UNIQUENESS OF TRIP-END DISTRIBUTION BY A GRAVITY MODEL

by

A.J. Goldman

Technical Report
to

Northeast Corridor Transportation Project

U.S. DEPARTMENT OF COMMERCE
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BY A GRAVITY MODEL(1)

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ABSTRACT

The "gravity model" approach, for passing from trip-start totals at sources and trip-end totals at sinks to a table of source-to-sink flow volumes, is shown to admit at most one solution. Thus a solution determined by some iterative method has intrinsic significance, independent of the particular calculation procedure used.

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1. INTRODUCTION

We begin with a brutally bare description of one version of the "gravity model" approach often used in analytical transportation studies to estimate the flows between origin-destination pairs. For our purposes it is unnecessary to discuss how this model appears as one of a battery of models, receiving its inputs from some, and providing its outputs to one or more others.

The data for the model consist of two sets of positive numbers, \( S_i \) and \( E_j \), and a table of positive numbers \( C_{ij} \). The intended interpretations are:

- \( S_i = \) number of trips starting at \( i \)-th source,
- \( E_j = \) number of trips ending at \( j \)-th sink,
- \( C_{ij} = \) "conductance" between \( i \)-th source and \( j \)-th sink.

The desired output of the model is a table of positive numbers \( T_{ij} \) satisfying the "accounting" equations

\[
\Sigma_j T_{ij} = S_i \quad (1.1)
\]

\[
\Sigma_i T_{ij} = E_j \quad (1.2)
\]

as well as the "gravity model" condition that

\[
T_{ij} = M_i M_j' C_{ij} \quad (1.3)
\]

for some sets of positive numbers \( M_i \) and \( M_j' \). As would be expected from (1.1) and (1.2), the intended interpretation is

\[
T_{ij} = \text{number of trips from } i\text{-th source to } j\text{-th sink.} \quad (1.4)
\]
It follows from (1.1) and (1.2) that the model will be consistent only if

\[ \Sigma_i S_i = \Sigma_j E_j. \]

We will assume that the problem data do indeed satisfy this condition, a natural one for a "closed system" in view of the interpretations of the \( S_i \) and \( E_j \).

A sufficiently strong obsession with the term "gravity" might lead one to think of \( M_i \) as a "generating mass" for the \( i \)-th source, and \( M'_j \) as an "attracting mass" for the \( j \)-th sink. And it is not uncommon to find \( M_i \) referred to as an "adjusted" or "corrected" version of \( S_i \), with similar language relating \( M'_j \) and \( E_j \). It seems to the writer that such usage, without compensating advantage, creates serious risks of confusion, e.g. suggesting (fallaciously) that \( M_i \) depends only on those problem data pertaining to the \( i \)-th source. The phrase "brutally bare" at the beginning of the paper was chosen, in part, to reflect a refusal to obscure the essentially artificial and arbitrary nature of the \( M_i \) and \( M'_j \).

Application of the model, in practice, involves an iterative procedure which when convergent does indeed yield a solution \( (T_{ij}, M_i, M'_j) \) of equations (1.1) through (1.3). This however raises the question...especially vexing in view of the peculiar roles of the \( M_i \) and \( M'_j \)...of whether some other iterative method (or a different set of initial values in the same iterative process) might not yield a different table of \( T_{ij} \). Unless uniqueness of the \( T_{ij} \) (i.e., independence of the method employed to calculate their values)
can be established, a charge of capriciousness can properly be lodged against making any practical use of...or attaching any physical significance such as (1.4) to...a set of $T_{ij}$-values obtained by some particular iterative method.

In this paper we shall show that the $T_{ij}$ are in fact unique. Despite the nonlinearity of the model (which is why an iterative solution procedure is required), the proof of uniqueness is pleasantly simple, involving only elementary algebra.

Note that the stickier question, of whether any solution necessarily exists, is not addressed. Note also that the assertion of uniqueness for the $T_{ij}$ is not extended to the $M_i$ and $M'_j$. Such an extension would clearly be incorrect, since one can multiply all $M_i$ by some positive factor and divide all $M'_j$ by the same constant without affecting the $T_{ij}$. This "degree of freedom" will be exploited below to simplify the uniqueness proof.
2. PROOF OF UNIQUENESS

The proof begins with the substitution of (1.3) into (1.1) and (1.2) respectively, yielding

\[ M_i = \frac{1}{\Sigma_j \left( \frac{C_{ij}}{S_i} \right)} M'_j, \quad (2.1) \]
\[ M'_j = \frac{1}{\Sigma_i \left( \frac{C_{ij}}{E_j} \right)} M_i. \quad (2.2) \]

Note that (2.1), (2.2) and (1.3) are equivalent to the original model.

From (2.1) and (1.3) we see that the \( T_{ij} \) and \( M_i \) will be uniquely determined once definite values for the \( M'_j \) are known.

Next substitute (2.1) into (2.2), yielding

\[ M'_j = \frac{1}{\Sigma_i \left[ \frac{(C_{ij}}{E_j}) / \Sigma_k \left( \frac{C_{ik}}{S_i} \right) M'_k \right]. \]

With the notation

\[ x_j = M'_j, \quad a_{ijk} = \frac{(C_{ik}}{S_i}) / \left( \frac{C_{ij}}{E_j} \right) > 0, \]

this takes the more palatable form

\[ x_j = \frac{1}{\Sigma_i \left( \frac{1}{\Sigma_k a_{ijk} x_k} \right). \quad (2.3) \]

If \( \bar{x} = (x_1, x_2, \ldots) \) is a positive solution of (2.3), then the same clearly holds for any positive scalar multiple of \( \bar{x} \). The uniqueness proof will clearly be complete if we can prove, conversely, that any two solutions \( \bar{x}, \bar{y} \) of (2.3) are proportional.
Assume, to the contrary, that positive solutions $\mathbf{x}$ and $\mathbf{y}$ are not proportional. The numbering can be chosen so that

$$\frac{x_1}{y_1} \leq \frac{x_k}{y_k} \quad (2.4)$$

for all $k$, and so strict inequality must hold for at least one $k$.

We can use the "degree of freedom" mentioned earlier to normalize $\mathbf{x}$ so that $x_1 = y_1$. This does not affect the numbering in (2.4), so we now have $x_k \geq y_k$ for all $k$, with strict inequality for at least one $k$. Then

$$\Sigma_k a_{ilk} x_k > \Sigma_k a_{ilk} y_k,$$

which implies

$$\frac{1}{\Sigma_i (1 / \Sigma_k a_{ilk} x_k)} > \frac{1}{(\Sigma_i (1 / \Sigma_k a_{ilk} y_k)}.$$

But since $\mathbf{x}$ and $\mathbf{y}$ both satisfy (2.3), the last inequality yields $x_1 > y_1$, contradicting the normalization $x_1 = y_1$. This completes the proof.