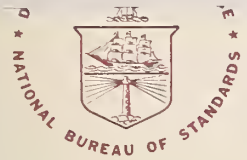

NBS TECHNICAL NOTE 1091

U.S. DEPARTMENT OF COMMERCE / National Bureau of Standards

Interelement Interactions in Phased Arrays: Theory, Methods of Data Analysis, and Theoretical Simulations

Lorant A. Muth

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[3]Located at Boulder, CO, with some elements at Gaithersburg, MD.

Interelement Interactions in Phased Arrays: Theory, Methods of Data Analysis, and Theoretical Simulations

Lorant A. Muth

Electromagnetic Fields Division
Center for Electronics and Electrical Engineering
National Engineering Laboratory
National Bureau of Standards
Boulder, Colorado 80303

U.S. DEPARTMENT OF COMMERCE, Malcolm Baldrige, Secretary

NATIONAL BUREAU OF STANDARDS, Ernest Ambler, Director

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CONTENTS

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Lorant A. Muth

Electromagnetic Fields Division
National Bureau of Standards
Boulder CO 80303

We review theoretically the effects of multiple reflections and mutual impedances in array environments and study possible methods of far-field pattern data analysis to recover interaction effects. We use theoretical expressions derived earlier to calculate in a two-element linear array the mutual-impedance matrix and effective excitations of elements as functions of interelement separation and n_{max}, the maximum mode number in the radiation pattern of the elements. Generalizations to two- and three-dimensional arrays are discussed.

Key words: data analysis; effective excitations; interelement scattering; multiple reflections; mutual impedance; phased arrays.

I. Introduction

To understand and to analyze interelement interactions in a phased array, we need to know the far-field patterns of each element[1] and of the full array. In this study we examine methods of analysis that seek to provide adequate information on interelement mutual-impedance and multiple-reflection processes to be able to synthesize far-field patterns of phased arrays from subarrays. We can then attempt to develop a procedure to obtain the far-field patterns of very large arrays constructed of subarrays whose patterns are well known. This problem is of current interest, and its solution, to our knowledge, has not been attempted before.

In broad perspective the solution to the problem is seen as follows:

A. The effect of mutual impedances on element excitations in the array environment has to be understood.

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[1] We will not be concerned here with the near-field measurement techniques and the preliminary data analysis that leads to the knowledge of the far fields. We assume that both the receiving and transmitting patterns of the elements and of the array as a whole can be determined using existing standard near-field measurement techniques.

B. The role played by multiple reflections and mutual impedances in producing elementary patterns in the open-circuited environment, which are different from free-space element patterns, has to be understood.

The two effects above are obviously related, but they are distinct. The first effect describes the interactions between the elements when they are all excited and radiating, and the second describes the interaction of a single radiating element with all the other open-circuited elements. The difference between these two effects can be understood immediately in the case of minimum-scattering antennas [1,2], where the elementary patterns are exactly the free-space patterns, but mutual-impedance effects are still present when the array is radiating. Both effects are simply a consequence of the fact that there is electromagnetic radiation everywhere in space, and, therefore, voltages are induced at each element of the phased array. An open-circuited antenna will, in fact, respond to incoming radiation according to its receiving pattern, but it will also immediately reradiate this incoming excitation according to its transmission and scattering patterns.

C. The open-circuited scattering matrix of each element that describes the scattered radiation in terms of the incoming radiation has to be known. The scattered radiation, when correctly combined with the free-space pattern of the radiating element, will give rise to the elementary patterns in the array environment.

The details of the general discussion above have been treated extensively in [1,2,3] and will not be repeated here.

D. The open-circuited scattering matrix of an element in an array environment has never been measured to our knowledge. It would be difficult to perform such a measurement in a facility that is not set up to do so, but in any case it would be costly to accomplish. Therefore, theoretical modeling of the scattering matrix is needed. Only a simple model that represents the phase shift of the signal as it enters and exits an open-circuited element can be written down immediately. Any further complexity in the scattering matrix will be due to the structural properties of the element in question and cannot be modeled easily without

supporting measurements. One can hope that such effects will be negligible and modeling will not be required, but only real measurements would shed some light on this matter.

E. As a consequence of the discussion in (D), it would be desirable to formulate the problem of interelement interactions in such a manner that the open-circuited scattering matrix does not have to be specified explicitly. One needs to formulate the measurement and computational procedures in terms of quantities that are easily measured or computed.

F. The problem of predicting the performance of large arrays from measurements performed on smaller subarrays made up of similar elements can be solved if the dependence of the effects in (A) and (B) on distance and the total number of elements in the entire array can be specified. Only theoretical modeling can be accomplished here, since the entire array is considered to be too large for deployment in existing measurement facilities. Essentially, the relevant effects fall off rapidly with distance; hence, adequate modeling might be possible.

In the body of the report we examine some important fundamentals of analysis of array patterns and interaction effects. We seek to discover through theoretical understanding and numerical simulations the information and measurements one needs to characterize interlement interactions in phased arrays.

First, we briefly review the theoretical fundamentals, emphasizing pattern addition with and without interaction effects. Second, we survey possible data analysis procedures using known information about the phase relationships between elements of arrays. We then use numerical simulations to test these procedures for numerical stability and for sensitivity to approximations. We conclude with a brief discussion of modeling of large arrays from data obtained from subarrays.

II. Theoretical Review

In our theoretical framework to describe interelement interaction processes we rely extensively on the scattering-matrix formalism as developed in [2] and [4]. A relevant theoretical extension of scattering-matrix concepts

applicable to arrays has been developed in [3]. We will use freely the material covered in these references, without going into detailed explanations of concepts.

We will rely primarily on the far-field pattern-addition formulas that give the resultant array pattern in terms of the individual patterns of the elements of the array, the phase shifts in the individual patterns as a function of location of the radiating elements, and the excitations of the elements [1,5]. For noninteracting elements, the pattern-addition formula is simply

$$\underline{F}^0(\theta,\phi) = \underline{f}^0(\theta,\phi) \sum_{n=1}^{N} \underline{a}_n^0 \, e^{-i\underline{k}\cdot\underline{D}_n}, \tag{1}$$

where the superscript zero indicates noninteracting elements, $\underline{F}$ is the array far-field pattern, $\underline{f}$ is the element pattern (assumed the same for all the elements), $\underline{a}_n$ is the excitation applied to the nth element, $\underline{D}_n$ is the position vector of the nth element, and $\underline{k}$ is the wave vector. The generalization of this formula to n different but still noninteracting elements is straightforward,

$$\underline{F}^0(\theta,\phi) = \sum_{n=1}^{N} \underline{a}^0{}_n \, \underline{f}_n^0 \, e^{-i\underline{k}\cdot\underline{D}_n}, \tag{2}$$

where $\underline{f}_n$ is the pattern of the nth element. In reality, of course, there could be significant interactions among the elements of an array. This will depend primarily on the individual element patterns and the relative locations of the elements. If we denote the presence of interactions by the symbol ^, then the array pattern is expressed as

$$\hat{\underline{F}}(\theta,\phi) = \sum_{n=1}^{N} \hat{\underline{a}}_n \hat{\underline{f}}_n \, e^{-i\underline{k}\cdot\underline{D}_n}, \tag{3}$$

where $\hat{\underline{f}}_n$ are now the elementary patterns in an open-circuited array environment, which, in general, are different from the element patterns in the noninteracting case: $\hat{\underline{f}}_n$ now include the effects of interelement scattering. One can obtain the effective excitations $\hat{\underline{a}}_n$ from the excitations $\underline{a}_n^0$ applied to the individual elements from

$$\hat{\underline{a}} = \underline{\underline{\Gamma}}(\underline{\underline{z}})\underline{a}^0, \tag{4}$$

where $\underline{\underline{\Gamma}}(\underline{\underline{z}})$ is the effective excitation matrix, and

$$\begin{aligned} \text{Tr } \hat{\underline{a}} &= (\hat{a}_1, \ldots \hat{a}_i, \ldots \hat{a}_N), \\ \text{Tr } \underline{a}^0 &= (a_1^0, \ldots a_i^0, \ldots a_N^0), \end{aligned} \tag{5}$$

where Tr denotes the transpose; here the effective excitation matrix $\underline{\underline{\Gamma}}(\underline{\underline{z}})$ is given by [2] ($\underline{\underline{I}}$ is the unit matrix)

$$\underline{\underline{\Gamma}}(\underline{\underline{z}}) = \underline{\underline{W}}\,(\underline{\underline{z}} + \underline{\underline{I}})^{-1}, \tag{6}$$

where

$$W_{ij} = (1 + z_{ij})\delta_{ij}, \quad 1 \leq i, j \leq N \tag{6a}$$

is a diagonal matrix, and

$$z_{ij} = z_{ij}\,(\underline{\underline{Z}}, \underline{\underline{S}}_0) \tag{6b}$$

are the elements of the mutual-impedance matrix $\underline{\underline{z}}$. As indicated, these quantities depend on the binary mode-mode mutual-impedance matrix $\underline{\underline{Z}}$ [1,3], and on the open-circuit scattering matrix $\underline{\underline{S}}_0$ [1,2].

We need to understand the structure of the elementary patterns $\hat{\underline{f}}_n$ and the effective excitation matrix $\underline{\underline{\Gamma}}$ in more detail to isolate effects accessible to measurements. Consider that element n is radiating (with unit strength) and all other elements are open circuited. Then the incoming radiation at element $m(\neq n)$ will induce currents that will radiate a pattern into all directions in space. We denote these induced patterns by $\hat{\underline{f}}_m^{(i)}$. The elementary pattern $\hat{\underline{f}}_n$ is then given by

$$\hat{\underline{f}}_n(\theta,\phi) = \underline{f}_n^0(\theta,\phi) + \sum_{m=1}^{N} \underline{f}_m^{(i)}(\theta,\phi)\, e^{-i\underline{k}\cdot\underline{D}_{mn}}. \tag{7}$$

This expression is similar to those in (1) and (2) and can be understood accordingly. If we neglect all but the first of the multiple reflections between the elements, we have in a first-order approximation that [1]

$$\hat{\underline{f}}_n(\theta,\phi) = \underline{f}_n^0(\theta,\phi) + \sum_{m=1}^{N}{}' \underline{f}_m^{(i)}(\theta,\phi)\, e^{-i\underline{k}\cdot\underline{D}_{mn}}, \tag{8}$$

where the prime on the sum indicates that the nth induced pattern originating from the radiating element is negligible. This is a higher order effect. One can write a first order approximation for the induced patterns emanating from elements m≠n in terms of the transmission characteristics $\underline{s}$ and $\underline{\underline{Z}}$, the mode-mode mutual-impedance matrix. For example, for two elements [1]

$$\underline{b}_1^{(i)} = -\tfrac{1}{2}\left(\underline{\underline{I}} - \underline{\underline{S}}_0^{(1)}\right) \underline{\underline{Z}}\, \underline{s}_2,$$

and

$$\underline{b}_2^{(i)} = -\tfrac{1}{2}\left(\underline{\underline{I}} = \underline{\underline{S}}_0^{(2)}\right) \underline{\underline{Z}}\, \underline{s}_1, \tag{9}$$

where $\underline{b}_m^{(i)}$ are the coefficients of the induced patterns $\underline{f}_m^{(i)}$, where element m is radiating. Corresponding expressions in an N-element environment could easily be derived [3]. This, however, would not be really useful, since any direct measurement of the induced pattern would yield the resultant pattern and not an approximation of it. In general, these approximations do not allow solving for the elements of $\underline{\underline{S}}_0$ without additional measurements. We can emphasize then that it is possible to obtain the coefficients of the scattered fields, but not the elements of the scattering matrix, from a single pattern measurement. Briefly, some important facts about the open-circuit scattering matrix $\underline{\underline{S}}_0$ are that it satisfies, in general, the relationships [1,3]

$$\underline{\underline{S}}_0^{\dagger}\, \underline{s}_{\beta\alpha} = \underline{s}_{\alpha\beta}, \tag{10a}$$

and

$$\underline{b}_\beta = \underline{\underline{S}}_0\, \underline{a}_\beta, \tag{10b}$$

where $\underline{s}_{\alpha\beta}$ ($\underline{s}_{\beta\alpha}$) give the transmitting (receiving) characteristics of the element, and $\underline{b}_\beta$ ($\underline{a}_\beta$) are the coefficients of the incoming (outgoing) waves, respectively. For reciprocal antennas a diagonal $\underline{\underline{S}}_0$ corresponds to introducing phase relationships into incoming waves, which then become time reversed and outgoing. Under these conditions different spherical modes are not mixed during scattering [1]. More detailed treatment of the scattering matrix is beyond the scope of this study.

To predict far-field patterns of very large antennas from measurements made on subarrays, the effective excitation matrix $\underline{\underline{\Gamma}}$ emerges as an important quantity. Equation (4) states that the effective excitations are linear

combinations of the excitations of the elements. For large interelement distances, or for large kD, $\underline{\underline{\Gamma}}$ approaches the unit matrix, since the off-diagonal elements z_{ij} approach zero in this limit. Thus,

$$\lim_{kD\to\infty} \frac{\partial \hat{a}_i}{\partial a_j} = \delta_{ij}. \tag{11}$$

This, however, does not immediately imply that the effect of the array environment on the applied excitations can be approximated by ignoring the contributions of all distant elements. In general, the condition of large interelement distance is not satisfied for realistic arrays. Only measurements or a detailed understanding of environmental effects as contained in the interaction matrix $\underline{\underline{\Gamma}}$ will reveal a valid approximation scheme. Environmental effects have been treated quantitatively in [3]. Here the relevant facts are that each element in the array at successively larger distances from the element under observation contributes less and less to altering the effective excitation. In addition, the contribution of the array environment to a binary interaction is a higher-order effect that can be neglected in a first-order approximation scheme. Therefore, we might be able to approximate the elements of the effective excitation matrix with free-space mutual impedances $\underline{\underline{z}}^0_{ij}$. We can represent these ideas as

$$\begin{aligned}
&\hat{z}_{ij}(N) = \hat{z}_{ij}(2) + \hat{\varepsilon}_{ij}(N-2) \\
&\hat{\varepsilon}_{ij}(N-2) \ll \hat{z}_{ij}(2) \\
&\hat{z}_{ij}(2) = z^0_{ij}(2) + \hat{\mu}_{ij}(2) \\
&\hat{\mu}_{ij}(2) \ll z^0_{ij}(2)
\end{aligned} \tag{12}$$

where the $\hat{}$ indicates that the quantity in question contains all interaction effects, the superscript indicates the number of elements interacting, and $\hat{\varepsilon}_{ij}$ and $\hat{\mu}_{ij}$ represent environmental contributions. In case this approximation scheme is invalid for a given array environment, a different approximation scheme can be obtained by replacing all the 2s with some integer $k \geq 3$, representing the number of elements undergoing significant interactions. The specific value of k will depend on $\underline{\underline{S}}_0$ and $\underline{f}^0_n$. A consistent criterion for

truncating the number of open-circuited elements whose contribution is included to synthesize the elementary pattern of a radiating element can be stated as

$$\left|f_n^{(2)}(\theta,\phi)\right| \cong \left|f_m^{(i)}(\theta,\phi)\right|, \tag{12a}$$

where $\underline{f}_n^{(2)}$ is the second-order induced-field pattern originating from the radiating element n, and $\underline{f}_m^{(i)}$ is the first-order induced-field pattern at the open-circuited element m, located at $\underline{D}_{mn}$. Condition (12a) will be satisfied if $\underline{D}_{mn}$ is large enough. More distant elements will contribute less than $\underline{f}_n^{(2)}$ to the overall pattern and, hence, need not be included in a first-order approximation scheme.

The mathematical formalism to isolate environmental effects has been presented in [3]. We need to solve (4) and (6) for $\hat{\underline{a}}$, assuming the $\underline{\underline{\Gamma}}$ and $\underline{a}^0$ are known, using the cyclic-product decomposition of $\underline{\underline{\Gamma}}$. We rewrite (4) and (6) as

$$\underline{\underline{Q}}\ (\underline{\underline{W}}^{-1}\ \hat{\underline{a}}) = \underline{a}^0 \tag{13}$$

where

$$\underline{\underline{Q}} = \underline{\underline{z}} + \underline{\underline{I}}. \tag{13a}$$

In general, in an N-element array, $\hat{\underline{a}}$ is given by

$$(\underline{\underline{W}}^{-1}\ \hat{\underline{a}})_i = (Q_{:ii})^{-1}\ \underline{\bar{Q}}_{:ii}, \tag{14}$$

where $Q_{:ii}$ and $\underline{Q}_{:ii}$ are cyclic-product decompositions [3] of $\underline{\underline{Q}}$. It is instructive to work out a few examples. For N=2 we get

$$\hat{a}_1(2) = \frac{a_1^0 - z_{12}\,(z_{22}+1)^{-1}\,a_2^0}{1 - (z_{11}+1)^{-1}\,z_{12}(z_{22}+1)^{-1}\,z_{21}}$$

and

$$\hat{a}_2(2) = \frac{a_2^0 - z_{21}(z_{11}+1)^{-1}\,a_1^0}{1 - (z_{22}+1)^{-1}\,z_{21}(z_{11}+1)^{-1}\,z_{12}}. \tag{14a}$$

We can write the first of these expressions as

$$\hat{a}_1(2) = \frac{a_1^0 + \bar{\Pi}_1^{(1)}(\underline{z}, a_2^0)}{1 + (z_{11} + 1)^{-1}\, \Pi_1^{(1)}(\underline{z})} \tag{14b}$$

where $\bar{\Pi}$ is the environmental effect. Careful examination of these equations and the expansion for N=3 shows the recursive structure of $\bar{\Pi}$. We can easily verify that

$$\bar{\Pi}_1(2) = \frac{z_{12}}{z_{22} + 1}\, \hat{a}_2(2;23) + \frac{z_{13}}{z_{33} + 1}\, \hat{a}_3(2;23) \tag{14c}$$

where the superscripts indicate the number of elements in the environment and the indices of the elements interacting, respectively. For four elements the expansions get progressively more complicated. We merely write the expressions at the next level of expansion of $\underline{Q}$ in (14).[2] For i=1,

$$\bar{\underline{Q}}_{:11} = a_1 - z_{12}\, Q_{11:22}^{-1}\, \bar{\underline{Q}}_{12:21} + z_{13}\, Q_{11:33}^{-1}\, \bar{\underline{Q}}_{13:31} - z_{14}\, Q_{11:44}^{-1}\, \bar{\underline{Q}}_{14:41}$$

$$Q_{:11} = (z_{11} + 1) - a_{12}\, Q_{11:22}^{-1}\, Q_{12:21} + z_{13}\, Q_{11:33}^{-1}\, Q_{13:31} - z_{14}\, Q_{11:44}\, Q_{14:41} \,. \tag{15}$$

Note that $W_{11} = z_{11} + 1$, which is the first term in $Q_{:11}$.

The effective excitation $\hat{\underline{a}}$ can be represented as

$$a_i^{(N)} = \frac{a_i^{50} + \bar{\Pi}_i^{(N-1)}(\underline{z}, a_j^5)}{1 + (z_{ii} + 1)^{-1}\, \Pi_i^{(N-1)}(\underline{z})}, \quad j \neq i, \tag{16}$$

where $\bar{\Pi}_i(\underline{z}, a_j)$ and $\Pi_i(\underline{z})$ represent environmental effects. Physical argument suggests that $\lim_{m\to\infty} \hat{a}_i^{(m)}$ exists; that is, the addition of a single element does not significantly affect the array environment beyond some N. We intend to use this fact when synthesizing patterns of very large arrays, where the effects of distant elements are not measured and are unknown. However, we will not examine this question further at this point.

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[2] The general N-element expressions can be written down using the expansion defined in Appendix B of [3].

III. Methods of Data Analysis

In this section we briefly examine methods of data analysis suggested by the equations and discussion presented in the previous section. Our purpose is to analyze simulated and real array data to isolate interaction effects such as scattering, mutual impedance, and changes in the element patterns of individual elements in the open-circuited array environment. We assume that the standard methods of near-field scanning and data analysis will provide far-field patterns that contain the effects we are trying to isolate and study. If this is not the case, we will have to examine current scanning and data-analysis techniques as applied to arrays anew.

We base our approach on one simple fact: all of the pattern equations in the last section are linear functions of the element-excitation vectors $\hat{\underline{a}}$ or $\underline{a}^0$ and also of the modal coefficients $\underline{b}$ that generate the far-field patterns. This suggests standard methods of analysis, such as least squares, wherein linear coefficients are recovered from pattern data, and a given (assumed) model of interaction between the elements of the array. Equation (3) is the most general statement of array pattern synthesis; eqs (1) and (2) are special cases. For purposes of analysis we rewrite (3) as a linear system as follows:

$$\begin{bmatrix} \hat{\underline{f}}_1(\theta_1,\phi_1) & \cdots & \hat{\underline{f}}_\ell(\theta_1,\phi_1)\, e^{-i\underline{k}\cdot\underline{D}_\ell} & \cdots & \hat{\underline{f}}_N(\theta_1,\phi_1)\, e^{-i\underline{k}\cdot\underline{D}_N} \\ \vdots & & & & \\ \hat{\underline{f}}_1(\theta_i,\phi_j) & \cdots & \hat{\underline{f}}_\ell(\theta_i,\phi_j)\, e^{-i\underline{k}\cdot\underline{D}_\ell} & \cdots & \hat{\underline{f}}_N(\theta_i,\phi_j)\, e^{-i\underline{k}\cdot\underline{D}_N} \\ \vdots & & & & \\ \hat{\underline{f}}_1(\theta_N,\phi_N) & \cdots & \hat{\underline{f}}_\ell(\theta_N,\phi_N)\, e^{-i\underline{k}\cdot\underline{D}_\ell} & \cdots & \hat{\underline{f}}_N(\theta_N,\phi_N)\, e^{-i\underline{k}\cdot\underline{D}_N} \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_i \\ \vdots \\ \hat{a}_N \end{bmatrix} = \begin{bmatrix} \hat{\underline{F}}(\theta_1,\phi_1) \\ \vdots \\ \hat{\underline{F}}(\theta_i,\phi_j) \\ \vdots \\ \hat{\underline{F}}(\theta_N,\phi_N) \end{bmatrix} \quad (17)$$

We have required here that the resultant array pattern, represented by $\hat{\underline{F}}(\theta,\phi)$ on the right, be known. In addition, the positions $\underline{D}_i$ of the elements must be specified, as well as each elementary pattern $\hat{\underline{f}}_i$. Equation (17) merely has to be reinterpreted if the elements are considered noninteracting, but distinct. Considerable simplification of (17) occurs if the elements are identical: according to (1) the array pattern can be divided by the element pattern (provided it is nonzero), in which case the matrix in (17) reduces to a matrix of the phase factors. In addition, one does not have to consider

each component separately. The solution of (17) gives the known excitations applied to the elements.

Thus this simple and straightforward analysis can be used to decide whether interaction effects are important: significant deviations from the known excitations will indicate importance of interaction effects in the array environment. One can then carry the analysis deeper. Each elementary pattern in (3) can be represented by a set of basis functions together with appropriate phase factors to indicate the position of the radiating element. A matrix can now be constructed whose rows are

$$\beta_1^{(1)} \ldots \beta_L^{(1)} \; \beta_1^{(2)} e^{-i\underline{k}\cdot\underline{D}_2} \ldots \beta_L^{(2)} e^{-i\underline{k}\cdot\underline{D}_2} \ldots\ldots \beta_1^{(N)} e^{-i\underline{k}\cdot\underline{D}_N} \ldots \beta_L^{(N)} e^{-i\underline{k}\cdot\underline{D}_N} \tag{18}$$

for each (θ_i, ϕ_j), $i = 1, M$; $j = 1, M'$ where the superscripts identify the array elements, the $\underline{\beta}_\ell$ are the far-field basis functions, L is the number of modes radiated, and the other symbols have been previously defined. Each point in the region of interest is represented by a row; the region need not be a full sphere, although the conditioning of the matrix will be influenced by the extent of the region. This analysis will yield coefficients that are the products of the effective excitations (already determined in the previous analysis) and of the transmission coefficients of each element. We can thus obtain the transmission coefficients of each element in the array environment, which is a function of the free-space transmission coefficients and of the scattered fields.

A special combination of the previous avenues of analysis arises in the case of one element radiating in the presence of another, open-circuited element. Equation (8) gives a first-order approximation of the pattern obtained in this situation, which in matrix representation is

$$\begin{bmatrix} f_1^0(\theta_1,\phi_1) & \beta_1^{(2)} e^{-i\underline{k}\cdot\underline{D}_2} \ldots\ldots & \beta_j^{(2)} e^{-i\underline{k}\cdot\underline{D}_2} \ldots\ldots & \beta_L^{(2)} e^{-i\underline{k}\cdot\underline{D}_2} \\ \vdots & & & \\ f_1^0(\theta_i,\phi_j) & \beta_1^{(2)} e^{-i\underline{k}\cdot\underline{D}_2} \ldots\ldots & \beta_j^{(2)} e^{-i\underline{k}\cdot\underline{D}_2} \ldots\ldots & \beta_L^{(2)} e^{-i\underline{k}\cdot\underline{D}_2} \\ \vdots & & & \\ f_1^0(\theta_N,\phi_N) & \beta_1^{(2)} e^{-i\underline{k}\cdot\underline{D}_2} \ldots\ldots & \beta_j^{(2)} e^{-i\underline{k}\cdot\underline{D}_2} \ldots\ldots & \beta_L^{(2)} e^{-i\underline{k}\cdot\underline{D}_2} \end{bmatrix} \begin{bmatrix} \hat{a}^{(1)} \\ b_1^{(2)} \\ \vdots \\ b_j^{(2)} \\ \vdots \\ b_L^{(2)} \end{bmatrix}. \tag{19}$$

Here we have omitted the right-hand side, which is exactly the same as in (17). All symbols have been previously defined. We allow for an effective excitation of the radiating element, which, however, should not be very different from the applied excitation in a first-order approximation. This serves to check the validity of our model.

Assuming that the above analyses can successfully provide information on induced patterns and excitations in simple configurations, we can proceed to obtain data to approximate the array environment of an N-element array.

IV. Experimental Procedure to Study Interaction Effects in an Array Environment

Briefly, the objective of the measurement procedure should be: 1) to verify eqs (12) and (14) by measuring the mutual impedance between two elements in various array environments, and 2) to compare the measurements to calculations. To accomplish this, both pattern measurement and port-side excitation measurements are needed. Multiple-reflection effects between the probe and the array environment, and on the port side, will have to be eliminated or corrected for during the analysis. The procedure is essentially the same for arrays of microstrip panels, waveguides, or horns. The following sequence of measurements is indicated:

A. Two-element study.

(1) Experimental measurements.

(a) Measure the free-space radiation pattern of both elements under study. If the elements are known to be identical within acceptable tolerances, then only one pattern measurement is needed. The elements should be tuned so that no port-side reflections occur when they radiate into free space.

(b) Measure the elementary pattern in the two-element array environment. This involves open circuiting one of the elements and exciting the other one. Ideally, elementary patterns for both elements should be measured to have some estimate of the variability within the array environment.

(c) Measure concurrently with (b) both the incoming and outgoing signals on the port sides of both antennas.

(d) Repeat steps (b) and (c) for various interelement distances that represent the positions of elements in the full array. Here, two-dimensional measurements are indicated if analysis of planar arrays is desired.

(2) Important effects and computations.

(a) From (A1a) and (A1b) above, we can observe the effects of interelement scattering and analyze the data for the coefficients of the main pattern as well as the induced patterns.

(b) From the data obtained in (A1c) and (A1d) above, the mutual impedance matrix can be obtained as a function of interelement distance in a two-element array. The measurements can be compared to computations of mutual impedance using various models for interelement scattering, such as a minimum scattering model and a model where the open-circuited scattering matrix is diagonal. These results can serve as an approximate binary interaction model in N-element arrays.

B. Three-element study.

(1) Experimental measurements.

(a) Measure the free-space radiation pattern of all elements under study. If the elements are known to be identical within acceptable tolerances, then only one pattern measurement is needed.

(b) Measure the elementary patterns in the three-element environment. This involves exciting one of the elements and open circuiting all the others. Ideally, all elementary patterns

should be measured to have some estimate of the variability within the array environment.

(c) Measure concurrently with (b) both the incoming and outgoing signals on the port sides of all antennas.

(d) Repeat steps (b) and (c) for various interelement distances. Two possibilities arise here: 1) only one of the elements is repositioned; and 2) both of the open-circuited elements are repositioned, but kept adjacent to each other. The first of these procedures will show a larger effect than the second.

(2) Important effects and computations.

(a) From (B1a) and (B1b) above, the effect of the open-circuited environment on the free-space element pattern can be observed. More specifically, the results here can be compared to the results in the two-element study to look for possible convergence in both the elementary patterns and in the coefficients of the scattered fields. This kind of analysis will allow us to specify the size of the environment that must be studied to accurately describe interelement interactions in large arrays. In particular, we will be able to decide whether nearest neighbor interactions or three-element subarrays are adequate to represent a full array. We might find that larger subarrays are necessary to correctly approximate the environment.

(b) From (B1c) and (B1d) the mutual-impedance matrix in a three-element environment can be constructed. Results can be compared to computed mutual-impedance matrices for various models of interaction. By comparing the mutual-impedance data obtained in the two cases indicated in (B1d), the relative importance of open-circuited elements in the environment will become apparent.

C. N-Element study.

In case the two- or three-element studies do not provide adequate information on convergence of elementary patterns and mutual impedances, the measurement above should be repeated with larger number of elements. We shall not outline here the steps involved as they are essentially the same as in (A) and (B) above.

V. Simulation Study

In our simulation study we had to satisfy three major concerns:

A. We verified that the computer codes were error free and, in fact, performed the desired tasks. Naturally, we could not proceed without first satisfying this requirement.

B. We ascertained that the matrix inversions required by the methods of analysis were stable computationally, so that the results could be considered reliable. To this purpose we have used the method of singular-value decomposition [6], a powerful method of matrix inversion that automatically provides full information on the conditioning of a matrix.

C. We simulated some of the most important effects described in the previous sections. Primarily we were concerned with illustrating interaction phenomena using simple theoretical models for our array elements. The following were studied:

(1) Two-element n-mode interactions. Here we calculated the

(a) mutual-impedance matrix as a function of interelement distance in a two-element array,

(b) effective excitations as functions of interelement spacing in a two-element array, and

(2) pattern synthesis and analysis for N>2 noninteracting elements.

In tables 1-4 (pp 24-28) we tabulated the elements of the mutual-impedance matrix for arrays of two identical elements, radiating with maximum mode numbers nmax = 1, 2, 3, and 4, respectively. The zeroth (no multiple reflections) and first order (only one reflection) approximations of each quantity are also presented. (Note that the zeroth order approximation to self impedances is 1.) In figures 1a, 1b, and 1c the mutual impedance z_{12} and the self-impedances z_{11} and z_{22} together with their approximations are plotted for $n_{max} = 4$, which is the most realistic case simulated. A few important observations can be made.

1. The first-order approximations are significantly better than the zeroth-order approximations for all but the largest separation of the elements. This discrepancy increases with nmax. Since the zeroth-order mutual impedance is the mutual impedance between minimum-scattering antennas [1], environmental effects are important even at relatively large distances for a radiation mode as low as nmax = 4. The last entry in table 4 will quickly substantiate this observation.

2. For small separation distances of the order of λ all approximations are largely in error, since the condition $kD \geq nmax$, defining the maximum mode number in the antenna pattern, is almost violated. For such separations the multiple-reflection effects due to the higher-order modes are exaggerated.

In tables 5-8 we show $\hat{\underline{a}}$, the effective excitation vector as a function of kD, when the two elements are excited in phase with unit amplitude. For $n_{max} = 4$, the data are plotted in figure 2. The original excitation is modified by the presence of mutual impedance and multiple reflections to result in a substantial change in amplitude and small induced imaginary components. As expected, multiple reflections are important between closely spaced elements, but the induced oscillations are still apparent at large separations, although the effect diminishes.

In tables 9-12 we show the effective excitation vector as a function of kD when the excitations of unit amplitude are 10 degrees out of phase. There

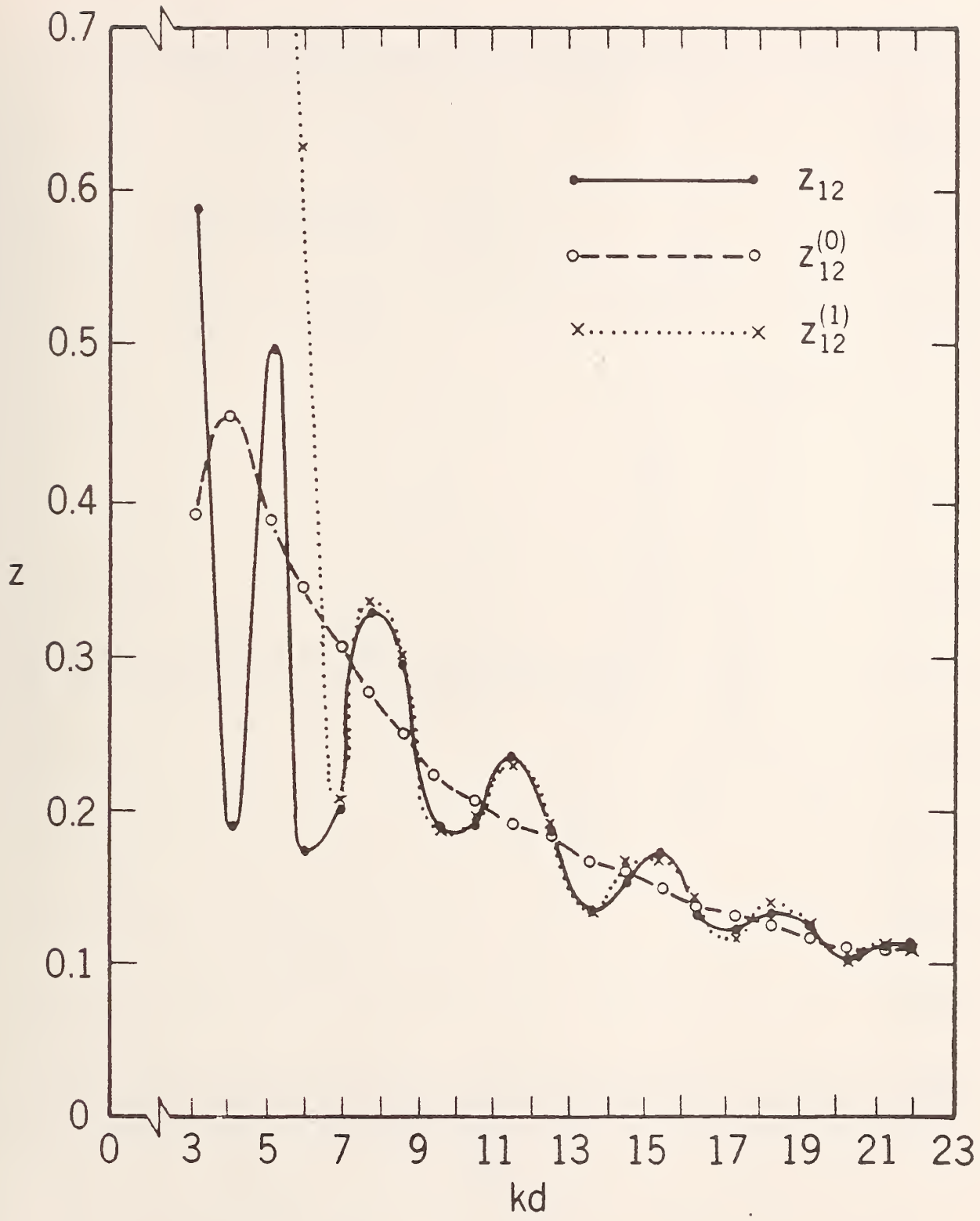

Figure 1a. The magnitude of the mutual-impedance z_{12} and its zeroth- and first-order approximations for $n_{max} = 4$ in a two-element array as a function of separation.

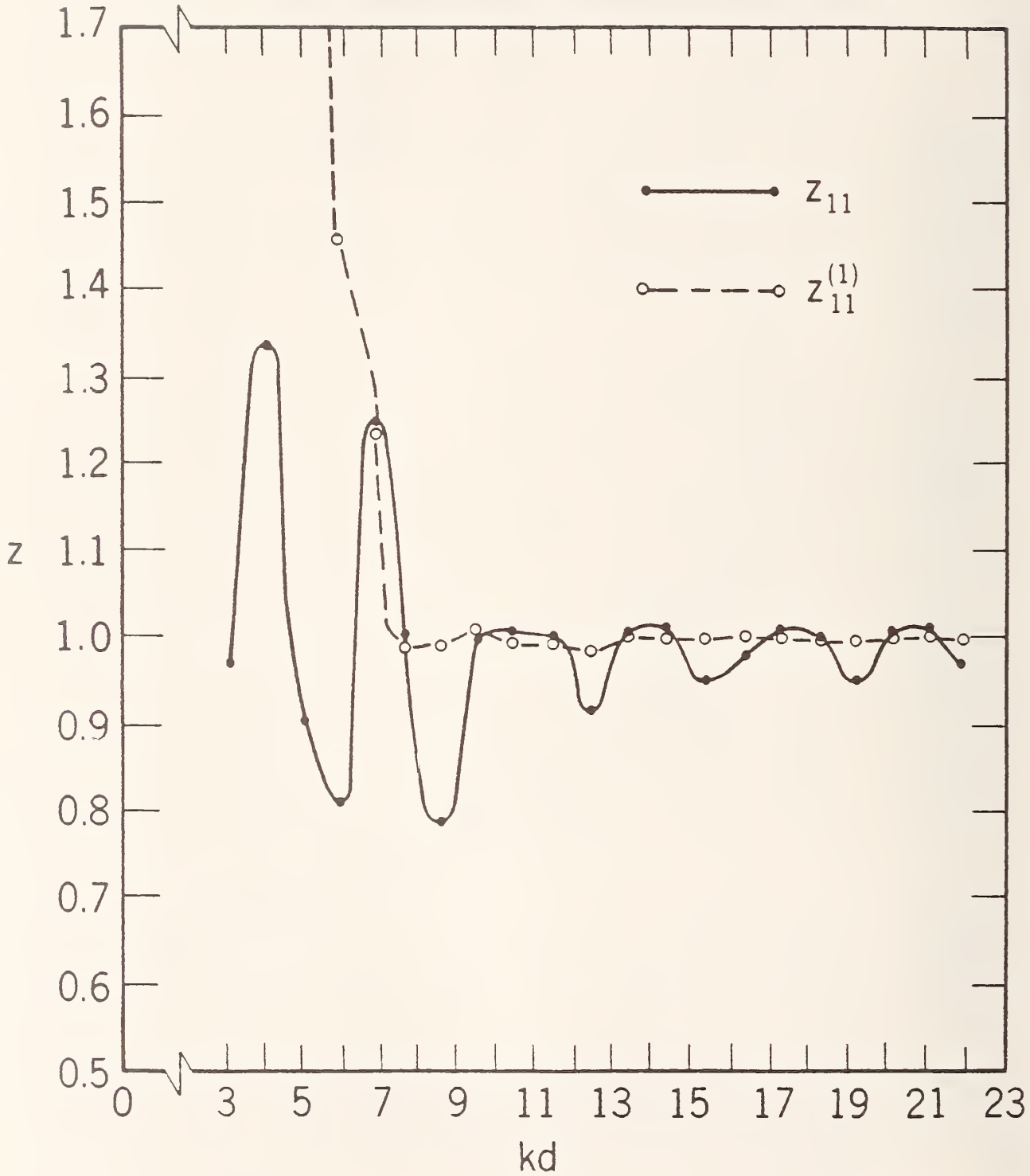

Figure 1b. The magnitude of the self-impedance z_{11} and its first-order approximation for $n_{max} = 4$ in a two-element array as a function of separation. The zeroth-order approximation is identically unity.

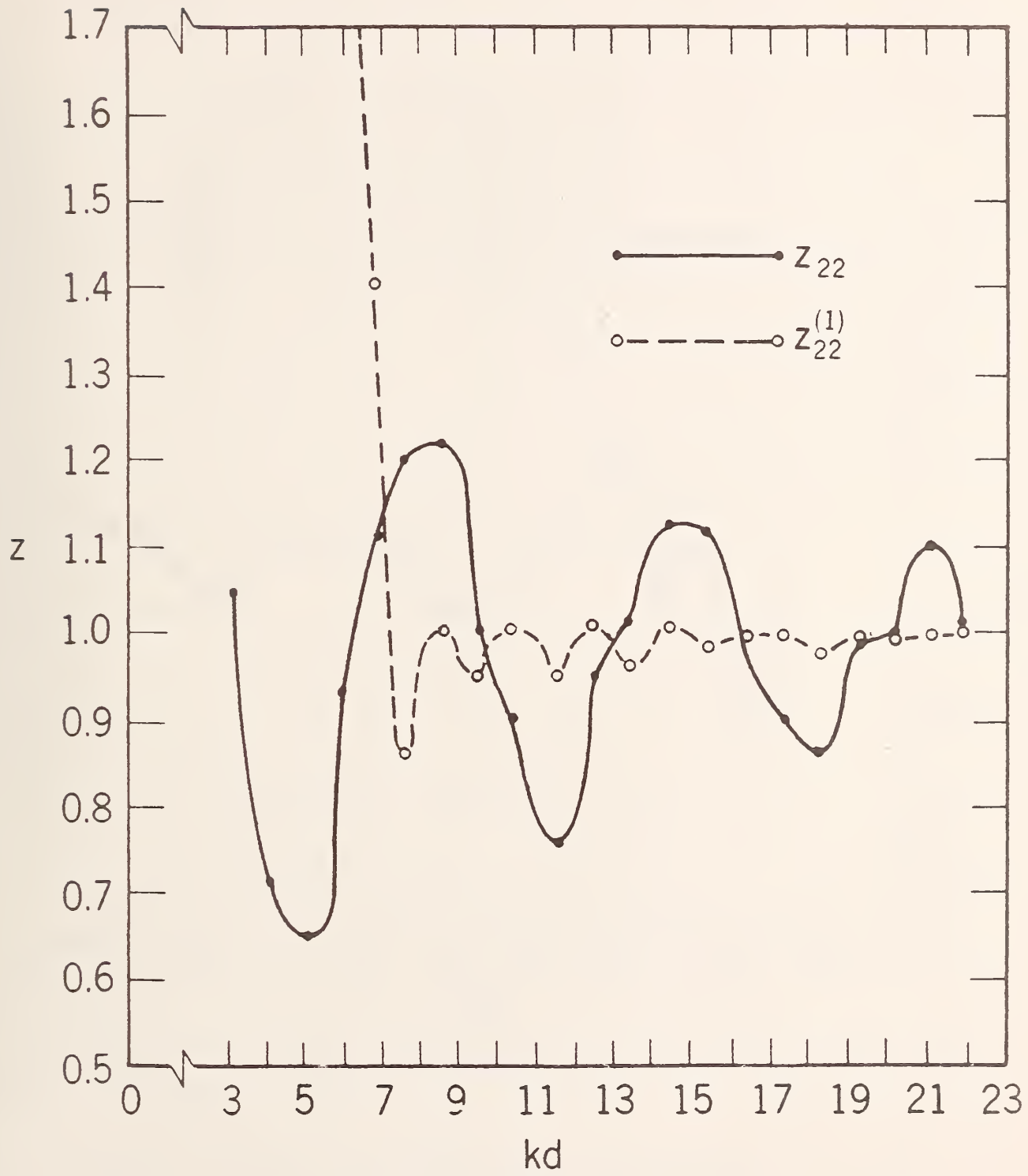

Figure 1c. The magnitude of the self-impedance z_{22} and its first-order approximation for $n_{max} = 4$ in a two-element array as a function of separation. The zeroth order approximation is identically unity.

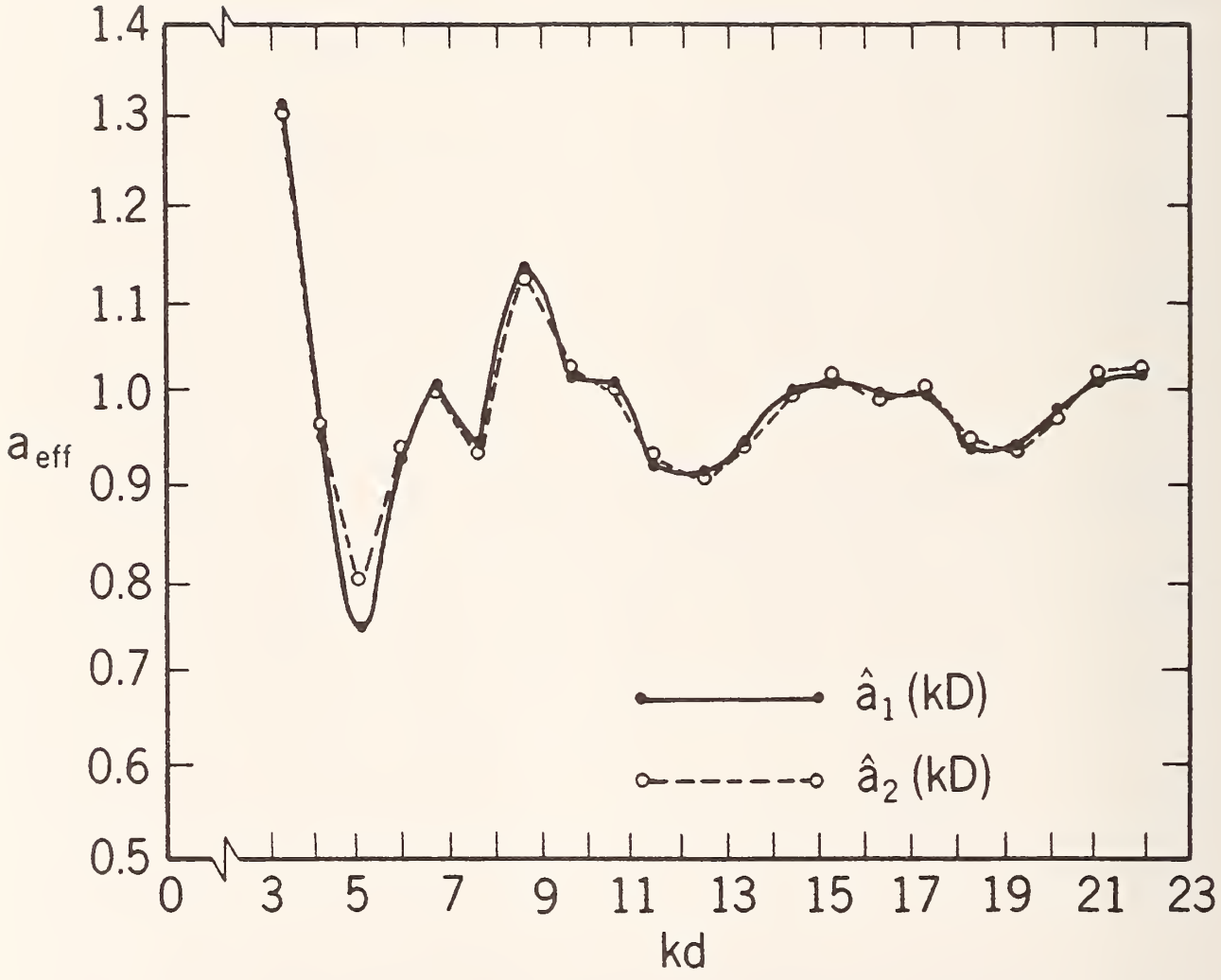

Figure 2. The magnitude of the effective excitation vector in a two-element array for $n_{max} = 4$ as a function of separation. The real excitation is unity and is altered by mutual-impedance effects.

seem to be no new qualitative features in the data when compared to the in-phase simulation.

In the pattern analysis simulation study we posed the following question: can the phased excitations of elements be recovered if both the array and element far-field patterns are known? We found that this inverse problem is stable for exact data. The three parameters of the study were: 1) N, the number of elements in the linear array situated on the z axis; 2) n_{max}, the maximum mode number radiated by the identical elements; and 3) NPOINTS, the number of points where the pattern is assumed to be known and equally spaced in the full range of θ and ϕ in spherical geometry. In table 13 we summarize the values chosen in the parameter space. Pattern inversions were performed for all possible combinations of values in the three dimensional parameter space. In table 14 we present some of the condition numbers and residuals obtained in the inversion of the pattern data as discussed in Section III. The data clearly show that the inversion procedure is stable.

VI. Generalization to Two- and Three-Dimensional Arrays

Up to now we applied our analysis to one-dimensional arrays. The generalization to two- and three-dimensional arrays is of practical interest and poses no new theoretical challenges or difficulties. Since the theory of interactions has been formulated using binary interaction matrices [3], the particular array geometry in which the two interacting elements are embedded is of no consequence; if the free-space far-field patterns are known in frames of reference fixed to the elements, the far-field patterns in other frames rotated with respect to the fixed frames can be calculated according to well known standard techniques [7]. In fact, using these ideas and a re-examination of antenna-scanning theories allows one to formulate a coordinate-independent scanning theory. This discussion is developed in somewhat more detail in the appendix.

VII. Summary, Conclusions, and Suggestions for Further Study

In this study we have discussed methods of analysis of far-field patterns of phased arrays both in the absence and presence of multiple reflections between elements. We have performed theoretical computations and obtained the

mutual-impedance matrix as a function of interelement spacing. We have learned that the matrix inversions needed in the course of these computations are stable as long as the condition $kd \geq n_{max}$ is not violated. We have also found that the first-order correction is an excellent approximation for the mutual impedances. This eliminates the need for the costly matrix inversion, unless extreme accuracy is demanded. For the self impedances, the first order approximation was found to be inadequate. We have also calculated the effective excitation of elements as a function of position. The presence of a small induced oscillating component is observed even at large distances. Simulations to recover the element excitations from exact array far-field data and elementary far-field data were successful.

The simulations described in this report can be considered to be phase I of a more complex study, wherein we would include

(a) more than two elements in the computation of mutual impedances and effective excitations,

(b) the effect of multiple reflections on the elementary patterns,

and, finally,

(c) computations generalized to two- or three-dimensional arrays.

We conclude by noting that no new theoretical results are needed to accomplish this more complex but more realistic phase II of our study.

VIII. References

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Table 1

The magnitude and phase (degrees) of the complex mutual-impedance z_{12} and its zeroth-and first-order approximations and self-impedances z_{11} and z_{22} and their first-order approximations* as functions of kD ($\pi \leq kD \leq 7\pi$) for $n_{max} = 1$

kD	z_{12}		$z_{12}^{(0)}$		$z_{12}^{(1)}$		z_{11}		$z_{11}^{(1)}$		z_{22}		$z_{22}^{(1)}$	
3.14	0.580	179.6	0.623	180.2	0.587	181.3	0.912	355.8	1.012	1.23	1.234	15.6	1.084	0.2
4.08	0.487	230.4	0.493	231.3	0.484	229.8	1.069	355.9	0.997	359.6	0.954	20.5	0.972	359.8
5.03	0.412	282.4	0.406	283.5	0.412	282.7	1.044	3.3	1.100	0.16	0.715	7.4	1.011	0.3
5.97	0.348	337.1	0.345	336.3	0.349	337.0	0.951	1.2	1.000	359.9	0.819	349.3	0.996	359.7
6.91	0.297	298.0	0.300	29.5	0.297	29.9	0.994	357.7	0.999	0.0	1.017	348.1	1.001	0.2
7.85	0.264	82.4	0.270	82.8	0.264	82.3	1.033	0.2	1.000	0.0	1.155	354.1	1.000	359.9
8.80	0.239	136.3	0.237	136.3	0.239	136.3	0.990	1.5	0.999	0.0	1.166	2.3	0.999	0.1
9.74	0.215	190.3	0.215	190.0	0.215	190.3	0.983	359.2	1.000	0.0	1.063	7.7	1.001	0.0
10.68	0.195	244.0	0.196	243.6	0.195	243.5	1.016	359.4	1.000	0.0	0.933	7.3	0.999	0.0
11.62	0.181	297.1	0.181	297.3	0.181	297.1	1.005	0.9	1.000	0.0	0.870	0.6	1.000	0.0
12.57	0.168	351.3	0.167	351.1	0.168	351.3	0.986	0.0	1.000	0.0	0.929	354.3	1.000	0.0
13.51	0.155	45.0	0.156	44.9	0.155	45.0	1.003	359.3	1.000	0.00	1.032	354.0	0.999	0.0
14.45	0.146	98.5	0.146	98.7	0.146	98.5	1.009	0.3	1.000	0.00	1.097	357.9	1.000	0.0
15.39	0.137	152.6	0.137	152.6	0.137	152.6	0.993	0.3	1.000	0.0	1.085	2.7	0.999	0.0
16.34	0.129	206.5	0.129	206.4	0.129	206.6	0.997	359.6	1.000	0.0	1.012	5.2	1.000	0.0
17.28	0.122	260.2	0.122	260.3	0.122	260.2	1.008	360.0	1.000	0.0	0.936	3.5	1.000	0.0
18.22	0.116	314.1	0.116	314.2	0.116	314.1	0.999	0.4	1.000	0.000	0.919	358.8	1.000	0.0
19.16	0.110	8.2	0.110	8.1	0.110	8.2	0.995	359.8	1.000	0.000	0.972	355.7	1.000	0.0
20.11	0.105	62.0	0.105	62.0	0.105	62.0	1.004	359.8	1.000	0.0	1.040	356.4	1.000	0.0
21.05	0.100	115.8	0.100	115.9	0.100	115.8	1.002	0.3	1.000	0.000	1.072	359.6	1.000	0.0
21.99	0.096	169.9	0.096	169.8	0.096	169.9	0.995	0.0	1.000	0.0	1.047	2.8	1.000	0.0

*The zeroth-order approximations $z_{11}^{(0)}$ and $z_{22}^{(0)}$ are always 1 independent of kD.

Table 2

The magnitude and phase (degrees) of the complex mutual-impedance z_{12} and its zeroth-and first-order approximations and self-impedances z_{11} and z_{22} and their first-order approximations* as functions of kD ($\pi < kD < 7\pi$) for $n_{max} = 2$

kD	z_{12}		$z_{12}^{(0)}$		$z_{12}^{(1)}$		z_{11}		$z_{11}^{(1)}$		z_{22}		$z_{22}^{(1)}$	
3.14	0.556	185.5	0.598	192.6	0.570	184.5	1.177	0.6	1.002	358.3	1.261	23.57	0.991	0.5
4.08	0.563	243.4	0.525	245.0	0.562	243.7	0.998	0.5	1.003	0.5	0.830	30.57	1.002	359.6
5.03	0.459	299.9	0.455	297.4	0.460	299.9	0.965	359.6	0.998	359.8	0.587	1.004	0.999	0.1
5.97	0.387	350.3	0.398	350.2	0.386	350.3	1.003	359.1	1.001	0.1	0.812	341.5	1.000	0.0
6.91	0.353	42.1	0.352	43.3	0.353	42.2	1.010	0.1	0.999	359.9	1.094	343.7	0.999	0.0
7.85	0.319	96.9	0.315	96.5	0.319	96.8	0.998	0.4	1.000	0.0	1.257	354.2	1.000	0.0
8.80	0.283	150.4	0.285	149.9	0.283	150.4	0.996	359.9	1.000	0.0	1.227	5.185	1.000	0.0
9.74	0.258	203.0	0.260	203.4	0.258	203.0	1.002	359.8	0.999	0.0	1.066	11.98	1.000	0.0
10.68	0.240	256.8	0.239	257.0	0.240	256.8	1.003	0.1	1.000	0.0	0.873	10.17	1.000	0.0
11.62	0.221	310.9	0.221	310.6	0.221	310.9	0.998	0.1	0.999	0.0	0.810	358.5	1.000	0.0
12.57	0.204	4.3	0.205	4.3	0.204	4.3	0.999	359.9	1.000	0.0	0.923	350.5	1.000	0.0
13.51	0.192	57.9	0.192	58.0	0.192	57.9	1.002	0.0	1.000	0.0	1.076	351.8	1.000	0.0
14.45	0.180	111.9	0.180	111.8	0.180	111.9	1.000	0.1	1.000	0.0	1.152	358.3	1.000	0.0
15.39	0.169	165.7	0.169	165.6	0.169	165.7	0.999	0.0	1.000	0.0	1.115	4.891	1.000	0.0
16.34	0.160	219.3	0.160	219.4	0.160	219.3	1.000	359.9	1.000	0.0	0.999	7.906	1.000	0.0
17.28	0.152	273.3	0.151	273.2	0.152	273.3	1.001	0.0	1.000	0.0	0.892	4.451	1.000	0.0
18.22	0.144	327.2	0.144	327.1	0.144	327.2	0.999	0.0	1.000	0.0	0.886	357.0	1.000	0.0
19.16	0.137	20.9	0.137	21.0	0.137	20.9	0.999	0.0	1.000	0.0	0.977	353.3	1.000	0.0
20.11	0.131	74.8	0.131	74.8	0.131	74.8	1.001	0.0	1.000	0.0	1.075	355.4	1.000	0.0
21.05	0.125	128.8	0.125	128.7	0.125	128.8	0.999	0.0	1.000	0.0	1.108	0.328	1.000	0.0
21.99	0.120	182.6	0.120	182.6	0.120	182.6	0.999	0.0	1.000	0.0	1.059	4.688	1.000	0.0

*The zeroth-order approximations $z_{11}^{(0)}$ and $z_{22}^{(0)}$ are always 1 independent of kD.

Table 3

The magnitude and phase (degrees) of the complex mutual-impedance z_{12} and its zeroth-and first-order approximations and self-impedances z_{11} and z_{22} and their first-order approximations* as functions of kD ($\pi \leq kD \leq 7\pi$) for $n_{max} = 3$

kD	z_{12}		$z_{12}^{(0)}$		$z_{12}^{(1)}$		z_{11}		$z_{11}^{(1)}$		z_{22}		$z_{22}^{(1)}$	
3.14	0.783	172.0	0.797	154.1	14.75	309.2	1.371	12.0	110.7	113.4	1.455	8.8	157.1	1.4
4.08	0.549	243.4	0.538	206.9	0.798	320.9	0.705	353.0	1.970	185.1	1.005	17.9	2.768	17.8
5.03	0.267	279.7	0.407	260.7	0.179	278.1	1.209	354.2	1.108	350.3	0.912	7.6	0.968	6.0
5.97	0.274	304.9	0.327	314.8	0.282	302.8	1.067	8.8	1.013	0.8	0.854	4.0	0.992	359.8
6.91	0.295	4.4	0.274	9.1	0.295	4.9	0.870	2.4	0.991	0.3	0.876	353.3	1.002	359.7
7.85	0.243	66.1	0.236	63.3	0.243	66.0	0.995	354.2	1.006	359.8	1.032	352.7	0.999	0.2
8.80	0.199	11.8	0.207	117.6	0.199	117.7	1.079	359.8	0.995	0.0	1.096	357.3	1.001	359.9
9.74	0.187	169.9	0.184	171.8	0.187	170.0	0.998	4.1	1.004	0.1	1.092	2.2	0.999	0.1
10.68	0.170	227.6	0.166	226.1	0.170	227.4	0.941	358.7	0.998	359.9	1.017	5.0	1.001	0.0
11.62	0.147	281.2	0.151	280.3	0.147	281.3	1.027	357.7	1.001	0.1	0.946	3.2	0.999	0.0
12.57	0.137	332.7	0.139	334.5	0.137	332.6	1.034	1.6	1.001	359.9	0.929	359.4	1.000	0.0
13.51	0.132	28.6	0.128	28.7	0.132	28.6	0.966	1.3	0.999	0.1	0.971	356.4	0.999	0.0
14.45	0.118	84.5	0.119	82.8	0.118	84.5	0.991	358.1	1.001	0.0	1.031	357.0	0.999	0.0
15.39	0.109	136.3	0.111	136.9	0.108	136.2	1.030	359.9	0.999	0.0	1.055	359.6	1.000	0.0
16.34	0.106	189.9	0.104	19.1	0.106	189.9	0.995	1.6	1.001	0.0	1.039	2.1	0.999	0.0
17.28	0.099	246.4	0.099	245.2	0.099	246.3	0.978	359.3	0.999	0.0	0.991	2.9	1.000	0.0
18.22	0.091	299.7	0.093	299.3	0.091	299.8	1.016	359.1	0.999	0.0	0.958	1.2	0.999	0.0
19.16	0.088	352.2	0.088	353.4	0.088	352.2	1.011	1.0	1.000	0.0	0.961	358.8	1.000	0.0
20.11	0.085	47.7	0.084	47.5	0.085	47.7	0.981	0.3	0.999	0.0	0.996	357.5	1.000	0.0
21.05	0.079	102.5	0.080	101.6	0.079	102.6	1.001	359.0	1.000	0.0	1.030	358.5	0.999	0.0
21.99	0.075	155.0	0.076	155.7	0.075	155.0	1.015	0.3	0.999	0.0	1.037	0.4	1.000	0.0

*The zeroth-order approximations $z_{11}^{(0)}$ and $z_{22}^{(0)}$ are always 1 independent of kD.

Table 4

The magnitude and phase (degrees) of the complex mutual-impedance z_{12} and its zeroth-and first-order approximations and self-impedances z_{11} and z_{22} and their first-order approximations* as functions of kD ($\pi \leq kD \leq 7\pi$) for $n_{max} = 4$

kD	z_{12}		$z_{12}^{(0)}$		$z_{12}^{(1)}$		z_{11}		$z_{11}^{(1)}$		z_{22}		$z_{22}^{(1)}$	
3.14	0.582	220.2	0.391	205.5	1412	49.4	0.975	17.5	901700	123.4	1.056	25.1	638400	210.3
4.08	0.194	293.6	0.426	253.2	48.86	58.8	1.343	27.2	1146	144.8	0.715	4.9	2542	180.8
5.03	0.494	339.0	0.384	303.0	4.421	88.0	0.904	346.0	19.66	161.5	0.646	334.2	63.12	216.6
5.97	0.162	322.7	0.341	354.3	0.624	155.5	0.804	14.2	1.471	270.4	0.934	352.0	5.300	310.1
6.91	0.204	87.3	0.303	46.4	0.204	344.6	1.259	348.1	1.223	0.2	1.136	347.4	1.410	20.4
7.85	0.336	70.9	0.273	98.9	0.344	84.2	1.012	6.8	0.992	359.6	1.209	349.2	0.873	359.8
8.80	0.293	165.5	0.248	151.7	0.295	160.8	0.889	4.1	0.983	1.2	1.252	10.0	1.057	358.1
9.74	0.189	209.5	0.227	204.8	0.184	213.4	0.999	353.8	1.018	359.0	1.001	10.4	0.960	1.7
10.68	0.189	248.2	0.209	258.0	0.194	245.7	1.069	359.6	0.983	0.5	0.909	8.9	1.036	359.0
11.62	0.236	306.2	0.193	311.4	0.231	308.1	1.020	4.1	1.014	359.9	0.769	359.4	0.968	0.4
12.57	0.186	16.6	0.180	4.9	0.190	15.1	0.922	359.1	0.989	359.9	0.972	349.0	1.026	0.2
13.51	0.142	58.7	0.168	58.4	0.139	59.5	1.029	357.1	1.007	0.3	1.074	353.8	0.981	359.4
14.45	0.158	103.1	0.158	112.0	0.161	102.9	1.042	1.3	0.997	359.7	1.139	357.6	1.011	0.7
15.39	0.173	167.5	0.149	165.6	0.170	167.3	0.966	2.3	0.999	0.3	1.126	6.1	0.996	359.2
16.34	0.132	226.2	0.141	219.3	0.134	226.7	0.982	357.6	1.002	359.8	0.971	7.3	0.999	0.6
17.28	0.121	269.2	0.133	273.0	0.120	268.5	1.035	359.5	0.997	0.1	0.906	3.7	1.005	359.6
18.22	0.137	322.1	0.127	326.7	0.138	322.7	1.004	2.0	1.003	0.0	0.880	357.1	0.993	0.2
19.16	0.127	25.8	0.121	20.5	0.127	25.3	0.967	359.5	0.997	359.9	0.996	352.9	1.006	0.0
20.11	0.106	75.7	0.115	74.3	0.105	76.0	1.015	358.7	1.001	0.1	1.074	356.3	0.995	359.9
21.05	0.110	123.3	0.111	128.1	0.110	123.2	1.019	0.9	1.000	359.9	1.102	0.2	1.002	0.2
21.99	0.115	182.8	0.106	181.9	0.114	182.7	0.979	0.8	0.999	0.1	1.057	5.2	1.000	359.8

*The zeroth-order approximations $z_{11}^{(0)}$ and $z_{22}^{(0)}$ are always 1 independent of kD.

Table 5

The magnitude and phase (degrees) of the effective excitation $\hat{\underline{a}}$ for $\pi \leq kD \leq 7\pi$ for a two-element array excited with unit amplitudes 10 degrees out of phase for $n_{max} = 1$.

kD	$\hat{a}_1(kD)$		$\hat{a}_2(kD)$	
3.14	1.369	359.6	1.412	7.4
4.08	1.172	12.9	1.179	21.1
5.03	0.907	12.9	0.973	20.5
5.97	0.831	0.6	0.879	15.3
6.91	0.911	354.4	0.870	7.6
7.85	1.004	353.3	0.954	2.9
8.80	1.092	335.9	1.074	4.0
9.74	1.112	1.7	1.120	10.3
10.68	1.033	5.7	1.054	14.7
11.62	0.940	4.3	0.973	14.7
12.57	0.920	359.5	0.927	11.6
13.51	0.961	356.5	0.938	7.6
14.45	1.021	356.3	0.996	57.6
15.39	1.065	358.6	1.058	7.5
16.34	1.057	2.1	1.065	11.2
17.28	1.000	3.7	1.019	13.3
18.22	0.951	1.9	0.969	12.7
19.16	0.951	358.9	0.948	10.2
20.11	0.985	357.3	0.967	7.7
21.05	1.028	357.7	1.013	7.2
21.99	1.049	359.9	1.048	9.0

Table 6

The magnitude and phase (degrees) of the effective excitation $\hat{\underline{a}}$ for $\pi \leq kD \leq 7\pi$ for a two-element array excited with unit amplitudes 10 degrees out of phase for $n_{max} = 2$.

kD	$\hat{a}_1(kD)$		$\hat{a}_2(kD)$	
3.14	1.337	0.3	1.341	8.9
4.08	1.153	18.0	1.157	25.2
5.03	0.817	12.1	0.913	20.2
5.97	0.821	357.2	0.855	14.0
6.91	0.927	352.5	0.855	5.5
7.85	1.035	352.4	0.982	0.5
8.80	1.128	356.6	1.122	4.4
9.74	1.128	3.5	1.140	12.1
10.68	1.012	7.6	1.039	16.5
11.62	0.906	4.0	0.946	15.0
12.57	0.910	358.0	0.908	10.8
13.51	0.971	355.3	0.934	6.1
14.45	1.042	355.9	1.017	4.8
15.39	1.086	359.3	1.084	8.0
16.34	1.060	3.4	1.072	12.5
17.28	0.982	4.6	1.006	14.2
18.22	0.932	1.4	0.951	12.7
19.16	0.947	357.8	0.936	9.4
20.11	0.995	356.5	0.970	6.6
21.05	1.044	357.6	1.031	6.7
21.99	1.061	0.6	1.063	9.6

Table 7

The magnitude and phase (degrees) of the effective excitation $\hat{\underline{a}}$ for $\pi \leq kD \leq 7\pi$ for a two-element array excited with unit amplitudes 10 degrees out of phase for $n_{max} = 3$.

kD	$\hat{a}_1(kD)$		$\hat{a}_2(kD)$	
3.14	1.453	356.0	1.441	0.7
4.08	1.094	16.8	1.153	27.04
5.03	0.955	7.6	0.985	16.61
5.97	0.898	5.9	0.953	16.44
6.91	0.873	357.1	0.865	11.52
7.85	0.975	353.7	0.933	4.28
8.80	1.057	355.6	1.028	4.5
9.74	1.098	359.7	1.096	8.0
10.68	1.050	4.2	1.070	13.3
11.62	0.972	4.1	0.995	14.0
12.57	0.935	1.1	0.951	12.3
13.51	0.952	357.6	0.938	9.0
14.45	1.004	356.7	0.983	6.7
15.39	1.045	358.2	1.033	7.5
16.34	1.052	1.0	1.056	10.0
17.28	1.012	2.8	1.027	12.5
18.22	0.970	2.0	0.984	12.4
19.16	0.957	359.8	0.960	10.8
20.11	0.978	358.0	0.966	8.6
21.05	1.015	357.9	1.001	7.7
21.99	1.037	359.4	1.032	8.8

Table 8

The magnitude and phase (degrees) of the effective excitation $\hat{\underline{a}}$ for $\pi \leq kD \leq 7\pi$ for a two-element array excited with unit amplitudes 10 degrees out of phase for $n_{max} = 4$.

kD	$\hat{a}_1(kD)$		$\hat{a}_2(kD)$	
3.14	1.298	12.6	1.338	19.4
4.08	0.938	5.6	0.998	14.6
5.03	0.747	358.3	0.829	15.9
5.97	0.925	1.7	0.951	14.0
6.91	1.019	354.7	0.986	4.7
7.85	0.983	352.1	0.902	2.0
8.80	1.147	358.1	1.160	5.8
9.74	1.085	3.3	1.093	12.4
10.68	1.026	5.7	1.044	14.6
11.62	0.903	5.0	0.953	15.6
12.57	0.928	357.2	0.911	9.6
13.51	0.978	356.4	0.954	7.0
14.45	1.027	356.1	1.001	5.5
15.39	1.088	359.4	1.088	7.9
16.34	1.042	3.2	1.054	12.5
17.28	0.990	3.7	1.009	13.3
18.22	0.936	1.7	0.958	12.8
19.16	0.954	357.8	0.940	9.1
20.11	0.997	357.1	0.978	7.3
21.05	1.035	357.7	1.021	7.1
21.99	1.058	0.5	1.061	9.6

Table 9

The magnitude and phase (degrees) of the effective excitation $\underline{a}$ for $\pi \leq kD \leq 7\pi$ for a two-element array excited with unit amplitudes in phase for $n_{max} = 1$

kD	$\hat{a}_1(kD)$		$\hat{a}_2(kD)$	
3.14	1.367	357.5	1.414	359.7
4.08	1.199	11.2	1.153	12.7
5.03	0.947	13.0	0.940	10.7
5.97	0.841	2.8	0.861	3.5
6.91	0.892	355.6	0.883	356.2
7.85	0.983	353.2	0.976	352.7
8.80	1.081	355.1	1.089	354.8
9.74	1.116	0.8	1.119	1.2
10.68	1.048	5.2	1.041	5.2
11.62	0.955	4.8	0.958	4.4
12.57	0.920	0.5	0.923	0.7
13.51	0.950	356.9	0.947	357.0
14.45	1.009	356.0	1.008	355.9
15.39	1.061	358.1	1.064	358.1
16.34	1.062	1.6	1.061	1.8
17.28	1.011	3.6	1.008	3.5
18.22	0.958	2.4	0.961	2.3
19.16	0.948	359.5	0.948	359.6
20.11	0.977	357.4	0.975	357.4
21.05	1.021	357.4	1.021	357.4
21.99	1.048	359.4	1.050	359.5

Table 10

The magnitude and phase (degrees) of the effective excitation $\hat{\underline{a}}$ for $\pi < kD < 7\pi$ for a two-element array excited with unit amplitudes in phase for $n_{max} = 2$

kD	$\hat{a}_1(kD)$		$\hat{a}_2(kD)$	
3.14	1.335	358.3	1.341	0.9
4.08	1.189	16.1	1.123	16.9
5.03	0.863	13.1	0.874	9.7
5.97	0.817	359.9	0.845	1.7
6.91	0.900	353.3	0.876	354.0
7.85	1.012	351.9	1.009	350.8
8.80	1.119	355.6	1.135	355.5
9.74	1.135	2.5	1.134	3.2
10.68	1.033	7.2	1.021	6.9
11.62	0.922	4.9	0.930	4.4
12.57	0.905	359.1	0.908	359.7
13.51	0.960	355.5	0.948	355.6
14.45	1.030	355.4	1.031	355.1
15.39	1.083	358.6	1.088	358.7
16.34	1.068	2.8	1.065	3.1
17.28	0.996	4.6	0.993	4.3
18.22	0.938	2.2	0.943	2.1
19.16	0.940	358.5	0.940	358.7
20.11	0.984	356.5	0.981	356.4
21.05	1.037	357.2	1.040	357.1
21.99	1.062	0.0	1.064	0.2

Table 11

The magnitude and phase (degrees) of the effective excitation $\hat{\underline{a}}$ for $\pi \leq kD \leq 7\pi$ for a two-element array excited with unit amplitudes in phase for $n_{max} = 3$

kD	$\hat{a}_1(kD)$		$\hat{a}_2(kD)$	
3.14	1.447	353.6	1.458	353.1
4.08	1.128	15.2	1.114	18.9
5.03	0.979	7.7	0.965	6.6
5.97	0.920	6.7	0.932	6.0
6.91	0.865	358.9	0.864	359.7
7.85	0.955	353.8	0.953	353.8
8.80	1.044	355.0	1.043	354.9
9.74	1.096	358.9	1.100	358.8
10.68	1.061	3.6	1.060	3.9
11.62	0.985	4.2	0.982	4.0
12.57	0.940	1.8	0.944	1.7
13.51	0.945	358.2	0.943	358.3
14.45	0.994	356.7	0.994	356.6
15.39	1.039	357.8	1.039	357.8
16.34	1.054	0.5	1.055	0.5
17.28	1.020	2.7	1.019	2.7
18.22	0.977	2.3	0.977	2.2
19.16	0.957	0.3	0.959	0.3
20.11	0.972	358.2	0.971	358.2
21.05	1.008	357.8	1.008	357.8
21.99	1.034	359.1	1.035	359.1

Table 12

The magnitude and phase (degrees) of the effective excitation $\hat{\underline{a}}$ for $\pi \leq kD \leq 7\pi$ for a two-element array excited with unit amplitudes in phase for $n_{max} = 4$

kD	$\hat{a}_1(kD)$		$\hat{a}_2(kD)$	
3.14	1.320	10.5	1.319	11.6
4.08	0.956	6.0	0.984	4.6
5.03	0.755	2.6	0.808	2.8
5.97	0.933	2.4	0.938	3.4
6.91	1.003	354.5	1.002	354.8
7.85	0.957	352.1	0.929	351.5
8.80	1.142	357.0	1.168	357.1
9.74	1.092	2.5	1.086	3.1
10.68	1.041	5.3	1.031	5.1
11.62	0.922	5.8	0.935	5.1
12.57	0.919	358.1	0.914	358.5
13.51	0.967	356.6	0.965	356.6
14.45	1.016	355.8	1.014	355.6
15.39	1.086	358.6	1.093	358.7
16.34	1.050	2.7	1.046	2.9
17.28	1.001	3.6	0.999	3.4
18.22	0.944	2.4	0.949	2.3
19.16	0.947	358.3	0.945	358.5
20.11	0.988	357.2	0.987	357.1
21.05	1.028	357.4	1.029	357.3
21.99	1.059	0.1	1.061	0.1

Table 13

The parameter space in the array inversion study.
Simulations were performed for all possible combinations of the parameters.

N	2 4 10 20
NMAX	1 2 3 4 10 15
NPOINTS*	p_0 p_0+2 $2p_0$ $3p_0$

* $p_0 = 2N+2$, where N is the number of elements in the array

Table 14

Some condition numbers and residuals
obtained in the pattern-inversion simulations.

N; NMAX; NPOINTS	Condition Number	Residual
2; 1; p_0	1.6	2×10^{-6}
2; 1; $3p_0$	1.6	5×10^{-5}
10; 1; p_0+2	2.5	3×10^{-3}
10; 10; $2p_0$	2.3	6×10^{-4}
20; 1; p_0	2.7	9×10^{-4}
20; 10; p_0	2.7	1×10^{-3}
20; 15; p_0	2.1	1×10^{-3}

Appendix A
Coordinate Independent Formulation of Scanning Theory

The ideas presented in Section VI can be extended to formulate a coordinate-independent scanning theory. We achieve conceptual simplicity and remove those features of the existing planar, cylindrical, and spherical theories that are dependent on the choice of coordinate system in which the measurements are taken and the data analysis is performed. What results is a formulation of antenna-antenna interactions specified in an interaction-coordinate system by the scalar distance d between the interacting antennas and by two sets of Euler angles of rotations that describe the orientations of the antennas with respect to initial orientations (the antenna-coordinate systems) in which the transmitting (receiving) characteristics of the antennas are assumed known. The antenna-coordinate system is a reference frame fixed to the antenna and moves with it as the antenna rotates. Physically, the interaction-coordinate system is the frame of reference in which the antennas interact. Figures 3a and 3b illustrate these two coordinate systems. Figure 3a shows the relationship of the test and probe antennas in the interaction coordinate system. The antennas are situated at the origins of two coordinate systems that can be brought to coincide with each other by a single translation; no rotations are involved. For simplicity we consider the antenna-coordinate systems attached to the antennas to coincide with the set of axes shown in figure 3a. In figure 3b the relationships between the interaction system (broken lines) and the antenna systems (solid lines) are shown. If the antenna characteristics are known in the antenna systems, they can be obtained in the interaction system, as discussed in some detail below. The fact that a sequence of measurements takes place on a planar, cylindrical, or spherical surface is an external constraint imposed by the needs of the experimentalist, that is, by the technological limitations of the data gathering instrumentation and the efficiency requirements of the numerical inversion process. For the moment we suspend these practical considerations to present an interaction formulation of antenna scanning theory. As we shall see current scanning procedures can be easily incorporated into the interaction point of view.

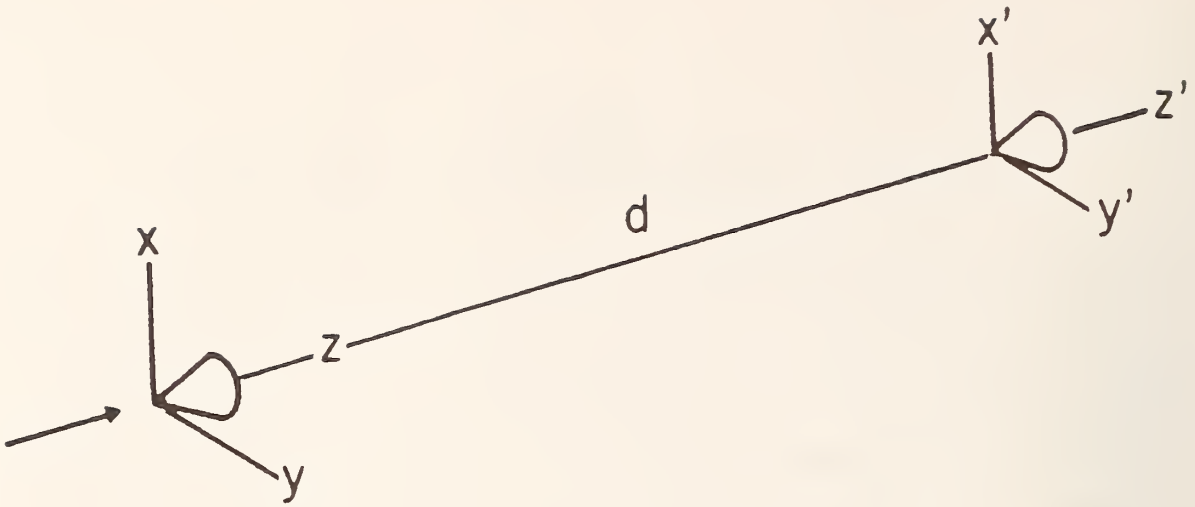

Figure 3a. The interaction-coordinate system in which the interaction between single-mode antennas is evaluated.

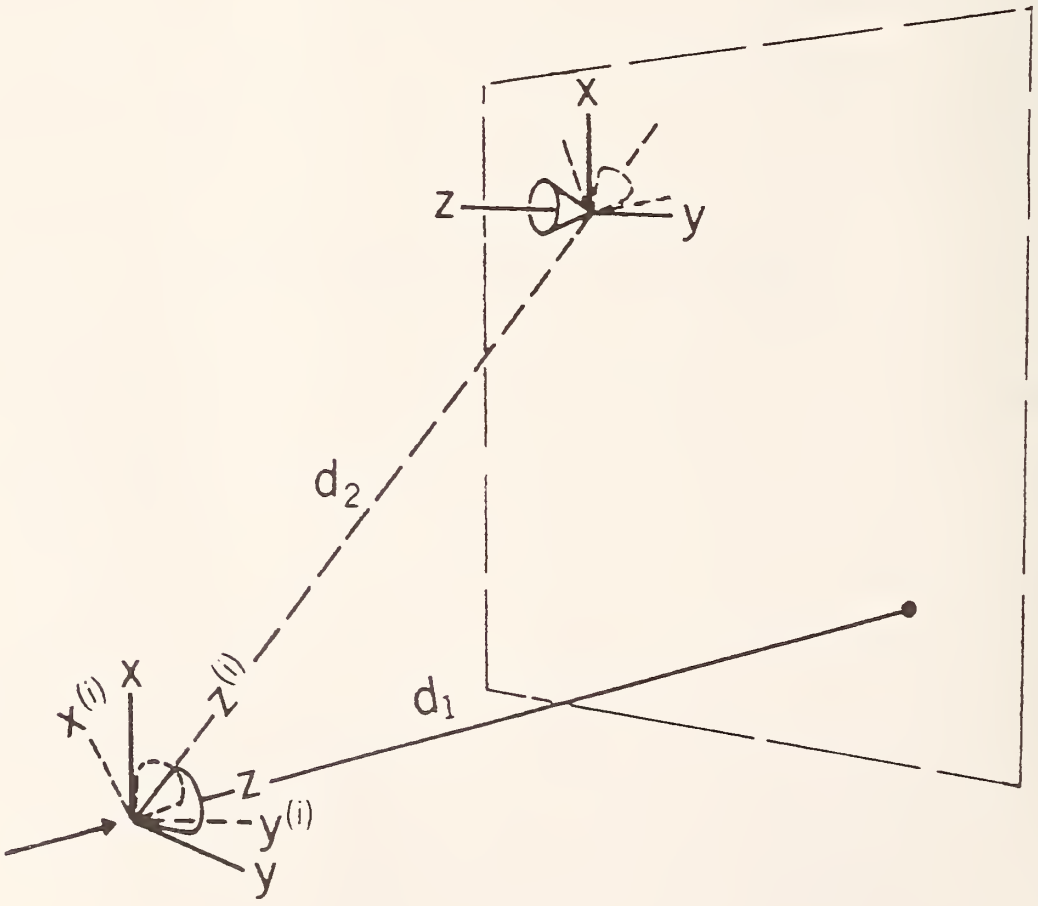

Figure 3b. The interaction-coordinate system embedded in the usual coordinate system of planar scanning. The antenna's coordinate system moves with the antenna as it is rigidly attached to it.

We can easily prove the following theorem:

A single measurement of a test antenna's performance at a given point in space a distance d from the probe can be represented as

$$m = \underline{p}^{\dagger}\ \underline{\underline{\chi}}(d)\ \underline{t}, \tag{A1}$$

where m (a complex scalar) is the result of measurement registered by the probe, $\underline{p}$ and $\underline{t}$ are column vectors of probe and test antenna coefficients, respectively, and $\underline{\underline{\chi}}(d)$ is the mode-mode coupling matrix, whose elements are coupling integrals evaluated for single-mode interactions (see eq (A6) below). All quantities in eq (A1) are evaluated in the interaction coordinate system. It is relatively straightforward to evaluate the matrix elements of $\underline{\underline{\chi}}$, as discussed in [1,3]. In principle, $\underline{\underline{\chi}}(d)$ is infinite dimensional; in practice, however, only a finite number of mode coefficients will be specified in $\underline{p}$ and $\underline{t}$.

In the antenna coordinate systems, the probe and test antenna far-field patterns $\underline{f}$ are given by

$$\underline{f}^{(p)} = \underline{\tilde{p}}\ \underline{e},$$

and

$$\underline{f}^{(t)} = \underline{\tilde{e}}\ \underline{t}, \tag{A2}$$

respectively, where $\underline{e}$ is an infinite dimensional column vector of spherical far-field basis functions. Then the dot product

$$\underline{f}_p^{*} \cdot \underline{f}_t = \underline{p}^{\dagger}\ \underline{\underline{\varepsilon}}_{ij}\underline{t} \tag{A3}$$

where $\underline{\underline{\varepsilon}}_{ij}$ are matrix elements defined as

$$\underline{\underline{\varepsilon}}_{ij} = \underline{e}_i^{*} \cdot \hat{\underline{e}}_j. \tag{A4}$$

With these definitions the coupling integral,

$$\int_0^{2\pi} \int_{i\infty}^{1} \underline{f}_p^{*} \cdot \underline{f}_t\, e^{-i\underline{k}\cdot\underline{d}}\, d(\cos\theta)\, d\phi, \tag{A5}$$

describing antenna-antenna interactions [1,3] in spherical coordinates using spherical basis functions can be written as in eq (A1), where

$$\chi_{ij} = \int_0^{2\pi} \int_{i\infty}^{1} \underline{\underline{\varepsilon}}_{ij} \, e^{-i\underline{k}\cdot\underline{d}} \, d(\cos\theta) \, d\phi \tag{A6}$$

which can be integrated exactly in the interaction coordinate system wherein the antennas are always on the z axis. If the antenna-coordinate systems do not coincide with the interaction coordinate system, the far-field coefficients in the interaction coordinates are given by

$$\underline{p}^{(I)} = \underline{\underline{D}} \, (\alpha\beta\gamma) \, \underline{p}^0 \tag{A7}$$

where $\underline{p}^0$ are the coefficients specified in the antenna frame, and $\underline{\underline{D}}$ $(\alpha\beta\gamma)$ represents a finite rotation of Euler angles α,β,γ [7] between the antenna and interaction frames. A similar expression holds for $\underline{t}^{(I)}$.

A sequence of measurements $\underline{\tilde{m}} = (m_1, m_2, \ldots m_i, \ldots m_n)$ can then be represented as

$$\underline{m} = \underline{\underline{P}} \, \underline{t}^0, \tag{A8}$$

where the rows of the matrix $\underline{\underline{P}}$ are given by

$$\underline{p}^{0\dagger} \, \underline{\underline{D}}^{\dagger}(\Omega_p) \, \underline{\underline{\chi}}(d) \, \underline{\underline{D}}(\Omega_t), \tag{A9}$$

where $\Omega_p(\Omega_t)$ are Euler angles of rotation of the probe (test) antenna. The far-field coefficients of the test antenna in the test antenna frame are then obtained by taking the inverse,

$$\underline{t}^0 = \underline{\underline{P}}^{-1} \, \underline{m}, \tag{A10}$$

if $\underline{\underline{P}}^{-1}$ exists.

Up to now no restrictions have been made as to how the sequence of measurements are to be made. If we wish to restrict ourselves to spherical scanning, then $d_i = d$, and the Euler angles corresponding to each measurement can be easily obtained in spherical geometry. For planar and cylindrical scanning, d_i is not constant; again the Euler angles can be easily obtained.

We illustrate these ideas with a simple example. Let the test and probe antennas be the set of dipoles

$$\tilde{\underline{p}}^0 = (p_1 \; p_0 \; p_1)$$

$$\tilde{\underline{t}}^0 = (t_1 \; t_0 \; t_1) \tag{A11}$$

giving in spherical coordinates the m = 1, 0, -1 components and, of course, n = 1. We will make measurements along an arc, i.e., $d_i = r$, the radius of the arc, such that only rotations about the x-axis will be needed to align the antenna coordinates with the interaction coordinates. The rotation matrix $d^{(1)}_{m'm}(\beta)$ is then given by [7]

$$d^{(1)}_{m'm}(\beta) = \begin{pmatrix} \frac{1}{2}(1 + \cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1 - \cos\beta) \\ -\frac{1}{\sqrt{2}}\sin\beta & \cos\beta & \frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1 - \cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1 + \cos\beta) \end{pmatrix}, \tag{A12}$$

and the interaction matrix is diagonal (no interaction between crossed dipoles)

$$\underline{\underline{\chi}}(r) = \begin{pmatrix} I_1 & & 0 \\ & I_0 & \\ 0 & & I_1 \end{pmatrix}. \tag{A13}$$

with $I_1(r) \neq I_0(r)$.

The rows of $\underline{\underline{P}}$ are then given by

$$m(\beta;r) = \underline{p}^{0\dagger} \, \underline{\underline{d}}^{\dagger}_{m'm}(\beta) \, \underline{\underline{\chi}}(r) \, \underline{\underline{d}}_{m'm}(\beta) \, \underline{t}_0. \tag{A14}$$

If we make measurements at $\beta = 0$, $\beta = \frac{\pi}{2}$ and at an arbitrary β, we get that

$$\underline{\underline{P}}(\beta;r) = \begin{pmatrix} I_1 p_1^* & I_0 \hat{p}_0^* & I_1 p_1^* \\ I_1 p_1^* & I_1 \hat{p}_0^* & I_1 p_1^* \\ q_1(\beta) & q_2(\beta) & q_3(\beta) \end{pmatrix}, \tag{A15}$$

where $\hat{p}_0 = \frac{p_0}{\sqrt{2}}$ and

$$\begin{aligned} q_1(\beta;r) &= I_1 p_1^* + \hat{p}_0^* (I_1 - I_0) \cos\beta \sin\beta \\ q_2(\beta;r) &= \hat{p}_0^* (I_1 \sin^2\beta + I_0 \cos^2\beta) \\ q_3(\beta;r) &= p_1^* I_1 - \hat{p}_0^* (I_1 - I_0) \cos\beta \sin\beta. \end{aligned} \tag{A16}$$

For $\underline{\underline{P}}^{-1}(\beta;r)$ to exist we must have $\det \underline{\underline{P}} \neq 0$. One can show that

$$\det \underline{\underline{P}} = -\frac{1}{2} p_1^* p_0^* p_0^* I_1 (I_0 - I_1)^2 \cos\beta \sin\beta \neq 0 \tag{A17}$$

except at $\beta = 0, \frac{\pi}{2}, \pi$, where measurements were already taken. Thus the third measurement can be at any angle sufficiently distinct from previous measurement points, and the probe-interaction matrix $\underline{\underline{P}}$ can be inverted and $\underline{t}_0$ recovered.

For completeness we conclude by briefly commenting on the transformation of the coupling integral expressed in Cartesian geometry as in [4] to the spherical coordinate form as seen in eq (A5). Once the spectral quantities are written in terms of far fields, the spherical representation can be obtained by a simple coordinate transformation

$$\begin{aligned} k_x &= k \sin\theta \cos\phi \\ k_y &= k \sin\theta \sin\phi \quad , \; k = \text{constant} \end{aligned} \tag{A18}$$

whose Jacobian is $k^2 \cos\theta \sin\theta$, i.e.,

$$dk_x dk_y = k^2 \cos\theta \sin\theta \, d\theta d\phi$$

or

$$dk_x dk_y = -\gamma d\gamma d\phi \tag{A19}$$

where $\gamma = k \cos\theta$. A more detailed derivation of eq (A5) will show that the factor γ in the Jacobian cancels due to the transformation to far-field quantities using the reciprocity relationships (see [8]).

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We review theoretically the effects of multiple reflections and mutual impedances in array environments and study possible methods of far-field pattern data analysis to recover interaction effects. We use theoretical expressions derived earlier to calculate in a two-element linear array the mutual-impedance matrix and effective excitations of elements as functions of interelement separation and n_{max}, the maximum mode number in the radiation pattern of the elements. Generalizations to two- and three-dimensional arrays are discussed.

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data analysis; effective excitations; interelement scattering; multiple reflections; mutual impedance; phased arrays.

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