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A Theory of Mutual Impedances and Multiple Reflections in an N-Element Array Environment

Lorant A. Muth

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Lorant A. Muth

Electromagnetic Fields Division
National Engineering Laboratory
National Bureau of Standards
U.S. Department of Commerce
Boulder, Colorado 80303

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L. Muth

Electromagnetic Fields Division
National Bureau of Standards
Boulder, Colorado

A general theoretical approach is formulated to describe the complex electromagnetic environment of an N-element array. The theory reveals the element-to-element interactions and multiple reflections within the array. From the formulation, it is found that the interaction between an excited element and an open-circuited element can be viewed as the sum of terms describing all possible signal paths within the array environment which start from the radiating element and terminate on the element under observation. Within all paths except the most direct one, multiple reflections between subgroups of elements take place. The resulting solution is highly structured and recursive and is discussed in detail in the text. Illustrative examples are provided to facilitate understanding of these ideas.

Key words: array antennas; array environment; coupling; multiple reflections; mutual impedance.

1. Introduction

In this paper we derive expressions for the mutual impedance between any two elements in an N-element array environment, where a single element is excited and N-1 elements are open circuited. Knowledge of these mutual impedances is necessary to accurately derive the radiation pattern of an N-element phased array from the free-space patterns of single elements. Such considerations are essential to accurately assess the performance of very large phased arrays, since mutual coupling between elements might affect the overall radiation pattern significantly.

The interaction between two antennas or two elements has been described in [1,2] and [3], respectively. In these references the presence of multiple reflections in the interaction is noted only briefly, and the observation is made that under certain well-defined mathematical conditions the closed form expression describing multiple reflections can be expanded in an infinite series.

The N-element interactions considered in this paper must be a complicated generalization of two-element interactions, since any subgroup of M elements can be thought of as a single element interacting with another element made up of the remaining N-M elements. Specifically one can consider a single radiating element interacting with the N-1 open circuited elements in the environment. Multiple reflections will then be present between the radiating element and the environment, just as in the 2-element model. Since the division of the N elements into a radiating component and an environment is arbitrary, it immediately follows that there must be multiple reflections within any subgroup of elements. However, there can be only a finite number of arbitrary groupings of N elements, hence the total set of interactions within N elements should be describable in a finite number of terms. We will show that each of these interaction terms corresponds to a possible signal path between the radiating element and the element under observation. Multiple reflections then occur within signal paths between any subgroup of elements.

To describe the above ideas mathematically we use the scattering matrix formalism to relate the incoming and outgoing waves at each of the N elements. We then derive an associated (N-1) x (N-1) linear system whose blocked elements are binary free space generalized current-current interaction matrices Λ_{ij}, and the elements of the solution vector are the (N-1) generalized currents describing the radiation pattern of each element. The binary interaction matrices are constructed from the free-space mode-mode mutual impedance matrices and the scattering matrices of the open circuited elements. To exhibit the electromagnetic interactions present among the elements of the phased array we solve this associated linear system in a manner that preserves the blocked matrix elements in the solution. The solution is then easily interpreted as a sequence of direct and multiple reflection interactions among the elements. To achieve this we developed a novel cyclic decomposition of the interaction matrix in terms of its blocked matrix elements $\mathbf{\Lambda}_{ij}$, and provide an inductive proof of the fact that the solution of the linear system is given by such a decomposition. Using this novel solution we study some elementary examples to demonstrate the effect of the environment on free space radiation patterns. Finally, the mutual impedances between elements z_{ij} are obtained in terms of the solutions of the interaction matrix $\mathbf{\Lambda}$. Some simple approximation schemes to the full solution are suggested to reduce the computational requirements in obtaining the mutual impedances between elements.

2. N Coupled Antennas or Elements

In Appendix A the basics of the scattering matrix formalism are reviewed and the symbols used here are defined. In the discussion below we denote the i-th antenna or array element by A_i, i = 1,N. The scattering matrix formalism gives the following matrix equations when A_1 is excited and A_2, ..., A_N are open circuited. For element 1, (considering the reflectionless case $S_{\alpha\alpha} = 0$)

$$\begin{pmatrix} \frac{1}{2}(v_1 - i_1) \\ \frac{1}{2}(\underline{V}_1 - \underline{I}_1) \end{pmatrix} = \begin{pmatrix} 0 & S^{(1)\dagger}_{\alpha\beta} \\ S^{(1)}_{\beta\alpha} & S^{(1)}_o - S^{(1)}_{\beta\alpha} S^{(1)\dagger}_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \frac{1}{2}(v_1 + i_1) \\ \frac{1}{2}(\underline{V}_1 + \underline{I}_1) \end{pmatrix} \qquad (1)$$

and the (N - 1) open circuited elements $(i_\ell = 0)$ are described by

$$\begin{pmatrix} \frac{1}{2} v_\ell \\ \frac{1}{2}(\underline{V}_\ell - \underline{I}_\ell) \end{pmatrix} = \begin{pmatrix} 0 & S^{(\ell)\dagger}_{\alpha\beta} \\ S^{(\ell)}_{\beta\alpha} & S^{(\ell)}_o - S^{(\ell)}_{\beta\alpha} S^{(\ell)\dagger}_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \frac{1}{2} v_\ell \\ \frac{1}{2}(\underline{V}_\ell + \underline{I}_\ell) \end{pmatrix} \qquad (2)$$

for ℓ = 2,N. Here

$$\frac{1}{2}(v_\ell + i_\ell) = a_\alpha^{(\ell)}$$

$$\frac{1}{2}(\underline{V}_\ell + \underline{I}_\ell) = \underline{a}_\beta^{(\ell)} \tag{3}$$

$$\frac{1}{2}(v_\ell - i_\ell) = b_\alpha^{(\ell)}$$

$$\frac{1}{2}(\underline{V}_\ell - I_\ell) = \underline{b}_\beta^{(\ell)}$$

and all symbols are defined in Appendix A.

In addition, the generalized voltages $\underline{V}_\ell$ are related to the generalized currents $\underline{I}_\ell$ by an impedance matrix Z [3]

$$\begin{pmatrix} \underline{V}_1 \\ \vdots \\ \underline{V}_\ell \\ \vdots \\ \underline{V}_N \end{pmatrix} = \begin{pmatrix} 1 & Z_{12} & \cdots & Z_{1\ell} & \cdots & Z_{1N} \\ \vdots & & \vdots & & & \vdots \\ Z_{\ell 1} & Z_{\ell 2} & \cdots & 1 & \cdots & Z_{\ell N} \\ \vdots & & \vdots & & & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{N\ell} & \cdots & 1 \end{pmatrix} \begin{pmatrix} -\underline{I}_1 \\ \vdots \\ -I_\ell \\ \vdots \\ -\underline{I}_N \end{pmatrix} \tag{4}$$

with $Z_{\ell k} = \tilde{Z}_{k\ell}$.

Here $Z_{k\ell}$ are free-space impedance matrices whose elements $\zeta_{ij}^{(k\ell)}$ are the free-space mode-mode mutual impedance integrals [3] (closely related to the coupling integral in [4]). Thus,

$$\zeta_{ij}^{(k\ell)} = 2 \int_0^{2\pi} d\phi \int_{-j\infty}^{1} e^{-jkD\cos\theta}\, \underline{f}_i^{(k)}\, \underline{f}_j^{*(\ell)}\, d(\cos\theta)$$

where $\underline{f}_i^{(k)}$, i=1,2 is the far field pattern of the i-th mode (corresponding to some set of mode numbers nm) radiated by antenna k, D is the distance between array elements k and ℓ, and * denotes the complex conjugate. For a detailed discussion of the method of evaluation of the above integral, see Appendix C. From the above matrices we obtain the following system of equations:

$$
\begin{aligned}
v_1 &= S^{(1)\dagger}_{\alpha\beta}\,(\underline{V}_1 + \underline{I}_1) + i_1 \\
v_\ell &= S^{(\ell)\dagger}_{\alpha\beta}\,(\underline{V}_\ell + \underline{I}_\ell), \quad \ell = 2,N
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\underline{V}_1 - \underline{I}_1 &= (S_o^{(1)} - S^{(1)}_{\beta\alpha} S^{(1)\dagger}_{\alpha\beta})\,(\underline{V}_1 + \underline{I}_1) + S^{(1)}_{\beta\alpha}\,(v_1 + i_1) \\
\underline{V}_\ell - \underline{I}_\ell &= S_o^{(\ell)}\,(\underline{V}_\ell + \underline{I}_\ell)\,, \quad \ell = 2,N
\end{aligned}
\tag{6}
$$

and

$$
\begin{aligned}
\underline{V}_1 + \underline{I}_1 &= -\,(Z_{12}\underline{I}_2 + \dots + Z_{1N}\underline{I}_N) \\
\underline{V}_\ell + \underline{I}_\ell &= -\,(Z_{\ell 1}\underline{I}_1 + \dots + Z_{\ell N}\underline{I}_N)'\,, \quad \ell = 2,N
\end{aligned}
\tag{7}
$$

In the last equation the diagonal term is missing on the right hand side, as indicated by the prime.

The equations for $v_1 - i_1$, $\underline{V}_1 - \underline{I}_1$ and $\underline{V}_1 + \underline{I}_1$ can be used to derive an expression for $\underline{I}_1$ in terms of i_1 and $\underline{I}_\ell$, $\ell = 2,N$. Thus, from (5) and (6)

$$\underline{V}_1 - \underline{I}_1 = S_o^{(1)}(\underline{V}_1 + \underline{I}_1) + 2S^{(1)}_{\beta\alpha} i_1$$

or

$$S^{(1)}_{\beta\alpha} i_1 = -\underline{I}_1 + \tfrac{1}{2}(\mathbf{1} - S_o^{(1)})(\underline{V}_1 + \underline{I}_1)$$

where $\mathbf{1}$ is the identity matrix. We now define free space binary current-current interaction matrices as

$$\Lambda_{ij} \equiv \frac{1}{2}\,(\mathbf{1} - S_o^{(i)})Z_{ij} \tag{8}$$

in terms of which the last expression, together with (7), give

$$\underline{I}_1 = -(\Lambda_{12}\underline{I}_2 + \ldots\ldots + \Lambda_{1N}\underline{I}_N) - S^{(1)}_{\beta\alpha} i_1. \tag{9}$$

Similarly, we get, in general, for the open circuited elements ℓ

$$\underline{I}_\ell = - (\Lambda_{\ell 1}\underline{I}_1 + \ldots + \Lambda_{\ell N}\underline{I}_N)', \quad \ell = 2,N \tag{10}$$

with the $\underline{I}_\ell$ term missing on the right side as indicated by the prime.

We can write the last expression as a (N-1) x (N-1) linear system

$$\begin{pmatrix} 1 & \Lambda_{23} & \cdots & \Lambda_{2\ell} & \cdots & \Lambda_{2N} \\ \Lambda_{32} & 1 & & \Lambda_{3\ell} & & \Lambda_{3N} \\ \vdots & & & \vdots & & \vdots \\ \Lambda_{\ell 2} & \cdots & & 1 & & \Lambda_{\ell N} \\ \vdots & & & \vdots & & \vdots \\ \Lambda_{N2} & \cdots & & \Lambda_{N\ell} & \cdots & 1 \end{pmatrix} \begin{pmatrix} \underline{I}_2 \\ \underline{I}_3 \\ \vdots \\ \underline{I}_\ell \\ \vdots \\ \underline{I}_N \end{pmatrix} = - \begin{pmatrix} \Lambda_{21}\underline{I}_1 \\ \Lambda_{31}\underline{I}_1 \\ \vdots \\ \Lambda_{\ell 1}\underline{I}_1 \\ \vdots \\ \Lambda_{N1}\underline{I}_1 \end{pmatrix} \tag{11}$$

or, with $\tilde{\underline{I}}_{1\ell} \equiv \Lambda_{\ell 1}\underline{I}_1$,

$$\Lambda \underline{I} = -\tilde{\underline{I}}_1 .$$

In sections 4 and 5 below we will show that the solution to this linear system can be written in the form

$$\underline{I}_j = -\Omega_{j1}\underline{I}_1, \qquad j = 2,N \tag{12}$$

where

$$\Omega_{j1} \equiv w_{jj}^{-1}\, w_{j1}$$

and w_{jj} represents the interaction of element j with itself in the presence of the open circuited environment, and w_{j1} represents the interaction of the radiating element 1 with element j in the presence of the open circuited environment. It will be seen that these $\mathbf{w}$ matrices have a high degree of structure that elucidates the interactions present in an N-element array. It will become apparent that $\boldsymbol{\Omega}_{j1}$ gives all possible interactions within the signal paths in the array from element 1 to element j, including multiple reflections between subgroups of elements along the paths.

In terms of $\boldsymbol{\Omega}_{j1}$, the generalized currents $\underline{I}_1$ in (9) and $\underline{V}_1 + \underline{I}_1$ and $\underline{V}_\ell + \underline{I}_\ell$ in (7) can be written as

$$\underline{I}_1 = -[\mathbf{1} - (\boldsymbol{\Lambda}_{12}\boldsymbol{\Omega}_{21} + \dots + \boldsymbol{\Lambda}_{1N}\boldsymbol{\Omega}_{N1})]^{-1} S^{(1)}_{\beta\alpha}\, i_1 \; . \tag{13}$$

$$\underline{V}_1 + \underline{I}_1 = (Z_{12}\boldsymbol{\Omega}_{21} + \dots\dots + Z_{1N}\boldsymbol{\Omega}_{N1})\underline{I}_1$$

$$\underline{V}_\ell + \underline{I}_\ell = (Z_{\ell 1}\boldsymbol{\Omega}_{11} + \dots Z_{\ell N}\boldsymbol{\Omega}_{N1})'\underline{I}_1 , \qquad \ell = 2,N \; . \tag{14}$$

From equations (5), (13) and (14) the mutual impedances $z_{11} \equiv v_1/i_1$ and $z_{\ell 1} \equiv v_\ell/i_1$ are then given by

$$z_{11} = 1 - S^{(1)\dagger}_{\alpha\beta}(Z_{12}\boldsymbol{\Omega}_{21} + \dots + Z_{1N}\boldsymbol{\Omega}_{N1}) \quad [\mathbf{1} - (\boldsymbol{\Lambda}_{12}\boldsymbol{\Omega}_{21} + \dots + \boldsymbol{\Lambda}_{1N}\boldsymbol{\Omega}_{N1})]^{-1} S^{(1)}_{\beta\alpha} \tag{15}$$

$$z_{\ell 1} = - S^{(\ell)\dagger}_{\alpha\beta}(Z_{\ell 1}\boldsymbol{\Omega}_{11} + \dots + Z_{\ell N}\boldsymbol{\Omega}_{N1})' \quad [\mathbf{1} - (\boldsymbol{\Lambda}_{12}\boldsymbol{\Omega}_{21} + \dots + \boldsymbol{\Lambda}_{1N}\boldsymbol{\Omega}_{N1})]^{-1} S^{(1)}_{\beta\alpha} \; . \tag{16}$$

Note that for N = 2 and $\boldsymbol{\Omega}_{11} = -1$ the results in [3] are recovered. We have now obtained expressions for z_{i1}, the mutual impedance between element i and the radiating element "1" in terms of the free space and array binary current-current interaction matrices $\boldsymbol{\Lambda}_{\ell k}$ and $\boldsymbol{\Omega}_{j1}$, respectively. As will be seen below $\boldsymbol{\Omega}_{j1}$ can also be interpreted as the total signal path interaction matrix between the radiating element and element j. In the next section we show how to obtain $\boldsymbol{\Omega}_{j1}$.

For completeness we derive expressions for the incoming $\underline{a}_\beta^{(\ell)}$ and outgoing $\underline{b}_\beta^{(\ell)}$ waves in terms of the excitation current i_1 of the radiating element. Since from (10) and (12)

$$\underline{I}_\ell \equiv \underline{a}_\beta^{(\ell)} - \underline{b}_\beta^{(\ell)} = (\sum_j \Lambda_{\ell j} \boldsymbol{\Omega}_{j1})' \underline{I}_1 \tag{17}$$

and (see Appendix A)

$$\underline{b}_\beta^{(\ell)} = S_o^{(\ell)} \underline{a}_\beta^{(\ell)} ,$$

one can easily show that

$$\underline{a}_\beta^{(\ell)} = \frac{1}{2}(\sum_j Z_{\ell j} \boldsymbol{\Omega}_{j1})' \underline{I}_1 \tag{18}$$

$$\underline{b}_\beta^{(\ell)} = \frac{1}{2} S_o^{(\ell)} (\sum_j Z_{\ell j} \boldsymbol{\Omega}_{j1})' \underline{I}_1 \tag{19}$$

which together with (13) give the incoming and outgoing waves at the ℓ-th open circuited antenna in terms of i_1, the excitation of the radiating element.

3. Explicit Solution to a 4-Element Problem

For 4 elements (element 1 radiating) equation (10) gives explicitly

$$\underline{I}_2 = -\frac{1}{2}(1 - S_o^{(2)}) (Z_{21}\underline{I}_1 + Z_{23}\underline{I}_3 + Z_{24}\underline{I}_4)$$

$$\underline{I}_3 = -\frac{1}{2}(1 - S_o^{(3)}) (Z_{31}\underline{I}_1 + Z_{32}\underline{I}_2 + Z_{34}\underline{I}_4) \tag{20}$$

$$\underline{I}_4 = -\frac{1}{2}(1 - S_o^{(4)}) (Z_{41}\underline{I}_1 + Z_{42}\underline{I}_2 + Z_{43}\underline{I}_3)$$

which is a 3 x 3 system in the unknowns $\underline{I}_2$, $\underline{I}_3$, $\underline{I}_4$; $\underline{I}_1$ is treated as known. We have explicitly exhibited the form of $\boldsymbol{\Lambda}_{ij}$. Note that $\underline{I}_j$ are vectors whose elements are given in terms of the modal coefficients of the radiation pattern of the antennas (see equations (17), (18), and (19)). All other quantities in (20) are matrices. Elements 2, 3, and 4 are open circuited; if the j-th antenna is a minimum scattering antenna ($\mathbf{S}_0^{(j)} = \mathbf{I}$) [1,7] then $\underline{I}_j = 0$ in (20), i.e., the antenna does not scatter the incoming radiation.

We can solve system (20) algebraically. To solve for $\underline{I}_2$ in terms of $\underline{I}_1$, for example, the subsystem $\underline{I}_3$ and $\underline{I}_4$ is expressed in terms of $\underline{I}_1$ and $\underline{I}_2$, which is then substituted into $\underline{I}_2$, and finally, $\underline{I}_2$ is obtained in terms of $\underline{I}_1$. The result is:

$$\left\{\begin{array}{c} \mathbf{1} \\ -\boldsymbol{\Lambda}_{23}(\mathbf{1} - \boldsymbol{\Lambda}_{34}\boldsymbol{\Lambda}_{43})^{-1}\boldsymbol{\Lambda}_{32} \\ +\boldsymbol{\Lambda}_{23}(\mathbf{1} - \boldsymbol{\Lambda}_{34}\boldsymbol{\Lambda}_{43})^{-1}\boldsymbol{\Lambda}_{34}\boldsymbol{\Lambda}_{42} \\ -\boldsymbol{\Lambda}_{24}(\mathbf{1} - \boldsymbol{\Lambda}_{43}\boldsymbol{\Lambda}_{34})^{-1}\boldsymbol{\Lambda}_{42} \\ +\boldsymbol{\Lambda}_{24}(\mathbf{1} - \boldsymbol{\Lambda}_{43}\boldsymbol{\Lambda}_{34})^{-1}\boldsymbol{\Lambda}_{43}\boldsymbol{\Lambda}_{32} \end{array}\right\} \underline{I}_2 = - \left\{\begin{array}{c} \boldsymbol{\Lambda}_{21} \\ -\boldsymbol{\Lambda}_{23}(\mathbf{1} - \boldsymbol{\Lambda}_{34}\boldsymbol{\Lambda}_{43})^{-1}\boldsymbol{\Lambda}_{31} \\ +\boldsymbol{\Lambda}_{23}(\mathbf{1} - \boldsymbol{\Lambda}_{34}\boldsymbol{\Lambda}_{43})^{-1}\boldsymbol{\Lambda}_{34}\boldsymbol{\Lambda}_{41} \\ -\boldsymbol{\Lambda}_{24}(\mathbf{1} - \boldsymbol{\Lambda}_{43}\boldsymbol{\Lambda}_{34})^{-1}\boldsymbol{\Lambda}_{41} \\ +\boldsymbol{\Lambda}_{24}(\mathbf{1} - \boldsymbol{\Lambda}_{43}\boldsymbol{\Lambda}_{34})^{-1}\boldsymbol{\Lambda}_{43}\boldsymbol{\Lambda}_{31} \end{array}\right\} \underline{I}_1 \tag{21}$$

or

$$\mathbf{W}_{22}\underline{I}_2 = -\mathbf{W}_{21}\underline{I}_1$$

$$\underline{I}_2 = -\boldsymbol{\Omega}_{21}\underline{I}_1 \tag{22}$$

where

$$\mathbf{\Omega}_{21} \equiv \mathbf{w}_{22}^{-1}\, \mathbf{w}_{21}.$$

Equation (21) or (22) gives the generalized current $\underline{I}_2$ at element 2 in terms of $\underline{I}_1$, the generalized current at the excited element 1. To obtain $\underline{I}_k$, $k \neq 1$, in general, one merely has to interchange all indices 2 and k. For example, let k = 3, then terms $\mathbf{\Lambda}_{23}$ and $\mathbf{\Lambda}_{34}\mathbf{\Lambda}_{42}$ become $\mathbf{\Lambda}_{32}$ and $\mathbf{\Lambda}_{24}\mathbf{\Lambda}_{43}$, respectively. Thus, all $\underline{I}_k$ can be written down after (21). Equation (22) generalizes to

$$\underline{I}_k = -\mathbf{\Omega}_{k1}\, \underline{I}_1 \tag{23}$$

where $\mathbf{\Omega}_{k1} \equiv \mathbf{w}_{kk}^{-1}\, \mathbf{w}_{k1}$, k = 2,4. One can easily generalize (23) to the case where an element ℓ other than element 1 is radiating. A mere interchange of indices ℓ and 1 will then yield $\mathbf{\Omega}_{k\ell}$ (the index 1 is now hidden in $\mathbf{\Omega}_{k\ell}$ and is not shown explicitly). Thus,

$$\underline{I}_k = -\mathbf{\Omega}_{k\ell}\, \underline{I}_\ell \tag{24}$$

where

$$\mathbf{\Omega}_{k\ell} = \mathbf{w}_{kk}^{-1}\, \mathbf{w}_{k\ell}$$

for $k,\ell = 1,4$; $\ell \neq k$.

Attempting an explicit solution of (10) for N > 4 leads one to unwieldy algebraic manipulations. A general solution of (11) for any N has been developed that gives $\mathbf{\Omega}_{\ell k}$ in terms of $\mathbf{\Lambda}_{\ell k}$, and will be discussed in section 5 below. The essential features of the general $\mathbf{w}_{\ell k}$ interaction matrices, however, are already apparent in $\mathbf{w}_{22}^{(4)}$ and $\mathbf{w}_{21}^{(4)}$ in (21). (Here we introduced the superscript notation to indicate the total number of elements in the array.) Therefore, we now turn our attention to both the mathematical structure and the physics contained in (21). Naturally, these two distinct points of view are fundamentally connected.

4. Signal Paths and Multiple Reflections in N-Element Arrays

Equation (21) gives a highly structured detailed description of the interaction between the open circuited element $(\underline{I}_2)$ and the radiating element $(\underline{I}_1)$ in terms of sums of products of binary current-current interaction matrices $\mathbf{\Lambda}_{ij}$. We denoted these sums in (22) by $\mathbf{w}_{22}$ and $\mathbf{w}_{21}$. For the moment let us treat $\mathbf{w}_{22}$ and $\mathbf{w}_{21}$ as independent entities describing how the radiating element (denoted by the rightmost index) interacts with the element under observation (denoted by the leftmost index). Then one can interpret the first term in each as a direct interaction term or possibly a zeroth order interaction term if the additional terms are small. Thus,

$$\mathbf{w}_{21}^{o} = \mathbf{\Lambda}_{21}$$

and $\mathbf{w}_{22}^{o} = 1$ for all N, the number of elements in the array. It then follows that

$$\underline{I}_2^{(N)} = -\mathbf{\Lambda}_{21}\underline{I}_1^{(N)} \tag{25}$$

could be a zeroth order approximation of the interaction between elements 1 and 2. This would strictly be true if all other elements in the array are minimum scattering elements. The additional terms in (21) give the environmental effects.

Let us denote the individual environmental terms in $\mathbf{w}_{21}$ and $\mathbf{w}_{22}$ by $\boldsymbol{\chi}_{21}$ and $\boldsymbol{\chi}_{22}$, and define $\boldsymbol{\varepsilon}_{k\ell}^{(M)}$ as the environmental interaction operator for an M-element environment, where M = N - 2. We can then write

$$\begin{aligned} \boldsymbol{\chi}_{21}^{(4)} &= \mathbf{\Lambda}_{2k}\ \boldsymbol{\varepsilon}_{k\ell}^{(2)}\ \mathbf{\Lambda}_{\ell 1} \\ \boldsymbol{\chi}_{22}^{(4)} &= \mathbf{\Lambda}_{2k}\ \boldsymbol{\varepsilon}_{k\ell}^{(2)}\ \mathbf{\Lambda}_{\ell 2} \end{aligned} \tag{26}$$

for ℓ,k = 3,4. In addition there are the direct interaction or zeroth order terms $\mathbf{\Lambda}_{22} = I$ and $\mathbf{\Lambda}_{21}$ in (21), which can now be written as

$$(\Lambda_{22} + \sum_{\ell,k=3}^{4} \Lambda_{2k}\, \varepsilon_{k\ell}^{(2)}\, \Lambda_{\ell 2})\underline{I}_2 = -(\Lambda_{21} + \sum_{\ell,k=3}^{4} \Lambda_{2k}\, \varepsilon_{k\ell}^{(2)}\, \Lambda_{\ell 1})\underline{I}_1 \quad . \tag{27}$$

These relationships clearly point out how the radiating element and the element under observation couple into the environment: the free space binary interaction matrices of the element under observation operate on the environmental operators and the product then operates on the free space binary interaction matrices of the radiating element.

The terms in (21) can be given a graphical interpretation as shown in figure 1. There are direct interaction lines representing the Λ_{ij} and closed loops representing the multiple reflections as expressed by inverses. Each diagram is labeled by the operator it represents. There are five diagrams, which is the total number of ways a line (representing an interaction) can be drawn from element 1 to element 2 directly or via the two element environment. The interaction of element 2 with itself can be represented similarly. For a larger array these diagrams become progressively more complex.

Equation (27) puts the radiating element (I_1) and the element (I_2) on equal footing. Both elements interact with the element under observation and the environment in similar manner, i.e., $\varepsilon_{k\ell}^{(2)}$ is the same on both sides of the equation. The minus sign, however, points to the fact that the open circuited element 2 is radiating only because element 1 is excited; if we differentiate (27) with respect to time we obtain a special case of Lenz's law [5].

Further detailed examination of the environmental operators $\varepsilon_{k\ell}^{(2)}$ in (26) reveals a structure that is amenable to physical interpretation. We note that the leading terms in $\varepsilon_{k\ell}^{(2)}$ are inverses, i.e., they are structured exactly as $\Omega_{21}^{(4)}$ in (22). Then we can write

$$\Omega_{\ell 1}^{(3)} \equiv \Upsilon_{\ell\ell}^{-1}\, \Upsilon_{\ell 1} \qquad \ell = 3,4 \tag{28}$$

indicating, as in (22), that the remaining coefficients really represent currents $\hat{\underline{I}}_3^{(3)}$ and $\hat{\underline{I}}_4^{(3)}$ in a three element environment. This can be verified by direct solution of (13) for a 3 element array. We see then that the interaction

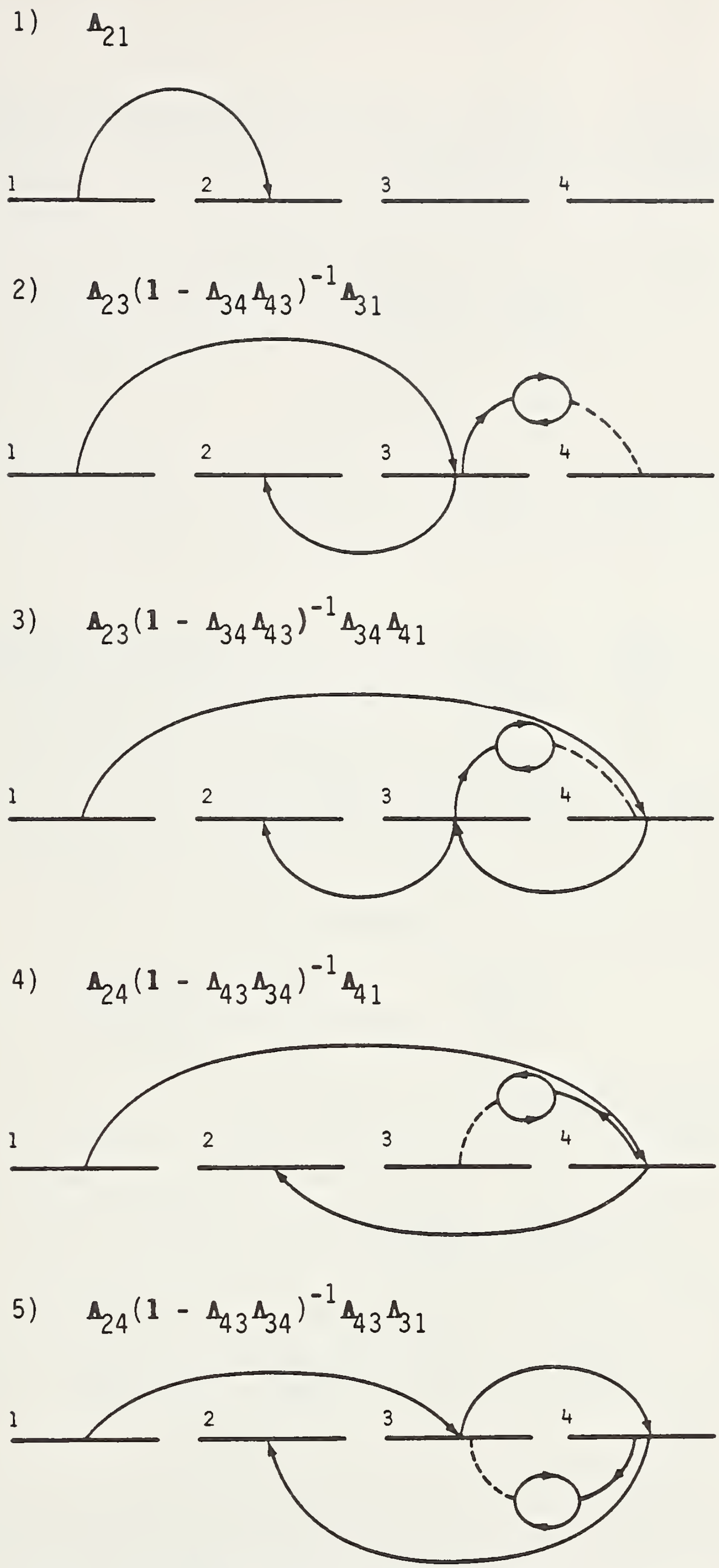

Figure 1. Possible signal paths and multiple reflections in a four element array.

of the environmental operator with the excited element simply gives the currents in the environment. We can write

$$\underset{-k}{\hat{I}}^{(3)} = -\Omega_{k1}^{(3)} \underset{-1}{I}^{(4)} = \sum_{\ell=3}^{4} \varepsilon_{k\ell}^{(2)} \Lambda_{\ell 1} \underset{-1}{I}^{(4)}$$

or

$$\Omega_{k1}^{(3)} = -\sum_{\ell=3}^{4} \varepsilon_{k\ell}^{(2)} \Lambda_{\ell 1}, \qquad k = 3,4 \quad . \tag{29}$$

Equation (21) can now be written as

$$\left(\Lambda_{22} - \Lambda_{23}\Omega_{32}^{(3)} - \Lambda_{24}\Omega_{42}^{(3)}\right)\underset{-2}{I}^{(4)} + \left(\Lambda_{21} - \Lambda_{23}\Omega_{31}^{(3)} - \Lambda_{24}\Omega_{41}^{(3)}\right)\underset{-1}{I}^{(4)} = 0$$

or $$w_{2k}^{(4)} = \Lambda_{2k} - \sum_{j=3}^{4} \Lambda_{2j}\Omega_{jk}^{(3)}, \qquad \text{for } k = 1,2 \ . \tag{30}$$

Equations (22) and (30) can be generalized as

$$w_{\ell k}^{(N)} = \Lambda_{\ell k} - \sum_{j=1}^{N}{}' \Lambda_{\ell j}\, \Omega_{jk}^{(N-1)} \tag{31}$$

where the prime indicates that $j \neq \ell,k$, and

$$w_{kk}^{(N)} \underset{-k}{I}^{(N)} + w_{k\ell}^{(N)} \underset{-\ell}{I}^{(N)} = 0$$

$$\Omega_{jk}^{(N-1)} = \left(w_{jj}^{(N-1)}\right)^{-1} w_{jk}^{(N-1)} \qquad j \neq \ell,k \tag{32}$$

is the array binary current-current interaction matrix in an (N-1) element array.

From (31) we can see that if the total number of terms in $\Omega_{jk}^{(N-1)}$ is t_{N-1} then $w_{\ell k}^{(N)}$ has

$$t_N = 1 + (N-2)t_{N-1}, \qquad N \geqslant 2 \tag{33}$$

terms, each representing a unique path of interaction between elements ℓ and k. Each of these terms can be represented diagramatically, as illustrated in figure 1 for N = 4. We note that the indices in the individual terms in (28), as in (21), occur in pairs, i.e., the left index of $\mathbf{\Omega}_{jk}$ equals the right index of $\mathbf{\Lambda}_{\ell j}$ as we go from right to left. Thus each term describes a continuous sequence of interactions.

Since the terms grow both in number and complexity as the number of elements N in the array increases, there is obviously a need for a general procedure to write down $\mathbf{\Omega}_{k\ell}^{(N)}$ in an N element environment. We turn to this problem next.

5. General N Element Interactions

In the previous section we discussed the structure of the binary array current-current interaction matrices $\mathbf{\Omega}_{ij}$ for a 4 element array, and the expressions were generalized to N elements. However, the explicit form of the environmental interaction matrices or of the $\mathbf{\Omega}_{\ell k}^{(N)}$ matrices for the general case still needs to be exhibited. In Appendix B we show that the solution to (11) is given by

$$\underset{-}{I}_{j}^{(N)} = \left(\Lambda_{:jj}\right)^{-1} \bar{\underset{-}{\Lambda}}_{:jj} \qquad j = 2,N \tag{34}$$

where $\mathbf{\Lambda}$ is the matrix defined in (11) and $\Lambda_{:jj}$, the cyclic decomposition of the matrix $\mathbf{\Lambda}$ about the element Λ_{jj}, is defined in (B1). This expression was developed with the intent to keep the elements $\mathbf{\Lambda}_{ij}$, which themselves are matrices, intact in the solution, rather than just treating (11) as a large linear system to be inverted to obtain the solution. In this manner the physics of interaction between the elements and the presence of multiple reflections is explicitly exhibited. These were discussed in some detail in the previous section.

Since the expansion as defined in (B1) is rather complicated, we illustrate it by writing the expansion of $\mathbf{\Lambda}$ in (11) for N = 3. $\Lambda_{22} = \mathbf{1}$ is the (1,1) element of the matrix and I(11;44) = 1. Hence,

$$\begin{pmatrix} 1 & \Delta_{23} & \Delta_{24} \\ \Delta_{32} & 1 & \Delta_{34} \\ \Delta_{42} & \Delta_{43} & 1 \end{pmatrix}_{:22} = 1 - \Delta_{23}M^{-1}_{22:33}M_{23:32} + \Delta_{24}M^{-1}_{22:44}M_{24:42} \quad (35)$$

where $M_{22:33}$ is the submatrix of $\Delta_{22} = 1$ obtained by deleting the first row and column and is expanded in a similar manner about element $\Delta_{33} = 1$. Thus,

$$M_{22:33} = \begin{pmatrix} 1 & \Delta_{34} \\ \Delta_{43} & 1 \end{pmatrix}_{:33} = 1 - \Delta_{34}\, M^{-1}_{33:44}M_{34:43} = 1 - \Delta_{34}\Delta_{43}$$

since $M^{-1}_{33:44} = 1$ and $M_{34:43} = \Delta_{43}$.

When the other terms in (35) are exhibited explicitly, the reader can easily verify that we recover the terms in (21). Furthermore, the recursive properties of the solution as discussed in the previous section become apparent from (34), (35) and the more general proof in Appendix B. The assumption here, of course, is that all inverses of the minors exist in (B1). Physically this means that there are multiple reflections among the elements whose indices appear in the inverses. If the inverse did not exist then there could be no multiple reflections. This could occur only if 2 or more elements occupied the same space, which, of course, is not true by definition of an array. Special resonant conditions that could lead to singular groupings of the interaction matrices are not considered at this point.

6. Conclusion, Work in Progress and Future Plans

We believe that we have achieved an in depth understanding of the electromagnetic interaction processes within an N-element array environment. Assuming that the transmitting characteristics of the elements $S^{(\ell)}_{\beta\alpha}$ have been measured and the open circuit scattering matrix S_0 is known, we now have a basis by which to calculate mutual impedances between elements and construct overall radiation patterns of an N-element array taking mutual impedances into account. Although the elements of S_0 have never been measured, one can make certain theoretical assumptions (see Appendix A and [3]) to perform sample calculations. The problem of measuring S_0 needs to be addressed seriously in the future.

Computer codes to evaluate the $\mathbb{A}$ and $\mathbf{\Omega}$ matrices defined earlier need to be developed. Computer codes to evaluate the mode-mode free space coupling coefficients, which are the elements of $\mathbf{\Lambda}$, are nearing the completion stage.* Once these codes are in production mode we will compute

(1) Mutual impedances between elements for various inter-element spacings,
   a) for various N (the number of elements)
   b) and for various free space patterns for the elements.

(2) Elementary patterns in an array environment.

(3) The overall radiation pattern of an array.

After completion of this stage of theoretical exploration, we will start work on the inverse aspect of the problem of creating overall radiation patterns from data obtained from subarrays. In addition, experimental work is being planned on a large array to verify the theory presented in this study.

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Appendix A

Brief Review of the Scattering Matrix Formalism

In the scattering matrix formalism incoming waves $\underline{a}$ are related to outgoing waves $\underline{b}$ by

$$\underline{b} = S\,\underline{a}, \tag{A1}$$

where S the scattering matrix of the antenna (valid for a single antenna or an array antenna) is given by

$$S = \begin{pmatrix} S_{\alpha\alpha} & S^{\dagger}_{\alpha\beta} \\ S_{\beta\alpha} & S_{\beta\beta} \end{pmatrix} \tag{A2}$$

and

$$\underline{b} = \begin{pmatrix} \underline{b}_{\alpha} \\ \underline{b}_{\beta} \end{pmatrix} \qquad \underline{a} = \begin{pmatrix} \underline{a}_{\alpha} \\ \underline{a}_{\beta} \end{pmatrix} .$$

In the above α refers to the port side and β to the space side of the antenna, $S_{\alpha\alpha}$ is the mutual coupling matrix of an array antenna or the input reflection coefficient of a single antenna, the row vector $S^{\dagger}_{\alpha\beta}$ contains the receiving coefficients, the column vector $S_{\beta\alpha}$ contains the modal transmitting coefficients and $S_{\beta\beta}$ gives the scattering coefficients of the antenna; $\underline{a}_{\alpha}(\underline{b}_{\alpha})$ are incoming (outgoing) waveguide mode coefficients, and $\underline{a}_{\beta}(\underline{b}_{\beta})$ are incoming (outgoing) wave mode coefficients. In the main text we consider only single mode waveguides, hence, a_{α} and b_{α} are scalars. The scattering matrix S is unitary [1,7], i.e.,

$$S^{\dagger}\, S = I \tag{A3}$$

where I is the unit matrix, which results in three independent conditions on the elements of S [7,8]. In addition, one can show that $S_{\beta\beta}$ can be written as $S_{\beta\beta} = S_o - S_{\beta\alpha}S^{\dagger}_{\alpha\beta}$ with $S^{\dagger}_o\, S_o = I$, and $S^{\dagger}_o\, S_{\beta\alpha} = S_{\alpha\beta}$. S_o is said to be the open circuit ($\underline{a}_{\alpha} = \underline{b}_{\alpha}$ or the current i=0) scattering matrix of the antenna which relates the incoming wave $\underline{a}_{\beta}$ to the outgoing wave $\underline{b}_{\beta}$ by $\underline{b}_{\beta} = S_o\underline{a}_{\beta}$. For further details the reader is referred to [1,2,7]. Theoretically all components of S except $S_{\alpha\alpha}$ are infinite dimensional, but in practice only finite dimensions can be considered. The level of truncation is in practice determined by the accuracy or noise level of the measuring system.

Linear combinations of the $\underline{a}$ and $\underline{b}$ wave amplitudes (α, β understood) can be formed to represent generalized voltages and currents [9]. Thus,

$$\underline{a} = \frac{1}{2}(\underline{V} + \underline{I})$$
$$\underline{b} = \frac{1}{2}(\underline{V} - \underline{I}) \ . \qquad (A4)$$

We make extensive use of this representation in the text.

Appendix B

Cyclic Product Decomposition of Matrix A

Given a square matrix **A** of order N, whose elements labeled $\mathbf{a}_{k\ell}$, $k,\ell=1$, are of order $1 \leqslant n < N$, then the cyclic product decomposition of matrix **A** about an element $\mathbf{a}_{kk'}$, denoted by $\mathbf{A}_{:kk'}$, is defined inductively as*

$$\mathbf{A}_{:kk'} = I(kk':\alpha\beta)\mathbf{a}_{kk'} + \sum_{\substack{\ell=1 \\ \ell\neq k'}}' I(k\ell:\alpha\beta)\mathbf{a}_{k\ell}\, \mathbf{M}^{-1}_{kk':\ell\ell}\, \mathbf{M}_{k\ell:\ell k'},\ k,\ k' = 1,N \qquad \text{(B1)}$$

where $\mathbf{A}_{:kk'}$ is of order n, $\mathbf{M}_{k\ell}$ is a submatrix of **A** of order (N-1) obtained by omitting the k-th row and ℓ-th column of **A**, and $\mathbf{M}_{k\ell:\ell k'}$ is the cyclic product decomposition of $\mathbf{M}_{k\ell}$ about the element $a_{\ell k'}$ (of order n) and $I(k\ell:\alpha\beta) = (-1)^{k+\ell+p+\varepsilon(k\ell:\alpha\beta)}$, where $\alpha\beta$ is the set of all the indices that were omitted from matrix **A** to form the submatrix $\mathbf{M}_{k\ell}$, and $\varepsilon(k\ell:\alpha\beta) = m$, where m is the number of times $k > \alpha$ and $\ell > \beta$ for all α and β, and p is the permutation index equal to the total number of times rows and columns have to be permuted to have their indices in increasing order. Note that initially $\alpha\beta$ is (N+1)(N+1). This complicated function of indices $I(k\ell:\alpha\beta)$ merely has the effect of alternating the sign of successive elements $a_{k\ell}$ in each submatrix according to their row and column positions. Note that if in the first step of the expansion $k = k'$, then we obtain products of terms with paired indices throughout the decomposition.

The definition (B1) defines the cyclic product decomposition of matrix **A** in terms of decompositions of submatrices of **A**. An important property of this inductive definition is that a decomposition about an element $\mathbf{a}_{kk'}$ will result in a sum of terms in each of which an element of column k' will appear as the last factor on the right. We note, without detailed proof, the following lemma:

Lemma 1: The cyclic product decomposition $\mathbf{A}_{:kk'}$ is of the following form

$$\mathbf{A}_{:kk'} = \mathbf{a}_{kk'} + \sum_{\substack{j=1 \\ j\neq k}}^{N} \mathbf{B}_j \mathbf{a}_{jk'} \qquad \text{(B2)}$$

*The reader is referred to the example in Section 5.

where B_j are matrices of order n obtained from the elements $a_{\ell k}$ of $\mathbf{A}$. Thus, the elements in column k' always appear on the right side in each term. The importance of this fact will be apparent in the theorem proven below.

A brief examination of (B1) should convince one of the correctness of lemma 1. The first term of an expansion about $a_{kk'}$ is an element of column k' by definition, and (except for a scalar factor in front of it) it stands by itself. The column index k' is not changed in the definition, that is, the first element of each successive decomposition of the submatrices $M_{k\ell}$, $\ell \neq k'$, will be an element from column k'.

<u>Theorem</u>: Given a linear system $\mathbf{A}\underline{\chi} = \underline{\beta}$ where $\underline{\tilde{\chi}} = (\underline{\chi}_1, \ldots, \underline{\chi}_N)$, $\underline{\tilde{\beta}} = (\underline{\beta}_1, \ldots, \underline{\beta}_N)$ and $\mathbf{A}_{ij} = \boldsymbol{\alpha}_{ij}$, i = 1,N; j = 1,N and $\underline{\chi}_i$ and $\underline{\beta}_i$ are n-dimensional vectors and $\boldsymbol{\alpha}_{ij}$ are nxn matrices. Then

$$\underline{\chi}_j = (\mathbf{A}_{:jj})^{-1}\, \underline{\bar{A}}_{:jj} \tag{B2}$$

where $\underline{\bar{A}}$ is obtained by replacing the j-th column of $\mathbf{A}$ by $\underline{\beta}$. Thus, this formula is seen to be a generalization of Cramer's rule to linear systems of vector variables and matrix coefficients.

We shall prove by induction the above formula for j = 1 only, since the result can be shown to follow for all j ≠ 1 by simple relabeling of terms after an even permutation of rows and columns of the linear system. The first step in an inductive proof is to show that the result is valid for some initial value of N, say N = 2. We will comment on this part of the proof at the end of this section, leaving some easily obtainable details to the reader to fill in.

Consider the linear system

$$\begin{array}{l}
\boldsymbol{\alpha}_{11}\underline{\chi}_1 + \boldsymbol{\alpha}_{12}\underline{\chi}_2 + \ldots + \boldsymbol{\alpha}_{1\ell}\underline{\chi}_\ell + \ldots + \boldsymbol{\alpha}_{1(N+1)}\underline{\chi}_{N+1} = \underline{\beta}_1 \\
\boldsymbol{\alpha}_{21}\underline{\chi}_1 + \ldots \qquad\qquad\qquad\qquad\quad + \boldsymbol{\alpha}_{2(N+1)}\underline{\chi}_{N+1} = \underline{\beta}_2 \\
\quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots \\
\boldsymbol{\alpha}_{\ell 1}\underline{\chi}_1 + \ldots \qquad + \boldsymbol{\alpha}_{\ell\ell}\underline{\chi}_\ell + \ldots + \boldsymbol{\alpha}_{\ell(N+1)}\underline{\chi}_{N+1} = \underline{\beta}_\ell \\
\quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots \\
\boldsymbol{\alpha}_{(N+1)1}\underline{\chi}_1 + \ldots + \boldsymbol{\alpha}_{(N+1)\ell}\underline{\chi}_\ell + \ldots + \boldsymbol{\alpha}_{(N+1)(N+1)}\underline{\chi}_{N+1} = \underline{\beta}_{N+1}
\end{array} \tag{B3}$$

or

$$\underline{A\chi} = \underline{\beta}$$

where $\underline{A}_{ij} = \alpha_{ij}$, $i = 1,N+1$, $j = 1,N+1$ are the blocked square matrix elements of **A**, and $\underline{\beta}_i$, $\underline{\chi}_i$ are vectors.

We form an NxN system from the above by omitting the first equation and transferring the $\alpha_{j1}\underline{\chi}_1$, $j = 2,N+1$ terms to the right hand side. We then get

$$\mathbf{M}_{11}\hat{\underline{\chi}} = \hat{\underline{\beta}} - \hat{\underline{\alpha}}_1\underline{\chi}_1 \quad . \tag{B4}$$

Here $\hat{\underline{\alpha}}_1$, is a column vector whose elements are α_{j1}, $j = 2$, N+1. The "^" indicates that we are dealing with the NxN reduced system, and $\mathbf{M}_{11}$ is the coefficient matrix of the reduced system and is also the submatrix of α_{11} of the full (N+1) x (N+1) coefficient matrix. We now assume that the solution to the reduced system is given by

$$\hat{\underline{\chi}}_j = \mathbf{M}^{-1}_{11:jj}\ \bar{\underline{M}}_{11:jj}, \quad j = 2,N+1 \tag{B5}$$

where $\bar{\mathbf{M}}$ is the matrix formed by replacing the j-th column of **M** by $\hat{\underline{\beta}} - \hat{\underline{\alpha}}_1\underline{\chi}_1$. We now permute column j of $\bar{\mathbf{M}}_{11}$ so that it appears as the first column. We denote the resultant matrix by $\bar{\mathbf{P}}_{11}$. Each exchange of columns introduces a factor of (-1) or altogether $(-1)^{(j-2)} = (-1)^j$. So,

$$\hat{\underline{\chi}}_j = \mathbf{M}^{-1}_{11:jj}\bar{\underline{P}}_{11:jj}(-1)^j . \tag{B6}$$

Since the right hand side has two terms

$$\hat{\underline{\chi}}_j = \mathbf{M}^{-1}_{11:jj}\ \bar{\underline{Q}}_{11:jj}\ (-1)^j + (-1)^{j+1}\mathbf{M}^{-1}_{11:jj}\ \bar{\underline{R}}_{11:jj}\underline{\chi}_1 \tag{B7}$$

where $\bar{Q}$ contains $\hat{\beta}$, and $\bar{R}$ contains $\hat{\alpha}_1$. On close examination we see that $\bar{\underline{R}}_{11:jj} = \underline{\mathbf{M}}_{1j:j1}$ of the full NxN system, where $\underline{\mathbf{M}}_{1j}$ is the minor of α_{1j}, and $\underline{Q}_{11:jj} = \underline{M}_{1j:j1}$ is the submatrix of α_{1j} of $\bar{\underline{A}}$ formed by replacing the first column of **A** by $\underline{\beta}$.

Hence,

$$\hat{\underline{\chi}}_j = \mathbf{M}^{-1}_{11:jj}\bar{\underline{M}}_{1j:j1}(-1)^j + \mathbf{M}^{-1}_{11:jj}\mathbf{M}_{1j:j1}(-1)^{j+1}\underline{\chi}_1,\ j = 2,N+1 \tag{B8}$$

and the other indices refer to the original full system. We now substitute $\hat{\underline{\chi}}_j$, j = 2,N+1 into the first equation of the full system

$$[\boldsymbol{\alpha}_{11} + \sum_j \alpha_{ij}\ \mathbf{M}^{-1}_{11:jj}\ \mathbf{M}_{1j:j1}\ (-1)^{(j+1)}]\ \underline{\chi}_1 = \underline{\beta}_1 + \sum_j \alpha_{1j}\mathbf{M}^{-1}_{11:jj}\ \bar{\underline{M}}_{1j:j1}(-1)^{j+1}$$

or, by definition,

$$\mathbf{A}_{:11}\underline{\chi}_1 = \bar{\underline{A}}_{:11} \tag{B9}$$

and

$$\underline{\chi}_1 = (\mathbf{A}_{:11})^{-1}\ \bar{\underline{A}}_{:11}\ . \tag{B10}$$

We still need to show that the solution to a 2 x 2 system is given by our formula. Let

$$\mathbf{A} = \begin{pmatrix} \boldsymbol{\alpha}_{11} & \boldsymbol{\alpha}_{12} \\ \boldsymbol{\alpha}_{21} & \boldsymbol{\alpha}_{22} \end{pmatrix}$$

Then $\underline{\chi}_1 = \left(\boldsymbol{\alpha}_{11} - \boldsymbol{\alpha}_{12}\ \boldsymbol{\alpha}^{-1}_{22}\ \boldsymbol{\alpha}_{21}\right)^{-1}\ \left(\underline{\beta}_1 - \boldsymbol{\alpha}_{12}\ \boldsymbol{\alpha}^{-1}_{22}\ \underline{\beta}_2\right)$ according to the expansion rule. This result can be easily verified by solving the 2 x 2 system directly by substitution. Similarly the result can be verified for $\underline{\chi}_2$. The proof by induction is now complete.

Appendix C
Evaluation of the Coupling Integral

The elements of the mutual impedance matrices $Z_{\ell k}$ between antennas ℓ and k in equation (4) are given by the mode-mode impedance integrals [3] (with $e^{-i\omega t}$ time convention)

$$\xi^{(k\ell)}_{nn',m} = 2 \int_0^{2\pi} d\phi \int_{+i\infty}^{1} \underline{\varepsilon}^{(k)}_{nm}(\theta,\phi) \quad \underline{\varepsilon}^{*(\ell)}_{n'm}(\theta,\phi)\ e^{+ikDx}dx \tag{C1}$$

where $x = \cos\theta$, D is the distance between antennas k and ℓ located on the z-axis, and $\underline{\varepsilon}_{nm}$ is the far field pattern of a single (even, odd) mode $\underline{e}^{(\ell,o)}_{nm}$ given by [3]

$$\underline{e}^{(e,o)}_{nm}(\theta,\phi) = -\frac{1}{N_{nm}} r\nabla_s P^m_n(x) \begin{Bmatrix}\cos m\phi \\ \sin m\phi\end{Bmatrix} \tag{C2}$$

where $N^2_{nm} = \dfrac{4\pi}{\Delta_{nm}} \dfrac{n(n+1)}{2n+1} \dfrac{(n+m)!}{(n-m)!}$ gives the normalization ($\Delta_{nm} = 1$, if $m = 0$, $\Delta_{nm} = 2$ otherwise), $P^m_n(x)$ are associated Legendre polynominals and the surface component of the gradient $\nabla_s \equiv \nabla - \hat{r}\ \ \nabla$ ($\hat{r}$ = unit vector), or in component form,

$$\nabla_s = \left(0, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \tag{C3}$$

Note that in (C1) only a single m index appears, explicitly stating that modes with $m \neq m'$ do not interact (the ϕ-integral vanishes). Coupling integrals between far fields $\bar{\underline{\varepsilon}}_{nm} = \hat{r} \times \underline{\varepsilon}_{nm}$ and $\underline{\varepsilon}_{nm}$ or between $\bar{\underline{\varepsilon}}_{nm}$ and $\bar{\underline{\varepsilon}}_{nm}$ can also be written analogously to (C1), but we will not do so, since the evaluation of all possible coupling integrals proceeds along the same lines.

To evaluate (C1)

(1) we perform the ϕ integral to obtain a factor of π

(2) we observe from the recursion relationships [10]

$$\frac{dP_n^m}{d\theta} = \frac{1}{2}\left[(n - m + 1)(m + n)P_n^{m-1} - P_n^{m+1}\right] \tag{C4}$$

$$\frac{m}{\sin\theta} P_n^m = \frac{1}{2}\cos\theta\left[(n - m + 1)(n+m)\ P_n^{m-1} + P_n^{m+1}\right] \tag{C5}$$

that $f(x) \equiv \underline{\varepsilon}_{nm} \cdot \underline{\varepsilon}^*_{n'm}$ can be written as products of $P_n^m\, P_{n'}^{m\pm1}$, and from Rodriguez's formula [6]

$$P_n^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell \tag{C6}$$

one can easily see that f(x) is a polynomial in x, and hence, is identical to a sum of Legendre polynomials $P_\ell(x)$

$$f(x) \equiv \sum_{\ell=|n-n'|}^{n+n'}{}' \alpha_\ell^m P_\ell(x) \tag{C7}$$

where the prime on the summation indicates that only even or odd ℓ-s occur in the sum.

The integral representation of spherical Hankel functions (for $e^{-i\omega t}$ convention) [10]

$$h_\ell^{(1)}(kD) = i^{-\ell}\int_{i\infty}^{1} P_\ell(x)\ e^{+ikDx}dx \tag{C8}$$

allows one to write the coupling integral as

$$\xi_{nn',m} = 2\sum_{\ell=|n-n'|}^{n+n'}{}' \alpha_\ell^m\, h_\ell^{(1)}(kD)\ . \tag{C9}$$

This series can be easily evaluated if α_ℓ^m are known.

The α_ℓ^m can be calculated theoretically from Clebsch-Gordon coefficients using the coupling rule for spherical harmonics [11], but a simple numerical procedure will also yield these coefficients. We generate f(x) on a discrete set of points x_i, i=1,L, and write (C7) as a matrix equation

$$\begin{pmatrix} P_o(x_1) & \cdots & P_\ell(x_1) & \cdots \\ P_o(x_k) & \cdots & P_\ell(x_k) & \cdots \\ P_o(x_L) & \cdots & P_\ell(x_L) & \cdots \end{pmatrix} \begin{pmatrix} \alpha_o \\ \vdots \\ \alpha_\ell \\ \vdots \\ \alpha_L \end{pmatrix} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_k) \\ \vdots \\ f(x_L) \end{pmatrix} \qquad (C10)$$

Since the Legendre polynomials are basic functions, the columns of the matrix in (C10) are essentially orthogonal, and the matrix will be well conditioned. Then (C10) can be easily inverted to obtain the α_ℓ^m. Numerical experiments yielded values of α_ℓ^m to machine accuracy.

In large arrays the asymptotic behavior of $\xi_{nn',m}$ will give the strength of interaction between distant elements. For x >> 1,ℓ the Hankel functions behave as [6]

$$h^{(1)}(x) \to (-i)^{\ell+1} \frac{e^{ix}}{x} \qquad (C11)$$

and, because of (C9), $|\xi_{nn',m}| \to \frac{1}{x}$.

For x << 1,ℓ, the real part of $h_\ell^{(1)}(x)$

$$j_\ell(x) \to 0 \qquad (C12)$$

and the imaginary part of $h_\ell^{(1)}(x)$

$$n_\ell(x) \to -\frac{1}{x^{\ell+1}} \to -\infty. \tag{C13}$$

For $\ell \approx x >> 1$ [12]

$$h_\ell^{(1)}(x) \to \frac{e^{i\pi/3}}{\ell^{5/6}} \quad . \tag{C14}$$

This last relationship shows that for elements whose radiation is bandlimited to $\ell \approx x = \frac{kD}{2}$, where D is the distance between elements, the mutual impedance between the highest modes present is of the order of unity. Hence, the asymptotic behavior of $n_\ell(x)$ for $x << 1$ is not troublesome.

From the above considerations one can deduce that for identical elements the off diagonal entries in (11) get progressively smaller. Hence, the linear system could be diagonally dominant, in which case a solution always exists [13]. The condition for strict diagonal dominance for a complex matrix **A** of order M is that [13]

$$|a_{ij}| > \sum_{\substack{j=1 \\ j \neq i}}^{M} |a_{ij}| \tag{C15}$$

where a_{ij} are the elements of **A**. It is easy to see from (C11)-(C14) that there is an interelement spacing D_0 at which the matrix **A** in (11) is diagonally dominant. For $x = kD_0$ (C15) can be written as

$$1 > \frac{\beta_i}{kD_0} \tag{C16}$$

where β_i is some constant such that $0 < \beta_i < \infty$. At this point we will consider only interelement spacings that satisfy (C16). The β_i could be obtained theoretically from the Clebsch-Gordon coefficients, i.e., from the α_ℓ^m in (C9), in which case D_0 would be expressed as a complicated but exact function of the mode numbers describing the element patterns.

Up to now we have assumed that the antennas in question are located on

Up to now we have assumed that the antennas in question are located on the z-axis. A two (or three) dimensionl generalization of the above procedure is easily achieved. The line connecting the two antennas is considered to be the z-axis, and the modes of the antennas expressed in the original reference frame fixed to the antennas are are expressed in this new rotated frame. This requires a change of basis, which is accomplished by the formula [11]

$$Y_{\ell m'}(\theta',\phi') = \sum_{m} D^{\ell}_{mm'}(\alpha\beta\gamma) Y_{\ell m}(\theta,\phi) \tag{C17}$$

when α, β, γ are Euler angles of rotation as defined in [11], and $D^{\ell}_{mm'}(\alpha\beta\gamma)$ is the rotation matrix in spherical basis. As an example, one can easily show that the field of an axial dipole, which is generated from $Y_{10}(\theta,\phi)$, when rotated through $\beta = -\frac{\pi}{2}$, will be generated from the real part of $Y_{\ell 1}(\theta,\phi) + Y_{\ell,-1}(\theta,\phi)$. These $Y_{\ell,\pm 1}(\theta,\phi)$ are in fact the modes present in the rotated frame.

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L. Muth

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11. ABSTRACT (A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here)

A general theoretical approach is formulated to describe the complex electromagnetic environment of an N-element array. The theory reveals the element-to-element interactions and multiple reflections within the array. From the formulation, it is found that the interaction between an excited element and an open-circuited element can be viewed as the sum of terms describing all possible signal paths within the array environment which start from the radiating element and terminate on the element under observation. Within all paths except the most direct one, multiple reflections between subgroups of elements take place. The resulting solution is highly structured and recursive and is discussed in detail in the text. Illustrative examples are provided to facilitate understanding of these ideas.

12. KEY WORDS (Six to twelve entries; alphabetical order; capitalize only proper names; and separate key words by semicolons)

array antennas; array environment; coupling; multiple reflections; mutual impedance

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