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A Direct Approach to the Derivation of Electric Dyadic Green's Functions

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+ Technical Note

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Arthur D. Yaghjian

A straightforward approach that does not require delta-function techniques is used to derive a generalized electric dyadic Green's function which remains valid within the source region. Although the electric field expressed by the dyadic Green's function proves to be unique, the exact form of the dyadic itself depends, in the source regions, upon the geometry of its "principal volume." The dependence on principal volume is determined explicitly, and the different Green's dyadics derived by a number of previous authors are shown to emerge merely through the appropriate choice for the principal volume. Moreover, delta-function techniques, which by themselves are shown to be inadequate to extract uniquely the proper electric dyadic Green's function in the source region, can be supplemented by a very simple procedure to yield unambiguously the correct Green's function and associated fields.

Key words: Dyadic Green's function; electromagnetic theory; principal values; source region.

INTRODUCTION

The problem we want to solve dates back to Maxwell himself [1]. Given Maxwell's equations in a homogeneous region of space, find the time harmonic electromagnetic (EM) fields in terms of the applied sources of current. The fields in an infinite region are assumed to obey the outward radiation condition, and if boundary surfaces are present, the tangential component of the electric or magnetic vector is assumed zero or otherwise specified on these surfaces.

One might think that the general solution to such a basic problem would have been investigated thoroughly and laid to rest long ago. And indeed this is the case for the solution to the fields outside the source points. However, at the points where sources are present there exist in the literature a number of different results to the same problems depending on the method of solution which is used to find the electric dyadic Green's function.[1] Yet uniqueness theorems ([2], Ch. VI) demand that there be just one correct solution for the fields even at the source points. Thus, it is the main purpose of this paper to first present a straightforward method for finding the correct solution to the electromagnetic fields inside as well as outside the source region, and second to explain and reconcile the discrepancies which exist between previous results.

The previous results have been obtained by two basic methods of solution to arrive at a dyadic Green's function, one, the integral formula method used, for example, by Stratton ([7], Secs. 3.4, 8.1), Wilcox [8], and Van Bladel [3a]; and two, the method of distributions with delta generating functions (hereinafter referred to merely as the delta-function method) used, for example, by Morse and Feshbach ([9], Ch. 13), Tai [4], Collin [10], Howard [5], and Rahmat-Samii [6].[2] A fundamental difference between the two methods involves the question of what we shall call the principal volume (principal area for two-dimensional problems discussed in Section V). For the integral formula method the geometry of the limiting principal volume which excludes the singularity of the Green's function must be stated explicitly to get the right answer when solving for the electric dyadic Green's function in the source region and the associated fields computed

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[1]For instance compare at the source points the free-space electric dyadic Green's function (and the EM fields computed from it) derived by Van Bladel [3a], Tai [4b] and Howard [5]; or compare the results of Tai [4c] with that of Rahmat-Samii [6] for a rectangular cavity. These are but a few of the number of instances which can be sited from the literature where the EM fields at the source points do not agree. Also, it should be emphasized that the reasons for the disagreement are much more fundamental than the error addressed and corrected by Tai [4a] of using only solenoidal wave functions in the expansion of the dyadic delta function.

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[2] Actually, in some of these and other papers on the subject, an approach is used which combines the integral formula and delta-function methods (see Section II). However, in most cases one method predominates, and in all cases the various steps in the particular approach taken can be classified as belonging to the integral formula method, the delta-function method, or both. It is appropriate to distinguish between the two methods because each one can be used separately to solve for the EM fields in terms of the current distribution.

from it [3a]. The delta-function method as applied in the past, however, does not require any mention of limit procedures involving the geometry of a principal volume. (It should be noted here that limit is used in the generalized sense which allows the value of the limit to depend on the geometry of the principal volume. This use is nothing more than the generalization to three dimensions of the one-dimensional concept of limit from the left and limit from the right ([11], p. 13).)

In the following sections the dependence of the electric dyadic Green's function on the geometry of the principal volume is verified and determined explicitly. It is further shown that delta-function techniques as they have been applied in the past suppress the specification of a principal volume and thus will not yield the correct fields at source points from the electric dyadic Green's function unless the suppressed principal volume is identified. And although the principal volume cannot be identified from the ordinary delta-function techniques alone, a surprisingly simple procedure can be used to supplement the delta-function method and properly incorporate the effect of the principal volume. A number of other conclusions are reached, some of the more important of which are summarized in the concluding Section VI.

Before entering the main body of the paper, a couple of other introductory discussions may prove helpful. The first has to do with the terminology "geometry of the principal volume," which is used repeatedly throughout the paper. Here "geometry" entails the specification of not only the shape of the surface of the principal volume (area in two dimensions) but also its position and orientation with respect to the singularity which it excludes. For example, it is not sufficient in order to specify the geometry of a rectangular principal volume to simply state that it is a rectangular box. In addition, the position of the singularity within the box and the orientation of the box with respect to the singularity in the coordinate system chosen for the problem must be specified. We will find notable exceptions to this rule, such as the spherical or cubical principal volume with singularity at the center, for which the dependence on principal volume remains independent of their orientation, but in general all three (the shape, position, and orientation of the principal volume) must be specified.

The second discussion has to do with the physical explanation of why the principal volume should affect the determination of electric field at source points. (This subject is treated in greater depth and detail in Section IV. 3.) Consider the hypothetical scheme for measuring the electrostatic field within a dielectric by removing an infinitesimal volume of dielectric material and inserting a point probe which measures the electric field. It is a wellknown phenomenon [15b] that this measured electric field, often referred to as the "local field," will depend on the geometry (shape, position with respect to the point probe, and orientation in the dielectric) of this deleted volume. The unique, mathematically defined macroscopic electric field is related to the local field by adding to the local field a polarization or source term which also depends on the geometry of the deleted volume. That is, both local field and source term depend on the geometry of the deleted volume, but their sum, which equals the macroscopic electric field, remains independent, as it should, of the deleted volume.

Now for time harmonic fields, a similar phenomenon occurs as part of the derivation of the electric dyadic Green's function in source regions of current, even when no polarization is present. At source points the electric dyadic Green's function separates into two parts, which correspond to the analogous local field and polarization or source term of electrostatics. In addition, the principal volume corresponds mathematically to the deleted volume of dielectric material. Both the local field and source term depend on the geometry of the principal volume, but their sum remains independent of the principal volume and equal to the unique, mathematically defined electric field. The analogy to electrostatics is valid in a physical measurement sense also. For if the principal volume of time harmonic current sources were actually removed physically and a point electric-field probe were inserted, the measured field would correspond to the local field, whose value again would depend on the geometry of the deleted volume. Section IV.3 demonstrates in detail the fundamental correspondence between electrostatic fields in dielectric media and electric dyadic Green's functions in source regions.

I. A STRAIGHTFORWARD SOLUTION FOR THE MAGNETIC AND ELECTRIC DYADIC GREEN'S FUNCTIONS IN FREE-SPACE

We begin with Maxwell's time harmonic equations in free-space written in terms of the electric field $\overline{E}$, magnetic induction $\overline{B}$, and applied current $\overline{J}$:

$$\nabla\times\overline{E} = i\omega\overline{B} \tag{1a}$$

$$\nabla\times\overline{B} = -i\omega\varepsilon_o\mu_o\overline{E} + \mu_o\overline{J}. \tag{1b}$$

As usual, ε_o and μ_o are the permittivity and permeability of free-space, and $e^{-i\omega t}$ time dependence has been suppressed. The SI system of units is used throughout. The $\overline{E}$ and $\overline{B}$ fields are restricted to satisfy the outgoing radiation condition at infinity, and $\overline{J}$ lies in a volume of finite extent. (In Section V two-dimensional problems are considered where the current $\overline{J}$ extends infinitely far in the ± z-direction.) By taking $\nabla\times$ (1) we find after a little manipulation the following vector Helmholtz equations for $\overline{B}$ and $\overline{E}$ alone:

$$\nabla^2\overline{B} + k^2\overline{B} = -\mu_o\nabla\times\overline{J} \tag{2a}$$

$$\nabla^2\left(\overline{E} - \frac{\overline{J}}{i\omega\varepsilon_o}\right) + k^2\left(\overline{E} - \frac{\overline{J}}{i\omega\varepsilon_o}\right) = \frac{\nabla\times\nabla\times\overline{J}}{i\omega\varepsilon_o} . \tag{2b}$$

$$(k = \omega\sqrt{\mu_o\varepsilon_o})$$

In free-space (2) also imply (1), and thus (1) and (2) are equivalent. That is, (2a) and (2b) are equivalent to Maxwell's equations. This familiar but rather surprising result, which depends strongly on the fields obeying the outward radiation condition, is proven in Appendix A.

Later in Section III, where we consider problems with boundary surfaces, the fields must satisfy in addition to the vector Helmholtz equations (2) the divergence requirements,

$$\nabla\cdot\overline{B} = 0 \quad \text{and} \quad \nabla\cdot\left(\overline{E} - \frac{\overline{J}}{i\omega\varepsilon_o}\right) = 0 , \tag{3a,b}$$

in order for (2) and (1) to be equivalent sets of equations. In the free-space problem, as Appendix A shows, (3) are derivable from (2) with the help of the radiation condition, and need not be stipulated as separate requirements.

The free-space solutions to (2) can be written down immediately as

$$\overline{B}(\overline{r}) = \frac{\mu_o}{4\pi} \lim_{V_\varepsilon \to 0} \int_{V_J - V_\varepsilon} [\nabla' \times \overline{J}(\overline{r}')]\psi(\overline{r},\overline{r}')dV' \tag{4a}$$

$$\overline{E}(\overline{r}) = -\frac{1}{4\pi i\omega\varepsilon_o} \lim_{V_\varepsilon \to 0} \int_{V_J - V_\varepsilon} [\nabla' \times \nabla' \times \overline{J}(\overline{r}')]\psi(\overline{r},\overline{r}')dV' + \frac{\overline{J}}{i\omega\varepsilon_o} \ . \tag{4b}$$

The scalar Green's function ψ is defined as

$$\psi(\overline{r},\overline{r}') = e^{ik|\overline{r}-\overline{r}'|}/|\overline{r}-\overline{r}'| \ ,$$

and V_J merely indicates a volume within which all sources ($\overline{J}$) are contained.

V_ε is the "principal volume" which excludes the singularity of the function ψ at $\overline{r}' = \overline{r}$. The geometry (shape, and position and orientation with respect to $\overline{r}$) of V_ε is specified for each $\overline{r}$ as the limit of V_ε approaches zero, i.e., shrinks down around the point $\overline{r}$. The principal volume V_ε may but need not be the same for each point $\overline{r}$. Because the limit of the principal value integrals in (4) do not depend upon the geometry of the principal volume V_ε, it is customary to omit any explicit reference to V_ε or the principal value sense in which the integrals are defined. However, if (4) are to be properly differentiated or transformed by integral formulas, both of which will be done below, it proves necessary and in some places crucial that the limiting process be retained explicitly [14]. This fact becomes clear immediately if we try to prove by direct substitution that the solutions (4) do indeed satisfy (2).

A mathematically rigorous proof that (4) are solutions to the Helmholtz equations (2) has been given by Kellogg ([12], Sec. VI.3) for $k = 0$ and by Müller ([2], Th. 7) for arbitrary $k > 0$. The usual textbook proof, by direct substitution bringing the ∇^2 operator under the integral and making use of the fact that

$$\nabla^2\psi + k^2\psi = -4\pi\delta(\bar{r}-\bar{r}'), \tag{5}$$

is simply not valid since the singular point $\bar{r}' = \bar{r}$ is excluded by the principal volume V_ε. The method of proof, by direct substitution can, however, be made sufficiently rigorous if it is realized that the principle volume V_ε as defined by the $\bar{r}'$ coordinates is a function of the observation point $\bar{r}$. In other words, the limits of integration for the triple integrals in (4) depend on the variable $\bar{r}$. And thus, when we differentiate with respect to $\bar{r}$, we must take this dependence into account and not merely interchange integration and differentiation. Appendix B shows that when done correctly the method of direct substitution yields the result that (4) satisfies (2).

Now (4) are solutions to Maxwell's equations in free-space in terms of the derivatives of $\bar{J}$, in particular, $\nabla\times\bar{J}$ and $\nabla\times\nabla\times\bar{J}$. A solution is desired in terms of $\bar{J}$ alone. To accomplish this we simply transform (4a) once and (4b) twice using standard integral formulas which convert a volume integral to a surface integral. Care must be taken to include the surface integral over V_ε which excludes the singularity.

Beginning with (4a) let

$$(\nabla'\times\bar{J})\psi = \nabla'\times(\bar{J}\psi) - \nabla'\psi\times\bar{J}, \tag{6}$$

and the integral in (4a) transforms as follows:

$$\int_{V_J - V_\varepsilon} (\nabla' \times \overline{J}) \psi \, dV' = - \int_{S_\varepsilon} \hat{n}' \times \overline{J} \psi \, dS' - \int_{V_J - V_\varepsilon} \nabla' \psi \times \overline{J} \, dV' \quad . \qquad (7)$$

($\hat{n}'$ is the outward normal to V_ε). The integral over S_J vanishes because this surface has been chosen to lie outside the source region. The integral in (7) over the surface S_ε of the principal volume also vanishes as $V_\varepsilon \to 0$ since ψ behaves as $\frac{1}{R'}$, but the surface integration behaves as R'^2 as $R' = |\overline{r}' - \overline{r}| \to 0$. Thus (4a) has been transformed to

$$\overline{B}(\overline{r}) = - \frac{\mu_o}{4\pi} \lim_{V_\varepsilon \to 0} \int_{V_J - V_\varepsilon} (\nabla' \times \psi \overline{\overline{I}}) \cdot \overline{J} \, dV' \ , \qquad (8)$$

where $\overline{\overline{I}}$ is the unit dyadic and use has been made of the identity

$$\nabla' \psi \times \overline{J} = (\nabla' \times \psi \overline{\overline{I}}) \cdot \overline{J} \ . \qquad (9)$$

The free-space magnetic dyadic Green's function ($\overline{\overline{G}}{}^o_m$) defined by

$$\overline{B}(\overline{r}) = \mu_o \lim_{V_\varepsilon \to 0} \int_{V_J - V_\varepsilon} \overline{\overline{G}}{}^o_m \cdot \overline{J} \, dV' \qquad (10)$$

is thus determined from (8),

$$\overline{\overline{G}}{}^o_m(\overline{r}, \overline{r}') = - \frac{1}{4\pi} \nabla' \times (\psi \overline{\overline{I}}), \qquad \overline{r}' \neq \overline{r} \ . \qquad (11)$$

It is a simple matter to show that because $\nabla' \psi$ behaves as $1/R'^2$ as $R' \to 0$ the volume integral in (10) remains independent of the geometry of the principal volume V_ε. Of course, this has to be the case since (10) was derived directly from (4a) whose volume integral is also independent of the geometry of V_ε.

The magnetic dyadic Green's function presents no special difficulties. It has been derived in free-space and for various interior and exterior regions by both integral formula and delta-function techniques, and the results agree consistently. Its derivation has been included here chiefly as an initial step for determining the electric dyadic Green's function from (4b).

Transformation of (4b) by the same technique applied to (4a) yields

$$\overline{E}(\overline{r}) = -\frac{1}{i\omega\varepsilon_o} \lim_{V_\varepsilon \to 0} \int_{V_J - V_\varepsilon} \overline{\overline{G}}{}^o_m \cdot (\nabla' \times \overline{J})\, dV' + \frac{\overline{J}}{i\omega\varepsilon_o} , \tag{12}$$

an equation which displays the same properties as (10). Still (12) does not give $\overline{E}$ in terms $\overline{J}$ alone but includes the derivatives $\nabla\times\overline{J}$ as well. To recast (12) in terms of $\overline{J}$ alone we need merely to transform the integral a second time. Substitution from the identity,

$$\overline{\overline{G}}{}^o_m \cdot (\nabla' \times \overline{J}) = -(\nabla' \times \overline{J}) \cdot \overline{\overline{G}}{}^o_m = -\nabla' \cdot (\overline{J} \times \overline{\overline{G}}{}^o_m) - \overline{J} \cdot \nabla' \times \overline{\overline{G}}{}^o_m , \tag{13}$$

into (12) and converting the divergence integral to a surface integral allows (12) to be written as

$$\overline{E}(\overline{r}) = \frac{1}{i\omega\varepsilon_o} \lim_{V_\varepsilon \to 0} \left(\int_{V_J - V_\varepsilon} (\nabla' \times \overline{\overline{G}}{}^o_m) \cdot \overline{J}\, dV' + \int_{S_\varepsilon} (\overline{\overline{G}}{}^o_m \times \hat{n}') \cdot \overline{J}\, dS' + \overline{J} \right) . \tag{14}$$

Use has been made of the fact that $\overline{\overline{G}}{}^o_m$ and $\nabla' \times \overline{\overline{G}}{}^o_m$ are antisymmetric and symmetric dyadics, respectively.

We shall now prove the very important result that, unlike in (7), the integration in (14) over the surface S_ε of the principal volume contributes to the electric field at source points. In addition, the value of this surface integral contribution depends upon the geometry of the principal volume. The value of the volume integral also varies with the geometry of the principal volume and in just the right way to keep the sum of the volume and surface integral contributions a unique value. That is, both the volume and surface integrals in (14) have individual values which depend on the geometry of the principal volume, but the variation in the volume integral is the negative of the surface integral and together they sum to give a unique $\overline{E}(\overline{r})$ independent of the geometry of the principal volume.

The surface integral in (14) can be rewritten with the help of (11) as

$$\lim_{V_\varepsilon \to 0} \int_{S_\varepsilon} (\overline{\overline{G}}{}^{o}_{m} \times \hat{n}') \cdot \overline{J}\, dS' = -\frac{1}{4\pi}\left[\lim_{V_\varepsilon \to 0} \int_{S_\varepsilon} (\overline{\overline{I}} \times \nabla'\psi)\times\hat{n}'\, dS'\right] \cdot \overline{J}\,. \tag{15}$$

When $\nabla'\psi$ is expressed explicitly as

$$\nabla'\psi = [ik - \frac{1}{R'}]\, \hat{e}_{R'}\, \frac{e^{ikR'}}{R'} \quad , \tag{16}$$

where $\hat{e}_{R'}$ is the unit vector, $\hat{e}_{R'} = \overline{R}'/R' = (\overline{r}'-\overline{r})/|\overline{r}'-\overline{r}|$, the right side of (15) reduces to

$$\frac{1}{4\pi}\left[\lim_{V_\varepsilon \to 0} \int_{S_\varepsilon} \frac{(\overline{\overline{I}} \times \hat{e}_{R'}) \times \hat{n}'}{R'^2}\, dS'\right]\cdot\overline{J}\,. \tag{17}$$

The ik term has vanished in the limit as $V_\varepsilon \to 0$. Further simplification results upon substitution of the identity

$$(\bar{\bar{I}}\times\hat{e}_{R'})\times\hat{n}' = \hat{n}'\hat{e}_{R'} - (\hat{n}'\cdot\hat{e}_{R'})\bar{\bar{I}} \quad , \tag{18}$$

into (17) to give

$$\left[\lim_{V_\varepsilon\to 0}\frac{1}{4\pi}\int_{S_\varepsilon}\frac{(\hat{n}'\hat{e}_{R'})}{R'^2}\,dS' - \bar{\bar{I}}\right]\cdot\bar{J} \ , \tag{19}$$

after the solid angle integration is set equal to 4π, i.e.,

$$\int_{S_\varepsilon}\frac{\hat{n}'\cdot\hat{e}_{R'}}{R'^2}\,dS' = \int_{S_\varepsilon} d\Omega' = 4\pi \ . \tag{20}$$

Substitution of (19) into (14) yields the final expression for the electric field in free-space in terms of $\bar{J}$,

$$\bar{E}(\bar{r}) = i\omega\mu_o \lim_{V_\varepsilon\to 0}\int_{V_J-V_\varepsilon}\bar{\bar{G}}^o_{eo}\cdot\bar{J}\,dV' + \frac{\bar{\bar{L}}\cdot\bar{J}}{i\omega\varepsilon_o} \quad , \tag{21a}$$

where

$$\bar{\bar{L}} \equiv \frac{1}{4\pi}\int_{S_\varepsilon}\frac{\hat{n}'\hat{e}_{R'}}{R'^2}\,dS' \quad , \tag{21b}$$

and

$$\bar{\bar{G}}^o_{eo} = -\frac{1}{k^2}\nabla'\times\bar{\bar{G}}^o_m = \frac{1}{k^2}\nabla\times\bar{\bar{G}}^o_m \quad , \ \bar{r}'\neq\bar{r} \ . \tag{21c}$$

As a brief review of the notation, $\bar{\bar{G}}^o_m$ is the magnetic dyadic Green's function given by (11), $\hat{n}'$ is the unit normal out of the principal volume V_ε, and $\hat{e}_{R'}$

is the unit vector pointing from $\bar{r}$ to $\bar{r}'$ (see Fig. 1). The limit as $V_\varepsilon \to 0$ is omitted in (21b) because the surface integral depends only on the geometry (shape, and position and orientation with respect to $\bar{r}$) of V_ε, not its size.

The conventional form [3b] of the free-space electric dyadic Green's function emerges as $\bar{\bar{G}}^o_{eo}$ when $\bar{\bar{G}}^o_m$ from (11) is put into (21c),

$$\bar{\bar{G}}^o_{eo} = \frac{1}{4\pi k^2} \nabla \times \nabla \times (\psi \bar{\bar{I}}) = \frac{1}{4\pi} [\bar{\bar{I}} + \frac{1}{k^2} \nabla\nabla]\psi, \quad \bar{r}' \neq \bar{r} \quad . \tag{22}$$

Equation (21a) confirms Van Bladel's result [3a] that the conventional electric dyadic Green's function is not sufficient at source points to determine the correct value of $\bar{E}(\bar{r})$. The extra dyadic $\bar{\bar{L}}$, which depends only on the geometry of the principal volume, is also required. The fact that the unique electric field at source points cannot be given by the volume integral alone becomes obvious when one investigates the volume integration near the singularity of $\bar{\bar{G}}^o_{eo}$ and finds that its contribution depends on the geometry of the principal volume. The dyadic $\bar{\bar{L}}$, which also depends on the geometry of the principal volume, compensates exactly to produce the unique $\bar{E}(\bar{r})$ regardless of the principal volume shape. This last statement can be proved directly from (21), but this is unnecessary if we realize that the right side of (21a) and (4b) are equal. Since (4b) is independent of the geometry of V_ε the same must be true of (21a).

In Section IV.1 the dyadic $\bar{\bar{L}}$ is proven to be a symmetric dyadic with unity trace and is evaluated for a number of different principal volumes. It is found that the different electric dyadic Green's functions derived by a number of authors [3a,4,6,8,10] emerge merely through the proper choice for the geometry of the principal volume. Except for Wilcox [8] and Van Bladel [3a], however, the electric dyadic Green's function was derived applying results from delta-function techniques, for which no mention of the geometry of the principal volume was made or appeared necessary. This apparent paradox introduces the next section, where it is shown that a basic assumption used in the ordinary application of delta-function techniques fails at the source points of the electric dyadic Green's function. Fortunately, a simple procedure is also found to remedy this failure.

II. THE INADAQUACIES OF DELTA-FUNCTION METHODS

The delta-function method alone can be applied to derive an expression for the magnetic or electric field from Maxwell's equations in terms of the source current $\overline{J}$. We shall briefly outline and discuss the shortcomings of this approach for electric dyadic Green's functions. But before doing so, a very common approach to the problem, which combines both an integral formula method and delta functions, will be discussed.

This latter approach begins with the dyadic Green's theorem,

$$\int_V [(\nabla\times\nabla\times\overline{P})\cdot\overline{\overline{Q}} - \overline{P}\cdot(\nabla\times\nabla\times\overline{\overline{Q}})]dV = \int_S \hat{n} \cdot [\overline{P} \times (\nabla\times\overline{\overline{Q}}) + (\nabla\times\overline{P}) \times \overline{\overline{Q}}]dS \ . \tag{23}$$

The vector $\overline{P}$ is then set equal to the electric field $\overline{E}$, which satisfies (2b), and the dyadic $\overline{\overline{Q}}$ is set equal to $\overline{\overline{G}}_e$, which satisfies the dyadic wave equation

$$\nabla\times\nabla\times\overline{\overline{G}}_e - k^2\overline{\overline{G}}_e = \delta(\overline{r}'-\overline{r})\overline{\overline{I}} \ . \tag{24}$$

In order to compare the results to those of Section I, assume the free-space problem; and thus $\overline{E}$ and $\overline{\overline{G}}_e$ will also satisfy the radiation condition at infinity. After some algebra, theorem (23) yields the familiar equation

$$\overline{E}(\overline{r}) = i\omega\mu_o \int_{V_J} \overline{\overline{G}}_e \cdot \overline{J} \, dV' \ . \tag{25}$$

(The surface integrals outside all sources vanish because of the radiation condition.) To solve for $\overline{E}$, given $\overline{J}$, (25) indicates we need only solve for $\overline{\overline{G}}_e$ in (24) under the condition of outward radiation at infinity. There appears no need for the mention of principal volumes.

The discrepancy between (25) and (21a) at source points is obvious. The answer to what has gone wrong lies in the use of Green's theorem (23). This identity requires that the vector $\overline{P}$ and dyadic $\overline{\overline{Q}}$ be continuous through their second derivatives. Sometimes this restriction is a mere formality, but in this case the singularity in $\overline{\overline{G}}_e$ is serious enough to render (23) invalid if $\overline{\overline{Q}}$ is set equal to $\overline{\overline{G}}_e$. In short, the left and right sides of (23) are not equal (if the singularity of $\overline{\overline{G}}_e$ is within the volume V), and the resultant equation (25) becomes invalid at the source points.

The derivation from theorem (23) can be made valid by excluding the singularity of $\overline{\overline{G}}_e$ with a small principal volume. Then instead of (25) emerging, the correct equation (21a) results (assuming the singularity of $\overline{\overline{G}}_e$ is determined or known a priori), eliminating any discrepancies between the two methods of derivations.

Next we consider the linear operator approach of delta-function theory to derive the Green's function [13]. This method does not explicitly depend upon Green's theorem (23) nor any similar integral formula. It begins with writing the equation for the electric field in operator form,

$$D(\overline{E}) = i\omega\mu_o \overline{J} \; , \tag{26}$$

where D is a linear differential operator (specifically $\nabla\times\nabla\times - k^2$). The inverse of (26), assuming it exists, can be taken and the following linear equation for $\overline{E}$ results:

$$\overline{E} = i\omega\mu_o D^{-1}[\overline{J}] \; . \tag{27}$$

Expressing $\overline{J}(\overline{r})$ in the dyadic delta function form,

$$\overline{J}(\overline{r}) = \int_{V_J} \delta(\overline{r}-\overline{r}') \; \overline{\overline{I}}\cdot\overline{J}(\overline{r}')dV' \; , \tag{28}$$

allows (27) to be written as

$$\overline{E}(\overline{r}) = i\omega\mu_o D^{-1} \int_{V_J} \delta(\overline{r}-\overline{r}')\ \overline{\overline{I}}\cdot\overline{J}(\overline{r}')dV' \quad . \tag{29}$$

<u>IF</u> we can bring the inverse linear operator inside the integral, (29) becomes

$$\overline{E}(\overline{r}) = \int_{V_J} D^{-1}\ [\delta(\overline{r}-\overline{r}')\overline{\overline{I}}]\cdot\overline{J}(\overline{r}')dV' \quad . \tag{30}$$

As usual, $D^{-1}\ [\delta(\overline{r}-\overline{r}')\overline{\overline{I}}]$ is denoted by $\overline{\overline{G}}_e$ and from (26) $\overline{\overline{G}}_e$ satisfies the equation

$$D(\overline{\overline{G}}_e) = \delta(\overline{r}-\overline{r}')\overline{\overline{I}} \ , \tag{31a}$$

or

$$\nabla\times\nabla\times\overline{\overline{G}}_e - k^2\overline{\overline{G}}_e = \delta(\overline{r}-\overline{r}')\overline{\overline{I}} \quad . \tag{31b}$$

As one can see, the same problem of the discrepancy between the equation (30) derived purely from the delta-function method, and the previous result (21a) has reared a second time. Once again the explanation of this discrepancy involves the singularity of the Green's function. For the delta-function derivation just outlined, this singularity interferes with the interchange of linear operators in a subtle yet crucial way. Specifically, in going from (29) to (30) the volume integral and linear operator D^{-1} were interchanged with respect to their order of operation. For most linear operators in circuit theory, for example, this interchange is perfectly valid. Also, for the linear operators associated with the magnetic field, or with the electric field away from sources, this interchange is perfectly valid. But for the electric field linear operator at source points, this interchange becomes invalid, i.e.,

$$D^{-1} \int_{V_J} \neq \int_{V_J} D^{-1} \quad . \tag{32}$$

An easy way to prove (32) is to take the correct inverse linear operator given in (21a) and demonstrate that the left and right sides of (32) are indeed unequal, the right side dependent upon the geometry of the principal volume, the left side not.

In summary, the interchange of linear operators basic to the ordinary delta-function method for solving differential equations becomes invalid at the source points of the electric dyadic Green's function, <u>and this is the fundamental reason for the discrepancies between the various electric dyadic Green's functions found in the literature</u>.

If the delta-function method is used in spite of this shortcoming, it often yields the electric dyadic Green's function corresponding to a particular principal volume, depending on what eigenfunctions are used and how they are manipulated. Unfortunately, the delta-function method does not reveal to what principal volume, if any, the resulting electric dyadic Green's function corresponds. That it often (not always) corresponds to a particular principal volume was an observation made only after evaluating $\bar{\bar{L}}$ in (21b) for a number of different geometries (see Section IV.2). In those cases where the electric dyadic Green's function derived by the delta-function method cannot be associated with a particular principal volume, the method has yielded bogus results which will give erroneous fields in the source region regardless of the principal volume chosen.

Fortunately, in all cases an extremely simple way, explained below, exists to remedy the delta-function method to include the proper effect of the principal volume.

But first, more discussion on the interchange of linear operators involved with the electric and magnetic Green's functions seems appropriate at this point. Consider equations (21). The electric dyadic Green's function can be written symbolically for all $\overline{r}'$ as

$$\overline{\overline{G}}_e^o = \frac{1}{k^2} [\nabla \times \overline{\overline{G}}_m^o - \delta(\overline{r}'-\overline{r})\overline{\overline{L}}] \ . \qquad (33)$$

It is emphasized that (33) has meaning only in the sense of (21a), since the first dyadic on the right side of (33) is not defined at $\overline{r}' = \overline{r}$ and the second dyadic is nonzero only at $\overline{r}' = \overline{r}$. For example, the usual delta-function result,

$$\overline{\overline{G}}_e = \frac{1}{k^2} [\nabla \times \overline{\overline{G}}_m^o - \delta(\overline{r}'-\overline{r})\overline{\overline{I}}] \ , \qquad (34)$$

is not equivalent to (33). But when (11) and (33) are manifested as (10) and (21), the resulting fields obey Maxwell's equation, provided the principal volume and the interchanges of differentiation and integration are handled properly as explained in Section I and Appendix B. Replacing (33) by (34), i.e., $\overline{\overline{L}}$ by $\overline{\overline{I}}$ in (21a), gives an erroneous electric field at the source points, since it can be shown that $\overline{\overline{L}} \neq \overline{\overline{I}}$ for any principal volume. (A quick way to prove that $\overline{\overline{L}}$ cannot equal $\overline{\overline{I}}$ is to use the results of Section IV.1 that the trace of $\overline{\overline{L}}$ must equal unity.)

Similarly, if the divergence of (33) is taken without regard to its symbolic nature, the result is

$$\nabla \cdot \bar{\bar{G}}_e^o = -\frac{1}{k^2} \nabla\delta(\bar{r}'-\bar{r}) \cdot \bar{\bar{L}} \tag{35}$$

rather than the familiar delta-function result obtained from (24),

$$\nabla \cdot \bar{\bar{G}}_e = -\frac{1}{k^2} \nabla\delta(\bar{r}'-\bar{r}) \ . \tag{36}$$

However, if the divergence of (21a) is taken, and the interchange of $\nabla\cdot$ and the integral is performed properly, the correct divergence expression for $\bar{E}$ is preserved; i.e.,

$$\nabla \cdot \bar{E} = \bar{J}/i\omega\varepsilon_o \ . \tag{37}$$

It should not be concluded from this section that delta-function techniques should be discarded as a way of finding the electric dyadic Green's function. On the contrary, the results of Section I summarized in (21) reveal how delta-function techniques for the electric dyadic Green's function at source points can be salvaged.

Away from the source points the delta-function method gives the proper results and at the source points (21) shows that the conventional electric Green's function requires but a minor modification. In particular, if we find the electric dyadic Green's function outside the sources by the delta-function method (or any other method), at the source points this same function can be utilized if: 1) the geometry (shape, and position and orientation with respect to the source point $\bar{r}$) of the principal volume is designated; and 2) $\bar{\bar{L}}$ for this principal volume is calculated from (21b). Nothing more is required.

Equations (21), on which the above discussion and conclusions were largely based, were derived for the free-space problem. In the following section the derivation and conclusions are shown to extend to problems with boundary conditions on finite surfaces as well.

III. THE SOLUTION IN THE PRESENCE OF BOUNDARIES

Equations (21) show that in the free-space problem the electric field can be found from the conventional electric dyadic Green's function plus a source dyadic $\bar{\bar{L}}$ which depends only on the geometry of the principal volume used in the evaluation of the volume integral. It is the purpose of this section to show that the same is true for problems involving boundary surfaces on which tangential $\bar{E}$ or $\bar{B}$ are zero (or otherwise specified). The excitation of resonant cavities and the scattering from perfectly conducting bodies are typical examples of such problems.

Consider the interior problem shown schematically in Fig. 2, and assume for the moment that tangential $\bar{E}$ or $\bar{B}$ is zero on the boundary. The applied current $\bar{J}$ lies within the volume V bounded by the perfectly electric or magnetic conducting surface S. Excitation at the resonant frequencies of the cavity is avoided. (The corresponding exterior problem, which has no real resonant frequencies, or problems with tangential $\bar{E}$ or $\bar{B}$ specified but nonzero, can be treated by the same analysis as the interior problem away from resonant frequencies, and thus need not be considered separately.)

The electric field within S obeys the vector wave equation

$$\nabla\times\nabla\times\bar{E} - k^2\bar{E} = i\omega\mu_o\bar{J} \, , \tag{38}$$

and, as the surface S is approached, the boundary condition for the tangential fields,

$$\bar{E}_{tan} = 0, \quad \text{or} \quad \bar{B}_{tan} = \frac{1}{i\omega}(\nabla\times\bar{E})_{tan} = 0 \quad . \tag{39a,b}$$

The solution to (38), which has been shown to exist uniquely away from resonant frequencies ([2], Ch. VI), can be divided into a particular solution $\overline{E}_p$ and a homogeneous solution $\overline{E}_h$, with $\overline{E} = \overline{E}^p + \overline{E}^h$. The particular solution is any solution which satisfies the same equation as (38) but without regard to the boundary conditions. The homogeneous solution satisfies the homogeneous vector wave equation

$$\nabla\times\nabla\times\overline{E}_h - k^2\overline{E}_h = 0 , \tag{40}$$

and the boundary condition on S

$$\overline{E}^h_{tan} = -\overline{E}^p_{tan} , \quad \text{or} \quad (\nabla\times\overline{E}^h)_{tan} = -(\nabla\times\overline{E}^p)_{tan} . \tag{41a,b}$$

For the particular solution one may choose the solution given in (21), i.e.,

$$\overline{E}^p(\overline{r}) = i\omega\mu_o \lim_{V_\varepsilon\to 0} \int_{V-V_\varepsilon} \overline{\overline{G}}^o_{eo} \cdot \overline{J}\, dV' + \frac{\overline{\overline{L}}\cdot\overline{J}}{i\omega\varepsilon_o} . \tag{42}$$

(It is a straightforward matter to show that (42) satisfies (38) as well as (2b).)

The solution to the homogeneous part $\overline{E}_h$ can be expressed in terms of a Green's function integrated over the boundary condition at the surface (see e.g., [4d], Sec. 17),

$$\overline{E}_h(\overline{r}) = \int_S \overline{\overline{G}}_S(\overline{r},\overline{r}') \cdot \overline{E}^p_{tan}(\overline{r}')dS' . \tag{43}$$

(The boundary condition (41a) has been chosen arbitrarily. A similar equation and subsequent analysis can be written for the boundary condition (41b).) The surface integration in (43) does not encounter any singularities of the Green's function $\bar{\bar{G}}_S(\bar{r},\bar{r}')$ since $\bar{r}$ remains inside S where $\bar{\bar{G}}_S$ is continuous, and thus no limiting principal values are required.

The field $\bar{E}^p_{tan}$ can be found from (42),

$$\bar{E}^p_{tan}(\bar{r}') = i\omega\mu_o \int_V [\bar{\bar{G}}^o_{eo}(\bar{r}',\bar{r}'')]_{tan} \cdot \bar{J}(\bar{r}'')dV'' . \tag{44}$$

The principal volume part of (42) is not contained in (44) since $\bar{r}''$ never equals $\bar{r}'$ if we assume the sources $\bar{J}$ are <u>within</u> S. Substitution of (44) into (43) gives the homogeneous solution in terms of $\bar{J}$,

$$\bar{E}_h(\bar{r}) = i\omega\mu_o \int_V \bar{\bar{G}}_h(\bar{r},\bar{r}') \cdot \bar{J}(\bar{r}')dS' . \tag{45}$$

The homogeneous dyadic Green's function $\bar{\bar{G}}_h$ is defined as

$$\bar{\bar{G}}_h(\bar{r},\bar{r}') = \int_S \bar{\bar{G}}_S(\bar{r},\bar{r}'') \cdot [\bar{\bar{G}}^o_{eo}(\bar{r}'',\bar{r}')]_{tan}\, dS'' , \tag{46}$$

which, by the standard theorems of advanced calculus ([11], Sec. 4-6), is a continuous function of $\bar{r}$ and $\bar{r}'$ since $\bar{\bar{G}}_S$ and $\bar{\bar{G}}^o_{eo}$ are continuous for $\bar{r}$ and $\bar{r}'$ within V.

When the homogeneous solution (45) and the particular solution (42) are added and $\bar{\bar{G}}_{eo}$ is defined as $\bar{\bar{G}}^o_{eo} + \bar{\bar{G}}_h$, an expression for $\bar{E}$ identical in form to (21a) emerges,

$$\overline{E}(\overline{r}) = i\omega\mu_o \lim_{V_\varepsilon \to 0} \int_{V-V_\varepsilon} \overline{\overline{G}}_{eo} \cdot \overline{J}\, dV' + \frac{\overline{\overline{L}}\cdot\overline{J}}{i\omega\varepsilon_o} \ . \tag{47a}$$

The singularity of $\overline{\overline{G}}_{eo}$ is the same as $\overline{\overline{G}}^o_{eo}$, and $\overline{\overline{L}}$ has the same definition as in (21b),

$$\overline{\overline{L}} \equiv \frac{1}{4\pi} \int_{S_\varepsilon} \frac{\hat{n}'\hat{e}_{R'}}{R'^2}\, dS' \ . \tag{47b}$$

The part of the electric dyadic Green's function denoted by $\overline{\overline{G}}_{eo}$ is given by

$$\overline{\overline{G}}_{eo} = \overline{\overline{G}}^o_{eo} + \overline{\overline{G}}_h, \quad \overline{r}' \neq \overline{r}, \tag{47c}$$

where $\overline{\overline{G}}^o_{eo}$ is the free-space dyadic Green's function (22) and $\overline{\overline{G}}_h$ is the solution to the homogeneous dyadic vector wave equation,

$$\nabla\times\nabla\times\overline{\overline{G}}_h - k^2\overline{\overline{G}}_h = 0 \quad , \tag{48a}$$

under the boundary condition

$$\hat{n}\times\overline{\overline{G}}_h = -\,\hat{n}\times\overline{\overline{G}}^o_{eo} \ , \tag{48b}$$

or

$$\hat{n} \times (\nabla\times\overline{\overline{G}}_h) = -\hat{n} \times (\nabla\times\overline{\overline{G}}^o_{eo}) \quad , \tag{48c}$$

depending upon whether tangential $\overline{E}$ or $\overline{H}$ is zero on the boundary, respectively.

From a delta-function point of view the $\bar{\bar{G}}_{eo}$ required in (47a) is simply the solution to

$$\nabla\times\nabla\times\bar{\bar{G}}_{eo} - k^2\bar{\bar{G}}_{eo} = \delta(\bar{r}-\bar{r}')\bar{\bar{I}} \quad , \tag{49a}$$

outside the singular point $\bar{r} = \bar{r}'$, and satisfying the Dirichlet or Neumann boundary conditions,

$$\hat{n}\times\bar{\bar{G}}_{eo} = 0 \quad \text{or} \quad \hat{n}\times(\nabla\times\bar{\bar{G}}_{eo}) = 0 \ . \tag{49b,c}$$

Equations (47) were derived for Dirichlet or Neumann boundary conditions and an interior region away from resonant frequencies. The same analysis and results apply to exterior regions and to any specified boundary conditions for which a unique solution exists.

The significance of (47a) lies in the fact that just as in the free-space equation (21a) the singularity of the dyadic $\bar{\bar{G}}_{eo}$ is excluded in that volume integration. That is, $\bar{\bar{G}}_{eo}$ need not be specified right at the singularity. However, to use (47a) the geometry of the principal volume must be designated and the extra dyadic $\bar{\bar{L}}$ determined from (47b). This is a small price to pay for the elimination of the problem of extracting the proper delta functions at the singularity (source point) of the electric dyadic Green's function--an effort which we have shown is futile in the sense that the problem cannot be solved uniquely by any means unless the geometry of the principal volume is designated.

Another advantage of not having to deal with the singularity of $\bar{\bar{G}}_{eo}$ in (47a) (except in terms of a principal volume integration) is that $\bar{\bar{G}}_{eo}$ can be expanded in solenoidal wave functions. That is, since $\nabla\cdot\bar{\bar{G}}_{eo} = 0$ for $\bar{r} \neq \bar{r}'$, the $\bar{\bar{G}}_{eo}$ under the volume integral of (47a) involves only solenoidal wave functions. In terms of the familiar $\bar{M}$, $\bar{N}$, and $\bar{L}$ notation for the Hansen vector wave functions found in a number of textbooks (e.g., Stratton [7], Morse & Feshbach [9], and Tai [4d]), this means that the $\bar{L}$ functions (not to be confused with the dyadic $\bar{\bar{L}}$) are not needed. Only the solenoidal eigenfunctions $\bar{M}$ and $\bar{N}$ are required in the expansion for $\bar{\bar{G}}_{eo}$ in (47a). If for a particular problem the electric dyadic Green's function is determined or found tabulated in the literature in terms of $\bar{M}$ and $\bar{N}$ functions alone (e.g., Tai's book [4d]), this is exactly the right $\bar{\bar{G}}_{eo}$ needed in (47a). At source points the expression for the electric field is amended simply by specifying the geometry of the principal volume chosen to evaluate the volume integral and by adding the extra term in (47a) involving the dyadic $\bar{\bar{L}}$. In the symbolic notation of previous authors the complete electric dyadic Green's function $\bar{\bar{G}}_e$ can be expressed as

$$\bar{\bar{G}}_e = \bar{\bar{G}}_{eo} - \frac{1}{k^2}\,\delta(\bar{r}'-\bar{r})\bar{\bar{L}} \quad . \tag{50}$$

As an example, suppose that the electric dyadic Green's function is wanted for a circular waveguide, and that the electric field is to be computed numerically from (47a) using a thin pencil shaped principal volume (centered on the singularity and aligned with the axis of the waveguide) as the patch which excludes the singularity of $\bar{\bar{G}}_{eo}$ (see Fig. 3). From Section 22, equation (5) of Tai's book [4d], $\bar{\bar{G}}_{eo}$ for this problem can be written down immediately,

$$\bar{\bar{G}}_{eo} = \frac{i}{4\pi} \sum_{n=o}^{\infty} \sum_{m=1}^{\infty} (2-\delta_o) \left[\frac{\bar{M}_{\underset{o}{e}n\mu}\, \bar{M}'_{\underset{o}{e}n\mu}}{\mu^2 I_\mu\, k_\mu} + \frac{\bar{N}_{\underset{o}{e}n\lambda}\, \bar{N}'_{\underset{o}{e}n\lambda}}{\lambda^2\, I_\lambda\, k_\lambda} \right] . \tag{51}$$

(The definitions of the functions and parameters in (51) are unimportant for the sake of this discussion but can be found, if desired, by referring to [4d].) The dyadic $\bar{\bar{L}}$ for a pencil-shaped principal volume is tabulated below in Section IV.2,

$$\bar{\bar{L}} = \frac{\bar{\bar{I}}_t}{2} \quad . \tag{52}$$

Equations (51) and (52) substituted into (47a) will then yield, for an applied current $\bar{J}$, the unique electric field everywhere inside the waveguide in the limit as a pencil-shaped "patch" (centered on the singularity and aligned with the z-axis) is shrunk down about the singularity.

In the following section, the source dyadic $\bar{\bar{L}}$ is proven to be a symmetric dyadic with unity trace; then it is determined analytically and tabulated for a number of principal volumes which might be commonly used as the volume "patch" which excludes the singularity in the numerical evaluation of (47a); and finally, a general mathematical and physical interpretation of $\bar{\bar{L}}$ is provided. In Section V, where two-dimensional problems are considered, the source dyadic is evaluated for a number of principal <u>areas</u> as well.

IV. SYMMETRY, EVALUATION, AND INTERPRETATION OF $\bar{\bar{L}}$

It has been shown that the dyadic Green's function equation (47a) for the electric field requires, at source points, the specification of the principal volume V_ε and the corresponding source dyadic $\bar{\bar{L}}$. Away from the source distribution $\bar{J}$, the $\bar{\bar{L}}$ term in (47a) vanishes and the remaining volume integral becomes independent of the geometry of the principal volume since the singularities of $\bar{\bar{G}}_{eo}$ would lie outside the source volume V_J. At the source points, the two terms on the right side of (47a) add to give a unique electric field $\bar{E}(\bar{r})$ independent of the geometry of the principal volume; yet the value of each term separately does depend upon the geometry of V_ε, and thus it must be specified in order to compute $\bar{E}(\bar{r})$ from (47a).

Once the geometry of V_ε (shape, and position and orientation with respect to the excluded singularity) is specified, $\bar{\bar{L}}$ is determined from (47b). The particular principal volume can be chosen arbitrarily according to the preference of the problem solver and the convenience with which it fits into the geometry of the problem and the associated computer programs. This section proves that $\bar{\bar{L}}$ is a symmetric dyadic with unity trace for any principal volume, and then evaluates and tabulates $\bar{\bar{L}}$ specifically for a sphere, right circular cylinder, rectangular box (includes the cube), and pillbox of arbitrary cross-section. Finally a general mathematical and physical interpretation is given for $\bar{\bar{L}}$.

IV.1. Symmetry of $\bar{\bar{L}}$

As a preliminary exercise, it will be proven that the dyadic $\bar{\bar{L}}$ is symmetric for an arbitrary principal volume; and, in addition, the trace of $\bar{\bar{L}}$ always equals unity. In rectangular unit vectors, $\bar{\bar{L}}$ takes the form, after dropping the primes on the variables in (47b),

$$\bar{\bar{L}} = \frac{1}{4\pi}\int_{S_\varepsilon} \frac{1}{R^3}\,[n_x x\hat{e}_x\hat{e}_x + n_x y\hat{e}_x\hat{e}_y + n_x z\hat{e}_x\hat{e}_z + n_y x\hat{e}_y\hat{e}_x + n_y y\hat{e}_y\hat{e}_y + n_y z\hat{e}_y\hat{e}_z + n_z x\hat{e}_z\hat{e}_x + n_z y\hat{e}_z\hat{e}_y + n_z z\hat{e}_z\hat{e}_z]\,dS. \tag{53a}$$

Consider the integral,

$$\int_{S_\varepsilon} \frac{(n_x y - n_y x)}{R^3}\,dS = \hat{e}_z \cdot \int_{S_\varepsilon} \hat{n} \times \frac{\bar{R}}{R^3}\,dS = \hat{e}_z \cdot \int_{S_\varepsilon + S_\infty} \hat{n} \times \frac{\bar{R}}{R^3}\,dS. \tag{53b}$$

The last equality holds because the integral over the surface at infinity is zero. Since there is no singularity of the integrand between S_ε and S_∞, the following vector integral formula applies

$$\int_{S_\varepsilon + S_\infty} \hat{n} \times \frac{\bar{R}}{R^3}\,dS = \int_{V_\infty - V_\varepsilon} \nabla \times \frac{\bar{R}}{R^3}\,dV. \tag{53c}$$

The curl of $\bar{R}/R^3$ is zero, leaving (53c) zero, and from (53b)

$$\int_{S_\varepsilon} \frac{n_x y}{R^3}\,dS = \int_{S_\varepsilon} \frac{n_y x}{R^3}\,dS. \tag{53d}$$

Substitution of (53d) and the corresponding results for $n_x z$ and $n_y z$ into (53a) proves the dyadic $\bar{\bar{L}}$ is indeed symmetric, i.e.,

$$\bar{\bar{L}} = \frac{1}{4\pi}\int_{S_\varepsilon} \frac{\hat{n}\hat{e}_R}{R^2}\,dS = \frac{1}{4\pi}\int_{S_\varepsilon} \frac{\hat{e}_R\hat{n}}{R^2}\,dS = [\bar{\bar{L}}]_{\text{transpose}}. \tag{53e}$$

A convenient way to summarize the proof is to say that the "vector of the dyadic $\bar{\bar{L}}$" has been proven zero and thus the dyadic is symmetric ([14], Sec. 101).

This symmetry displayed by $\bar{\bar{L}}$ holds for principal volumes of arbitrary geometry and is confirmed for the specific principal volumes considered below. The coordinate axes are so positioned and oriented in the following examples that $\bar{\bar{L}}$ takes the form of a diagonal dyadic as well.

Finally, we note from (53a) that the trace of $\bar{\bar{L}}$ given by

$$\text{trace } (\bar{\bar{L}}) = \frac{1}{4\pi} \int_{S_\varepsilon} \frac{\hat{n}\cdot\hat{e}_R}{R^2}\, dS, \tag{54a}$$

is merely the normalized solid angle integration over a closed surface and thus is always equal to unity, i.e.,

$$\text{trace}(\bar{\bar{L}}) = 1. \tag{54b}$$

IV.2. Evaluation of $\bar{\bar{L}}$ for Various Principal Volumes.

1) Sphere

For the sphere with origin (singularity) at the center, $\hat{n}' = \hat{e}_{R'}$, and (47b) becomes, after dropping the primes on the variables,

$$\bar{\bar{L}} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \hat{e}_R\, \hat{e}_R \sin\theta \, d\theta \, d\phi \quad .$$

The angles θ and ϕ are the usual spherical coordinates. Expressing $\hat{e}_R$ in rectangular unit vectors,

$$\hat{e}_R = \hat{e}_x \sin\theta \cos\phi + \hat{e}_y \sin\theta \sin\phi + \hat{e}_z \cos\theta \quad ,$$

allows the integration to be performed. Upon integration all but the diagonal terms vanish, leaving the familiar results [3a,8] for a spherical principal volume,

$$\underset{\text{(sphere)}}{\bar{\bar{L}}} = \frac{1}{3}\left(\hat{e}_x\hat{e}_x + \hat{e}_y\hat{e}_y + \hat{e}_z\hat{e}_z\right) = \frac{\bar{\bar{I}}}{3} \quad . \tag{55}$$

2) Right Circular Cylinder

Consider the right circular cylinder with origin at the center and its axis aligned with the z-direction. The edge of the cylinder makes an angle θ_o with this axis. The unit vectors $\hat{e}_R$ and $\hat{n}$ expressed in a useful combination of cylindrical and spherical coordinates are

$$\hat{e}_R = \hat{e}_\rho \sin\theta + \hat{e}_z \cos\theta$$

$$\hat{n} = \begin{cases} \pm\hat{e}_z & \text{on top and bottom of cylinder} \\ \hat{e}_\rho & \text{on side of cylinder} \end{cases}$$

The element of surface area dS can also be written in terms of R, θ and ϕ,

$$dS = \begin{cases} \dfrac{R^2 \sin\theta}{|\cos\theta|}\, d\theta\, d\phi & \text{on top and bottom} \\ R^2\, d\theta\, d\phi \text{ on side} \end{cases}$$

Substitution of these relations into (47b) allows the integration to be performed after letting $\hat{e}_\rho = \hat{e}_x \cos\phi + \hat{e}_y \sin\phi$. $\bar{\bar{L}}$ for the cylinder emerges as

$$\underset{\text{(right circular cylinder)}}{\bar{\bar{L}}} = (1 - \cos\theta_o)\hat{e}_z\hat{e}_z + \frac{1}{2}\cos\theta_o\, \bar{\bar{I}}_t \quad , \tag{56a}$$

where $\bar{\bar{I}}_t$ is the unit dyadic transverse to the z direction. In rectangular and cylindrical coordinates, e.g., $\bar{\bar{I}}_t = \hat{e}_x\hat{e}_x + \hat{e}_y\hat{e}_y = \hat{e}_\rho\hat{e}_\rho + \hat{e}_\phi\hat{e}_\phi$.

If the circular cylinder is made into a circular pencil-shaped principal volume ($\theta_o = 0$), $\bar{\bar{L}}$ becomes

$$\underset{\text{(circular pencil)}}{\bar{\bar{L}}} = \bar{\bar{I}}_t/2 \quad . \tag{56b}$$

Or if the circular cylinder is flattened to a pillbox ($\theta_o = 90°$),

$$\underset{(\text{circular pillbox})}{\bar{\bar{L}}} = \hat{e}_z \hat{e}_z \quad , \tag{56c}$$

which checks with the results below for a pillbox of arbitrary shape (not necessarily circular).

3) <u>Rectangular Box</u>

Assume that the rectangular box has its origin at the center of an xyz coordinate system and that the edges of the box lie parallel to these axes. Let the length of the sides be denoted by a, b, and c, respectively. Then for the side of the box in the plane z = c/2, we have

$$\hat{n}\,\hat{e}_R = \frac{\hat{e}_z}{R}\,[x\,\hat{e}_x + y\,\hat{e}_y + \frac{c}{2}\,\hat{e}_z] \quad ,$$

and for the opposite side in the plane z = - c/2,

$$\hat{n}\,\hat{e}_R = -\,\frac{\hat{e}_z}{R}\,[x\,\hat{e}_x + y\,\hat{e}_y - \frac{c}{2}\,\hat{e}_z] \quad .$$

The contribution to $\bar{\bar{L}}$ from these two sides is then

$$\frac{1}{4\pi}\,\hat{e}_z\hat{e}_z \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \frac{c}{R^3}\,dS = \frac{\hat{e}_z\hat{e}_z}{4}\,\Omega_z \quad ,$$

where Ω_z is twice the solid angle subtended by the origin at the center of the box and a side of the box perpendicular to the z-direction.

Since the same analysis applies to the x and y directions, we can write down $\bar{\bar{L}}$ immediately as

$$\underset{(\text{rectangular box})}{\bar{\bar{L}}} = \frac{1}{4\pi}\left[\Omega_x \hat{e}_x\hat{e}_x + \Omega_y \hat{e}_y\hat{e}_y + \Omega_z \hat{e}_z\hat{e}_z\right] , \tag{57a}$$

where Ω_x, Ω_y, and Ω_z are the solid angles subtended by the two sides perpendicular to the x, y, and z direction respectively ($\Omega_x + \Omega_y + \Omega_z = 4\pi$). If c is allowed to approach zero, the rectangular box goes to a rectangular pillbox, $\Omega_x = \Omega_y = 0$, $\Omega_z = 4\pi$, and $\bar{\bar{L}}$ becomes

$$\underset{(\text{rectangular pillbox})}{\bar{\bar{L}}} = \hat{e}_z\hat{e}_z , \tag{57b}$$

which checks with (56c) and the general pillbox result below.

If a is set equal to b and both a and b approach zero, then the rectangular box becomes pencil shaped with square cross-section, $\Omega_z = 0$, $\Omega_x = \Omega_y = 2\pi$, and $\bar{\bar{L}}$ becomes

$$\underset{(\text{square pencil})}{\bar{\bar{L}}} = \frac{\bar{\bar{I}}_t}{2} , \tag{57c}$$

which is identical to the circular pencil result of (56b).

If a, b, and c are all set equal to produce a cubical principal volume, $\Omega_x = \Omega_y = \Omega_z = 4\pi/3$, and $\bar{\bar{L}}$ becomes

$$\underset{(\text{cube})}{\bar{\bar{L}}} = \bar{\bar{I}}/3. \tag{57d}$$

Note that the source dyadic $\bar{\bar{L}}$ for a cube and sphere are identical. This could have been predicted from the symmetry in both cases. Symmetry also predicts the fact that $\bar{\bar{L}} = \bar{\bar{I}}/3$ for any orientation of the cube (and, of course, the sphere) as long as the origin (singularity) remains at the center.

4) <u>Pillbox Of Arbitrary Cross-section</u>

Finally consider the pillbox shaped principal volume. By pillbox we mean a disk or flattened right cylinder (not necessarily circular). In computing

$\bar{\bar{L}}$, the contribution from the infinitesimally thin sides of the pillbox is zero. The contribution from the top and bottom is just

$$\bar{\bar{L}}_{\text{(pill box)}} = \frac{2}{4\pi} \int_0^{2\pi} \int_0^{\pi/2} \hat{e}_z \hat{e}_z \sin\theta \, d\theta \, d\phi = \hat{e}_z \hat{e}_z \quad . \tag{58}$$

Regardless of the shape of the pillbox principal volume, as long as the top and bottom approach each other about the singularity (origin) faster than the sides, $\bar{\bar{L}}$ is simply the unit dyad along the axis of the pillbox.

The $\hat{e}_z \hat{e}_z$ dyad in (58) is a very familiar term. It is the same result that Tai [4a], Collin [10], and Rahmat-Samii [6] got for the extra source term using delta-function techniques to find the electric dyadic Green's function in rectangular waveguides. Although no mention was made to principal volumes in their work, it is not surprising that they obtained the results corresponding to a disc-shaped principal volume with axis along the z-direction, since each used an eigenfunction expansion which preferred or "favored" the z-axis ([9], p. 1850).

If a nonfavoring or symmetric eigenfunction expansion is used to solve a rectangular problem, one might expect to get the symmetric result ($\bar{\bar{L}} = \bar{\bar{I}}/3$) displayed by the cubical principal volume. And indeed this proves true in Rahmat-Samii's analysis of the rectangular cavity [6] (assuming he omitted a factor of 1/3 in the second term of his equation (27)). In Tai's analysis of the rectangular cavity [4c], he again favors the z-direction in the eigenfunction expansion and again gets the $\hat{e}_z \hat{e}_z$ result.

All of these comparisons illustrate the important result that the simple process of tabulating the values of $\bar{\bar{L}}$ for different principal volumes allows us to explain many of the differences between the results of previous authors for the electric dyadic Green's function at source points. The results of the integral formula method of Wilcox [8] and Van Bladel [3a], who carefully specify a spherical principal volume, and the results of the delta-function method can be reconciled through the proper choice of principal volume and corresponding $\bar{\bar{L}}$.

The values of $\bar{\bar{L}}$ for the different principal volumes considered above and their correspondence to results of previous authors is summarized in Table I. The values of $\bar{\bar{L}}$ for all of these and many other simple geometries could probably be determined from symmetry considerations alone, but this intriguing topic will not be treated in this paper.

IV.3. A Mathematical and Physical Interpretation of $\bar{\bar{L}}$

Finally, before leaving the subject of the source dyadic $\bar{\bar{L}}$ it is appropriate to give a mathematical and physical interpretation into the character of $\bar{\bar{L}}$. From a mathematical point of view, $\bar{\bar{L}}$ defined by (47b) has the form of a generalized or dyadic solid angle normalized by $1/4\pi$. That is, if a dot product were placed between $\hat{n}'$ and $\hat{e}_{R'}$ in (47b), the integration would be that of a solid angle integrated over a closed surface, resulting in a value of 4π steradians, regardless of the geometry of the principal volume. (We noted this fact above in Eq. (54a) when proving the trace of $\bar{\bar{L}}$ is always unity.) With no dot between $\hat{n}'$ and $\hat{e}_{R'}$, the integration can still be considered to have units of steradians, but the result is the symmetric dyadic $\bar{\bar{L}}$ whose value does depend on the geometry of the principal volume. Still, an appropriate name for $\bar{\bar{L}}$ might be the "normalized dyadic solid angle."

The dyadic $\bar{\bar{L}}$ has an interesting physical interpretation. Suppose one were to measure the electric field at a point within a current distribution by removing an infinitesimally small volume of current and inserting an ideal point probe. The measured field would be that given by (47a) but without the $\bar{\bar{L}}$

term, since the current $\overline{J}$ at this point has been removed. This measured or "local" field would also depend upon the shape of the infinitesimal volume and its relative position and orientation with respect to the point probe, because we have shown that each term separately in (47a) (and thus the first term) displays said dependence. <u>Thus $\overline{\overline{L}}$ determines the perturbation in electric field caused by the hypothetical measurement scheme of removing an infinitesimally small volume of current V_ε of given shape, and position and orientation with respect to an ideal point probe, which measures the electric field</u>.

The perturbation of electric field caused by the removal of an infinitesimal volume of dielectric material is a familiar phenomenon in electrostatics [15]. And, in fact, if the electrostatic field is correctly expressed in terms of polarization $\overline{P}$ within dielectric material, a source dyadic identical to $\overline{\overline{L}}$ appears. A brief derivation begins with the electrostatic field $\overline{E}_o$ given by the equation

$$\nabla \times \nabla \times \overline{E}_o = 0,$$

or

$$\nabla^2 \overline{E}_o = \nabla(\nabla \cdot \overline{E}_o) = \frac{-\nabla(\nabla \cdot \overline{P})}{\varepsilon_o} = \frac{-\nabla \times \nabla \times \overline{P}}{\varepsilon_o} - \frac{\nabla^2 \overline{P}}{\varepsilon_o},$$

or

$$\nabla^2 \left(\overline{E}_o + \frac{\overline{P}}{\varepsilon_o}\right) = -\nabla \times \nabla \times \frac{\overline{P}}{\varepsilon_o}, \tag{59}$$

assuming the free charge is zero. Equation (59) is the same as (2b) with the $-i\omega\overline{P}$ replacing $\overline{J}$ and ω=o. The analysis of Section I applies to (59), giving the following result for $\overline{E}_o$,

$$\overline{E}_o = \frac{1}{4\pi\varepsilon_o} \lim_{V_\varepsilon \to 0} \int_{V_p - V_\varepsilon} \overline{\overline{G}}_o \cdot \overline{P} dV' - \frac{\overline{\overline{L}} \cdot \overline{P}}{\varepsilon_o}, \tag{60a}$$

where

$$\overline{\overline{G}}_o \equiv \nabla\nabla \left(\frac{1}{|\overline{r}-\overline{r}'|}\right), \tag{60b}$$

and $\overline{\overline{L}}$ is the same dyadic defined above for the time harmonic fields.

The quantity $\overline{E}_o + \frac{\overline{\overline{L}} \cdot \overline{P}}{\varepsilon_o}$ is a generalization of what is sometimes referred to as the "local field" in electrostatics [15]. For a spherical principal volume with the probe at the center, $\overline{\overline{L}} = \overline{\overline{I}}/3$ and the familiar local field $\overline{E}_o + \overline{P}/3\varepsilon_o$ within a dielectric emerges. For time harmonic fields the analogous local field is given by $\overline{E} - \frac{\overline{\overline{L}} \cdot \overline{J}}{i\omega\varepsilon_o}$, which also becomes $\overline{E} + \frac{\overline{\overline{L}} \cdot \overline{P}}{\varepsilon_o}$ if equivalent polarization current $-i\omega\overline{P}$ replaces the free current $\overline{J}$.

In the following section, two-dimensional problems are considered and the source dyadic for different principal areas are evaluated.

V. TWO-DIMENSIONAL PROBLEMS

So far only three-dimensional (3-D) problems have been considered, i.e., where the current sources $\bar{J}$ were contained in a volume of finite extent. Physically, of course, these are the only problems one encounters. But from a mathematical point of view it is often convenient to model certain problems by a two-dimensional (2-D) geometry where the sources extend to ± infinity in a certain direction (here chosen to be the z-direction).

The analysis is completely analogous to the 3-D case. It begins with Maxwell's equations (1) to derive the Helmholtz equations (2). In free-space a procedure similar to that given in Appendix A, but using ordinary Bessel functions instead of spherical Bessel functions, proves that (2) are equivalent to (1), and thus the solution to (2) can be concentrated on without further reference to (1). But just as in the 3-D case, if boundary surfaces are present on which tangential electric or magnetic fields are given, (2) must include (3) as separate conditions to be equivalent to Maxwell's equations (1).

Since $\bar{J}$ is a constant function of z for 2-D problems (although $\bar{J}$ can still have a z-component), the z part of the volume integration in (4) can be performed. Specifically,

$$\int_{-\infty}^{\infty} \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} dz' = \pi i\, H_o^{(1)}(k|\bar{t}-\bar{t}'|)\ , \quad \bar{t} \neq \bar{t}', \tag{61}$$

$$\bar{r} = \bar{t} + z\,\hat{e}_z$$

$$\bar{r}' = \bar{t}' + z'\hat{e}_z$$

where $H_o^{(1)}$ is the Hankel function of the first kind. Using (61) in (4) transforms the 3-D solution (4) into the 2-D solution,[3]

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[3]Even though (61) diverges at $\bar{t} = \bar{t}'$, it can be shown that it is permissible to elongate the principal volume in (4) to ± infinity in the z-direction. Thus, $\bar{t}$ never has to equal $\bar{t}'$ and the principal volume reduces to a principal area.

$$\bar{B}(\bar{t}) = \frac{\mu_o i}{4} \lim_{A_\varepsilon \to 0} \int_{A_J - A_\varepsilon} (\nabla_t' \times \bar{J}(\bar{t}'))\, H_o^{(1)}(k|\bar{t}-\bar{t}'|)\,dA' \tag{62a}$$

$$\bar{E}(\bar{t}) = -\frac{1}{4\omega\varepsilon_o} \lim_{A_\varepsilon \to 0} \int_{A_J - A_\varepsilon} (\nabla_t' \times \nabla_t' \times \bar{J}) H_o^{(1)}(k|\bar{t}-\bar{t}'|)\,dA' \quad . \tag{62b}$$

In (62) the singularity of $H_o^{(1)}$ is logarithmic, and the principal volume converts to a principal area which excludes this singularity. The area integral extends over a large enough area A_J to cover all 2-D source current $\bar{J}$. The logarithmic singularity is weak enough so that the value of the integrals in (62) do not depend upon the geometry of the principal area. But, as in the 3-D case, the limit process is retained explicitly to enable the desired differentiation and integral transformations to be performed correctly on (62). That (62) is indeed the free-space 2-D solution to (2) can be verified in a manner directly analogous to the procedure used in Appendix B.

Equation (62a) can be transformed once and (62b) twice using the 2-D equivalents of the integral transformations performed in Section I. Such a procedure yields the following 2-D equations analogous to (10) and (21):

$$\bar{B}(\bar{t}) = \mu_o \lim_{A_\varepsilon \to 0} \int_{A_J - A_\varepsilon} \bar{\bar{g}}_m^o \cdot \bar{J}\, dA' \tag{63a}$$

$$\bar{E}(\bar{t}) = i\omega\mu_o \lim_{A_\varepsilon \to 0} \int_{A_J - A_\varepsilon} \bar{\bar{g}}_{eo}^o \cdot \bar{J}\, dA' + \frac{\bar{\bar{\ell}} \cdot \bar{J}}{i\omega\varepsilon_o} \tag{63b}$$

$$\bar{\bar{\ell}} \equiv \frac{1}{2\pi} \int_{C_\varepsilon} \frac{\hat{n}'\hat{e}_{T'}}{T'}\, dc' \quad . \tag{63c}$$

The magnetic integral (63a) is again independent of the geometry of the principal area and the 2-D magnetic dyadic Green's function is

$$\bar{\bar{g}}_m^o = \frac{i}{4} \nabla_t' \times (H_o^{(1)} \bar{\bar{I}}_t) \quad . \tag{64}$$

The electric field equation (63b) again has two parts, the conventional integral part and the source dyadic term. The value of each of these depends on the geometry (shape, and position and orientation with respect to $\bar{t}$) of the principal area A_ε, but the sum does not, and produces a unique electric field $\bar{E}(\bar{t})$. The conventional 2-D electric dyadic Green's function can be written

$$\bar{\bar{g}}_{eo}^o = -\frac{1}{k^2} \nabla_t' \times \bar{\bar{g}}_m^o = \frac{-i}{4k^2} \nabla_t \times \nabla_t \times (H_o^{(1)} \bar{\bar{I}}_t) = \frac{-i}{4}(\bar{\bar{I}}_t + \frac{1}{k^2} \nabla_t \nabla_t) H_o^{(1)} \quad . \tag{65}$$

The extra 2-D source dyadic term $\bar{\bar{\ell}}$ defined by (63c) is the obvious analogue of the three-dimensional dyadic $\bar{\bar{L}}$ defined in (21b) or (47b). The line integral in (63c) is performed over the curve C_ε bounding the principal area A_ε. The outward normal to this curve is denoted by $\hat{n}'$, and $\hat{e}_{T'}$ is the unit vector from the singularity point within A_ε to the integrating points on the curve C_ε (see Fig. 4). Just like $\bar{\bar{L}}$, the 2-D dyadic $\bar{\bar{\ell}}$ can be shown to be a symmetric dyadic with unity trace.

If a dot product were placed between the $\hat{n}'$ and the $\hat{e}_{T'}$ in (63c) the result of the integral would merely be the 2π angle swept out by the closed curve C_ε. Whereas $\bar{\bar{L}}$ could be described as a "normalized dyadic solid angle," $\bar{\bar{\ell}}$ can be described even more simply as a "normalized dyadic angle." The analogous physical interpretation of $\bar{\bar{\ell}}$ is that it determines the perturbation in electric field caused by the hypothetical scheme of removing an infinitesimally thin current volume (extending from $z = +\infty$ to $-\infty$) with cross-sectional area A_ε of given shape, and position and orientation with respect to an ideal point probe, which measures the electric field. The equations (63) derived for free-space can be generalized to include boundaries as was done in Section III for 3-D problems.

In fact the 2-D results are so closely related and analogous to the 3-D results that it becomes unnecessary to discuss the 2-D problems in great detail. It suffices to say that like the 3-D case the 2-D electric dyadic Green's function requires the specification of a principal area, and can be conveniently expressed by the conventional solenoidal Green's dyadic plus an extra source dyadic $\bar{\bar{\ell}}$ which simply and unambiguously provides the correct fields at the source points. The normalized dyadic angle $\bar{\bar{\ell}}$ has been determined from (63c) and tabulated in Table II for some common principal areas or 2-D patches which might be used, for example, in a computer evaluation of (63b).

VI. SUMMARY

The intent of this paper was motivated by the conflicting results which exist in the literature for the electric dyadic Green's function and the fields computed from it in the source region. Different authors have obtained different results for the same problems depending on the method of solution which was used. Yet fundamental theorems of EM theory assures the existence of a unique solution for these problems even at source points. Moreover, the disagreement is much more fundamental than the error addressed and corrected by Tai [4a] of using only solenoidal wave functions in the eigenfunction expansion of the dyadic delta function.

The various methods of solution reduce to just two basically different approaches, the integral formula approach and the delta-function approach. The two different approaches or methods of solution often yield different results at the source points of the electric dyadic Green's function. The concept of principal volume was introduced, and it was shown that the fundamental difference between the two basic methods involved the specification of this principal volume. The integral formula method requires the explicit specification of the geometry of the principal volume while the delta-function method does not.

To shed light on this discrepancy, a simple, straightforward method of solution, which began with the known solution to the scalar wave equation, was used to obtain the correct electric dyadic Green's function and associated field for free-space inside and outside the source region. (Later the derivation was extended to problems with boundary conditions on finite surfaces as well.) It was found that this electric dyadic Green's function actually required the specification of two dyadics, the conventional dyadic outside the source point and a symmetric, unity trace, source dyadic $\bar{\bar{L}}$ which depended solely on the geometry of the principal volume. (However, the electric field computed from these two dyadics remains independent of the geometry of the principal volume - as it must.) Thus, the results of the integral formula method were confirmed, and the delta-function method was investigated to determine the reason for its apparent failure to reveal the principal volume and thus give unique fields at the source points.

The problem with applying the ordinary delta-function techniques to obtain the electric dyadic Green's functions was traced to a very basic assumption in the theory that the order of two linear operators could be interchanged. This interchange is valid for most linear circuit problems, for the magnetic dyadic Green's function, and for the electric dyadic Green's function away from source points. But at the source point of the electric dyadic Green's function the interchange becomes invalid; i.e., the expressions before and after the interchange are not equal, one depending on the geometry of the principal volume, the other not.

If these shortcomings of ordinary delta-function techniques are ignored, the result often (not always) coincides with that of a particular principal volume, depending on the eigenfunctions which are used to expand the electric dyadic Green's function. However, the particular principal volume must be identified explicitly if the Green's function is to be used to compute the unique electric field from applied current sources. And the ordinary delta-function techniques do not identify the principal volume to which its solution corresponds.

Fortunately, it was shown that the delta-function techniques could be remedied in a very simple way to assure the proper electric field at source points as well. Once the electric dyadic Green's function is found outside the source point (by the delta-function method or by any other method), then the proper modification at the source point can be made by

1) designating the geometry (shape, and position and orientation with respect to the source point) of the principal volume and

2) computing the symmetric source dyadic $\bar{\bar{L}}$ (from (47b)) corresponding to this principal volume.

Nothing more is required.

Moreover, since the conventional dyadic Green's function is required only outside the singularity (source point) when using the modified delta-function method just described, only the solenoidal vector wave functions ($\bar{M}$ and $\bar{N}$) are required in our eigenfunction expression of the Green's function. The need and contribution at source points of the nonsolenoidal vector wave functions ($\bar{L}$) is completely replaced by the contribution of the source dyadic $\bar{\bar{L}}$, which is determined from the geometry of the principal volume alone.

The consequences of this result are significant. If we can find the electric dyadic Green's function valid outside the source region tabulated in the literature (e.g., in Tai's book [4d]), then the proper modification at source points is made by simply specifying the principal volume (or "patch" used in the computer algorithm) and evaluating the corresponding dyadic $\bar{\bar{L}}$. A specific example for circular waveguide is given in Section III.

The value of $\bar{\bar{L}}$ was determined and tabulated for a number of common principal volumes or "patches," including the sphere, circular cylinder, rectangular box, and pillbox. The results of many previous authors for the electric dyadic Green's function were shown to correspond to different principal volumes (see Table I). That is, the apparent discrepancies between the results obtained by a number of authors using different methods of solution for the electric dyadic Green's function at source points are explained and reconciled merely through the proper choice of principal volume and evaluation of the corresponding source dyadic $\bar{\bar{L}}$.

An appropriate mathematical interpretation of $\bar{\bar{L}}$ was found to be that of a "normalized dyadic solid angle." An interesting physical interpretation of $\bar{\bar{L}}$ stems from the fact that it determines the perturbation in electric field caused by the hypothetical measurement scheme of removing an infinitesimal volume of current of given shape, and position and orientation with respect to an ideal point probe, which measures the electic field. It was shown further that the same source dyadic $\bar{\bar{L}}$ can be used to express the local electrostatic field within a dielectric.

Finally, in Section V, two-dimensional problems were considered and an analysis similar to the three-dimensional case was performed. The 2-D results were completely analogous to the 3-D results, with cylindrical Hankel functions replacing the spherical waves, and principal areas merely replacing principal volumes.

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APPENDIX A

EQUIVALENCE OF MAXWELL'S EQUATIONS AND THE HELMHOLTZ EQUATIONS FOR $\overline{E}$ AND $\overline{B}$

The vector Helmholtz equations (2) of the main text were derived in the usual manner from Maxwell's equations (1). Here we shall derive (1) starting with (2) in free-space, and prove in a very simple fashion, the often assumed result that the wave equations (2) and Maxwell's equations (1) are equivalent in the free-space region. In addition, it will be shown that the result extends to regions with boundary surfaces only if the fields in (2) are made to obey the supplementary divergence conditions (3).

Begin by taking the divergence of (2a) and realizing that the $\nabla\cdot$ and ∇^2 operators commute, to give

$$\nabla^2(\nabla\cdot\overline{B}) + k^2(\nabla\cdot\overline{B}) = 0 \quad . \tag{A1}$$

According to Theorem 13 of Müller [2], the twice continuously differentiable solution to the differential equation (A1) can be expanded in a complete set of spherical harmonics $Y_{\ell m}(\theta,\phi)$ and spherical Bessel functions $j_\ell(kr)$,

$$\nabla\cdot\overline{B} = \sum_{\ell,m} C_{\ell m}\, Y_{\ell m}(\theta,\phi) j_\ell(kr) \quad , \tag{A2}$$

where the $C_{\ell,m}$ are coefficients which do not depend upon r, θ, or ϕ.

In free-space $\overline{B}$ and thus $\nabla\cdot\overline{B}$ must satisfy the radiation condition as $r \to \infty$, i.e.,

$$\nabla\cdot\overline{B} \underset{r\to\infty}{\sim} C(\theta,\phi)e^{ikr}/r \text{ or faster.} \tag{A3}$$

As $r \to \infty$, the spherical Bessel functions behave as sin kr/r or cos kr/r . And since the $Y_{\ell m}(\theta,\phi)$ are orthogonal functions, the only way (A2) can equal (A3) as

$r \to \infty$ is for all the $C_{\ell m}$ and thus $\nabla\cdot\overline{B}$ to be zero. That is, the radiation condition (A3) in the free-space problem has obviated the need to specify

$$\nabla\cdot\overline{B} = 0 \quad , \tag{A4a}$$

as a separate condition to the Helmholtz equation (2a). In a similar fashion (2b) and the radiation condition on $\overline{E}$ insures

$$\nabla \cdot [\overline{E} - \frac{\overline{J}}{i\omega\varepsilon_o}] = 0 \quad . \tag{A4b}$$

Still it remains to prove that (2) imply (1). To do this, rewrite (2), utilizing the zero divergences (A4), in vector wave equation form

$$\nabla\times\nabla\times\overline{B} - k^2\,\overline{B} = \mu_o\,\nabla\times\overline{J} \tag{A5a}$$

$$\nabla\times\nabla\times\overline{E} - k^2\,\overline{E} = i\omega\mu_o\,\overline{J} \quad . \tag{A5b}$$

Take $\nabla\times$ the second equation and equate the result to $\nabla\times\overline{J}$ in the first equation. After some rearranging and use of (A4a) we find

$$\nabla^2[\nabla\times\overline{E} - i\omega\overline{B}] + k^2[\nabla\times\overline{E} - i\omega\overline{B}] = 0 \quad . \tag{A6}$$

Because $\overline{E}$ and $\overline{B}$ must satisfy the radiation condition, so also must $\nabla\times\overline{E} - i\omega\overline{B}$ decay as fast or faster than the radiation condition demands. Thus, the same analysis we used to show from (A1) and (A3) that $\nabla\cdot\overline{B} = 0$ implies, from (A6) and the radiation condition, that

$$\nabla\times\overline{E} - i\omega\overline{B} = 0 \quad . \tag{A7a}$$

Substitution of (A7a) into (A5b) gives the second Maxwell equation

$$\nabla\times\overline{B} + i\omega\varepsilon_o\mu_o\overline{E} - \mu_o\overline{J} = 0 \quad . \tag{A7b}$$

Consequently, in the free-space problem where $\overline{E}$ and $\overline{B}$ satisfy the radiation condition, we have proved that (2) and (1) of the main text are equivalent sets of equations.

In regions where there are boundary surfaces on which tangential $\overline{E}$ or $\overline{B}$ are specified, the same proof of equivalence fails to apply. The main reason for the failure is that (A1) no longer implies that $\nabla\cdot\overline{B} = 0$ if boundary conditions other than the radiation condition are present. Thus the divergence conditions (A4) (eqs. (3) of the main text) on the fields must be included as separate requirements supplementing (2).

If the solutions to (2) are made to satisfy the divergence conditions (3), (A5) are again implied by (2) and the following equations similar to (A6) can be obtained:

$$\nabla\times\nabla\times[\nabla\times\overline{E} - i\omega\overline{B}] - k^2[\nabla\times\overline{E} - i\omega\overline{B}] = 0 \qquad \text{(A8a)}$$

$$\nabla\times\nabla\times[\nabla\times\overline{B} + i\omega\varepsilon_o\mu_o\overline{E} - \mu_o\overline{J}] - k^2[\nabla\times\overline{B} + i\omega\varepsilon_o\mu_o\overline{E} - \mu_o\overline{J}] = 0 . \qquad \text{(A8b)}$$

If on the boundary surfaces the tangential fields are zero,

$$\overline{B}_{tan} = \frac{1}{i\omega}(\nabla\times\overline{E})_{tan} = 0 \quad \text{or} \quad \overline{E}_{tan} = \frac{-1}{i\omega\varepsilon_o\mu_o}(\nabla\times\overline{B})_{tan} = 0, \qquad \text{(A9a,b)}$$

then, as a consequence of the theorems in Ch. VII of Müller [2], the only solution to (A8) and (A5) for an exterior region or an interior region away from resonant frequencies is the trivial solution; and once again Maxwell's equations are implied. That is, (2) plus (3) are equivalent to (1) for problems with surfaces on which $\overline{E}_{tan}$ or $\overline{B}_{tan}$ are zero. Also, by a similar proof, the same results can be extended to given tangential boundary conditions other than zero.

APPENDIX B

A DIRECT PROOF FOR THE SOLUTION TO THE WAVE EQUATION

As part of this work on tracing down the inconsistencies among different methods for finding the electric dyadic Green's function at source points, it became clear that a simple, yet sufficiently rigorous proof, by direct substitution was desirable to verify the familiar free-space solution to the wave equation. In short, a straightforward proof is desired to verify that

$$\phi(\bar{r}) = -\frac{1}{4\pi} \lim_{V_\varepsilon \to 0} \int_{V-V_\varepsilon} \frac{f(\bar{r}')}{|\bar{r}'-\bar{r}|} e^{ik|\bar{r}'-\bar{r}|} dV' \tag{B1}$$

is indeed the outgoing wave solution to the wave equation

$$\nabla^2\phi + k^2\phi = f(\bar{r}) \ . \tag{B2}$$

Such a proof leads to an understanding of when and why it is or is not permissible to interchange the order of the differentiations and integrations.

Of course, a rigorous proof that (B1) satisfies (B2) uniquely can be found in Ch. VI of Kellogg [12] for $k = 0$, and in Ch. I of Müller [2] for $k > 0$ and outgoing radiation, but the proofs are indirect and presented from basically a mathematician's point of view, which can be somewhat prohibitive.

Also, it should be mentioned that strictly speaking the integral formula method used in Sections 3.4 and 8.1 of Stratton [7] does not constitute a proof that (B1) is the solution to (B2). Stratton's analysis says that <u>if</u> a solution to (B2) exists, then it can be manipulated using integral formulas to prove that (B1) satisfies (B2). Stratton's analysis is a subtle uniqueness proof, but existence remains to be proven. The books by Kellogg and Müller are testimonies to the necessity of such existence theorems.

As a trivial example to illustrate this point, suppose we assume initially that a positive real number solution exists to the equation $x^2 = -1$. We can then square this equation, find $x = 1$, and erroneously assume that this is the unique solution. The error lies in not first proving the existence of a solution. If we do not prove existence initially, the solution obtained through implication must always be checked by substituting back in the original equation. Exactly the same limitation applies to the analysis in Sections 3.4 and 8.1 of Stratton. It does not constitute a proof that (B1) satisfies (B2).

The usual textbook method of proving (B1) satisfies (B2) consists in substituting (B1) into (B2) bringing the ∇^2 operator through the limit and the integral sign, and letting

$$\begin{aligned} \nabla^2\psi + k^2\psi &= -4\pi\ \delta(\bar{r}'-\bar{r}) \\ \psi &= e^{ik|\bar{r}'-\bar{r}|}/|\bar{r}'-\bar{r}| \ . \end{aligned} \tag{B3}$$

This proof is valid away from the source point, i.e., where $f(\bar{r}) = 0$, but at source points it does not recover the function $f(\bar{r})$ unless the limiting process is carelessly ignored. In other words, the limit excludes the singularity of ψ, and thus the delta function in (B3) is never encountered. Away from the source points no singularities exist, the limit process becomes superfluous, and the ∇^2 operator and the integration can be interchanged rigorously by the standard Leibnitz rule of advanced calculus (e.g., [11], Sec. 4-12) which apply to continuously differential integrands.

Actually, one might think that Leibnitz's rule could be applied at source points as well, since the singularity is excluded by the limit process. Obviously this is not the case since, as explained above, such a procedure does not recover the source function $f(\bar{r})$.

One tempting explanation of the trouble is that the differentiation and limit process cannot be interchanged; i.e., is it true that

$$\nabla^2 \lim_{V_\varepsilon \to 0} \int_{V-V_\varepsilon} = \lim_{V_\varepsilon \to 0} \nabla^2 \int_{V-V_\varepsilon} \quad . \tag{B4}$$

The answer is not obvious since such a theorem is not true in general. However, it can be proven true for the particular type of singular integrals (B1) involved here. (The proof will not be included here because we eventually use an approach which does not require such a theorem.)

Thus, we can write

$$\nabla^2 \phi = -\frac{1}{4\pi} \lim_{V_\varepsilon \to 0} \nabla^2 \int_{V-V_\varepsilon} \frac{f(\bar{r})}{|\bar{r}'-\bar{r}|} e^{ik|\bar{r}'-\bar{r}|} dV' \quad . \tag{B5}$$

Now the essential reason why ∇^2 and the integration cannot be interchanged immediately in (B5) becomes clear. The limits of integration depend upon the variable of differentiation $\bar{r}$. Since V_ε encloses the point $\bar{r}$, with respect to the integration variable $\bar{r}'$ the surface equation for V_ε is a function of $\bar{r}$.

Fortunately, there is a disarmingly simple approach to circumvent this problem. Return to the original equation (B1) and make the change of integrating variable

$$\bar{r}'' = \bar{r}' - \bar{r} \quad , \tag{B6}$$

converting (B1) to

$$\phi(\bar{r}) = -\frac{1}{4\pi} \lim_{V_\varepsilon \to 0} \int_{V-V_\varepsilon} \frac{f(\bar{r}'' + \bar{r})}{r''} e^{ikr''} dV''. \tag{B7}$$

No longer is the equation for the surface of V_ε a function of the position vector $\overline{r}$ (with respect to the new integration variable $\overline{r}''$). That is, the limits of integration in (B7) are not a function of the $\overline{r}$ coordinates. Furthermore, since the value of the integral in (B7) does not depend upon the geometry of the principal volume, if we choose a spherical principal volume and write dV" in spherical coordinates,

$$dV'' = (r'')^2 \sin \theta'' \, d\theta'' \, d\phi'' \, dr'', \tag{B8}$$

the limit process is accomplished inherently, and (B7) becomes

$$\phi(\overline{r}) = -\frac{1}{4\pi} \int_V f(\overline{r}'' + \overline{r}) e^{ikr''} r'' \sin \theta'' \, d\theta'' \, d\phi'' \, dr'' \quad . \tag{B9}$$

Provided the source function f has well defined derivatives, Liebnitz's rule permits (B9) to be differentiated by bringing the differentiation under the integral sign. Specifically,

$$\nabla^2 \phi(\overline{r}) = -\frac{1}{4\pi} \int_V \nabla^2 f(\overline{r}'' + \overline{r}) e^{ikr''} r'' \sin \theta'' \, d\theta'' \, d\phi'' \, dr'' \quad . \tag{B10}$$

If the volume integral in (B10) is returned, remembering that the limiting process must also be reinstated explicitly to exclude the singularity (since integral formulas which will be used to transform (B11) below are valid only if the limit process is retained explicitly), (B10) becomes

$$\nabla^2 \phi(\overline{r}) = -\frac{1}{4\pi} \lim_{V_\varepsilon \to 0} \int_{V-V_\varepsilon} \frac{\nabla''^2 f(\overline{r}'' + \overline{r})}{r''} e^{ikr''} dV'' \quad . \tag{B11}$$

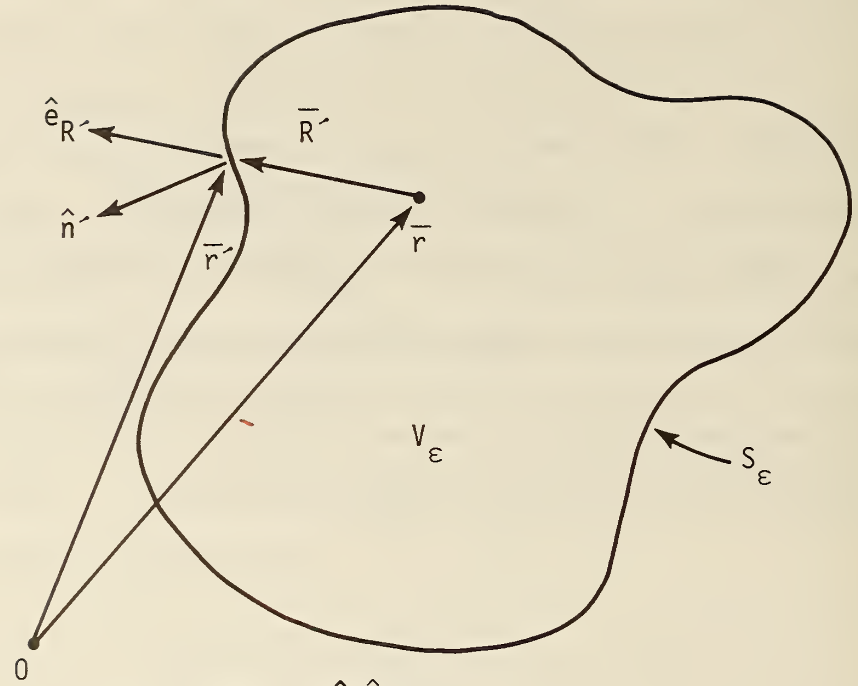
Use has been made of the fact that

$$\nabla f(\overline{r}'' + \overline{r}) = \nabla'' f(\overline{r}'' + \overline{r}) \quad . \tag{B12}$$

The rest of the proof follows exactly the procedure used in Section I of the main text. Equation (B11) is transformed by the divergence theorem twice, carefully retaining the surface integral about the principal volume V_ε, to yield the desired wave equation,

$$\nabla^2\phi = - k^2\phi + f(\bar{r}) \quad . \tag{B13}$$

In summary then, a simple method by direct substitution can be used to prove (B1) satisfies (B2). The method consists in making the change of integration variable (B6), which allows the ∇^2 operator to be taken inside the integral sign with impunity and operate only on the source function f. The resulting integral is transformed twice by the divergence theorem and the solution to the wave equation is verified.



$$\overline{\overline{L}} = \frac{1}{4\pi} \int_{S_\varepsilon} \frac{\hat{n}'\hat{e}_{R'}}{R'^2} \, dS'$$

Figure 1. Notation associated with a principal volume V_ε used to define the dyadic $\overline{\overline{L}}$.

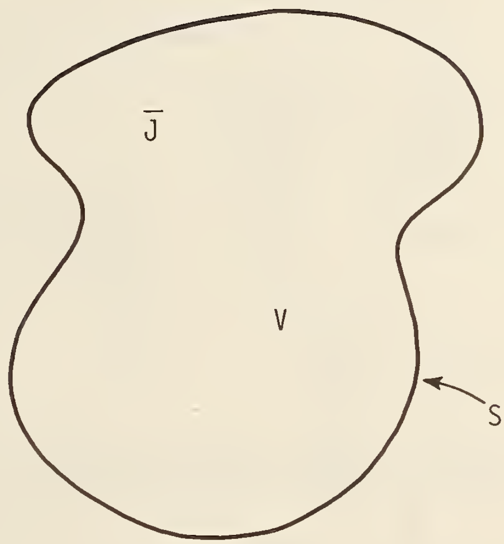


Figure 2. Interior region V with boundary surface S on which $\overline{E}_{tan}$ or $\overline{B}_{tan}$ is specified.

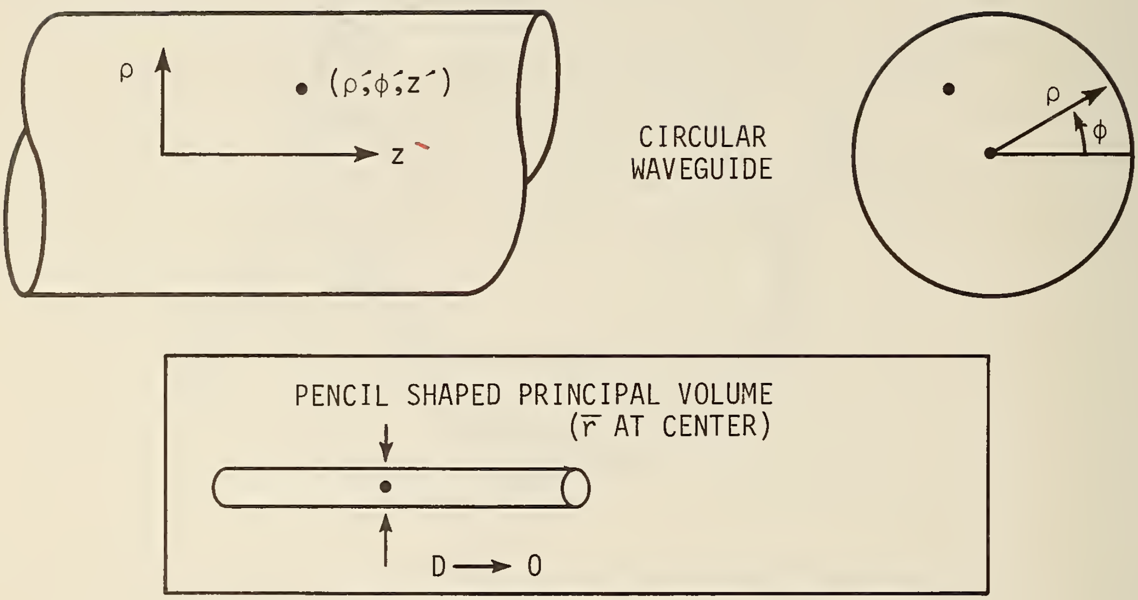
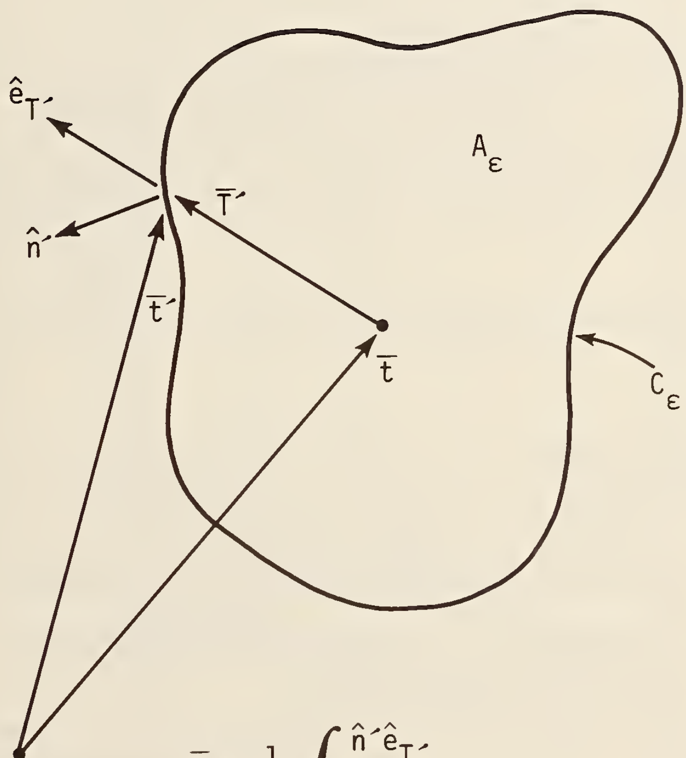


Figure 3. Circular waveguide with pencil-shaped principal volume.



$$\overline{\overline{\ell}} = \frac{1}{2\pi} \int_{C_\varepsilon} \frac{\hat{n}'\hat{e}_{T'}}{T'}\, dc'$$

Figure 4. Notation associated with a principal area A_ε used to define the 2-D dyadic $\overline{\overline{\ell}}$.

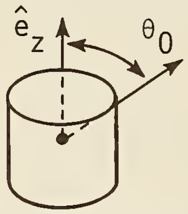
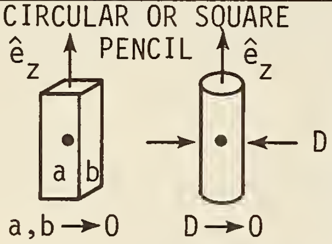
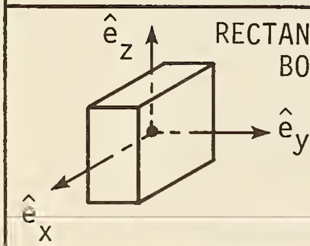
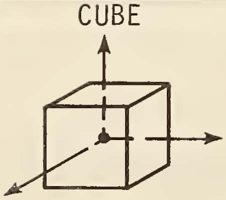
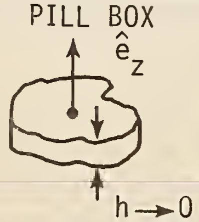
PRINCIPLE VOLUME ($\bar{r}$ AT CENTER)	$\bar{\bar{L}}$	PROBLEM & AUTHOR(S)
SPHERE 	$\frac{\bar{\bar{I}}}{3}$	FREE-SPACE WILCOX (1957) VANBLADEL (1961)
RIGHT CIRCULAR CYLINDER 	$(1-\cos\theta_0)\hat{e}_z\,\hat{e}_z + \tfrac{1}{2}\cos\theta_0\,\bar{\bar{I}}_t$	
CIRCULAR OR SQUARE PENCIL 	$\frac{\bar{\bar{I}}_t}{2}$	
RECTANGULAR BOX 	$\frac{1}{4\pi}(\Omega_x\hat{e}_x\hat{e}_x + \Omega_y\hat{e}_y\hat{e}_y + \Omega_z\hat{e}_z\hat{e}_z)$ $(\Omega_x + \Omega_y + \Omega_z = 4\pi)$ Ω_x, Ω_y, AND Ω_z ARE TWICE THE SOLID ANGLE SUBTENDED BY A SIDE $\perp$ TO THE x, y, and z, DIRECTION RESPECTIVELY.	
CUBE 	$\frac{\bar{\bar{I}}}{3}$	RECTANGULAR CAVITY RAHMAT-SAMII (1975)
PILL BOX 	$\hat{e}_z\,\hat{e}_z$	RECTANGULAR WAVEGUIDE AND CAVITY TAI (1973, 1976) WAVEGUIDE ONLY COLLIN (1973) RAHMAT-SAMII (1975)

Table I. Tabulation of source dyadic $\bar{\bar{L}}$, and correspondence to previous authors.

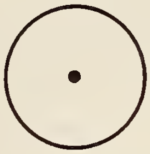
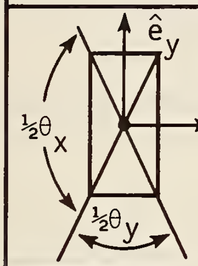
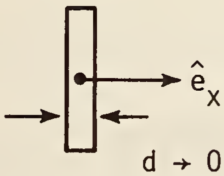
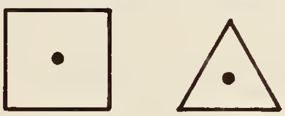
PRINCIPAL AREA ($\bar{t}$ AT CENTER)	$\bar{\bar{\ell}}$
CIRCLE 	$\frac{\bar{\bar{I}}_t}{2}$
 RECTANGLE	$\frac{1}{2\pi}(\theta_x \hat{e}_x \hat{e}_x + \theta_y \hat{e}_y \hat{e}_y)$ $(\theta_x + \theta_y = 2\pi)$
SLIT 	$\hat{e}_x \ \hat{e}_x$
SQUARE AND EQUILATERAL TRIANGLE 	$\frac{\bar{\bar{I}}_t}{2}$

Table II. Tabulation of 2-D source dyadic $\bar{\bar{\ell}}$.

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16. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

A straightforward approach that does not require delta-function techniques is used to derive a generalized electric dyadic Green's function which remains valid within the source region. Although the electric field expressed by the dyadic Green's function proves to be unique, the exact form of the dyadic itself depends, in the source regions, upon the geometry of its "principal volume." The dependence on principal volume is determined explicitly, and the different Green's dyadics derived by a number of previous authors are shown to emerge merely through the appropriate choice for the principal volume. Moreover, delta-function techniques, which by themselves are shown to be inadequate to extract uniquely the proper electric dyadic Green's function in the source region, can be supplemented by a very simple procedure to yield unambiguously the correct Green's function and associated fields.

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