{-i\pi/3}\, w_1'(-r)$$

$$w_1'(re^{-i2\pi/3}) = e^{+i\pi/3}\, w_2'(r) \qquad w_2'(re^{-i2\pi/3}) = 2e^{-i\pi/6}\, v'(r)$$

Derivatives:

If w(t) is any solution of the differential equation $\frac{d^2 w}{dt^2} - tw = 0$, then

$$w^{(2)}(t) = t\, w(t) \qquad \left(\text{Note that } w^{(n)}(t) = \frac{d^n}{dt^n} w(t)\right)$$

$$w^{(3)}(t) = w(t) + t w'(t)$$

$$w^{(4)}(t) = t^2 w(t) + 2w'(t)$$

$$w^{(5)}(t) = 4t\, w(t) + t^2 w'(t)$$

$$w^{(6)}(t) = (t^3 + 4)\, w(t) + 6t\, w'(t)$$

$$w^{(7)}(t) = 9t^2 w(t) + (t^3 + 10)\, w'(t)$$

$$w^{(8)}(t) = (t^4 + 28t)\, w(t) + 12t^2 w'(t)$$

$$w^{(9)}(t) = (16t^3 + 28)\, w(t) + (t^4 + 52t)\, w'(t)$$

$$w^{(10)}(t) = (t^5 + 100t^2)\, w(t) + (20t^3 + 80)\, w'(t)$$

$$w^{(11)}(t) = (25\,t^4 + 280\,t)\,w(t) + (t^5 + 160\,t^2)\,w'(t)$$

$w^{(2)}(0) = 0$ $\qquad$ $w^{(5)}(0) = 0$ $\qquad$ $w^{(8)}(0) = 0$

$w^{(3)}(0) = w(0)$ $\qquad$ $w^{(6)}(0) = 4\,w(0)$ $\qquad$ $w^{(9)}(0) = 28\,w(0)$

$w^{(4)}(0) = 2\,w'(0)$ $\qquad$ $w^{(7)}(0) = 10\,w'(0)$ $\qquad$ $w^{(10)}(0) = 80\,w'(0)$

A Taylor Series Expansion:

If $w(t)$ is any solution of $\dfrac{d^2 w}{dt^2} - tw = 0$, then

$$w(t + h) = \alpha(t, h)\,w(t) + \beta(t, h)\,w'(t)$$

$$w'(t + h) = \alpha'(t, h)\,w(t) + \beta'(t, h)\,w'(t)$$

where

$$\alpha(t, h) = 1 + t\,a_2(h) + a_3(h) + t^2\,a_4(h) + 4\,t\,a_5(h) + (t^3 + 4)\,a_6(h)$$

$$+ 9\,t^2\,a_7(h) + (t^4 + 28\,t)\,a_8(h) + (16\,t^3 + 28)\,a_9(h)$$

$$+ (t^5 + 100\,t^2)\,a_{10}(h) + (25\,t^4 + 280\,t)\,a_{11}(h) + \ldots$$

$$\beta(t, h) = a_1(h) + t\,a_3(h) + 2\,a_4(h) + t^2\,a_5(h) + 6\,t\,a_6(h) + (t^3 + 10)\,a_7(h)$$

$$+ 12\,t^2\,a_8(h) + (t^4 + 52\,t)\,a_9(h) + (20\,t^3 + 80)\,a_{10}(h) + (t^5 + 160\,t^2)\,a_{11}(h) + \ldots$$

$$\alpha'(t,h) = t\,a_1(h) + a_2(h) + t^2 a_3(h) + 4t\,a_4(h) + (t^3+4)\,a_5(h) + 9t^2\,a_6(h)$$

$$+ (t^4 + 28t)\,a_7(h) + (16t^3 + 28)\,a_8(h) + (t^5 + 100t^2)\,a_9(h)$$

$$+ (25t^4 + 280t)\,a_{10}(h) + \ldots$$

$$\beta'(t,h) = 1 + t\,a_2(h) + 2\,a_3(h) + t^2\,a_4(h) + 6t\,a_5(h) + (t^3 + 10)\,a_6(h)$$

$$+ 12t^2\,a_7(h) + (t^4 + 52t)\,a_8(h) + (20t^3 + 80)\,a_9(h) + (t^5 + 160 + t^2)\,a_{10}(h) + \ldots$$

and $$a_k(h) = \frac{h^k}{k!} \quad (k = 0, 1, 2, \ldots) \; .$$

Note that $$a_k(h) = \frac{h}{k}\, a_{k-1} \quad (k = 1, 2, 3, \ldots) \; .$$

Formulas Involving Complex Arguments:

Let $z = x + iy$ where x, y are real. If $w(z)$ is any solution of the differential equation

$$\frac{d^2 w}{dz^2} - z\,w = 0,$$

the following expressions are fairly convenient for computation when y is small and x is not too large:

$$w(z) = [\theta(x,y)\,w(x) + \phi(x,y)\,w'(x)] + i\,[\xi(x,y)\,w(x) + \eta(x,y)\,w'(x)]$$

where

$$\theta(x, y) = 1 - \frac{x y^2}{2} + \frac{x^2 y^4}{24} - \frac{(x^3+4) y^6}{720} + \frac{(x^4+28x) y^8}{40,320} - \frac{(x^5 + 100 x^2) y^{10}}{3,628,800} + \ldots$$

$$\phi(x, y) = \frac{y^4}{12} - \frac{x y^6}{120} + \frac{x^2 y^8}{3360} - \frac{(x^3 + 4) y^{10}}{181,440} + \ldots$$

$$\xi(x, y) = -\frac{y^3}{6} + \frac{x y^5}{30} - \frac{x^2 y^7}{560} + \frac{(4 x^3 + 7) y^9}{90,720} - \frac{(5 x^4 + 56 x) y^{11}}{7,983,360} + \ldots$$

$$\eta(x, y) = y - \frac{x y^3}{6} + \frac{x^2 y^5}{120} - \frac{(x^3 + 10) y^7}{5040} + \frac{(x^4 + 52 x) y^9}{362,880} - \frac{(x^5 + 160 x^2) y^{11}}{39,916,800} + \ldots$$

and

$$w'(z) = [\theta'(x, y)\, w(x) + \phi'(x, y)\, w'(x)] + i\,[\xi'(x, y)\, w(x) + \eta'(x, y)\, w'(x)]$$

where

$$\theta'(x, y) = -\frac{y^2}{2} + \frac{x y^4}{6} - \frac{x^2 y^6}{80} + \frac{(4 x^3 + 7) y^8}{10,080} - \frac{(5 x^4 + 56 x) y^{10}}{725,760} + \ldots$$

$$\phi'(x, y) = 1 - \frac{x y^2}{2} + \frac{x^2 y^4}{24} - \frac{(x^3+10) y^6}{720} + \frac{(x^4 + 52 x) y^8}{40,320} - \frac{(x^5 + 160 x^2) y^{10}}{3,628,800} + \ldots$$

$$\xi'(x, y) = x y - \frac{x^2 y^3}{6} + \frac{(x^3 + 4) y^5}{120} - \frac{(x^4 + 28 x) y^7}{5040} + \frac{(x^5 + 100 x^2) y^9}{362,880} - \ldots$$

$$\eta'(x, y) = -\frac{y^3}{3} + \frac{x y^5}{20} - \frac{x^2 y^7}{420} + \frac{(x^3 + 4) y^9}{18,144} - \ldots$$

Note: An interesting discussion of Airy functions and relations among the many differing notations is found in an unpublished report (dated Dec. 1959) by Nelson A. Logan of the Lockheed Aircraft Co., Sunnyvale, Calif.

Appendix C

A Note on the Conductivity of the Lower Ionosphere at VLF

In the theory of ionospheric propagation of radio waves, it is nearly always assumed that the collision frequency of electrons with neutral particles is independent of electron energy. In fact, the Appleton-Hartree equations were developed on this basis. It has been suggested recently [Alpert et al. 1953; Sen and Wyller, 1960; Phelps, 1960] that the calculation of the propagation constant for a weakly ionized medium should take into account this energy dependence. While the theory has been so generalized by Sen and Wyller [1960] and Johler and Harper [1962], it seems worthwhile to present a somewhat simplified account of the consequences of a linear dependence of the collision frequency on electron energy. Furthermore, this sheds some light on the validity of describing the electrical characteristics of the lower ionosphere in terms of a conductivity.

W. P. Allis [1956] and others have related the components of the dielectric tensor to integrals involving the angular frequency ω, the gyro frequency of electrons ω_H, the electron plasma frequency, the electron density N, the normalized electron energy distribution function f_o, the frequency of mementum transfer collisions of electrons with gas molecules $\nu(u)$. When a constant and uniform magnetic field is applied in the z direction the dielectric constant of the ionized medium is a tensor of the form

$$(\epsilon) = \begin{pmatrix} \epsilon' & -iq & 0 \\ iq & \epsilon' & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$\epsilon' = \tfrac{1}{2}(\epsilon_L + \epsilon_R)$$

$$q = \tfrac{1}{2}(\epsilon_L - \epsilon_R)$$

$$\epsilon'' = \epsilon_P .$$

In this, the dielectric constants ϵ_L, ϵ_R and ϵ_P may be written in the form

$$\epsilon_L = \epsilon(\omega + \omega_H)$$

$$\epsilon_R = \epsilon(\omega - \omega_H)$$

$$\epsilon_P = \epsilon(\omega)$$

where, according to Molmud [1959],

$$i\,\epsilon(\Omega)\,\omega = \sigma(\Omega) = -\frac{4\pi}{3}\,\epsilon_o\,\omega_o^2 \int_0^\infty \frac{u^{3/2}}{\nu(u) + i\,\Omega}\,\frac{\partial f_o}{\partial u}\,du .$$

In the above, $\sigma(\Omega)$ is a generalized conductivity which is a function of a generalized frequency Ω.

In the case where the electrons are in thermal equilibrium with the gas, the energy distribution of the electrons is Maxwellian and given by

$$f_o = (e/\pi k T)^{3/2}\, e^{-eu/kT} .$$

In the case of weakly ionized dry air, the collision frequency $\nu(u)$ is now believed to be approximately proportional to the electron energy u. In fact, Phelps [1960] has shown that, for low energy electrons in nitrogen,

$$\nu(u) \cong 1.2 \times 10^{-7} N(N_2) u \text{ sec}^{-1}$$

where u is in electron-volts/c. c. For present purposes, we will just set $\nu(u) = a u$ where a is a constant. Furthermore, a normalized collision frequency ν_1 is chosen such that

$$\nu_1 = a k T/e$$

where k is the usual Boltzmann constant, T is the absolute temperature in degrees Kelvin and e is the electronic charge.

Using the simplifications described in the preceding paragraph, it is seen that

$$\sigma(\Omega) = \frac{4\pi}{3} \epsilon_o \omega_o^2 \left(\frac{a}{\pi \nu_1}\right)^{3/2} \int_0^\infty \frac{u^{3/2}}{a u + i\Omega} e^{-u a/\nu_1} d\left(\frac{u a}{\nu_1}\right) .$$

This is essentially the formulas given by Phelps [1960], Sen and Wyller [1960] and others. As they have indicated, the integral can be expressed in terms of the "semi-conductor integral" $E_p(x)$ defined by

$$E_p(x) = \frac{1}{(p!)} \int_0^\infty \alpha^p (\alpha^2 + x^2)^{-1} e^{-\alpha} d\alpha .$$

These have been tabulated by Dingle, Arndt and Roy [1956] for integral and half-integral values of p in the range -1/2 to + 5. It easily follows that

$$\sigma(\Omega) = \frac{\epsilon_o \, \omega_o^2}{\nu_1} \left[\frac{5}{2} E_{5/2}(x) - i \, x \, E_{3/2}(x) \right]$$

where $x = \Omega/\nu_1$.

If, on the other hand, $\nu(u)$ had been replaced by a constant ν_o, as is conventionally done, we would have arrived at the standard result

$$\sigma(\Omega) = \frac{\epsilon_o \, \omega_o^2}{\nu_o + i \, \Omega} .$$

This can be written in the form

$$\sigma(\Omega) = \epsilon_o \, \omega_r \, \frac{1}{1 + i \, \beta}$$

where $\beta = \dfrac{\Omega}{\nu_o}$ and $\omega_r = \dfrac{\omega_o^2}{\nu_o}$.

The parameter ω_r occurs often in the theory of VLF propagation which is usually formulated on the assumption of an energy independent collision frequency. Furthermore, at VLF $\omega << \omega_H$ so that

$$\beta \cong \pm \frac{\omega_H}{\nu_o} .$$

The parameter ω_H/ν_o also occurs consistently in the presentation of theoretical results. Thus, it appears that a convenient way to illustrate the implications of the energy dependent collision frequency is to define effective values $(\omega_r)_e$ and β_e as follows

$$\sigma(\Omega) = \epsilon_o (\omega_r)_e \frac{1}{1 + i\,\beta_e} \;.$$

Thus

$$(\omega_r)_e = \omega_{r_1}\, a_e(x)$$

where
$$\omega_{r_1} = \frac{\omega_o^2}{\nu_1}$$

and
$$a_e(x) = \frac{\frac{5}{2}\, E_{3/2}(x)}{\left[\frac{5}{2}\, E_{5/2}(x)\right]^2 + \left[x\, E_{3/2}(x)\right]^2} \;.$$

Also, it is seen that

$$\beta_e(x) = \frac{x\, E_{3/2}(x)}{\frac{5}{2}\, E_{5/2}(x)}$$

where, as above, $x = \dfrac{\Omega}{\nu_1}$.

Using the numerical values of $E_{3/2}(x)$ and $E_{5/2}(x)$ given by Dingle, Arndt and Roy [1956], the values of $\alpha_e(x)$ and $\beta_e(x)$ are plotted as a function of x and given in Table C-1. It is seen that for small values of x, $\alpha(x)$ asymptotically approaches the constant value of 2/3, whereas $\beta_e(x)$ is asymptotically approaching the value 2x. On the other hand, for large values of x, the respective asymptotes are 0.4 and x/2.5.

If the collision frequency was chosen to be independent of energy, the corresponding values $\alpha_e(x)$ and $\beta_e(x)$ would be simply 1.0 and x, respectively. It is thus concluded that a linear energy dependence for the collision frequency is not going to lead to any essential modifications to the theory of VLF propagation. In fact, most of the numerical results on the characteristics of the VLF modes can be adapted directly to the energy dependent case if ω_r and the ratio ω_H/ν are given their more general meaning.

Table C-1

x	$\alpha_e(x)$	$\beta_e(x)$
0.01	0.66546	0.01755
0.02	0.66363	0.03322
0.03	0.66164	0.04776
0.05	0.65738	0.07451
0.1	0.64665	0.13256
0.2	0.62691	0.22772
0.4	0.59561	0.37771
0.6	0.57184	0.50127
0.8	0.55300	0.61080
1.0	0.53771	0.71148
1.2	0.52487	0.80578
1.6	0.50441	0.98308
2.0	0.48921	1.15113
2.5	0.47428	1.35171
3.0	0.46293	1.54763
3.5	0.45416	1.74084
4.0	0.44701	1.93203
5.0	0.43617	2.31099
6.0	0.42862	2.69037
7.0	0.42315	3.07048
8.0	0.41911	3.45146
9.0	0.41579	3.83384
10.0	0.41360	4.22111
11.0	0.41177	4.60788
12.0	0.41011	4.99500
13.0	0.40893	5.38480
14.0	0.40783	5.77448
15.0	0.40679	6.16362
16.0	0.40626	6.55735
17.0	0.40545	6.94835
20.0	0.40395	8.12884

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Ionosphere Research and Propagation. Low Frequency and Very Low Frequency Research. Ionosphere Research. Prediction Services. Sun-Earth Relationships. Field Engineering. Radio Warning Services.

Radio Propagation Engineering. Data Reduction Instrumentation. Radio Noise. Tropospheric Measurements. Tropospheric Analysis. Propagation-Terrain Effects. Radio-Meteorology. Lower Atmosphere Physics.

Radio Standards. High Frequency Electrical Standards. Radio Broadcast Service. Radio and Microwave Materials. Atomic Frequency and Time Interval Standards. Electronic Calibration Center. Millimeter-Wave Research. Microwave Circuit Standards.

Radio Systems. High Frequency and Very High Frequency Research. Modulation Research. Antenna Research. Navigation Systems.

Upper Atmosphere and Space Physics. Upper Atmosphere and Plasma Physics. Ionosphere and Exosphere Scatter. Airglow and Aurora. Ionospheric Radio Astronomy.

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