NATIONAL BUREAU OF STANDARDS REPORT

9645

UNIQUENESS OF TRIP-END DISTRIBUTION BY A GRAVITY MODEL

by

A.J. Goldman

Technical Report

to

Northeast Corridor Transportation Project

U.S. DEPARTMENT OF COMMERCE

NATIONAL BUREAU OF STANDARDS

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ABSTRACT

The "gravity model" approach, for passing from trip-start totals at sources and trip-end totals at sinks to a table of source-to-sink flow volumes, is shown to admit at most one solution. Thus a solution determined by some iterative method has intrinsic significance, independent of the particular calculation procedure used.

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1. INTRODUCTION

We begin with a brutally bare description of one version of the "gravity model" approach often used in analytical transportation studies to estimate the flows between origin-destination pairs. For our purposes it is unnecessary to discuss how this model appears as one of a battery of models, receiving its inputs from some, and providing its outputs to one or more others.

The data for the model consist of two sets of positive numbers, S_i and E_j, and a table of positive numbers C_{ij}. The intended interpretations are

S_i = number of trips starting at i-th source,

E_j = number of trips ending at j-th sink,

C_{ij} = "conductance" between i-th source and j-th sink.

The desired output of the model is a table of positive numbers T_{ij}, satisfying the "accounting" equations

$$\Sigma_j T_{ij} = S_i \tag{1.1}$$

$$\Sigma_i T_{ij} = E_j \ , \tag{1.2}$$

as well as the "gravity model" condition that

$$T_{ij} = M_i M_j' C_{ij} \tag{1.3}$$

for some sets of positive numbers M_i and M_j' . As would be expected from (1.1) and (1.2), the intended interpretation is

$$T_{ij} = \text{number of trips from i-th source to j-th sink.} \tag{1.4}$$

It follows from (1.1) and (1.2) that the model will be consistent only if

$$\Sigma_i S_i = \Sigma_j E_j .$$

We will assume that the problem data do indeed satisfy this condition, a natural one for a "closed system" in view of the interpretations of the S_i and E_j .

A sufficiently strong obsession with the term "gravity" might lead one to think of M_i as a "generating mass" for the i-th source, and M_j' as an "attracting mass" for the j-th sink. And it is not uncommon to find M_i referred to as an "adjusted" or "corrected" version of S_i, with similar language relating M_j' and E_j . It seems to the writer that such usage, without compensating advantage, creates serious risks of confusion, e.g. suggesting (fallaciously) that M_i depends only on those problem data pertaining to the i-th source. The phrase "brutally bare" at the beginning of the paper was chosen, in part, to reflect a refusal to obscure the essentially artificial and arbitrary nature of the M_i and M_j' .

Application of the model, in practice, involves an iterative procedure which when convergent does indeed yield a solution (T_{ij}, M_i, M_j') of equations (1.1) through (1.3). This however raises the question...especially vexing in view of the peculiar roles of the M_i and M_j'...of whether some <u>other</u> iterative method (or a different set of initial values in the same iterative process) might not yield a different table of T_{ij} . Unless <u>uniqueness</u> of the T_{ij} (i.e., independence of the method employed to calculate their values)

can be established, a charge of capriciousness can properly be lodged against making any practical use of...or attaching any physical significance such as (1.4) to...a set of T_{ij}-values obtained by some particular iterative method.

In this paper we shall show that the T_{ij} are in fact unique. Despite the nonlinearity of the model (which is why an iterative solution procedure is required), the proof of uniqueness is pleasantly simple, involving only elementary algebra.

Note that the stickier question, of whether any solution necessarily exists, is not addressed. Note also that the assertion of uniqueness for the T_{ij} is not extended to the M_i and M_j'. Such an extension would clearly be incorrect, since one can multiply all M_i by some positive factor and divide all M_j' by the same constant without affecting the T_{ij}. This "degree of freedom" will be exploited below to simplify the uniqueness proof.

2. PROOF OF UNIQUENESS

The proof begins with the substitution of (1.3) into (1.1) and (1.2) respectively, yielding

$$M_i = 1/\Sigma_j \left(C_{ij} / S_i \right) M'_j \,, \tag{2.1}$$

$$M'_j = 1/\Sigma_i \left(C_{ij} / E_j \right) M_i \,. \tag{2.2}$$

Note that (2.1), (2.2) and (1.3) are equivalent to the original model. From (2.1) and (1.3) we see that the T_{ij} and M_i will be uniquely determined once definite values for the M'_j are known.

Next substitute (2.1) into (2.2), yielding

$$M'_j = 1/\Sigma_i [(C_{ij} / E_j) / \Sigma_k (C_{ik} / S_i) M'_k] \,.$$

With the notation

$$x_j = M'_j \;, \quad a_{ijk} = (C_{ik} / S_i) / (C_{ij} / E_j) > 0 \,,$$

this takes the more palatable form

$$x_j = 1/\Sigma_i (1 / \Sigma_k a_{ijk} x_k) \,. \tag{2.3}$$

If $\vec{x} = (x_1, x_2, \ldots)$ is a positive solution of (2.3), then the same clearly holds for any positive scalar multiple of $\vec{x}$. The uniqueness proof will clearly be complete if we can prove, conversely, that <u>any</u> two solutions $\vec{x}$, $\vec{y}$ of (2.3) are proportional.

Assume, to the contrary, that positive solutions $\vec{x}$ and $\vec{y}$ are <u>not</u> proportional. The numbering can be chosen so that

$$x_1/y_1 \leq x_k/y_k \tag{2.4}$$

for all k, and so strict inequality must hold for at least one k.

We can use the "degree of freedom" mentioned earlier to normalize $\vec{x}$ so that $x_1 = y_1$. This does not affect the numbering in (2.4), so we now have $x_k \geq y_k$ for all k, with strict inequality for at least one k. Then

$$\Sigma_k \, a_{ilk} x_k > \Sigma_k \, a_{ilk} y_k \, ,$$

which implies

$$1/\Sigma_i (1 \, / \, \Sigma_k \, a_{ilk} x_k) > 1 \, / \, (\Sigma_i (1 \, / \, \Sigma_k \, a_{ilk} y_k) \, .$$

But since $\vec{x}$ and $\vec{y}$ both satisfy (2.3), the last inequality yields $x_1 > y_1$, contradicting the normalization $x_1 = y_1$. This completes the proof.

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